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Carathéodory Sets in the Plane

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Abstract

This work is devoted to the class of sets in the complex plane which nowadays are known as Carathéodory sets, more precisely speaking, as Carathéodory domains and Carathéodory compact sets. These sets naturally arose many times in various research areas in Real, Complex and Functional Analysis and in the Theory of Partial Differential Equations. For instance, the concept of a Carathéodory set plays a significant role in such topical themes as approximation in the complex plane, the theory of conformal mappings, boundary value problems for elliptic partial differential equations, etc. The first appearance of Carathéodory domains in the mathematical literature (of course, without the special name at that moment) was at the beginning of the 20th century, when C. Carathéodory published his famous series of papers about boundary behavior of conformal mappings. The next breakthrough result which was obtained with the essential help of this concept is the Walsh–Lebesgue criterion for uniform approximation of functions by harmonic polynomials on plane compacta (1929). Up to now the studies of Carathéodory domains and Carathéodory compact sets remains a topical field of contemporary analysis and a number of important results were recently obtained in this direction. Among them one ought to mention the results about polyanalytic polynomial approximation, where the class of Carathéodory compact sets was one of the crucial tools, and the results about boundary behavior of conformal mappings from the unit disk onto Carathéodory domains. Our aim in the present memoir is to give a survey on known results related with Carathéodory sets and to present several new results concerning the matter. Starting with the classical works of Carathéodory, Farrell, Walsh, and passing through the history of Complex Analysis of the 20th century, we come to recently obtained results, and to our contribution to the theory.

Keywords. Carathéodory domain, Carathéodory compact set, conformal mapping, uniform approximation, pointwise approximation, L^p-approximation, complex polynomials, complex rational functions, harmonic functions, polyanalytic functions

Mathematics Subject Classification (2020). Primary 30-02; Secondary 30C20, 30E10, 30H99, 54F15

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Contents

Preface

In this work we are dealing with the class of sets in the complex plane that nowadays are called Carathéodory sets. We recall that a bounded domain in the complex plane is called a Carathéodory domain, if its boundary coincides with the boundary of the unbounded connected component of the complement to its closure. A compact set is called a Carathéodory compact set, if its boundary coincides with the boundary of its polynomial convex hull, that is the union of this compact set and all bounded connected components of its complement.

Our aim is to give a survey on results related with the class of Carathéodory sets, as well as to establish some new properties of these sets. The concept of Carathéodory set turned out to be closely related to many other topics in Complex Analysis. Among them it is worth emphasizing certain topics in planar topology, conformal mappings, theory of Hardy and Bergman spaces, orthogonal polynomials, pointwise approximation by polynomials in the complex variable, uniform approximation by holomorphic and harmonic functions, Szegő's theorem and so on. Then, to make the memoir more self-contained, we have included some concepts and results from these related areas. Many of them are certainly known to the specialist in the corresponding areas but not to readers working in other areas of analysis. We have tried to write this survey in such a way that it will be accessible to a wide mathematical audience. Of course we targeted mostly on complex analysts, but we hope that mathematicians working in different areas will find in our survey some interesting topics.

An overlapping between the theory of Carathéodory sets with different branches of real and complex analysis makes it difficult to fix a solid, consistent and reasonable system of notation. Thus, the notation that comes from the theory of conformal maps is different from the corresponding notation coming from the theory of spaces of analytic functions (Hardy spaces, Bergman spaces, etc.), or from the theory of orthogonal polynomials. Even in the research papers in one topic the notation is changing. Through the survey we have tried to use our own unified system of notation which seems to us adequate to the topic under consideration. However, occasionally we have used the same notation as the original sources, in order to help the reader to compare both expositions.

In this survey we have included all results that we know, where the notion of Carathéodory set has some relation, perhaps a very small one. In order to be self-contained we have included complete proofs or their sketched versions for the majority of results, even if they are regarded as classical and well-known. If a result is included in the text without proof (neither completed nor sketched), we not only give a corresponding reference, but also explain the principal ideas underlying its proof. Moreover, a number of specific references are given in order that the interested reader can delve into the study of each of the topics covered. Also we have included several new results, and in such a case the corresponding proofs are provided with all necessary details. Occasionally we have included new variations of background results with the aim of contributing to a better understanding of the theory. Moreover, we have paid attention to constructing, to mentioning or to referring to various examples, so that the geometric behavior of sets under consideration becomes more clear. In this subject there are still many open questions, and we will mention some of them.

We have tried to follow the historical order of exposition in each section, and we have included some historical notes. This can be useful to understand the theory, to compare chronologically the different lines of research and to realize that occasionally someone has made contributions to the theory without knowing previous results closely related to theirs. We believe that a survey of this type will be useful as a first step to clarify and unify the world of Carathéodory sets.

Let us present a brief overview of the history of the investigations of Carathéodory sets. As far as we know, the first Carathéodory sets to be introduced were the Carathéodory domains. At [20, page 136] in 1912 the first nontrivial example of such domains was presented. Later domains of this kind were used at the end of the 1920s in the work [132] by J. Walsh on approximation of functions by harmonic polynomials. However, the special name for this class of domains was not assigned at that time, the name "Carathéodory domains" for this class was given much later. Then, Walsh encouraged his student O. J. Farrell to continue the line of research related to harmonic approximation and properties of conformal maps of planar domains, and Farrell in the 1930s proved several new results about Carathéodory domains (see [44–46] and the corresponding discussion in what follows).

It seems that Farrell's works and results were forgotten for thirty years in the occidental school until the moment of publication of [112] about the pointwise convergence of polynomials in a complex variable. However, it seems, that some ideas about Carathéodory sets were presented in the Russian school of complex analysis during the 1930s and 1940s. A remarkable contribution was made by A. I. Markushevich in his thesis [84], where he considered polynomial approximations in the space of square integrable holomorphic functions in Carathéodory domains. Let us note that Markushevich at that time also did not use the special name for such domains. Perhaps the first (as far as we know) occurrence of the term "Carathéodory domain" in the mathematical literature was in 1939 in the paper [71] by M. V. Keldysh, which also studied polynomial approximation in spaces of square integrable holomorphic functions on compact sets in the complex plane.

With a great deal of certainty, one can assume that the origins of this name are related to the fundamental works [20–22] by C. Carathéodory on conformal maps, which were published in 1912 and 1913 and in which domains such as a cornucopia (see the domain G_1 in Figure 2 below) appeared for the first time in mathematical literature. We will discuss this picture in the chapter on the properties of the conformal maps of Carathéodory domains. It will become clear that the term "Carathéodory domain" is fairly appropriate and adequate.

One ought to make here the following remark. Taking into account the historical circumstances and given the contribution of the aforementioned mathematicians it would be more fair to use the name "Walsh–Farrell–Carathéodory domains" for the class of such domains. But nowadays it makes no sense to adopt this term, because the current terminology is already fixed in the literature. However, it is interesting to have this in mind. In this connection it is worth noticing that neither Farrell in his later paper [48], nor Walsh in his book [134] used any special name for this class of domains.

Later on the class of Carathéodory domains, already with its specific name, may be found, for instance, in the following papers [64, 112, 122, 123] and books [34, 85]. The book [34] is the unique source that we are aware of, where a Carathéodory domain is not assumed to be bounded. The reader interested in the topics on holomorphic and harmonic approximation on unbounded sets may refer, for example, to [5] and [57]. The concept of a K-set defined in [5] may seem interesting concerning our context.

Several topics, where the concept of a Carathéodory domain plays a crucial role were intensively developed in the 1960s–1980s. Let us mention, for instance, the studies of generators for algebras of functions. As was shown in the work [120] by D. Sarason, and in several subsequent papers, the concept of a Carathéodory domain turned out to be closely related to such topics.

Besides, in the 1950s–1960s, the concept of a Carathéodory compact set was introduced and substantially used in a series of works about approximation of functions by rational functions and polynomials in a complex variable. The first occurrence of the term "Carathéodory compact set" itself was, as far as we know, in the work [123] by S. O. Sinanyan who obtained several results about approximation by holomorphic and harmonic polynomials on Carathéodory compact sets in the L^p-norm, $1 \leqslant p < +\infty$. These results of Sinanjan generalize the previous results by Farrell and Markushevich about similar approximation on Carathéodory domains. However, the concept of a Carathéodory compact set was used previously by E. Bishop (see [13–15]) in his studies of measures orthogonal to algebras of polynomials on such compact sets; he called such compact sets "balanced compact sets". For some unclear reasons these important papers of Bishop have been rarely mentioned thereafter, so the name of balanced sets (in both cases of open and compact sets) has been only occasionally used in what follows. Moreover, certain of Bishop's results were rediscovered later by different authors (highly likely completely independently) in similar or slightly different settings.

For the reader's convenience, and mostly for didactic purposes, we will include in the exposition some general results from complex analysis, approximation theory and planar topology. We will use the special symbol ¶ (and write, for instance, Theorem¶)

to highlight the results of the following three kinds: new results about Carathéodory sets which are obtained here for the first time, recent results by the authors concerning the matter, and results which may be regarded as valuable modifications or refinements of certain known results about Carathéodory sets.

This survey was written during several visits of the second author to the Departament de Matemàtiques de la Universitat Autònoma de Barcelona. He is very grateful to the people in this department for the hospitality and excellent scientific atmosphere therein.

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Barcelona–Moscow

Basic notation and definitions

In what follows we will use the standard notation $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$, for sets of complex, real, integer and positive integer numbers, respectively. Moreover, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For two points $z_1, z_2 \in \mathbb{C}$ we denote by $[z_1, z_2]$ the straight line segment from z_1 to z_2. Furthermore, we denote by $D(a, r)$ the open disc with center at some point $a \in \mathbb{C}$ and radius $r > 0$ so that $D(a, r) = \{z : |z - a| < r\}$. We also put $\mathbb{D} := D(0, 1)$ and $\mathbb{T} := \{z : |z| = 1\}$ so that $\mathbb{D}$ and $\mathbb{T}$ are the open unit disk and the unit circle in $\mathbb{C}$, respectively.

We will denote by $\jmath$ the function such that $\jmath(z) = z$ without paying attention to its domain of definition that will be always clear from the context upon every appearance of the function $\jmath$.

Let $\mathrm{Area}(E)$ stand for the area of a measurable set $E \subset \mathbb{C}$, while the integral of a measurable function f over a measurable set E against the planar Lebesgue measure will be denoted by $\int_E f(z)\, dA(z)$.

Let us also denote by $m_{\mathbb{T}}$ the normalized Lebesgue measure on $\mathbb{T}$, so that for $\zeta = e^{i\vartheta}$ one has $dm_{\mathbb{T}}(\zeta) = \frac{1}{2\pi} d\vartheta$. Accordingly, for a rectifiable curve γ in $\mathbb{C}$ defined on a closed interval I one writes $dz|_{\gamma}$ (or dz if there are no doubts what γ one deals with) for the measure on $\gamma^* := \gamma(I)$ which acts (as a functional in the space of continuous complex valued functions on γ) by the formula $f \mapsto \int_{\gamma} f(z)\, dz$.

We will use the usual abbreviations a.a. and a.e. for the sentences "almost all" and "almost everywhere". In all cases when the corresponding measure is not mentioned explicitly, it will be completely clear from the context.

For a set $E \subset \mathbb{C}$ we will denote by $\overline{E}$, ∂E, $\mathrm{Int}(E) = E^{\circ}$, and $E^{\complement} = \mathbb{C} \setminus E$ the closure, the boundary, the interior, and the complement of E, respectively (all these topological entities are considered with respect to the topology of $\mathbb{C}$). As usual, a domain will mean a connected open set.

Let $\mathbb{C}_{\infty}$ stand for the standard one-point compactification of $\mathbb{C}$. The boundary of a set E in the topology of $\mathbb{C}_{\infty}$ will be denoted by $\partial_{\infty} E$.

If E is a subset of $\mathbb{C}$, we say that E *separates the plane* if the set $\mathbb{C} \setminus E$ is not connected. In what follows the word *component* will always mean *connected component*.

Let us recall, that a *Jordan curve* Γ (in the literature it is called sometimes a simple closed curve) is a homeomorphic image of the unit circle $\mathbb{T}$. By virtue of the classical Jordan curve theorem (see, for instance, [19, page 102]) the set $\Gamma^{\complement}$ is not connected. It consists of two components. The bounded component of the set $\Gamma^{\complement}$ is called a *domain bounded by* Γ and it will be denoted by $D(\Gamma)$. The unbounded component of $\Gamma^{\complement}$ will be denoted by $\Omega_{\infty}(\Gamma)$. It also follows from the Jordan curve theorem that $\Gamma = \partial D(\Gamma) = \partial \Omega_{\infty}(\Gamma)$.

By an *arc*, we will mean a homeomorphic image of $[0, 1]$.

For a function (or a measure) f and for a set E we will denote by $f|_E$ the restriction of f to E. For a set of functions $\mathcal{F}$ we will put $\mathcal{F}|_E = \{f|_E : f \in \mathcal{F}\}$.

In what follows we will denote by (a_n) a sequence (of objects a_n of any nature), and the index n will run over $\mathbb{N}_0$ or $\mathbb{N}$. The convergence of any sequence (a_n) will be always considered as $n \to \infty$.

Also we need to introduce several sets and spaces of functions which will be used in what follows. We will denote by $\mathcal{P}$ the set of polynomials in a complex variable, and by $\mathcal{R}$ the set of all complex rational functions defined on $\mathbb{C}_\infty$. The set of all poles of a given function $g \in \mathcal{R}$ will be denoted by $\{g\}_\infty$.

For a closed set $X \subset \mathbb{C}$ let $C(X)$ be the space of all bounded and continuous complex functions on X. This space is a Banach space with respect to the standard uniform norm $\|\cdot\|_X$, which is defined for $f \in C(X)$ as follows: $\|f\|_X = \sup_{z \in X} |f(z)|$. The space of all real-valued functions from $C(X)$ will be denoted by $C(X, \mathbb{R})$.

Let (g_n) be some sequence of functions defined on a compact set $X \subset \mathbb{C}$. Occasionally (in some bulky sentences) we will write $g_n \rightrightarrows g$ on X in order to say that this sequence converges uniformly on X to g, but more often we will write such a fact in a more traditional way, i.e., in the form "$g_n \to g$ uniformly on X". Let now (g_n) be a sequence of functions defined on an open set U. We will say that g_n converges to some function g *locally uniformly in* U, if for each compact subset $K \subset U$ we have $g_n \to g$ uniformly on K. This fact will be denoted as "$g_n \to g$ locally uniformly in U", or, in the short form, as "$g_n \rightrightarrows g$ locally in U".

For an open set Ω let $H(\Omega)$ and $H^\infty(\Omega)$ be the spaces of all *holomorphic* and *bounded holomorphic* functions in Ω, respectively. For a function $f \in H^\infty(\Omega)$ we put $\|f\|_{\infty,\Omega} = \|f\|_\Omega := \sup_{z \in \Omega} |f(z)|$. In what follows the space $H^\infty(\mathbb{D})$ will be simply denoted by H^∞, and the norm $\|\cdot\|_{\infty,\mathbb{D}}$ will be denoted by $\|\cdot\|_\infty$. For a compact set K let $H(K)$ be the space of restrictions to K of functions in $H(G)$, where G is some open set that contains K.

Furthermore, the symbol $\mathrm{Har}(\Omega)$ will denote the space of all *harmonic* functions in Ω. In different contexts we will consider real or complex valued harmonic functions, and in all cases when it will not be clear which class of functions we are dealing with we will use the more accurate notation $\mathrm{Har}(\Omega, \mathbb{R})$ and $\mathrm{Har}(\Omega, \mathbb{C})$ for the respective classes of harmonic functions.

Chapter 1

Definitions and topological properties of Carathéodory sets

In this chapter, we define the classes of Carathéodory sets which we are dealing with, and explore topological properties of such sets.

1.1 Definitions and first examples

Take a compact set $K \subset \mathbb{C}$. The open set $\mathbb{C} \setminus K$ has at most a countable number of bounded open connected components $\Omega_j = \Omega_j(K)$, $j \in I$, where $I = I(K)$ is some set of indices, and one component $\Omega_\infty(K)$ which is unbounded. The domain $\Omega_\infty(K)$ is called the *outer domain* of K. It will be convenient to put $\Omega'_\infty(K) := \Omega_\infty(K) \cup \{\infty\}$.

The set $\partial_{\text{ext}} K := \partial\Omega_\infty(K)$ is traditionally called the *external boundary* of K. The set

$$\partial_{\text{int}} K := \partial K \Big\backslash \Big(\partial\Omega_\infty(K) \cup \bigcup_{j\in I} \partial\Omega_j\Big)$$

is called the *inner boundary* of K. Thus,

$$\partial K = \partial_{\text{int}} K \cup \partial_{\text{ext}} K \cup \bigcup_{j\in I} \partial\Omega_j(K).$$

Accordingly, for an open set $U \subset \mathbb{C}$ we set

$$U_\infty := \Omega_\infty(\bar{U}), \quad U'_\infty := \Omega_\infty(\bar{U}) \cup \{\infty\}. \tag{1.1}$$

One of the most significant entities for our further considerations will be the concept of the *polynomial convex hull* of a set. Let us recall that for a bounded set $E \subset \mathbb{C}$ its polynomial convex hull, denoted by $\widehat{E}$ or $E^\wedge$, is defined as follows:

$$\widehat{E} = \Big\{z \in \mathbb{C} : |p(z)| \leqslant \sup_{w\in E} |p(w)|,\ p \in \mathcal{P}\Big\}.$$

A bounded set E is called *polynomially convex* if $E = \widehat{E}$.

Observe that always $E \subset \widehat{E} = (\widehat{E})^\wedge$ and the set $\widehat{E}$ is closed. Moreover, it can be easily verified that

$$(\bar{E})^\wedge = \widehat{E}$$

for any bounded set $E \subset \mathbb{C}$. This equality will be frequently and implicitly used in what follows.

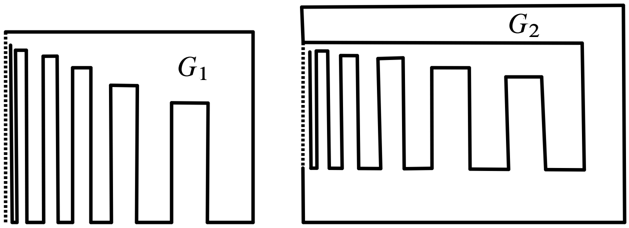

Figure 1. G_1 is a Carathéodory domain, while G_2 is not.

If K is a compact subset of $\mathbb{C}$ then the maximum modulus principle for holomorphic functions and the classical Runge approximation theorem yield

$$\widehat{K} = K \cup \bigcup_{j \in I(K)} \Omega_j(K),$$

and $\partial \widehat{K} = \partial_{\mathrm{ext}} K$. Thus, $\widehat{K}$ is the union of K and all bounded components of $\mathbb{C} \setminus K$.

In what follows we will be dealing with several kinds of Carathéodory sets, namely, with Carathéodory open sets (in particular, with Carathéodory domains), and Carathéodory compact sets. Despite the fact, that the principal ideas underlying these concepts are the same, it is convenient to define them separately.

Definition 1.1. A set $G \subset \mathbb{C}$ is called a Carathéodory open set if it satisfies the following conditions:

(1) G is nonempty, open and bounded;

(2) $\partial G = \partial_{\mathrm{ext}}(\partial G)$.

A connected open Carathéodory set G is called a Carathéodory domain.

Since $G_\infty = \Omega_\infty(\partial G)$, condition (2) in Definition 1.1 also means that $\partial G = \partial G_\infty$.

A very simple example of a Carathéodory domain is provided by any Jordan domain, that is a domain of the form $D(\Gamma)$ for an arbitrary Jordan curve Γ. It follows directly from Definition 1.1 that the domains $D = \mathbb{D} \setminus [0, 1)$, $D_2 = \mathbb{D} \setminus \overline{D}(\frac{1}{2}, \frac{1}{2})$, and $D_3 = \mathbb{D} \setminus [-\frac{1}{2}, \frac{1}{2}]$ are not Carathéodory domains. In the picture in Figure 1 one can see two more complicated examples.

Notice, that for a Carathéodory domain G the set $\mathbb{C} \setminus \overline{G}$ may be disconnected. The "outer cornucopia", which is a ribbon which winds around $\overline{\mathbb{D}}$ and accumulates to $\mathbb{T}$, see the domain G_1 at Figure 2 below, gives an example of such behavior. Observe that the domain G_2 in Figure 2 is not a Carathéodory domain. This domain will be useful for certain further considerations.

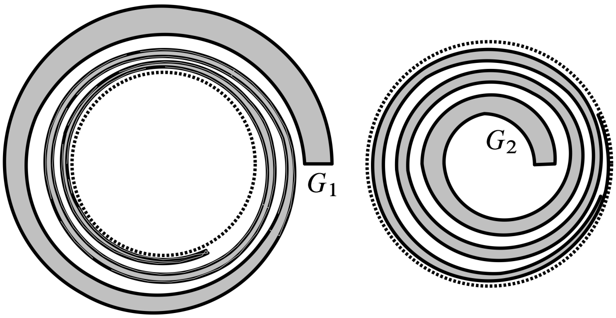

Figure 2. A cornucopia G_1 and an "inner snake" domain G_2.

In fact, the set $\mathbb{C} \setminus \overline{G}$ (for a Carathéodory domain G) can be even infinitely connected as shown by the infinite cornucopia given in Figure 3.

Definition 1.2. A nonempty compact set $K \subset \mathbb{C}$ is said to be a Carathéodory compact set, if $\partial K = \partial \widehat{K}$.

1.2 Properties of connectivity

We will explore in this section certain properties of connectivity of Carathéodory sets. These properties are not only of interest in their own right in the general context of the theory of Carathéodory sets, but they will be used repeatedly (but sometimes implicitly) in what follows.

We recall that an open set U is simply connected if and only if the set $\mathbb{C}_\infty \setminus U$ is connected. The following result is easy to prove.

Proposition 1.3. *Let K be a Carathéodory compact set, and let U be a Carathéodory open set. Then, the following hold.*

(a) *If $K^\circ = \emptyset$, then any compact subset $Y \subset K$ is also a Carathéodory compact set;*

(b) *If a compact set Y is the union of some components of K, then Y is a Carathéodory compact set;*

(c) *If Ω is the union of some components of U, then Ω is a Carathédoroy open set.*

For a bounded open set U let us introduce the concept of its Carathéodory hull, which is yet another variety of the concept of a Carathéodory set.

Definition 1.4. Let $U \neq \emptyset$ be a bounded open set. The set $U^* := \operatorname{Int}(\widehat{U})$ is called the Carathéodory hull of U.

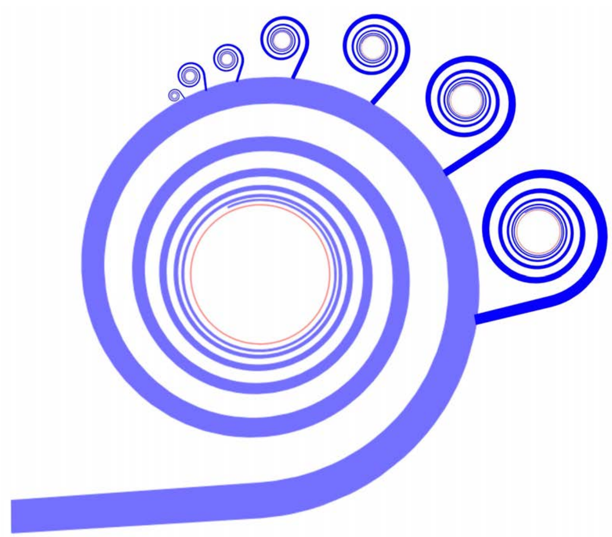

Figure 3. An infinite cornucopia.

For example, in Figure 3 the set G^* is the union of the cornucopia domain G itself (the domain shown in blue in this picture) and all disks, where this cornucopia accumulates. For the inner snake at the right-hand side in Figure 2, the set G_2^* is the small open disc, where the cornucopia is included.

Proposition¶ 1.5. *For Carathéodory open sets, the following holds.*

(1) *Every Carathéodory open set U is simply connected.*

(2) *If G is a Carathéodory domain, then G is a component of G^*. Conversely, for any bounded open set B, each component of B^* is a Carathéodory domain.*

Proof. Take a Carathéodory open set U and assume that it is not simply connected. Then, there exists a component V of U which is not simply connected. In such a case the set $\mathbb{C}_\infty \setminus V$ is not connected. Then, there exist two closed sets $X, Y \subset \mathbb{C}_\infty \setminus V$ such that

$$X \cap Y = \emptyset, \quad X \neq \emptyset, \quad Y \neq \emptyset, \quad \mathbb{C}_\infty \setminus V = X \cup Y. \tag{1.2}$$

Assume that $\infty \in Y$, then X is a compact subset of $\mathbb{C}$. Then,

$$\Omega_\infty(\bar{V}) \subset \mathbb{C}_\infty \setminus \bar{U} \subset X \cup Y.$$

Since $\Omega_\infty(\bar{V})$ is a connected set, then (1.2) yields

$$\Omega_\infty(\bar{V}) \subset Y \Rightarrow \partial\Omega_\infty(\bar{V}) \subset Y. \tag{1.3}$$

It holds, moreover, that $\partial X \subset \partial V$. Indeed, for any $w_0 \in \partial X$ and $\delta > 0$ with $\delta < \operatorname{dist}(X, Y)$, one has

$$D(w_0, \delta) \cap X \neq \emptyset, \quad \emptyset \neq D(w_0, \delta) \cap X^{\complement} = D(w_0, \delta) \cap (Y \cup V) = D(w_0, \delta) \cap V.$$

So, $w_0 \in \partial V$. Since V is a Carathéodory domain, the properties (1.2) and (1.3) imply

$$\emptyset \neq \partial X \subset \partial U \cap \partial X = \partial \Omega_\infty(\overline{V}) \subset Y \cap X = \emptyset,$$

which gives a contradiction. Thus, any Carathéodory open set U is simply connected as it is claimed.

Let now $K = \widehat{G}$. Since $G \subset \operatorname{Int}(\widehat{K})$, let V be the component of $\operatorname{Int}(\widehat{K})$ such that $G \subset V$. It needs to be shown that $G = V$. Assume that there is a point $z_1 \in V \setminus G$. Let $z_0 \in G$ and take a polygonal line $L \subset V$ such that $z_0, z_1 \in L$ (it is possible since V is a domain). Then, there exists a point w such that

$$w \in \partial G \cap L \subset \partial G \cap V \subset \partial \widehat{G} \cap \operatorname{Int}(\widehat{G}) = \emptyset,$$

which is a contradiction. ∎

Proposition¶ 1.6. *The following properties are satisfied.*

(1) *If U is a Carathéodory open set, then $U = \operatorname{Int}(\overline{U})$.*

(2) *If G is a simply connected domain such that the set $\overline{G}$ does not separate the plane, then G is a Carathéodory domains if and only if $G = \operatorname{Int}(\overline{G})$.*

Proof. It is clear that $U \subset \operatorname{Int}(\overline{U})$. Assume that there exists a point $z \in \operatorname{Int}(\overline{U}) \setminus U$. Then, for some $\delta > 0$, it holds that $D(z, \delta) \subset \overline{U}$, and hence $z \in \partial U = \partial \Omega_\infty(\overline{U})$. So, $\emptyset \neq D(z, \delta) \cap (\mathbb{C} \setminus \overline{U}) \subset \overline{U} \cap (\mathbb{C} \setminus \overline{U}) = \emptyset$, which is a contradiction. Thus, (1) is proved.

We are going now to prove the second part of the proposition. Since $\overline{G}$ does not separate the plane, one has that $\Omega_\infty(\overline{G}) = \mathbb{C} \setminus \overline{G}$. Then,

$$\partial \Omega_\infty(\overline{G}) = \partial(\mathbb{C} \setminus \overline{G}) = \partial \overline{G} = \overline{\overline{G}} \setminus \operatorname{Int}(\overline{G}) = \overline{G} \setminus G = \partial G,$$

and the proof is completed. ∎

Note that if K is a Carathéodory compact set, then K may be different from $\overline{\operatorname{Int}(K)}$.

The domain G_2 at Figure 2 is an "inner snake" which is not a Carathéodory domain. It gives an example showing that the converse assertion to the part (1) of Proposition 1.6 is not true, and also that the hypothesis in the part (2) of this Proposition that G does not separate the plane is essential.

The concept of a Carathéodory domain is not topologically invariant. For example one can consider the domain $f(G_2)$, where G_2 is presented in the picture in Figure 2 and $f(z) = 1/(z-a)$ with a being the center of the center of the disk in which G_2 accumulates. In order for the Carathéodory property for a given domain to be preserved by a homeomorphism of G, some additional assumptions on this homeomorphism are required. The next result generalizes [39, Theorem 2], where the hypotheses that the set $\overline{G}$ does not separate the plane is additionally assumed.

Theorem¶ 1.7. *Let G be a Carathéodory domain. Assume that* $f\colon \widehat{G} \to \mathbb{C}$ *is a continuous injection. Then,* $f(G)$ *is a Carathéodory domain. If* $\overline{G}$ *does not separate the plane then* $\overline{f(G)}$ *also does not separate the plane.*

Proof. For a subset $A \subset \widehat{G}$ the continuity of f and compactness of $f(\overline{A})$ imply that $f(\overline{A}) = \overline{f(A)}$. Let now $Y := f(\overline{G}) = \overline{f(G)}$.

In view of the theorem on the invariance of open sets (see, for instance [94, page 122] or [78, page 475]) the function f maps open sets in $\mathbb{C}$ to open sets in $\mathbb{C}$, in particular the set $f(G)$ is a domain.

Assume now that G is such that $\widehat{G} \neq \overline{G}$. Then, for any bounded component B of $\mathbb{C} \setminus \overline{G}$ its image $f(B)$ coincides with some bounded component Ω of $\mathbb{C} \setminus Y$.

Let us prove this claim. Take such B. Then, the set $f(B)$ is a domain and $f(B) \cap Y = f(B \cap \overline{G}) = \emptyset$ since f is injective. So, $f(B)$ has to be included into some component of $\mathbb{C} \setminus Y$. Assume that $f(B) \subset \Omega_\infty(Y)$. In this case, one can take a point $a \in B$ and find some infinite polygonal line $L \subset \Omega_\infty(Y)$ joining $f(a)$ with ∞. Clearly $L \cap f(B) \neq \emptyset$ and $L \cap f(B)^{\complement} \neq \emptyset$, since $f(B)$ is a bounded set. Then, there exists a point $b \in L \cap \partial f(B) \subset \Omega_\infty(Y) \cap Y = \emptyset$, which gives a contradiction. Therefore, there is a bounded component Ω of $\mathbb{C} \setminus Y$ such that $f(B) \subset \Omega$. Thus, $f(\overline{B}) \subset \overline{\Omega}$.

Now, let us assume, that $f(B) \neq \Omega$. In such a case one can take a point $a' \in \Omega \setminus f(B)$ and a point $b' \in f(B)$. If $\Lambda \subset \Omega$ is a polygonal line joining a' with b', one has $\emptyset \neq \Lambda \cap \partial f(B) \subset \Omega \cap Y = \emptyset$, which, again, is a contradiction. Therefore, $f(B) = \Omega$ and, hence, $f(\overline{B}) = \overline{\Omega}$, as it was claimed.

Going further, let Ω be a bounded component of the set $\mathbb{C} \setminus Y$. Then, $\partial\Omega \subset \partial Y$. Put $F := f^{-1}(\partial\Omega)$ so that F is a compact subset of ∂G. Thus, $\widehat{F} \subset \widehat{G}$ and $f(\widehat{F}) = \overline{\Omega}$ by the previous arguments.

Now, we have to prove that

$$f(G) = \operatorname{Int}(\overline{f(G)}). \tag{1.4}$$

By part (1) of Proposition 1.6 we obtain

$$f(G) = f(\operatorname{Int}(\overline{G})) \subset f(\overline{G}) = \overline{f(G)}.$$

But $f(\operatorname{Int}(\overline{G}))$ is an open set, so $f(G) \subset \operatorname{Int}(\overline{f(G)})$. In order to verify the opposite inclusion it suffices to observe that

$$f^{-1}(\operatorname{Int}(\overline{f(G)})) = f^{-1}(\operatorname{Int}(f(\overline{G}))) \subset \operatorname{Int}(f^{-1}(f(\overline{G}))) = \operatorname{Int}(\overline{G}) = G.$$

Thereafter, using (1.4) we have

$$\partial\overline{f(G)} = \overline{\overline{f(G)}} \setminus \mathrm{Int}(\overline{f(G)}) = \overline{f(G)} \setminus f(G) = \partial f(G). \tag{1.5}$$

Assume now that $f(G)$ is not a Carathéodory domain. Using (1.5) again, one has

$$\partial f(G) = \partial\overline{f(G)} = \partial_{\mathrm{int}} Y \cup \partial_{\mathrm{ext}} Y \cup \bigcup \partial\Omega_j(Y) \neq \partial_{\mathrm{ext}} Y,$$

where $\{\Omega_j = \Omega_j(Y)\}$ is the collection (nonempty in the case under consideration) of components of the set $\mathbb{C} \setminus Y$.

Therefore, for every bounded component B of the set $\mathbb{C} \setminus \overline{G}$ there exists a point $z \notin \partial_{\mathrm{ext}} B$ but $z \in \partial_{\mathrm{int}} B \cup \bigcup \partial\Omega_j$. This implies that there exists a point $z_1 \in \partial\Omega_j$ for some index j such that $z_1 \notin \partial_{\mathrm{ext}} B$. Put now $M := \partial B$ and $K := \partial\Omega_j$, so that K is a component of M. By Zoretti's theorem [136, page 109] there exists a Jordan curve Υ that encloses $\partial\Omega_j$ and such that $M \cap \Upsilon = \emptyset$ while $d(\Upsilon, K) < \varepsilon$ for some sufficiently small ε. Then, $\Upsilon \subset f(G)$, $f^{-1}(\Omega_j) = G_j$ and $\Upsilon_1 = f^{-1}(\Upsilon)$ is a Jordan curve on G such that Υ_1 encloses ∂G_j. But it is a contradiction because Υ_1 separates points of $\Omega_\infty(\overline{G})$, which are in the bounded component of $\mathbb{C} \setminus \Upsilon_1$, from ∞.

If $\widehat{G} = \overline{G}$ then, by the theorem of invariance by homeomorphisms on $\mathbb{C}_\infty$ [78, page 550], the set $\mathbb{C} \setminus Y$ is also connected. In this case, the proof may be completed using Proposition 1.6 (part (2)) and (1.4). ∎

Proposition¶ 1.8. *Let K be a compact subset of $\mathbb{C}$. Then, K is a Carathéodory compact set if and only if*

$$\mathrm{Int}(\widehat{K}) = \mathrm{Int}(K) \cup \bigcup G_j,$$

where $\{G_j\}$ is the collection of all bounded components of the set $\mathbb{C} \setminus K$.

Proof. In the case that $K = \widehat{K}$ (that is for compact sets which do not separate the plane) there is nothing to prove. Assume that K is a now a general Carathéodory compact set. Since $\mathrm{Int}(K) \cup \bigcup G_j \subset \widehat{K}$, then

$$\mathrm{Int}(K) \cup \bigcup G_j \subset \mathrm{Int}(\widehat{K}).$$

Let $w \in \mathrm{Int}(\widehat{K})$ and take $\varepsilon > 0$ such that $D(w,\varepsilon) \cap \Omega_\infty(K) = \emptyset$. Then, $D(w,\varepsilon) \cap \partial K = \emptyset$. This means that $D(w,\varepsilon) \subset K$ or $D(w,\varepsilon) \subset K^{\mathsf{C}} \setminus \Omega_\infty(K) = \bigcup G_j$. Then, $w \in \mathrm{Int}(K)$ or $w \in G_k$ for some index k.

Conversely, assume that $\mathrm{Int}(\widehat{K}) = \mathrm{Int}(K) \cup \bigcup G_j$, then

$$\partial\widehat{K} = \widehat{K} \cap \left(\mathrm{Int}(K) \cup \bigcup G_j\right)^{\mathsf{C}} = \widehat{K} \cap \left(\bigcup G_j\right)^{\mathsf{C}} \cap \mathrm{Int}(K)^{\mathsf{C}} = K \cap \mathrm{Int}(K)^{\mathsf{C}} = \partial K.$$

Therefore, K is a Carathéodory compact set and the proof is completed. ∎

Ending this section let us provide yet other clear relations among Carathéodory sets.

Proposition¶ 1.9. *The following statements hold.*

(1) *If K is a Carathéodory compact set with $K^\circ \neq \emptyset$, then K° is a Carathéodory open set.*

(2) *If U is a Carathéodory open set, then $\bar{U}$ is a Carathéodory compact set.*

1.3 Accessible boundary points

In this section we recall the concept of an accessible boundary point and present certain properties of accessible points on boundaries of Carathéodory domains.

Definition 1.10. Let U be an open set in $\mathbb{C}$, and let $a, b \in \partial U$.

(1) An arc $\mathcal{E}$ beginning at some point $w \in U$, ending at a, and such that $\mathcal{E} \setminus \{a\} \subset U$ is called *an end-cut* of U (or in U).

(2) An arc $\mathcal{C}$ beginning at a, ending at b, and such that $\mathcal{C} \setminus \{a, b\} \subset U$ is called *a cross-cut* of U (or in U).

The following fact may be found in [94, page 145].

Theorem 1.11. *Let G be a domain in $\mathbb{C}$.*

(1) *If G is simply connected and if $\mathcal{C}$ is a cross-cut in G, then $G \setminus \mathcal{C} = G_1 \cup G_2$, where G_1 and G_2 are disjoint simply connected domains.*

(2) *If for each cross-cut $\mathcal{C}$ in G the set $G \setminus \mathcal{C}$ is not connected, then G is a simply connected domain.*

The next definition is a small refinement of the definition given in [136, page 111].

Definition 1.12. (1) Let X be a subset of $\mathbb{C}$. A point $a \in \partial X$ is said to be accessible from X, if there exists some end-cut $\mathcal{E}$ of X ending at a.

(2) Let G be a simply connected domain in $\mathbb{C}$. A point $z \in \partial G$ is accessible from, at least, two sides of G, if there exists a cross-cut $\mathcal{C}$ in G with endpoints $a, b \in \partial G$, such that $z \notin \{a, b\}$, $z \in \partial G_1$, $z \in \partial G_2$, where $G \setminus \mathcal{C} = G_1 \cup G_2$, and the point z is accessible both from G_1 and from G_2.

For a simply connected domain G in $\mathbb{C}$ we put

$$\partial_a G := \big\{z \in \partial G : z \text{ is accessible from } G\big\}.$$

It is natural to call the set $\partial_a G$ the accessible part of the boundary of G. The set $\partial_a G$ is always dense in ∂G. This follows from the fact, easy to prove, that the set of points which are accessible by segments (as end-cuts) is dense. In [89] it was proved the important fact that $\partial_a G$ is a Borel set for every domain G.

Definition 1.13. Let M be a connected set and $w \in M$. The point w is called a cut point of M if the set $M \setminus \{w\}$ is not connected. The point w is called an end point of M, if there exists a sequence (U_n) of (circular) neighborhoods of w such that $\operatorname{diam}(U_n) \to 0$, as $n \to \infty$, and the set $\partial U_n \cap M$ consists of a single point for each n.

Proposition¶ 1.14. *If G is a Carathéodory domain, then ∂G does not have points which are accessible from both sides of G. Moreover, ∂G has neither cut points nor end points.*

Proof. Assume that a point z is an accessible point from both sides of G and let $\mathcal{C}$ be a cross-cut of G with endpoints a and b such that $z \neq a$, $z \neq b$, satisfying all requirements of Definition 1.12. For $s = 1, 2$, let $\mathcal{E}_s$ be two end-cuts in G_s starting at some points $z_s \in G_s$ and ending at the point z. Since G is a domain, let $L \subset G$ be a polygonal line joining the points z_1 and z_2 such that $L \cap \mathcal{E}_s \subset \{z_1, z_2\}$ for each $s = 1, 2$. Then, $\Upsilon := \mathcal{E}_1 \cup \mathcal{E}_2 \cup L$ is a Jordan curve that separates a and b. If Ω_1 and Ω_2 are the components of $\mathbb{C} \setminus \Upsilon$ we may assume that $a \in \Omega_1$, $b \in \Omega_2$. Since G is a Carathéodory domain then $\Omega_s \cap \Omega_\infty(\overline{G}) \neq \emptyset$ for each s. So,

$$\emptyset \neq \Upsilon \cap \Omega_\infty(\overline{G}) \subset (G \cup \{z\}) \cap (\mathbb{C} \setminus \overline{G}) = \emptyset,$$

which gives the desired result.

Assume now that $w \in \partial G$ is a cut point. Then, $\partial G = M_1 \cup \{w\} \cup M_2$ with $\overline{M_1} \cap M_2 = \overline{M_2} \cap M_1 = \emptyset$. By the separation theorem (see [136, page 108]) applied to the sets $A = \overline{M_1}$ and $B = \overline{M_2}$ there exists a Jordan curve $\Upsilon \subset G \cup \{w\}$ that separates M_1 and M_2. After that the proof may be completed as it was for accessible points.

Finally, if $w \in \partial G$ is an end point then, by its definition, w is the limit of some sequence (ζ_n) of points which are cut points of ∂G. However, this sequence cannot exist, therefore such a point w does not exist. ∎

The next result was obtained in [26] but here we prove it in a more simple manner.

Proposition¶ 1.15. *Let G be a Carathéodory domain and let B be a bounded component of $\mathbb{C} \setminus \overline{G}$. Then, the set $\partial_a G \cap \partial B$ consists of at most one point.*

Proof. Assume the opposite, which means, that there exists a cross-cut $\mathcal{C} \subset B \cup \{\zeta_1, \zeta_2\}$, $\zeta_1 \neq \zeta_2$, $\zeta_1, \zeta_2 \in \partial_a G \cap \partial B$. Let G_1 and G_2 be two simply connected domains such that $G \setminus \mathcal{C} = G_1 \cup G_2$. Then, take a line R, orthogonal to the segment $[\zeta_1, \zeta_2]$ and passing through the middle point of this segment. Denote by $R^\pm$, respectively, two rays of R starting at the middle point of $[\zeta_1, \zeta_2]$. Then, take the last point $\zeta_3 \in R \cap \partial B$ in the direction of the ray R^+, and the last point $\zeta_4 \in R \cap \partial B$ in the direction of the ray R^-. All points ζ_j, $j = 1, 2, 3, 4$ are different. Put $C_0 = \mathcal{C}$, $C_1 = \overline{G}$, then $C_0 \cap C_1 = \{\zeta_1, \zeta_2\}$ is not connected, then, by the second theorem of Janiszewski (see [78, page 506]), the continuum $C_0 \cup C_1$ separates the plane. So, $\mathbb{C} \setminus (C_0 \cup C_1) = U_1 \cup U_2$, where U_1 and U_2 are two open sets, with $G_j \subset U_j$ for $j = 1, 2$. In each of two small discs $D(\zeta_k, \delta)$, for $k = 3, 4$, there exists a point $z_j \in \Omega_\infty(\overline{G}) \cap U_j \neq \emptyset$ for $j = 1, 2$. These facts together with $\Omega_\infty(\overline{G}) = (\Omega_\infty(\overline{G}) \cap U_1) \cup (\Omega_\infty(\overline{G}) \cap U_2)$ give a contradiction. ∎

Example 2 in [26] shows a Carathéodory domain G such that the set $\partial_a G \cap \partial B$ is a singleton. A more informative example is given in Example 2.20 in Chapter 2, see Figure 6 below.

We mention here that the authors of [26] were unaware at that moment of the result proved in [38, page 172]. The aforementioned result says, in the notations of Proposition 1.15, that $\partial_a G \cap \partial_a B$ is either empty or consists of a single point. The difference of considering $\partial_a B$ in place of ∂B allows the author of [38] to argue more directly. But this difference is essential, because of Example 2.20. Let us also refer Proposition 2.19, where additional information is presented concerning the matter.

Corollary¶ 1.16. *If G is a Carathéodory domain such that $\partial G = \partial_a G$, then the set $\mathbb{C} \setminus \overline{G}$ is connected.*

Corollary¶ 1.17. *If W_1 and W_2 are two different components of a Carathéodory open set U, then $\partial_a W_1 \cap \partial_a W_2$ consists of at most one point.*

We end this section with the next result, which will be used several times in what follows.

Proposition¶ 1.18. *For every Carathéodory compact set X there exists a Carathéodory continuum Y such that $X \subseteq Y$ and $X^\circ = Y^\circ$.*

Proof. In order to prove this assertion we consider for each integer $k \geqslant 1$ the family $\mathcal{D}_k$ of the dyadic squares of the generation k, i.e.,

$$\mathcal{D}_k = \left\{ Q = \left[\frac{j_1}{2^k}, \frac{j_1+1}{2^k}\right) \times \left[\frac{j_2}{2^k}, \frac{j_2+1}{2^k}\right) : j_1, j_2 \in \mathbb{Z} \right\}.$$

Define the subfamily $\mathcal{D}_k(X)$ consisting of all squares $Q \in \mathcal{D}_k$ such that $X \cap \overline{Q} \neq \emptyset$, put $F_k := \bigcup_{Q \in \mathcal{D}_k(X)} Q$ and suppose $F_{k,1}, \ldots, F_{k,r_k}$ to be the closures of the polynomial hulls of the components of F_k. In such a case one has that $X \subset F_{1,1}^\circ \cup \cdots \cup F_{1,r_1}^\circ$. For each k and $j = 1, \ldots, r_k$ we choose a point $z_{k,j} \in \partial F_{k,j}$. Set $F_k^* := \bigcup_{j=1}^{r_k} F_{k,j}$. Denote by $I_{k+1,j}$ the set of indexes $s = 1, \ldots, r_k$ such that $F_{k+1,s} \subset F_{k,j}$ and set $F_{k+1,j}^* := \bigcup_{s \in I_{k+1,j}} F_{k+1,s}$.

In what follows by a tree we mean a connected polygonal line T such that $\mathbb{C} \setminus T$ is connected.

Let us construct a sequence of trees (T_k) with $T_{k-1} \subset T_k$ by induction. Take a point $z \notin X$ and choose a tree T_1 such that T_1 connects z with all points $z_{1,j}$, $j = 1, \ldots, r_1$ and such that the set $\mathbb{C} \setminus (F_1^* \cup T_1)$ is connected. Suppose now that the trees $T_1, \ldots, T_k$ are already constructed. Let us show how to construct the tree T_{k+1}. Since

$F_{k,j}$ for $j = 1, \dots, r_k$ contains a finite number of $\{F_{k+1,s}\}$ (where $s = 1, \dots, r_{k+1}$), we can choose a new tree $T_{k,k+1,j}$ that connects $z_{k,j}$ with all $z_{k+1,s}$ for $s \in I_{k+1,j}$ such that the domain $G_k := \mathbb{C} \setminus (T_k \cup Y_k)$, where $Y_k = \bigcup_{j=1}^{r_k} (F^*_{k+1,j} \cup T_{k,k+1,j})$ is simply connected. Now, we put $T_{k+1} = T_k \cup (\bigcup_{j=1}^{r_k} T_{k,k+1,j})$.

Finally, we take $T = \overline{\bigcup_{k=1}^{\infty} T_k}$ and let $Y = X \cup T$. Then, Y is a compact set such that $X^\circ = Y^\circ$. Since all G_k are simply connected domains and $\mathbb{C} \setminus Y = \bigcup_k G_k$, then Y is connected and finally, Y is a Carathéodory compact set because of the fact that $\partial Y = \partial X \cup T$. ■

Chapter 2

Carathéodory sets and conformal maps

2.1 Some background on conformal maps

Let B and G be domains in $\mathbb{C}$. One says that a function f maps B conformally onto G (respectively, into G) if f is holomorphic and injective in B, and $f(B) = G$ (respectively, $f(B) \subset G$). The Riemann mapping theorem is the starting point of all studies of conformal maps. Let us recall some historical remarks concerning the Riemann theorem since they are important for better understanding the role of Carathéodory's ideas and results. B. Riemann enunciated his outstanding theorem on conformal maps in his dissertation in 1851. The Riemann theorem says that, if G is a simply connected domain such that $G \neq \mathbb{C}$ and $G \neq \emptyset$, then there exists a conformal map f from $\mathbb{D}$ onto G. If $a \in G$ and $\vartheta \in [0, 2\pi)$ are fixed, then there exists a unique conformal map f satisfying the normalization conditions $f(0) = a$ and $\arg f'(0) = \vartheta$. If $\vartheta = 0$, the corresponding f is called the *Riemann mapping function* (with respect to a). Notice, that the proof given by Riemann contained a gap which was eliminated later on by D. Hilbert and other authors. The standard modern proof was developed by R. Riesz and L. Fejér and was published by T. Radó in 1923. It may be found, for instance, in [33, Chapter vii] and in [61, page 30]. Montel's theory of normal families of holomorphic functions plays a crucial role in this proof. Also we refer to [19, page 298], where one can find a constructive proof of the Riemann mapping theorem made by P. Koebe and C. Carathéodory.

If f is the Riemann map from $\mathbb{D}$ onto G and $g = f^{-1}$, the number $R = 1/g'(a)$ is called the *conformal radius* of G we respect to a. The function $g_0 := g/g'(a)$ defined on G maps G conformally onto $D(0, R)$ and satisfies the normalization conditions $g_0(a) = 0$ and $g_0'(a) = 1$. Sometimes this function is more easily handled than the Riemann map. For instance, the function g_0 possesses several minimality properties, one of which is given by the following proposition.

Proposition 2.1. *Let $G \neq \mathbb{C}$ be a simply connected domain and let $a \in G$. Then, the function g_0 defined above is the unique solution to the extremal problem*

$$\int_G |g_0'|^2 \, dA = \inf\left\{\int_G |h'|^2 \, dA : h \in H(G),\ h(a) = 0,\ h'(a) = 1\right\} = \pi R^2.$$

The details of the proof may be found in [61, page 55]. It is appropriate to recall that the standard area formula (see, e.g., [43, page 96]) yields that for any measurable set $E \subset G$, for each function $h \in H(G)$ (not necessarily univalent) and for every real nonnegative measurable function F defined on $h(E)$ it holds

$$\int_{h(E)} F(z)\, n(h,z)\, dA(z) = \int_E F(h(w))\, |h'(w)|^2\, dA(w), \tag{2.1}$$

where $n(h,a)$ stands for the number of points of $h^{-1}(\{a\})$, for each $a \in \mathbb{C}$. The above formula also holds for complex measurable functions F, provided that one of its two entries is well defined.

One of the central questions in the theory of conformal maps which is of high importance for our considerations is the study of the behavior of a conformal map $f\colon \mathbb{D} \to G$ near a point $\zeta \in \partial\mathbb{D}$. In a general sense this behavior is given by the concept of a *prime end*. We denote by $\operatorname{diam}^{\#}$ the diameter of sets in the spherical metric in $\mathbb{C}_\infty$ (see [104, page 1]).

Definition 2.2. Let G be a simply connected domain. We call a sequence of cross-cuts $(\mathcal{C}_n)$ a *null-chain* of G if

(i) $\overline{\mathcal{C}}_n \cap \overline{\mathcal{C}}_{n+1} = \emptyset$ for each $n = 0, 1, 2, \ldots$;

(ii) $\mathcal{C}_n$ separates $\mathcal{C}_0$ and $\mathcal{C}_{n+1}$ for each $n = 1, 2, 3, \ldots$;

(iii) $\operatorname{diam}^{\#}(\mathcal{C}_n) \to 0$ as $n \to \infty$.

If ∂G is bounded one can replace $\operatorname{diam}^{\#}$ with the Euclidean diameter.

Let us recall the notion of equivalence of null-chains. We say that two null-chains $(\mathcal{C}_n)$ and $(\mathcal{C}'_n)$ are equivalent if, for every large number m, there exists a number n such that $\mathcal{C}'_m$ separates $\mathcal{C}_n$ from $\mathcal{C}_0$, and $\mathcal{C}_m$ separates $\mathcal{C}'_n$ from $\mathcal{C}'_0$. The equivalence classes of null-chains with respect to this relation are called the *prime ends* of G. Let us denote the set of all prime ends of G by $\mathfrak{Pr}(G)$. It is possible to define a topology on the set $\mathfrak{Pr}(G)$ such that $G \cup \mathfrak{Pr}(G)$ become compact. The next result is one of the keystones in the conformal mapping theory, it is known as *Carathéodory prime ends theorem.*

Theorem 2.3. *Let f maps $\mathbb{D}$ conformally onto a bounded simply connected domain G. There exists a homeomorphism*

$$\hat{f}\colon \overline{\mathbb{D}} \to G \cup \mathfrak{Pr}(G)$$

which extends f (that is $f(z) = \hat{f}(z)$ for $z \in \mathbb{D}$) and for any $\zeta \in \mathbb{T}$ and for any null-chain $(\mathcal{C}_n)$ representing the prime end $\hat{f}(\zeta)$ the sequence $f^{-1}(\mathcal{C}_n)$, for sufficiently large n, forms a null-chain in $\mathbb{D}$ separating 0 from ζ.

In the simplest case that $G = \mathbb{D}$, the set $\mathfrak{Pr}(\mathbb{D})$ is homeomorphic to $\mathbb{T}$. Going further we need to recall some notation whose detailed account may be found in [104, Section 2.5] and [32, Chapter 9]. The *impression* of the prime end $\hat{f}(\zeta)$ (which also is the cluster set of f at the point ζ) is the set

$$I(\hat{f}(\zeta)) = I(f,\zeta) = C(f,\zeta) = \bigcap_{r>0} \overline{f(D(\zeta,r)\cap\mathbb{D})},$$

while the set of principal points of $\hat{f}(\zeta)$ is the set

$$\Pi(\hat{f}(\zeta)) = \Pi(f,\zeta) = C_{[0,\zeta)}(f,\zeta) = \bigcap_{0<r<1} \overline{f([r\zeta,\zeta))}.$$

The global cluster set of f is defined as

$$C(f) = \bigcap_{n\geqslant 2} \overline{f(\{z\in\mathbb{D} : |z| > (n-1)/n\})}.$$

In terms of these sets the prime ends of f are classified as follows:

First kind: $\Pi(f,\zeta)$ is a singleton and $\Pi(f,\zeta) = I(f,\zeta)$;
Second kind: $\Pi(f,\zeta)$ is a singleton, but $\Pi(f,\zeta) \neq I(f,\zeta)$;
Third kind: $\Pi(f,\zeta)$ is not a singleton and $\Pi(f,\zeta) = I(f,\zeta)$;
Fourth kind: $\Pi(f,\zeta)$ is not a singleton, but $\Pi(f,\zeta) \neq I(f,\zeta)$.

In the case that ∂G is a Jordan curve we have the following result, which is often called *Carathéodory–Osgood–Taylor theorem* (in several textbooks this theorem is also called *Carathéodory extension theorem for Jordan domains*).

Theorem 2.4. *Let f map $\mathbb{D}$ conformally onto a bounded domain G. The following conditions are equivalent:*

(i) *f has a continuous injective extension to a mapping from $\overline{\mathbb{D}}$ onto $\overline{G}$;*

(ii) *∂G is a Jordan curve;*

(iii) *∂G is locally connected and has no cut points.*

We refer the reader, depending on his expertise, to [19, page 309], [77, Chapter II], or [104, Chapter 2], where several proofs of this theorem with different levels of details may be found. The question when f has a continuous extension to $\overline{\mathbb{D}}$, perhaps without injectivity, was also solved by Carathéodory. The next result is referred as *Carathéodory continuity theorem.*

Theorem 2.5. *Let f map $\mathbb{D}$ conformally onto a bounded domain G in the complex plane. Then, the following four conditions are equivalent:*

(i) *f has a continuous extension to a mapping from $\overline{\mathbb{D}}$ onto $\overline{G}$;*

(ii) *there exists a continuous map ψ on $\mathbb{T}$ such that $\psi(\mathbb{T}) = \partial G$;*

(iii) *the set ∂G is locally connected;*

(iv) *the set $\mathbb{C}\setminus G$ is locally connected.*

The implication (iii)⇒(ii) in Theorem 2.5 is a special case of the Hahn–Mazurkiewicz theorem (see [94, page 59]). The implication (ii)⇒(iii) is a general fact on continuous images of locally connected compacta. A complete proof of the other equivalences may be found, for instance, in [104] or in [94].

The next question which is natural to pose for an arbitrary conformal map f is the question whether the boundary values f exist on the boundary of the domain, where f is defined except, may be, some "relatively small" set. This question may be solved in different manners, depending on the tools used. We will need the following result concerning the matter. Let us recall the definition of Hardy spaces in $\mathbb{D}$. For $p > 0$ the space $H^p = H^p(\mathbb{D})$ consists of all functions $f \in H(\mathbb{D})$ such that $M_p(f) < \infty$, where

$$M_p(f) = \sup_{r \to 1} \int_{\mathbb{T}} |f(r\zeta)|^p \, dm_{\mathbb{T}}(\zeta).$$

For all $p > q > 0$ the inclusions $H^\infty \subset H^p \subset H^q \subset N$ hold, where $N = N(\mathbb{D})$ is the Nevanlinna class in $\mathbb{D}$. We recall that any function $f \in N(\mathbb{D})$ has the form $f = f_1/f_2$, where $f_1, f_2 \in H^\infty(\mathbb{D})$.

Given a point $\zeta \in \mathbb{T}$ and β with $0 < \beta < \pi/2$, then the Stolz angle $S_\zeta(\beta)$ is the set

$$\{z \in \mathbb{D} : |\arg(1 - \overline{\zeta} z)| < \beta, |z - \zeta| < 2\cos\beta\}.$$

Let now h be a function from $\mathbb{D}$ to $\mathbb{C}_\infty$. One says that h has angular limit (or, in other words, boundary value) at the point $\zeta \in \mathbb{T}$, if for each $\beta \in (0, \pi/2)$ the limit

$$\lim_{S_\zeta(\beta) \ni z \to \zeta} h(z)$$

exists and is independent on β. This common value is denoted by $h(\zeta)$.

Theorem 2.6. *The following statements hold.*

(1) *Let $h \in N(\mathbb{D})$. Then, the angular limit $h(\zeta) \neq \infty$ exists for $m_{\mathbb{T}}$-a.a. $\zeta \in \mathbb{T}$.*

(2) *If f maps $\mathbb{D}$ conformally into $\mathbb{C}$, then $f \in H^p$ for every $p < 1/2$, and therefore $f(\zeta) \neq \infty$ exists for a.a. $\zeta \in \mathbb{T}$.*

(3) *Moreover, if f maps $\mathbb{D}$ conformally into $\mathbb{C}$, then the boundary values $f(\zeta)$ exist for all $\zeta \in \mathbb{T}$, except a set of logarithmic capacity zero.*

We are not providing any special reference for these results, the interested reader can follow, for instance, [104, Theorems 1.7, 8.2, 9.19, and Corollary 2.17], as well as the explanation given in [105, Chapter ii, Sections 1 and 2]. In fact, one important ingredient here is the classical Fatou's theorem that says that any function $f \in H^\infty$ has a.e. on $\mathbb{T}$ finite angular limits.

Definition 2.7. For a function $f \in H^\infty$ let $F(f)$ be the set of all points $\zeta \in \mathbb{T}$ for which the boundary value $f(\zeta)$ exists. This set is called a *Fatou set* of f.

Carathéodory in [20] considered sequences (G_n) of simply connected domains and studied when the sequence of the corresponding Riemann maps converges in some sense. We ought to recall some results from this subject.

Definition 2.8. Let (G_n) be a sequence of domains (not necessarily simply connected) and assume that there exists $a \in \bigcap_{n=1}^{\infty} G_n$. One says that G_n converges to a set G in the sense of the kernel convergence with respect to a, and G is the kernel of this sequence, if one of the two following conditions holds.

(1) If there exists $\rho > 0$ such that $D(a, \rho) \subset G_n$ for all sufficiently large n, then G must be a domain, $a \in G$, $G \neq \mathbb{C}$, and the following two conditions must be satisfied:

(1a) if $w \in G$ then there exists $\varepsilon > 0$ such that $D(w, \varepsilon) \subset G_n$ for large n,

(1b) if $w \in \partial G$, then $w = \lim w_n$ for some sequence of points (w_n) such that $w_n \in \partial G_n$ for each n.

(2) If the previous condition (1) is not satisfied then $G = \{a\}$.

This convergence is well defined, but it clearly depends on the choice of the given point a. In the case that G_n converges to G with respect to a in the sense of kernel convergence we will write $G_n \to G$ with respect to a. If it is clear from the context what a we are dealing with we will simply write $G_n \to G$.

The notion of kernel convergence has several surprising properties, for instance it underlies several deep results about convergence of sequences of conformal maps. The following Carathéodory kernel convergence theorem shows the relations between concepts of kernel convergence and locally uniform convergence of the corresponding conformal maps in the case of simply connected domains.

Theorem 2.9. *Let (G_n) be a sequence of simply connected domains, $G_n \neq \mathbb{C}$, and let a be a point such that $a \in G_n$ for each n. Let f_n be a conformal map from $\mathbb{D}$ onto G_n such that $f_n(0) = a$, $f_n'(0) > 0$. Then,*

$$f_n \rightrightarrows f \text{ locally in } \mathbb{D} \quad \text{if and only if} \quad G_n \to G \text{ with respect to } a, \tag{2.2}$$

where f and G are defined as follows: if $G = \{a\}$ then f is the constant function, so that $f(z) = a$ for all z; while in the case that $G \neq \{a\}$, so that the domain G must be simply connected and $G \neq \mathbb{C}$, the function f is the conformal map from $\mathbb{D}$ onto G with the normalization $f(0) = a$ and $f'(0) > 0$.

Moreover, in the case that G is a simply connected domain and $G_n \to G$ with respect to a, it holds that $f_n^{-1} \rightrightarrows f^{-1}$ locally in G.

In the proof of this theorem several important tools of the theory of conformal maps are used, let us notice, for instance Hurwitz's and Montel's theorems, Koebe's distortion theorem, etc. The proof of Carathéodory kernel convergence theorem may be found in many sources, see for example [61, page 54] or [104, page 14].

Let now (G_n) be a sequence of domains which converges to a Jordan domain G in the sense of kernel convergence with respect to some point a. In this case, Walsh (see [129, 130] as well as [134, pages 32–34]) was able to obtain a more strong result. Notice that this theorem is of high importance, but nowadays it seems to be almost forgotten and did not appear in the mathematical literature during many decades.

Theorem 2.10. *Let G be a Jordan domain, $a \in G$ and let (G_n) be a sequence of simply connected domains satisfying $\overline{G} \subset G_n$ such that $\overline{G_{n+1}} \subset G_n$ for all n and $G_n \to G$ with respect to a. Let ψ_n be the conformal map from G_n onto G normalized by the conditions $\psi_n(a) = a$ and $\psi_n'(a) > 0$. Then, $\psi_n(z) \rightrightarrows z$ on $\overline{G}$.*

It is not clear, whether it is possible to extend this theorem for more wide class of domains. The next question looks quite reasonable.

Question I. Will the statement of Theorem 2.10 hold in the case that G is a Carathéodory domain with accessible boundary, that is $\partial G = \partial_a G$?

2.2 Carathéodory domains and conformal maps

The reason that Carathéodory paid attention to the domains which nowadays are called by his name is shown in the next result. As far as we know, the paper [20] contains the first occurrence of the cornucopia (see, for instance, the domain G_1 in Figure 2), in the mathematical literature.

Theorem 2.11. *Let $G \neq \emptyset$ be a bounded simply connected domain. Then, G is a Carathéodory domain if and only if there exists a sequence (Γ_n) of Jordan curves such that*

$$\Gamma_n \subset \Omega_\infty(\overline{G}), \quad \overline{D(\Gamma_{n+1})} \subset D(\Gamma_n)$$

for each n, and $D(\Gamma_n) \to G$ as $n \to \infty$ with respect to any fixed point $a \in G$. This equivalence does not depend on the choice of a.

If G is a Carathéodory domain, let g_n be the conformal map from $D(\Gamma_n)$ onto $\mathbb{D}$ with the normalization $g_n(a) = 0$ and $g_n'(a) > 0$, and let g be the conformal map from G onto $\mathbb{D}$ with the same normalization. Then, $g_n \rightrightarrows g$ locally in G as $n \to \infty$. In fact,

$$\overline{G} \subset W := \bigcap_{n=1}^{\infty} D(\Gamma_n), \tag{2.3}$$

but it can happen that $\overline{G} \neq W$.

Proof. Let us take a nonempty bounded simply connected domain G satisfying all conditions of the theorem. In order to prove that G is a Carathéodory domain, let us take an arbitrary point $w \in \partial G$. By condition (1b) of Definition 2.8 there exist a sequence of points (w_n) such that $w_n \in \partial D(\Gamma_n) = \Gamma_n \subset \Omega_\infty(\overline{G})$, and $w_n \to w$ as $n \to \infty$. Also there exists another sequence (w_n') such that $w_n' \in G \subset \mathbb{C} \setminus G_\infty(\overline{G})$ and $w_n' \to w$ as $n \to \infty$. So, $w \in \partial\Omega_\infty(\overline{G})$. Since G is simply connected, then G is a Carathéodory domain by definition.

Assume now that G is a Carathéodory domain. The domain $G_\infty' = \Omega_\infty(\overline{G}) \cup \{\infty\}$ is simply connected in $\mathbb{C}_\infty$. So, one can take a conformal map h from $\mathbb{D}$ onto G_∞' with the normalization $h(0) = \infty$. Let us now define $\Gamma_n := h(\{t : |t| = n/(n+1)\})$. Then, each Γ_n is a Jordan curve such that $\overline{G} \subset \overline{D(\Gamma_{n+1})} \subset D(\Gamma_n)$. Since $(D(\Gamma_n))$ is a decreasing sequence of domains it converges to the component of $\bigcap_{n=1}^{\infty} D(\Gamma_n)$ that contains a, which is G. The remaining conclusions follow from Theorem 2.9. ∎

Let now G be a Carathéodory domain, and let f be some conformal map from $\mathbb{D}$ onto G. Take a point $w \in \partial G$. According to [104, Proposition 2.14] one has $w \in \partial_a G$ if and only if there exists a curve $\gamma\colon [0,1] \to \overline{\mathbb{D}}$ having the properties $\gamma(s) \in \mathbb{D}$ for $s \in [0,1)$ and $\gamma(1) = t$ for some $t \in \partial\mathbb{D}$, such that $\lim_{s\to 1^-} f(\gamma(s)) = w$. Moreover, it follows that $t \in F(f)$ and $f(t) = w$.

Proposition¶ 2.12. *Let G be a Carathéodory domain, and $w \in \partial_a G$. Then, there exists a unique point $t \in F(f)$ such that $f(t) = w$.*

Proof. The existence of two points t and $t' \neq t$ such that $\varphi(t) = \varphi(t') = w$ would imply that the point w is accessible from both sides of G. But Proposition 1.14 says that the boundary of the (Carathéodory domain) G does not have points which are accessible points from both sides of G. ∎

Corollary¶ 2.13. *Let G be a Carathéodory domain. Then, ∂G is locally connected if and only if ∂G is a Jordan curve. In particular, if ∂G is rectifiable then ∂G is a Jordan curve.*

Proof. Assume that ∂G is locally connected and take a conformal map f from $\mathbb{D}$ onto G. By Theorem 2.5, f has a continuous extension to $\overline{\mathbb{D}}$. Let $f_1\colon \partial\mathbb{D} \to \partial G$ be the restriction of such extension. By Proposition 2.12, f_1 is injective, and since it is defined in a compact set, then f_1 is a homeomorphism from $\partial\mathbb{D}$ onto ∂G, so that ∂G is a Jordan curve. The second assertion is a consequence of the general fact that a continuum with finite length is locally connected. ∎

A Carathéodory domain may have prime ends of all four kinds, as it can be seen at Figure 4, where the domain G_1 gives the desired example.

The class of non-degenerate continua E possessing the property that there exists a bounded univalent function f in $\mathbb{D}$ and a point $\zeta \in \mathbb{T}$ such that $C(f,\zeta) = E$, was studied and characterized in details, see for instance, [30, Proposition 5]. Next result establishes some restriction to the size of $C(f,\zeta)$ when f is the Riemann map onto some Carathéodory domain.

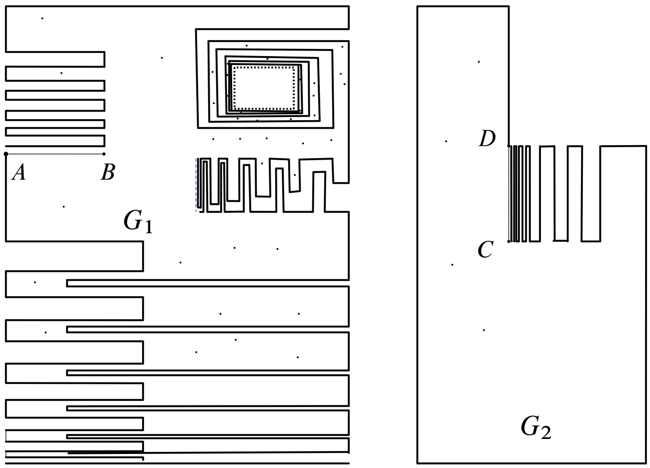

Figure 4. The Carathéodory domain G_1 has prime ends of all four kinds.

Proposition¶ 2.14. *The following properties are satisfied.*

(1) *If G is a Carathéodory domain and f is a conformal map from $\mathbb{D}$ onto G, then the set $C(f,\zeta)$ is a Carathéodory continuum for each point $\zeta \in \partial\mathbb{D}$.*

(2) *Conversely, if K is a Carathéodory continuum, then there exists a Carathéodory domain G and a conformal map $f\colon \mathbb{D} \to G$ such that f has a continuous extension to $\overline{\mathbb{D}} \setminus \{1\}$ and $C(f,1) = \Pi(f,1) = \partial K$.*

Proof. (1) Fix a point $\zeta \in \partial\mathbb{D}$ and let $z \in K := C(f,\zeta)$. Since $K \subset \partial G = \partial G_\infty$, then there exists a sequence (z_n) such that $z_n \in G_\infty$ for all n, and $z = \lim_{n\to\infty} z_n$. Each point z_n can be joined to ∞ by an infinite polygonal line L such that $L \subset \mathbb{C} \setminus G \subset \mathbb{C} \setminus K$, so $z_n \in \Omega_\infty(K)$. Therefore, $\partial K = K \subset \partial\Omega_\infty(K) \subset K$.

(2) Let K be such that $\partial\Omega_\infty(K) = \partial K$, then $\Omega' = \Omega_\infty(K) \cup \{\infty\}$ is a simply connected domain in $\mathbb{C}_\infty$. Let $h\colon \mathbb{D} \to \Omega'$ be a conformal map such that $h(0) = \infty$. Going further let us take an open ribbon $S \subset \mathbb{D}$ which spirals to $\mathbb{T}$ and such that $0 \notin \overline{S}$. Let $\psi\colon \mathbb{D} \to S$ be a conformal map such that $C(\psi,1) = \Pi(\psi,1) = \mathbb{T}$. Then, $G = h \circ \psi(\mathbb{D})$ is the desired domain. In fact, it is clear that $f = h \circ \psi$ has a continuous extension to $\overline{\mathbb{D}} \setminus \{1\}$. Moreover, take $w \in \partial_a\Omega' \subset \partial K$, and let $\mathcal{E}$ be an end-cut ending at w. Then, $h^{-1}(\mathcal{E})$ is an end-cut that ends at some point of ∂G. Then, $h^{-1}(\mathcal{E})$ cuts infinitely many points of S and of $\mathbb{D} \setminus S$. Then, $w \in \partial G_\infty \cap \partial G$. This situation holds for each point of a dense set, then $\partial K \subset \partial G_\infty \cap \partial G$. Thus,

$$\partial G_\infty = (\partial G \cap \Omega') \cup \partial K \subset \partial G \cup \partial K,$$

which means that G is a Carathéodory domain. ∎

Let us recall that a continuum K is said to be *indecomposable* if it cannot be written in the form $K = M \cup N$, where M and N are proper subcontinua of K (for more information about this notion see [78, Chapter V]) and [68, Section 3.8]. A Carathéodory continuum can be indecomposable. One of the simplest example of such continua is the Knaster buckethandle, see [78, Example 1, page 204]. Let us denote this continuum by K_b. Applying Proposition 2.14 we can see that there exists a Carathédory domain G and a conformal map $f : \mathbb{D} \to G$ such that f has a continuous extension to $\overline{\mathbb{D}} \setminus \{1\}$ and $\Pi(f, 1) = C(f, 1) = K_b$ is an indecomposable continuum. A related example is given in [29, Proposition 4] but therein the set $\Pi(f, 1)$ is a singleton, $C(f, 1) = \partial G$ and ∂G is an indecomposable continuum however G is not Carathéodory domain. Thus, to obtain a more involved example it is necessary to have some free space between G and $\Omega_\infty(\overline{G})$. This can be done using the construction of the Lakes of Wada, see [68, Section 3.8]. We need to make some modification of this construction for further considerations.

Example 2.15. Consider the compact set

$$X_0 = \left\{z : -2 \leqslant \operatorname{Re} z \leqslant 4, |\operatorname{Im} z| \leqslant \frac{3}{2}\right\} \Big\backslash \left(D\Big(-1, \frac{1}{2}\Big) \cup D\Big(1, \frac{1}{2}\Big) \cup D\Big(3, \frac{1}{2}\Big)\right).$$

To preserve the poetic flavor of the original example, we will imagine that X_0 is an island in the ocean and the small discs are three lakes, the first one having blue water, the second one green, while the third one red. Let us dig a system of canals in X_0 following the next procedure. For $k \in \mathbb{N}$ define the system of time moments $t_k = (k-1)/k$, and the sequence of distances $d_k = 1/k$, so that $t_k \to 1$ and $d_k \to 0$ as $k \to \infty$. Let V_1 be the canal (considering as an open set) that brings water from the ocean to every point of the land within distance d_1 of every point of X_0, and let $X_1 = X_0 \setminus V_1$. At the time moment t_2 let V_2 be the canal that brings water from the blue lake to every dry point within distance d_2 of every point of X_1. The first steps of this construction is illustrated by Figure 5. For time moments t_3 and t_4 let us do the same, but using water from the green lake and from the red lake, respectively. Thereafter let us repeat this cycle of construction of canals infinitely many times until we arrive to the time $t = 1$. It is possible to make this construction in such a way that the entrances to the canals in the blue lake are two sequence of open intervals on $\partial D(-1, \frac{1}{2})$ which are mutually disjoint and accumulate only at the points $-\frac{1}{2}$ and $-\frac{3}{2}$. Then, take W_{blue}, a simply connected domain formed by the blue lake together all canals with blue water, and put $X := \partial W_{blue}$.

Let us accent some properties of the domain W_{blue} constructed in Example 2.15. Denote by W_{green} the union of the green lake with all of the canals starting therein and by W_{red} the respective union for the corresponding red lake, then W_{green} and W_{red} are simply connected bounded components of the set $\mathbb{C} \setminus \overline{W_{blue}}$. The construction of the

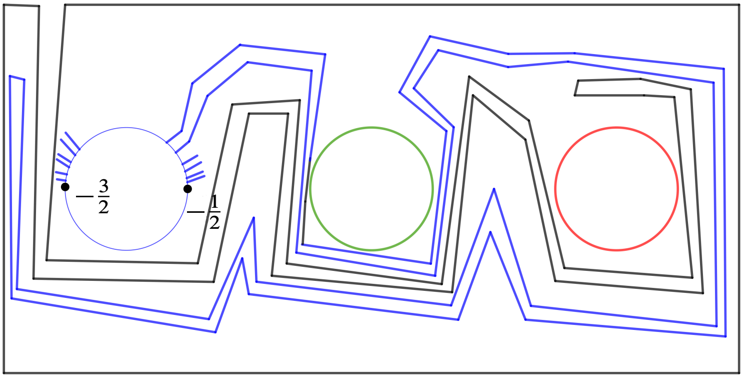

Figure 5. The first steps of the construction of the compact set $X = \partial W_{blue}$.

canals from the ocean implies that

$$\partial W_{blue} = \partial W_{green} = \partial W_{red} = \partial \Omega_\infty(\overline{W_{blue}}),$$

which yields that W_{blue} is a Carathéodory domain. Furthermore, ∂W_{blue} is a indecomposable continuum. If f is a conformal map from $\mathbb{D}$ onto W_{blue}, then f is continuous on $\overline{\mathbb{D}}$ except two points, says ζ_1 and ζ_2, where $C(f,\zeta_1) = C(f,\zeta_2) = X = \partial W_{green} = \partial W_{red}$, while $C(f,\zeta) \subset X$ for all $\zeta \in \mathbb{T}$.

By a suitable modification of the construction given in Example 2.15 it is possible to obtain a Carathéodory domain $G = f(\mathbb{D})$, for a conformal mapping f such that the (closed) set $T(f) = \{\zeta \in \mathbb{T} : C(f,\zeta) = \partial G\}$ is infinite. By a certain theorem by Rutt (see more details in [29]) in the case that the set $T(f)$ is not empty, the set ∂G is an indecomposable continuum, or the union of two indecomposable continua. Moreover, if ∂G is indecomposable, then $T(f) \neq \emptyset$. We do not know how big the set $T(f)$ may be for a Carathéodory domain in a general situation. We do not even know the answer to the following question.

Question II. Whether there exists a Carathéodory domain for which the set of prime ends of the first kind would be empty (so that the respective conformal map f cannot be continuously extended to any point of $\mathbb{T}$)?

The usual example of a domain of such kind (see, for example, [32, page 184]) is clearly not a Carathéodory domain.

Examples of this kind should not surprise the reader, since they are quite natural in a certain sense. To see this we need to use some results from plane topology.

Definition 2.16. One says that a set $E \subset \mathbb{C}$ possesses the *non-separation property* if for each closed subset $F \subset E$, such that $F \neq E$, the set $E^{\complement} \cup F$ is connected, that is the set $E \setminus F$ does not separate the plane. Otherwise, one says that E possesses the *separation property*.

Every Jordan curve possesses the non-separation property, but is it possible to assert the converse? This question was an open problem for some time at the beginning of the XX century. Its solution allows us to state the following result related with Carathéodory domain, which needs to be compared with [39, Proposition 10].

Theorem¶ 2.17. *Let K be a Carathéodory compact set. Then, one of the two following mutually exclusive conditions is fulfilled:*

(1) *K possesses the separation property;*

(2) *$K = \partial G$ for each component G of $\mathbb{C} \setminus K$.*

Moreover, if $\mathbb{C} \setminus K$ contains only two components, then K is a Jordan curve, while in the case that $\mathbb{C} \setminus K$ contains at least three components, the set K is an indecomposable continuum or the union of two indecomposable continua.

Proof. If $K^\circ \neq \emptyset$, then the condition (1) holds for such K. Therefore, let us consider the compact sets K possessing the non-separation property and having empty interior. If the set $\mathbb{C} \setminus K$ has only one component, then $K = \partial\Omega_\infty(K)$. Assume now that the set $\mathbb{C} \setminus K$ has a bounded component. Then, $\partial G \subset K$. If $K \neq \partial G$, then ∂G separates the plane, and hence $K = \partial G$.

To show that the conditions (1) and (2) are exclusive let us assume that the compact set K satisfy (2) and let F be some closed subset of K, different of K. Then, the set

$$K^{\mathsf{C}} \cup F = \bigcup_G G \cup F,$$

where G runs over all components of the set $\mathbb{C} \setminus K$, is connected because $G \subset G \cup F \subset \overline{G}$ and both sets G and $\overline{G}$ are connected. So, K does not satisfy the condition (1).

Assume now that $K^{\mathsf{C}} = G \cup \Omega_\infty(K)$, where G is simply connected domain. If the set $\partial G = K$ is not locally connected then there exists a sequence (F_n) of mutually disjoint closed sets with $F_n \subset K$, and a closed set $F \subset K$, $F \neq K$, such that $F_n \to F$ in the Hausdorff metric. Then,

$$\overline{K \setminus F} = K \quad \text{and} \quad K^{\mathsf{C}} \cup F = (G \cup F) \cup \Omega_\infty(K)$$

with $(\overline{G \cup F}) \cap \Omega_\infty(K) = \emptyset$. Then, K possesses the separation property, which is impossible. Therefore, ∂G is locally connected, which yields, according to Corollary 2.13, that ∂K is a simple closed curve.

In the case that there exist three components of the set $\mathbb{C} \setminus K$, or more, than its common boundary is K and we need to refer the theorem stated in [78, page 590] in order to finish the proof. ∎

Let $\mathcal{E} \subset \mathbb{D}$ be an end-cut ending at some point $\zeta \in \mathbb{T}$, and let $f \in C(\mathbb{D})$. The cluster set $C_{\mathcal{E}}(f, \zeta)$ of f following $\mathcal{E}$ is defined as follows:

$$C_{\mathcal{E}}(f, \zeta) = \bigcap_{n=1}^{\infty} \overline{f\left(\left\{z \in \mathcal{E} : |z - \zeta| < \frac{1}{n}\right\}\right)}.$$

This set does not depend on the choice of the initial point of $\mathcal{E}$, so we can always assume that the initial point of $\mathcal{E}$ is the origin. It is easy to prove, and it is well-known (see, for instance, [32, Theorems 4.6 and 4.7]) that $C(f, \zeta) = C_{\mathcal{E}}(f, \zeta)$ for some $\mathcal{E}$. Moreover, by definition $\Pi(f, \zeta) = C_{[0,\zeta]}(f, \zeta)$.

The following result was communicated to us by Ch. Pommerenke.

Proposition¶ 2.18. *Let G be a Carathéodory domain, and let f be a conformal map from $\mathbb{D}$ onto G. Assume that there exist two points, say ζ_1 and ζ_2, in $\mathbb{T}$ such that $\zeta_1 \neq \zeta_2$ and for each $j = 1, 2$ there is an end-cut $\mathcal{E}_j$ in $\mathbb{D}$ ending at ζ_j and possessing the property*

$$C_{\mathcal{E}_1}(f, \zeta_1) \cup C_{\mathcal{E}_2}(f, \zeta_2) \subset E,$$

for some continuum $E \subset \partial G$. Then, for one of the open arcs Υ of $\mathbb{T} \setminus \{\zeta_1, \zeta_2\}$ one has

$$I(f, \zeta) \subset E \tag{2.4}$$

for each point $\zeta \in \Upsilon$.

Proof. We may assume that $\mathcal{E}_1 \cap \mathcal{E}_2 = \{0\}$. Take $F = \overline{f(\mathcal{E}_1)} \cup \overline{f(\mathcal{E}_2)} \cup E$. Then, F is a continuum that separates the plane. Let V be the bounded component of $F^{\complement}$ such that $f(\mathcal{E}_1) \cup f(\mathcal{E}_2) \subset \overline{V}$. Let $U \subset \mathbb{D}$ be the domain whose boundary is $\overline{\mathcal{E}_1} \cup \overline{\mathcal{E}_2} \cup \overline{\Upsilon}$, where Υ is one the arc of $\mathbb{T} \setminus \{\zeta_1, \zeta_2\}$ chosen in such a way that $f(U) \subset V$.

Let us now assume that (2.4) is false. Take a point $\zeta' \in \Upsilon$ and a sequence (z_n) such that $z_n \in U$ and $z_n \to \zeta'$ as $n \to \infty$ such that the sequence $(f(z_n))$ converges as $n \to \infty$ to some point $w \in (\overline{V} \setminus E) \cap \partial G$. So, there exists a closed disc $\overline{D(w, r)} \subset V$ such that $\overline{D(w, r)} \cap F = \emptyset$. Then, take a point $\alpha \notin \overline{G} \cap D(w, r)$. Since G is Carathéodory, there exists an infinite polygonal line $\mathcal{L} \subset \mathbb{C} \setminus \overline{G}$ that starts at α and goes to ∞. But therefore

$$\mathcal{L} \subset V \cup (\mathbb{C} \setminus \overline{V}), \quad \mathcal{L} \cap V \neq \emptyset, \quad \mathcal{L} \cap (\mathbb{C} \setminus \overline{G}) \neq \emptyset.$$

which gives a contradiction since $\mathcal{L}$ is connected. ∎

In the case that $I(f, \zeta_1) \cap I(f, \zeta_2) \neq \emptyset$ we can take as a candidate for E in the previous proposition the continuum $I(f, \zeta_1) \cup I(f, \zeta_2)$. For example, for the domains G_1 and G_2 in Figure 4 one can take as E the segments $[A, B]$ and $[C, D]$, respectively.

Under the assumptions of Proposition 1.15 it is possible to say more about the cluster set in the special point.

Proposition¶ 2.19. *Let G be a Carathéodory domain, and f be a conformal map from $\mathbb{D}$ onto G. Assume that B is a bounded component of $\mathbb{C} \setminus \overline{G}$ such that $\partial_a G \cap \partial B = \{w\}$. Let $\zeta \in \mathbb{T}$ be such that $f(\zeta) = w$. Then, $\partial B \subset I(f, \zeta)$.*

Proof. For simplicity let us assume that $\zeta = 1$, so that f has the radial limit w at 1. For $r \in (0, 1)$ let $\ell(r)$ stands for the length of the set $f(\{z \in \mathbb{D} : |z-1| = r\})$. One of key points in the theory of conformal maps is the fact that

$$\int_0^1 \frac{\ell(r)^2}{r}\, dr < +\infty,$$

see [104, Proposition 2.2]. Then, there exists a sequences of cross-cuts $\mathcal{C}_n = f(\{z \in \mathbb{D} : |z-1| = r_n\})$ such that $\ell(r_n) \to 0$ as $n \to \infty$. Each cross-cut $\mathcal{C}_n$ joins some point $\alpha_n \in \partial G$ with another point $\beta_n \in \partial G$, cuts the image $f([0, 1])$ in one point, and, finally, $\mathcal{C}_n$ tends to $\{w\}$. For each n take ε_n such that

$$\left(\overline{D(\alpha_n, \varepsilon_n)} \cup \overline{D(\beta_n, \varepsilon_n)}\right) \cap \partial B = \emptyset,$$

and $\varepsilon_n \to 0$ as $n \to \infty$. Going further we cover ∂B by a finite sequence of closed discs of radius ε_n in such a way that centers of these disks belong to G_∞. We can joint the centers of the constructed disks by polygonal lines in order to obtain a new polygonal line $\mathcal{L}_n \subset G_\infty$ such that $\mathcal{L}_n \cap \overline{D(\alpha_n, \varepsilon_n)} \neq \emptyset$ and $\mathcal{L}_n \cap \overline{D(\beta_n, \varepsilon_n)} \neq \emptyset$, and the compact set $\mathcal{L}_n \cup \overline{D(\alpha_n, \varepsilon_n)} \cup \overline{D(\beta_n, \varepsilon_n)} \cup \mathcal{C}_n$ separates the plane into two components. Denote by W_n the corresponding bounded component. This process can be done in such a way, that, moreover, $W_{n+1} \subset W_n$. Then, $f(\{z \in \mathbb{D} : |z-1| < r_n\}) \subset W_n$. Taking into account the fact $\partial B \subset C(f)$, we conclude that

$$\partial B \subset \overline{f(\{z \in \mathbb{D} : |z-1| < r_n\})}$$

for each r_n. So, $\partial B \subset I(f, 1)$. ■

In general, in the above proposition the set $I(f, \zeta)$ is much bigger than ∂B.

The next example is new, but it is based on ideas of [26, Example 2]. This example shows that in the framework of hypotheses of Proposition 2.19 it can happen that $\partial_a G \cap \partial B = \{w\}$, but $\partial_a G \cap \partial_a B = \emptyset$, and B has different impressions of inaccessible points from B.

Example 2.20. Take $Q = \{z : 0 \leqslant \operatorname{Im} z < \pi,\ 0 < \operatorname{Re} z < 2\}$ and let $I_1, I_2, \ldots$ be a sequence of intervals

$$I_n = [ia_n, ib_n], \quad a_1 = 2, \quad a_n < b_n < a_{n+1} < \pi, \quad \lim_{n\to\infty} a_n = \pi.$$

Let $J_1, J_2, \ldots$ be a sequence of intervals

$$J_n = [ia'_n, ib'_n], \quad b'_1 = 1, \quad a'_n < b'_n, \quad b'_{n+1} < a'_n, \quad \lim_{n\to\infty} a'_n = 0.$$

Let, for each $n \geqslant 1$,

$$A_n = \{z : \operatorname{Im} z \in I_n,\ 0 \leqslant \operatorname{Re} z \leqslant 1\}, \quad z_n = \frac{1}{n+2} + i\frac{a_{n+1}+b_n}{2},$$
$$B_n = \{z : -1 < \operatorname{Re} z < 0,\ \operatorname{Im} z \in (ia'_n, ib'_n)\}, \quad z'_n = -\frac{n}{n+1} + i\frac{a'_n + b'_n}{2},$$
$$\tilde{Q} = \left(\{z \in Q : \operatorname{Im} z \geqslant \operatorname{Re} z\} \setminus \bigcup_{n=1}^{\infty} A_n\right) \cup \bigcup_{n=1}^{\infty} B_n,$$
$$F = \partial\tilde{Q} \setminus ((0, 2+2i] \cup [2+2i, 2+\pi i] \cup (1+\pi i, 2+\pi i]).$$

Let now $L_1, L_2, \ldots$ be a sequence of mutually disjoint closed intervals over the segment

$$\left\{z : 0 < \operatorname{Re} z < \frac{3}{2}, \operatorname{Im} z = \operatorname{Re} z\right\}$$

such that $L_n \to 0$.

Let $S_1 \subset \tilde{Q}$ be narrow enough closed ribbon starting at L_1, entering in B_1 until the point $z'_1 \in S_1$, continuing thereafter and finishing at z_1, always without crossing the line $\{z : \operatorname{Im} z = a_2\}$. Assume that $S_1, S_2, \ldots, S_n$ are already constructed. Then, $S_{n+1} \subset \tilde{Q}$ is a narrow enough closed ribbon starting on L_{n+1} with the following properties:

(i) $z'_{n+1} \in S_{n+1}$;

(ii) S_{n+1} is always in the left-hand side of S_n. In particular, $S_{n+1} \cap S_j = \emptyset$ for each $j \leqslant n$;

(iii) S_{n+1} ends at the point z'_{n+1} without crossing the line $\{z : \operatorname{Im} z = a_{n+2}\}$.

(iv) $d_{\mathcal{H}}(\partial S_{n+1}, F) < \min\{1/n, d_{\mathcal{H}}(\partial S_n, F)\}$, where $d_{\mathcal{H}}$ is the Hausdorff distance.

This process can be continued indefinitely. Then, take

$$W_+ = \operatorname{Int}\left(\bigcup_{n=1}^{\infty} S_n \cup \{z \in Q : \operatorname{Re} z \geqslant \operatorname{Im} z\}\right).$$

Define, finally, $G = \exp(W_+ \cup W_- \cup (0,2))$, where W_- denotes the reflection of W_+ over the real axes. Now, the point $w = 1$ is an accessible point from G and $\mathbb{C} \setminus \overline{G} = B$ is a bounded component from which $[b, a)$ and $(c, 1]$ are inaccessible from B.

In Figure 6 the third step of the construction of W_+ was shown.

Figure 7 shows the domain G and the component B, this picture can help the reader to get a better understanding of the constructed domain.

Let now g map a given domain G conformally onto $\mathbb{D}$. The question whether g has a continuous extension to $\overline{G}$, or not, is also very interesting and important, however it usually not included in textbooks and courses on conformal maps.

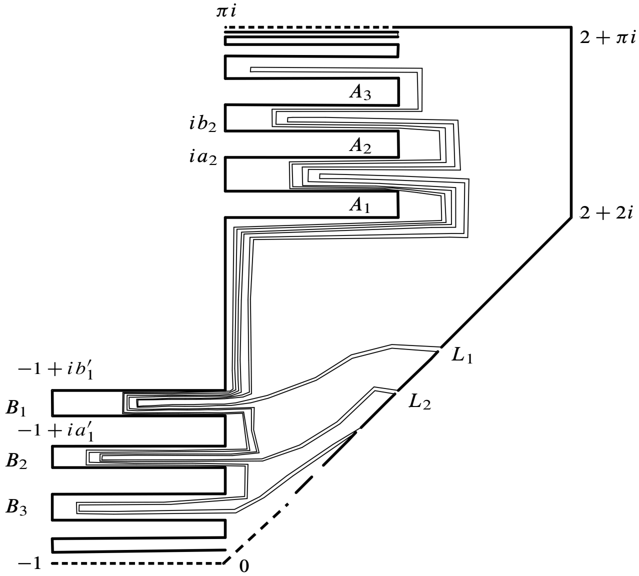

Figure 6. The third step of the construction of W_+.

Definition 2.21. Let G be a simply connected domain, and let f be a conformal map from $\mathbb{D}$ onto G. A point $w \in \partial G$ is said to be simple in the sense of Carathéodory if the set $\{\zeta \in \mathbb{T} : w \in C(f, \zeta)\}$ is a singleton.

The concept of a simple point in the sense of Carathéodory is independent of the choice of f. For the domain G_1, see Figure 4, all points in the arc $[A, B)$ are not simple in the sense of Carathéodory, while all other points in ∂G_1 are simple in this sense. To avoid confusion with other uses of the term "simple point" (see, for example, [115, Chapter 14]), we decide to use the term "simple in the sense of Carathéodory".

The next result was obtained in [44], it gives the criterion for continuity of g. However, this characterization is not completely topological. A proof can be found in [85].

Proposition 2.22. *Let G be a bounded simply connected domain and let g map G conformally onto $\mathbb{D}$. A continuous extension $\tilde{g}\colon \overline{G} \to \overline{\mathbb{D}}$ of g exists if and only if each point $w \in \partial G$ is a simple point in the sense of Carathéodory.*

In other words, the existence of a continuous extension of g is equivalent to the statement that distinct prime ends have disjoint impressions. Figure 1 can help to better understanding the previous result since there exists a continuous extension of g for G_1, but not for G_2.

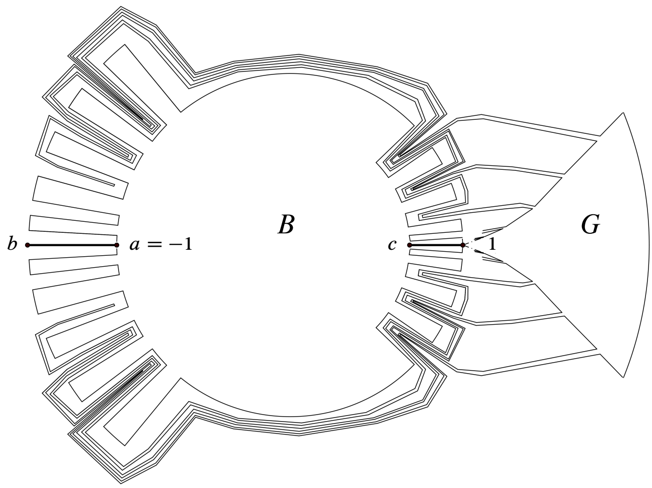

Figure 7. Inaccessible points from the bounded component B.

Proof. For the proof of necessity let us assume that some point $w \in \partial G$ is not simple. Then, $w \in C(f, \zeta_1) \cap C(f, \zeta_2)$, where $f = g^{-1}$ and ζ_1 and $\zeta_2 \neq \zeta_1$ are two points in $\mathbb{T}$. One can find two sequences, say (z_n) and (z_n'), such that $z_n \to \zeta_1$, $z_n' \to \zeta_2$, while $f(z_n) \to w$ and $f(z_n') \to w$. In this case, the continuity of $\tilde{g}$ would imply that $\tilde{g}(w) = \lim z_n = \zeta_1$ and $\tilde{g}(w) = \lim z_n' = \zeta_2$ which is a contradiction.

The proof of sufficiency. Let us define $\tilde{g}(w) = \zeta$, where $\zeta \in \mathbb{T}$ is the unique point such that $w \in C(f, \zeta)$ in the case that $w \in \partial G$, while $\tilde{g}(w) = g(w)$ for $w \in G$. The continuity property of $\tilde{g}$ is not difficult, but some arguments from the theory of cluster sets are needed for the proof. ■

Furthermore, in [44] Farrell proved the following result, which is related to the theorem about kernel convergence.

Theorem 2.23. *Let G be a Carathéodory domain such that each point in ∂G is simple in the sense of Carathéodory. Let $z_0 \in G$ and let (G_n) be a sequence of bounded simply connected domains, such that*

$$\overline{G} \subset G_{n+1} \subset \overline{G_n},$$

for $n \geqslant 1$, and $G_n \to G$ with respect to z_0. For $n \geqslant 1$, let g_n be the conformal map from G_n to $\mathbb{D}$ such that $g_n(z_0) = 0$ and $g_n'(z_0) > 0$. Denote by $\tilde{g}$ the extension of the conformal map from G onto $\mathbb{D}$ to $\overline{G}$ with $\tilde{g}(z_0) = 0$ and $\tilde{g}'(z_0) > 0$.

Then, $g_n \rightrightarrows \tilde{g}$ on $\overline{G}$.

Assume now that G is a Carathéodory domain, $z_0 \in G$ and let f be the conformal map from $\mathbb{D}$ onto G with the normalization $f(0) = z_0$ and $f'(0) > 0$. Furthermore, let $f^{-1}\colon G \to \mathbb{D}$ be the corresponding inverse map. The next result is a refinement of [26, Theorem 1].

Theorem¶ 2.24. *Let G be a Carathéodory domain and let (J_n) be a sequence of Jordan curves such that $D(J_n) \to G$ with respect to some point $z_0 \in G$ and $\overline{G} \subset D(J_n)$, $\overline{D(J_n)} \subset D(J_{n-1})$ for each $n > 1$. Let $f_n\colon \overline{\mathbb{D}} \to \overline{D(J_n)}$ be the extension of the respective conformal map with the normalization $f_n(0) = z_0$ and $f_n'(0) > 0$ inherited from f. Then, the following hold.*

(1) *If $\mathcal{E}$ is an end-cut in G, then f_n^{-1} converges uniformly on $\mathcal{E}$ to f^{-1}, in particular, $f_n^{-1}(z) \to f^{-1}(z)$ for each point $z \in \partial_a G$;*

(2) *If W is a bounded component of $\mathbb{C} \setminus \overline{G}$, then $|f_n^{-1}| \to 1$ uniformly on $\overline{W}$. However, in general it is not true that f_n^{-1} converges to some constant on $\overline{W}$.*

Since the proof of this theorem is essentially the same as the respective proof in [26], we present here only its sketch which highlights the keynote steps.

Sketch of the proof of Theorem 2.24. Without loss of generality we may also assume that $\mathcal{E}$ starts at the point z_0. Let now $b_0 \in \partial_a G$ be the end point of $\mathcal{E}$. Put $\varrho := f^{-1}(\mathcal{E})$ so that ϱ is an arc in $\mathbb{D} \cup \{\zeta_0\}$, where $\zeta_0 = f^{-1}(b_0)$, passing from 0 to ζ_0.

For each $m \in \mathbb{N}$ we consider a point $b_m \in J_m$ which is a nearest point to b_0. For each $m \geqslant 1$ we put $\mathcal{E}_m := \mathcal{E} \cup [b_0, b_m]$ and $\varrho_m := f_m^{-1}(\mathcal{E}_m)$. Let $\zeta_m = f_m^{-1}(b_m)$ and note, that each $\varrho_m = f_m^{-1}(\mathcal{E}) \cup f_m^{-1}([b_0, b_m])$ is the union of two consecutive arcs in $\mathbb{D} \cup \{\zeta_m\}$. It is clear, that the sequence (ϱ_m) accumulates to some subset Λ of $\overline{\mathbb{D}}$. It means that Λ is the set of all points $w \in \overline{\mathbb{D}}$ such that there exists a sequence (w_{m_j}) of points such that $w_{m_j} \in \varrho_{m_j}$ and $w_{m_j} \to w$ as $j \to \infty$.

The set Λ possesses some special properties. Namely, one has

(i) Λ is a continuum;

(ii) $\Lambda \subset \varrho \cup \mathbb{T}$;

(iii) $\varrho \subset \Lambda$;

(iv) The set $\Lambda \cap \mathbb{T}$ is connected.

Therefore, $\Lambda = \varrho \cup \gamma$, where γ is some closed subarc of $\mathbb{T}$. In order to prove the first assertion we need to show that $\Lambda = \varrho$ or, in other words, that $\gamma = \{\zeta_0\}$.

Let w_m' be a nearest point of the set ϱ_m to t_0 and let ϱ_m' be the subcontinuum $f_m^{-1}(\mathcal{E}_m')$, where $\mathcal{E}_m'$ is the segment $[f_m(w_m'), b_m]$ in the case when $f_m(w_m') \notin \mathcal{E}$, or the set $\mathcal{E}_m'' \cup [b_0, b_m]$ otherwise, where $\mathcal{E}_m''$ is the subarc of $\mathcal{E}$ that joints the points $f_m(w_m')$ and b_0.

We have that $f_m(w_m') \to b_0$ as $m \to \infty$ and therefore $\operatorname{diam}(f_m(\varrho_m')) \to 0$ as $m \to \infty$.

Notice that ϱ_m' is either an arc or the union of two consecutive arcs. Then, applying [103, Theorem 9.2] to ϱ_m', or to each of the arcs that form ϱ_m', we conclude, that $\operatorname{diam}(\varrho_m') \to 0$ as $m \to \infty$, which means, that $\zeta_m \to \zeta_0$ as $m \to \infty$ and hence, $\gamma = \{\zeta_0\}$.

We are going to prove the assertion of the part (2). Assume that $|f_n^{-1}|$ does not converge uniformly to 1 on $\overline{W}$. Then, there exist a sequence (z_k) in W and a subsequence $(f_{n_k}^{-1})$ such that $|f_{n_k}^{-1}(z_k)| \leqslant r < 1$ for all k. Let $w_k := f_{n_k}^{-1}(z_k)$. Taking a subsequence of (w_k) if it is necessary we may assume that $w_k \to w_0$, $|w_0| \leqslant r < 1$. Since f_{n_k} converge uniformly on the compact set $\bigcup_{k=0}^{\infty}\{w_k\}$ to f we have

$$f(w_0) = \lim_{k\to\infty} f_{n_k}(w_k) \in \overline{W}.$$

But $f(w_0) \in G$ and $\overline{W} \cap G = \emptyset$, so that we arrive to a contradiction.

For the last assertion we must consider Example 2.15, where $W_{green} \cup W_{red} \subset D(J_n)$. Then, the sequence (f_n^{-1}) has two accumulation points, say ζ_1 and ζ_2 with the notation in the aforementioned example. To prove this some arguments are needed. However, we omit them, because we believe that this help will be enough for the reader. ∎

Corollary¶ 2.25. *Let G be a Carathédory domain. Then, f and g can be extended to Borel measurable functions (denoted also by f and g) on $\mathbb{D} \cup F(f)$ and $G \cup \partial_a G$, respectively, and such that*

$$g(f(\xi)) = \xi \quad \text{for all } \xi \in F(f),$$
$$f(g(\zeta)) = \zeta \quad \text{for all } \zeta \in \partial_a G.$$

The domain G_2 in Figure 4, which is not a Jordan domain, has the property $\partial_a G_2 = \partial G_2$. For such domains one has the following corollary.

Corollary¶ 2.26. *Let G be a Carathéodory domain such that $\partial_a G = \partial G$, and let f be some conformal map from $\mathbb{D}$ onto G. Then, f^{-1} can be extended to $\overline{G}$ and this extension belongs to the first Baire class on $\overline{G}$.*

Notice, that this corollary generalizes the Carathéodory extension theorem to the case that the domain under consideration is a Carathéodory domain with accessible boundary. It is clear, that this class of domain is substantially wider than the class of Jordan domain.

Chapter 3

Uniform and pointwise approximation on Carathéodory sets

3.1 Uniform approximation by polynomials

Problems on approximation of analytic functions by polynomials and rational functions were always of special importance during the development of contemporary analysis, but they have attracted special attention after the classical results about approximation in the complex domain obtained by Weierstrass and Runge at the end of 19th century. Let us recall, that Weierstrass proved that any continuous function defined on [0, 1] may be uniformly approximated on this segment by a sequence of polynomials. The *Runge's theorem* is as follows.

Theorem 3.1. *Let $K \subset \mathbb{C}$ be a compact set, and let $E \subset \mathbb{C}_\infty \setminus K$ be a set which contains, at least, one point of each component of $\mathbb{C}_\infty \setminus K$. If $f \in H(K)$, then for every $\varepsilon > 0$ there exists a rational function R with poles on E such that $\|f - R\|_K < \varepsilon$.*

This theorem was published in 1885, [116], the same year as the aforesaid result by Weierstrass. There are several proofs of Runge's theorem, see, for instance, [118, pages 171–177] for the proof which is close to the original one. See also [115, Chapter 13] for the proof using certain functional analysis methods, and [33, Chapter VIII] for a more direct and elementary proof.

The following properties follow directly from Runge's theorem. Let Ω be an open set in $\mathbb{C}$, let E be a set which contains one point of each component of $\mathbb{C}_\infty \setminus \Omega$. Then, for every function $f \in H(\Omega)$ one can find a sequence (R_n) of rational functions with poles lying only in E, such that $R_n \rightrightarrows f$ locally in Ω. In the special case when the set $\mathbb{C}_\infty \setminus \Omega$ is connected (note that this means that Ω is a simply connected set, but not necessarily a connected one), one can take $E = \{\infty\}$ and a sequence (K_n) of compact subsets of Ω such that $\bigcup_{n=1}^{\infty} K_n = \Omega$, and thus obtain a sequence (P_n) of polynomials such that $P_n \rightrightarrows f$ locally in Ω. Let us observe that the set $\mathbb{C}_\infty \setminus \Omega$ may have uncountably many components: for instance one can consider $\Omega = \mathbb{C} \setminus K$, where $K \subset [0, 1]$ is the linear $1/3$-Cantor set.

Note that the condition that the set $\mathbb{C}_\infty \setminus \Omega$ is connected cannot be relaxed in the latter statement. Namely, one has the following theorem.

Theorem 3.2. *Let $U \subset \mathbb{C}$ be an open set, and assume that for every $f \in H(U)$ there exists a sequence (P_n) of polynomials such that $P_n \rightrightarrows f$ locally in U. Then, the set $\mathbb{C}_\infty \setminus U$ is connected.*

Indeed, assume that the set $\mathbb{C}_\infty \setminus U$ is not connected, then $\mathbb{C}_\infty \setminus U = K \cup Y$, where K is a compact subset of $\mathbb{C}$, Y is closed set, $\infty \in Y$, and $K \cap Y = \emptyset$. By the *separation theorem*, see [136, page 108], there exists a Jordan curve $J \subset U$ such that $K \subset D(J)$. Let $a \in K$, then the function $h(z) = 1/(z-a)$ cannot be approximated by a sequence of polynomials uniformly on J. Indeed, let $C = \sup\{|z-a| : z \in J\}$ and $\rho = 1/(2C)$. If there exists $P \in \mathcal{P}$ such that $\|h - P\|_J < \rho$, then the inequality

$$|1 - p(z)(z-a)| < \rho|z-a| \leqslant \frac{1}{2}$$

holds for all $z \in J$. Therefore, by the maximum modulus principle, this inequality also holds for $z = a$, but this is a contradiction.

For further considerations we need to introduce several algebras of functions. Let $K \subset \mathbb{C}$ be a compact set. Denote by $P(K)$ the algebra of all functions which can be approximated uniformly on K by polynomials, so that $P(K)$ is the closure in $C(K)$ of the subspace $\mathcal{P}|_K$. Next, let $R(K)$ be the algebra consisting of all functions which can be approximated uniformly on K by rational functions with poles lying outside K. Furthermore, we put $A(K) = C(K) \cap H(K^\circ)$. It is clear that

$$P(K) \subset R(K) \subset A(K) \subset C(K). \tag{3.1}$$

All aforesaid algebras $A(\cdot)$, $R(\cdot)$ and $P(\cdot)$ may be defined in the same way for any closed subset of $\mathbb{C}_\infty$.

It can be readily verified that $P(\overline{\mathbb{D}}) = A(\overline{\mathbb{D}})$ and $R(\mathbb{T}) = C(\mathbb{T})$. Furthermore, the equality $P(K) = C(K)$ implies that the set $\mathbb{C} \setminus K$ is connected, while the Runge's theorem says that $P(K) = R(K)$ whenever the set $\mathbb{C} \setminus K$ is connected.

The question on for which compact sets K the approximation property $P(K) = A(K)$ is satisfied is quite natural. The investigation of this question was started in the 1920s by J. L. Walsh, who dealt with two important cases when K is the closure of a generic Jordan domain, and when K is a closed arc. In [129–131] Walsh proved several results, and his most general statement in this topic is as follows (for proofs and further details see [134, Chapter II]).

Theorem 3.3. *Let $Y \subset \mathbb{C}_\infty$ be a closed set such that ∂Y is a finite union of Jordan curves or closed arcs, no two of which have more than finitely many common points. Then, $A(Y) = R(Y)$. More precisely, let $E \subset \mathbb{C}_\infty \setminus Y$ be a set that contains at least one point of each component of $\mathbb{C}_\infty \setminus Y$. Then, for every $f \in A(Y)$ there exists a sequence (F_n) of rational functions with poles lying outside E such that $F_n \rightrightarrows f$ on Y.*

Let us comment how this result was proved in a particular case. If $Y = \overline{G}$ for some Jordan domain G, the proof runs as follows. Take a function $f \in A(Y)$ and a sequence (G_n) of Jordan domains such that $G_n \to G$. Let ψ_n be conformal map from G_n onto G as it was considered in Theorem 2.10. Then, each function $f \circ \psi_n$ is holomorphic in some neighborhood of Y (each function in its own one). Next, given an arbitrary $\varepsilon > 0$ Runge's theorem implies that there exists a polynomial P_n such that $|f(\psi_n(z)) - P_n(z)| < \varepsilon/2$ for all $z \in Y$. Finally, the fact that $\psi_n(z) \rightrightarrows z$ on Y and the uniform continuity of f on Y yield that $|f(\psi_n(z)) - f(z)| < \varepsilon/2$ for $z \in Y$.

The topological conditions imposed in Theorem 3.3 turned out to be not essential, since the following result was established in 1931 by F. Hartogs and A. Rosenthal, see [62]. If $K \subset \mathbb{C}$ is a compact set such that $\mathrm{Area}(K) = 0$, then $R(K) = C(K)$. What about compact sets X with empty interior for which $R(X) \neq C(X)$? Let us recall that the first example of such kind was constructed by A. Roth [110, page 97]. It was a compact set of the form $X = \overline{\mathbb{D}} \setminus \bigcup_{n \geqslant 1} D_n$, where each $D_n \Subset \mathbb{D}$ is some appropriately chosen open disk. The principal idea underlying Roth's example construction turned out to be crucial for a number of further constructions of examples of the failure of approximation. Let us note the construction of this kind given in [56, page 26]. In view of the shape of this compact set X, all such examples are called nowadays a "Swiss cheeses" or "Champagne bubbles".

Later on Walsh encouraged his student O. J. Farrell to study the problem of polynomial approximation to a function f holomorphic in a domain G but not necessarily continuous in $\overline{G}$ (but assuming only that f is bounded in G) and gave him some ideas how to proceed in this case. Farrell in [44] considered the problem on uniform approximation by polynomials of a conformal map from G onto the unit disk. As far as we know this is the second paper in the mathematical literature, where the notion of Carathéodory domain is important.

Theorem 3.4 (Farrell). *Let G be a bounded simply connected domain in $\mathbb{C}$, and let g map G conformally onto $\mathbb{D}$. Then, g has a continuous extension $\tilde{g}$ to $\overline{G}$ and $\tilde{g}$ may be approximated by polynomials uniformly on $\overline{G}$ if and only if G is a Carathéodory domain and all points in ∂G are simple in the sense of Carathéodory.*

Proof. Assume that the desired $\tilde{g}$ exists and that it can be approximated by polynomials uniformly on $\overline{G}$. Then, $\tilde{g}$ is continuous and Proposition 2.22 implies that all points in ∂G are simple. Moreover, $|\tilde{g}(w)| = 1$ for each $w \in \partial\Omega_\infty(\overline{G})$, then $|g(w)| < 1$ for $w \in \widehat{G} \cap \overline{G}$. So, $\partial G = \partial\Omega_\infty(\overline{G})$.

Conversely, fix a point $z_0 \in G$, take a sequence (G_n) of Jordan domains converging to G with respect to z_0 (see Theorem 2.11) and the corresponding sequence (g_n) of conformal maps from G_n onto $\mathbb{D}$. Since all point in ∂G are simple, then g has a continuous extension $\tilde{g}$ to $\overline{G}$, and in view of Theorem 2.23 for a given $\varepsilon > 0$ there exists such n that $|\tilde{g}(z) - g_n(z)| < \varepsilon$ for all $z \in \overline{G}$. Since $g_n \in H(G_n)$, it follows from Runge's approximation theorem that there exists $P_n \in \mathcal{P}$ such that $\|P_n - g_n\|_{\overline{G}} < \varepsilon$. Then, $\|\tilde{g} - P_n\|_{\overline{G}} < 2\varepsilon$ as desired. ∎

Furthermore, [44, Theorem IV] may be stated as follows.

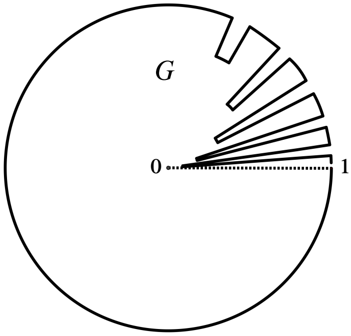

Figure 8. A counterexample to the opposite inclusion in (3.2).

Theorem 3.5. *Let G be a Carathéodory domain such that all points in ∂G are simple in the sense of Carathéodory, and let f be some conformal map from $\mathbb{D}$ onto G. Then,*

$$P(\overline{G}) \supset \{h \in A(\overline{G}) : h \text{ is constant on } I(\hat{f}(\zeta)) \text{ for each } \zeta \in \mathbb{T}\}. \tag{3.2}$$

Proof. Denote by B the set in the right-hand side of (3.2) and take $h \in B$. Put $F = h \circ f$. Then, F has a continuous extension $\widetilde{F}$ to $\overline{\mathbb{D}}$ because h is constant in each set $C(f, \zeta)$, $\zeta \in \mathbb{T}$. Then, $\widetilde{F} \in A(\overline{\mathbb{D}})$. Given $\varepsilon > 0$ let $P \in \mathcal{P}$ be such that $\|\widetilde{F} - P\|_{\overline{\mathbb{D}}} < \varepsilon$. Let $g = f^{-1}$, and let $\tilde{g}$ be the continuous extension of g to $\overline{G}$. Put $z = \tilde{g}(w)$ for $w \in \overline{G}$. Then,

$$|h(w) - P(\tilde{g}(w))| = |\widetilde{F}(z) - P(z)| < \varepsilon$$

for each $w \in \overline{G}$. Since $\tilde{g} \in P(\overline{G})$, then $P \circ \tilde{g} \in P(\overline{G})$. So, $h \in P(\overline{G})$. ■

The opposite inclusion in (3.2) is not true in the general case. To construct a direct example, let us consider a sequence

$$1 > a_1 > b_1 > a_2 > b_2 > \cdots a_n > b_n > a_{n+1} > \cdots > 0$$

such that $a_n \to 0$, and define the domain G, see Figure 8, in such a way that

$$\overline{G} = \overline{\mathbb{D}} \Big\backslash \bigcup_{n=1}^{\infty} \left\{ z = re^{it} : \frac{1}{n} < r \leqslant 1, t \in (b_n, a_n) \right\}.$$

It is clear that the constructed domain G is a Carathéodory domain. The function $h(z) = \sqrt{1-z}$, defined on $\overline{G}$, belongs to $P(\overline{G})$, but it is not constant in $[0, 1]$ which is the impression of some prime end.

Remark¶ 3.6. The set of the right-hand side of (3.2) seems to be very small in the case that f is not continuous on $\overline{\mathbb{D}}$. It looks quite plausible that this set is equal to the set of functions $F \circ f^{-1}$, where $F \in A(\overline{\mathbb{D}})$.

The following two theorems obtained by Lavrentiev [79] and Keldysh [72], respectively, turned out to be important milestones on the way of studying the problem of polynomial approximation on compact sets in the complex plane. In what follows they will be called *Lavrentiev's theorem* and *Keldysh's theorem*, respectively.

Theorem 3.7. *Let $K \subset \mathbb{C}$ be a compact set. Then, $P(K) = C(K)$ if and only if $K^\circ = \emptyset$ and $\mathbb{C} \setminus K$ is connected.*

Theorem 3.8. *Let $G \subset \mathbb{C}$ be a bounded domain. Then, $P(\overline{G}) = A(\overline{G})$ if and only if the set $\mathbb{C} \setminus \overline{G}$ is connected.*

Finally, the problem on characterization of such compact sets $K \subset \mathbb{C}$ for which it holds $P(K) = A(K)$ was completely solved by S. N. Mergelyan in 1952, see [90]. The following theorem summarize several Mergelyan's statements, it will be called *Mergelyan's theorem* in what follows.

Theorem 3.9. *Let $K \subset \mathbb{C}$ be a compact set.*

(1) *$P(K) = A(K)$ if and only if the set $\mathbb{C} \setminus K$ is connected.*

(2) *If $\mathbb{C} \setminus K$ has finitely many components, then $A(K) = R(K)$.*

(3) *Assume that there exists a decreasing sequence (δ_n) with $\delta_n \to 0$ such that for each point $b \in \partial K$ there exist an arc $\gamma_n \subset D(b, \delta_n) \cap K^{\complement}$ and a number $r_n > 0$ such that $\operatorname{diam}(\gamma_n) \geqslant r_n$. Let $f \in A(K)$ and let $\omega(f, \cdot)$ denotes its modulus of continuity. If*

$$\liminf_{n\to\infty} \omega(f, \delta_n) \left(\frac{\delta_n}{r_n} \right)^2 = 0, \tag{3.3}$$

then for every $\varepsilon > 0$ there exist $F \in \mathcal{R}$ with $\{F\}_\infty \subset K^{\complement}$ such that

$$\|f - F\|_K < \varepsilon.$$

Notice that the part (3) of Mergelyan's theorem yields that $R(K) = A(K)$ whenever all components of $\mathbb{C} \setminus K$ have diameter bigger than some given number $\delta > 0$.

Several proofs of Mergelyan's theorem may be found in the literature, see, for instance, [115, Chapter 20], [55, Chapter III], [32, Section 8.6], and [134, Appendix I]. Moreover, in [24] one can find the dual proof of this theorem, due to L. Carleson, see also [125, Chapter V].

Observe that using the ideas underlying the proof of Runge's theorem the statement of the part (3) of Theorem 3.9 may be improved in such a way that all poles of F can be chosen to belong to some prescribed set containing a point of each component of $\mathbb{C}_\infty \setminus K$.

Theorem 3.10. *Let K be a Carathèodory compact set, then $R(K) = A(K)$.*

Proof. For each $\delta > 0$ let $a \in \partial K = \partial\Omega_\infty(K)$. Then, take $a' \in \Omega_\infty(K)$ such that $|a - a'| < \delta/2$. Then, a' can be joined to ∞ by some infinite polygonal line L. The part of L that contains a' and ends in the first point, where L exists $\overline{D(a,\delta)}$ is an arc with diameter bigger than $\delta/2$. So, for each sequence (δ_n) in conditions of part (3) in Theorem 3.9 one can take $r_n = \delta/2$ and hence (3.3) holds for each function $f \in A(K)$. Then, $R(K) = A(K)$. ∎

Corollary 3.11. *Let U be a Carathéodory open set, then $R(\partial U) = C(\partial U)$.*

The problem on characterization of those compact sets K for which it holds $R(K) = A(K)$ was solved in 1967 by A. G. Vitushkin in terms of the analytic capacity of the sets $D(a,r) \setminus K$ and $D(a,r) \setminus K^\circ$. We are not going to enter this topic, and we refer to [128] and [56, Chapter VIII] for the corresponding explanation. But one ought to pay attention to the following thing. For proving his result Vitushkin have proposed and elaborate the special approach to approximation, which is based on localization of singularities of the function being approximate, and further approximation of each localized functions. Using this approach one can obtain another proof of Theorem 3.10 without using Mergelyan's theorem. An example of the proof of such kind (in a different situation of approximation by polyanalytic rational functions) may be found in [28, Proposition 2.5]. In view of this it would be interesting to obtain the proof of Theorem 3.10 that avoids both the application of Mergelyan's theorem and Vitushkin's localization technique, at least in the case that $K = \overline{G}$ for a Carathéodory domain G.

3.2 Uniform harmonic approximation

An investigation of the problem on approximation of continuous functions by harmonic ones was started by Walsh in the 1920s. For an open set U let $\mathrm{Har}(U) = \mathrm{Har}(U, \mathbb{R})$ be the set of all real harmonic functions on U. Next, for a compact set $K \subset \mathbb{C}$ we denote by $\mathrm{Har}(K)$ the set of functions $u|_K$, where $u \in \mathrm{Har}(V)$ for some (depending on u) open set V containing K, and by $\overline{\mathrm{Har}(K)}$ the closure of $\mathrm{Har}(K)$ in $C(K)$. Then, $\overline{\mathrm{Har}(K)} \subset C(K) \cap \mathrm{Har}(K^\circ)$. By definition, a harmonic polynomial is $\operatorname{Re} P$, where $P \in \mathcal{P}$. For example the real polynomial $x^3 - 3xy^2 + x^2 + 2xy - y^2$ is harmonic, since it is a real part of $z^3 + (1-i)z^2$. Here, and in the sequel a real polynomial means a polynomial in two real variables x and y with real coefficients. A good reference for study of harmonic functions from the point of view of complex analysis is the book [107].

Let us also recall that a domain $G \subset \mathbb{C}$ is called n-connected, if the set $\mathbb{C}_\infty \setminus G$ has n components. A domain G is called finitely connected, if it is n-connected for some integer $n \geqslant 1$. Notice also that if G is a domain in $\mathbb{C}$, while K is some component of the set $\mathbb{C}_\infty \setminus G$ and K does not contain ∞, then K needs to be a compact subset of $\mathbb{C}$.

A bit of background about Dirichlet problem and harmonic measure

Let us recall some facts about harmonic functions, harmonic measure and the Dirichlet problem that we will use in what follows. Let U be a non-empty bounded open set of $\mathbb{C}$ and $f : \partial U \to \mathbb{R} \cup \{\pm\infty\}$ be an arbitrary function. Following the traditional terminology we will call such f a *boundary function*. Let us denote by $\overline{\mathcal{U}}_f$ the set of all functions h which are superharmonic or identically equal to $+\infty$ in each component of U with $\liminf_{y\to x} h(y) \geqslant f(x)$ for all $x \in \partial U$, and which are bounded from below on U. Furthermore, let $\overline{\mathcal{H}}_f$ be the function defined as follows: $\overline{\mathcal{H}}_f = \inf\{h : h \in \overline{\mathcal{U}}_f\}$. One says that $\overline{\mathcal{H}}_f$ is the upper solution of the generalized Dirichlet problems in U for the boundary function f. Next, similarly, one can define the set $\underline{\mathcal{U}}_f$ as the set of all functions h which are subharmonic or identically equals $-\infty$ in each component of U with $\limsup_{y\to x} h(y) \leqslant f(x)$ for all $x \in \partial U$, and bounded from above on U. Using this set we define the function $\underline{\mathcal{H}}_f := \sup\{h : h \in \underline{\mathcal{U}}_f\}$. Such function $\underline{\mathcal{H}}_f$ is called the lower solution of the generalized Dirichlet problem in U with the boundary function f. These definitions, as well as proofs of almost all results mention here in connection with Dirichlet problem may be found in [67, Chapter 8]. If $\overline{\mathcal{H}}_f = \underline{\mathcal{H}}_f$ and if both these functions are harmonic on U, then f is called a *resolutive boundary function*, while the function $\mathcal{H}_f = \overline{\mathcal{H}}_f = \underline{\mathcal{H}}_f$ is called the *solution of Dirichlet problem* with boundary function f (or, shortly, *Dirichlet solution* for f). The corresponding method to obtain a harmonic function from a boundary function f is called the *Perron–Wiener–Brelot* method.

Wiener's theorem says that any function $f \in C(\partial U)$ is a resolutive boundary function. Having this in mind we have the following statement, see [67, Lemma 8.12].

Lemma 3.12. *For $z \in U$ and $f \in C(\partial U)$ let $L_z(f) = \mathcal{H}_f(z)$. Then, L_z is a positive linear functional on the space $C(\partial U)$ and there exists a unique Borel probability measure μ_z on ∂U such that for all $z \in U$ and $f \in C(\partial U)$ it holds*

$$\mathcal{H}_f(z) = L_z(f) = \int f \, d\mu_z.$$

Moreover, we have (see [67, Theorem 8.14]).

Lemma 3.13. *If W is a component of U, then the class of Borel subsets of ∂U of μ_z-measure zero is independent of $z \in W$.*

Now, for $z \in U$ we define the set $\mathcal{F}_z$ of all sets having the form $(E \setminus N) \cup (N \setminus E)$ with $E \subset \partial U$ and $N \subset B$, where N and B are Borel sets such that $\mu_z(B) = 0$. Then, $\mathcal{F} := \bigcap_{z\in\partial U} \mathcal{F}_z$ is a σ-algebra containing all Borel subsets of ∂U, and the measure μ_z can be uniquely extended to $\mathcal{F}$.

Definition 3.14. The measure μ_z defined above is called the *harmonic measure* on ∂U relative to U and z, and it will be denoted in what follows by $\omega(z, \cdot, U)$.

Using Lemma 3.13 and the standard Radon–Nikodym theorem one can see that for every component W of U the measures $\omega(z_1, \cdot, U)$ and $\omega(z_2, \cdot, U)$ are mutually absolutely continuous for any points $z_1, z_2 \in W$. Moreover, the Radon–Nikodym derivative $h := d\omega(z_1, \cdot, U)/d\omega(z_2, \cdot, U)$ satisfies

$$\omega(z_1, \cdot, U) = h \cdot \omega(z_2, \cdot, U), \quad \text{and} \quad C^{-1} \leqslant |h(z)| \leqslant C \quad \text{for a.a. } z \in \partial W, \tag{3.4}$$

where $C > 0$ is some constant depending on z_1, z_2, W and U. Furthermore, $\omega(z, \cdot, U)$ has no atoms for each $z \in U$.

A keynote property of the harmonic measure is the following result.

Theorem 3.15. *Let U be a non-empty bounded open set. A boundary function f is resolutive if and only if it is $\omega(z, \cdot, U)$-integrable for some $z \in U$. If f is resolutive, then for all $z \in U$ it holds*

$$\mathcal{H}_f(z) = \int_{\partial U} f(\zeta)\, d\omega(z, \zeta, U).$$

To study the behavior of $\mathcal{H}_f(z)$ when $z \to \zeta \in \partial U$ we need the notion of regular point. Recall that a point $\zeta \in \partial U$ is said to be a *regular point*, if $\lim_{z\to\zeta} \mathcal{H}_f(z) = f(\zeta)$ for every function $f \in C(\partial U)$. A bounded set U is said to be *regular* (or *Dirichlet*) *open set*, if every point of ∂U is a regular one.

There are several sufficient conditions to conclude that a given point is regular, for example if there is a (half-opened) segment $[a, \zeta) \subset \mathbb{C} \setminus U$ with $\zeta \in \partial U$. However, the more useful condition is the following one given by A. Lebesgue.

Theorem 3.16. *Let $\zeta \in \partial U$ be such a boundary point that there exists a continuum $\mathcal{L}$ (consisting of more than one point) such that $\mathcal{L} \setminus \{\zeta\} \subset \mathbb{C} \setminus U$. Then, ζ is a regular point. In particular, if U is a simply connected set, then it is a Dirichlet open set.*

The proof of this theorem may be found in [33, Chapter X].

It follows from this theorem that any nonempty bounded open set $U \subset \mathbb{C}$ such that no component of ∂U reduces to a singleton is a Dirichlet set. For such open set U and for a function $f \in C(\partial U)$ let us define

$$\hat{f}(z) = \begin{cases} f(z) & \text{if } z \in \partial U, \\ \mathcal{H}_f(z) & \text{if } z \in U. \end{cases} \tag{3.5}$$

Then, $\hat{f} \in C(\overline{U}) \cap \operatorname{Har}(U)$. Moreover, for every $z \in U$ it holds

$$\hat{f}(z) = \int_{\partial U} f(\zeta)\, d\omega(z, \zeta, U). \tag{3.6}$$

In fact, we have the following corollary.

Corollary¶ 3.17. *Let U be a Carathéodory open set. Then, all points in ∂U are regular. So, U is a Dirichlet open set. Moreover, if B is a bounded connected component of $\mathbb{C} \setminus U$, then $\omega(z, \partial B, U) = 0$ for every $z \in U$.*

Uniform approximation by harmonic functions

Let us start with one suitable generalization of the fact, that the open connected set U is simply connected if and only if for every function $h \in H(U)$ there exists a sequence (P_n) of polynomials such that $P_n \rightrightarrows h$ locally in U. We have

Theorem 3.18. *Let G be a finitely connected domain, let E_j, $j = 1, \dots, N$, $N \geqslant 1$, are all bounded components of $\mathbb{C} \setminus G$, and let $a_j \in E_j$ for each $j = 1, \dots, N$. Then, any function $u \in \operatorname{Har}(G)$ can be uniquely expressed in G in the form*

$$u(z) = \operatorname{Re} h(z) + \sum_{j=1}^{N} c_j \log |z - a_j|, \quad z \in G, \tag{3.7}$$

where $h \in H(G)$ and $c_1, c_2, \dots, c_N$ are real numbers.

Furthermore, let $K \subset \mathbb{C}$ be a compact set, and let $G_1, G_2, \dots$ be all bounded components of the set $\mathbb{C} \setminus K$ (if exist). Let $a_j \in G_j$ for each j. Then, the set of functions of the form (3.7), where h runs over $R(K)$ and $c_j \in \mathbb{R}$, is dense in $\overline{\operatorname{Har}(K)}$. In particular, if $\mathbb{C} \setminus K$ is connected, then the harmonic polynomials are dense in $\overline{\operatorname{Har}(K)}$.

The first part of this theorem is a very classical result, it is known by the name of Logarithmic Conjugation theorem. However, it is not clear what is the most relevant reference to it prior to the paper [6], where one can find the history, the direct proof, and several consequences of this result. It seems that the first occurrence of the aforementioned result in the mathematical literature was in [132], but the assumption that the domain under consideration has analytic boundary was made therein.

The result of the second part of Theorem 3.18 is not a difficult fact, its detailed proof may be found in [18, Section 3.4]. Note, that this result can be proved using duality arguments as follows. Take a real valued measure μ on K which is orthogonal to the functions $\operatorname{Re} h$, $h \in R(K)$, and $\log |z - a_j|$ for all indices j. One can check that for the logarithmic potential of μ

$$\breve{\mu}(w) = \int \log |z - w| \, d\mu(z),$$

which is defined a.e. in $\mathbb{C}$, one has $\breve{\mu}(w) = 0$ for each $w \notin K$. This fact together with the formula

$$\int g \, d\mu = \frac{1}{2\pi} \int \Delta g \, \breve{\mu} \, dA,$$

which is valid for all compactly supported functions g of class C^2, implies that μ is orthogonal to $\overline{\mathrm{Har}(K)}$ (the symbol Δ stands, as usual, for the Laplace operator).

We are going now to proceed with the Walsh–Lebesgue theorem, which is one of the most famous and most important results about approximation of functions by harmonic polynomials. The name of Walsh–Lebesgue theorem is associated in the literature to several related results. In order to be more clear we present here three such results. The first one was proved in [132]. Later on L. Carleson in [24] made a new proof because he says that the original proof is not complete. Walsh repeatedly said in [132, 133] that his proofs are based on Lebesgue's important work [80]. This explains the reason why the name "Walsh–Lebesgue theorem" was subsequently adopted for the next Theorems 3.19, 3.21, 3.22, and 3.23.

Theorem 3.19 (Walsh–Lebesgue theorem; the first of such name). *Let $K \subset \mathbb{C}$ be a compact set with connected complement. Then, for every function $u \in C(\partial K, \mathbb{R})$ there exists a sequence (P_n) of harmonic polynomials such that $P_n \rightrightarrows u$ on ∂K.*

Scheme of the proof. Let (K_n) be a sequence of compact sets, each of which has a boundary consisting of a finite number of C^1-smooth Jordan curves, such that $K_{n+1} \subset K_n^\circ$ and

$$K = \bigcap_{n=1}^{\infty} K_n.$$

Each continuous function on ∂K can be approximated uniformly on ∂K by C^1-smooth functions. Then, one can assume that $u \in C^1(\mathbb{C})$. In each domain K_n° take u_n to be the solution of the Dirichlet problem with boundary data $u|_{\partial K_n}$. Each set K_n° is simply connected, then each function u_n is the real part of some holomorphic function f_n. Each of these functions f_n can be approximated by polynomials in view of Runge's theorem. The real part of these polynomials are harmonic polynomials, and they converge uniformly on ∂K to u_n. It remains to show that $u_n \rightrightarrows u$ on ∂K. This fact is a keynote point of the proof, and it is a consequence of the following lemma due to A. Lebesgue.

Lemma 3.20. *Let $K \subset \mathbb{C}$ be a compact set, and let (K_n) be such sequence of compact sets that ∂K_n consists of a finite number of smooth closed curves, $K_{n+1} \subset K_n^\circ$, and $\bigcap_{n=1}^{\infty} K_n = K$. Let $u \in C^1(\mathbb{C})$, and let u_n be the harmonic extension of $u|_{\partial K_n}$ to K_n°. If each $z \in \partial K$ satisfies the condition*

$$\int_S \frac{dr}{r} = +\infty, \tag{3.8}$$

where

$$S = \{r \in (0, +\infty) : \partial D(z, r) \cap K^{\complement} \neq \emptyset\},$$

then $u_n \rightrightarrows u$ on ∂K.

The detailed proof of this lemma may be found in [56, pages 35–36]. The condition (3.8) is called *Lebesgue's condition*. ■

Another proof of Theorem 3.19 was given in [24, pages 168–171]. This proof follows the pattern of the proof of the part (2) of Theorem 3.18.

Theorem 3.21 (Walsh–Lebesgue theorem; the second of such name). *Let $K \subset \mathbb{C}$ be a compact set such that the set $\mathcal{B}_K$ of all bounded components of $\mathbb{C} \setminus K$ is not empty. Let E be a set that contains one point for each $G \in \mathcal{B}_K$. Suppose that*

(a) *the set $\mathcal{B}_K$ is finite, and $u \in C(\partial K, \mathbb{R})$, or*

(b) *each component of K is finitely connected, and $u \in C(K, \mathbb{R}) \cap \operatorname{Har}(K^\circ)$.*

Then, u can be approximated uniformly on K by functions of the form (3.7) *with such $h \in \mathcal{R}$ that all poles of h are inside E, the points $a_j \in E$ and $c_j \in \mathbb{R}$*

Scheme of the proof. The proof of item (a) is given in [18, page 191] using the theory of representing measure for $R(K)$. For item (b) we follow the outline proposed by Walsh. Take a closed disc $\overline{D}$ with $K \subset \overline{D}$ and a continuous function u_0 defined on $\overline{D}$ that extends u. Then, there exists a real polynomial P that differs from u_0 by less than a given $\varepsilon > 0$. The next step is to construct a decreasing sequence (S_j) of closed sets, each of which is bounded by a finite number of non-intersecting Jordan polygonal lines (with wedges parallel to coordinate axis), such that $K = \bigcap_{j=1}^{\infty} S_j$. Let now h_j be the solution for the Dirichlet problem on $\operatorname{Int}(S_j)$ with the boundary function $P|_{\partial S_j}$. Then, $h_j \rightrightarrows P$ on ∂K. Then, take $k \in \mathbb{N}$ such that the difference between h_k and P is less than ε on ∂K. But h_k can be uniformly approximated on K by a function of such kind that were considered in Theorem 3.18 (for the points of E). It remains to modify this approximating function in such a way to settle its singularities to the given points in E. Then, the approximation is obtained on ∂K, but since $u \in C(K) \cap \operatorname{Har}(K^\circ)$, the approximation also holds on K. ■

Next result is stated in [133, page 518] and it is the oldest result were the notion of Carathéodory set plays a role. It can be proved using Theorem 3.19.

Theorem 3.22 (Walsh). *Let $G \subset \mathbb{C}$ be a bounded simply connected domain, and let K be a compact set in $\mathbb{C}$. Then, the following statements hold.*

(a) *Each function $u \in C(\overline{G}, \mathbb{R}) \cap \operatorname{Har}(\operatorname{Int}(\overline{G}))$ can be uniformly approximated on $\overline{G}$ by harmonic polynomials if and only if G is a Carathéodory domain.*

(b) *Each function $g \in C(K, \mathbb{R})$ can be uniformly approximated on K by harmonic polynomials if and only if K is a Carathéodory compact set and $K^\circ = \emptyset$.*

Proof. Let us prove the statement of part (a). For proving the statement of part (b) see the next theorem.

Assume that G is a Carathéodory domain, then $\mathrm{Int}(\overline{G}) = G$, and so, $u \in \mathrm{Har}(G)$. Put $K = \widehat{G}$. If the set $\mathbb{C} \setminus \overline{G}$ is connected then applying Theorem 3.19 we obtain a sequence (u_n) of harmonic polynomials such that $u_n \rightrightarrows u$ on $\partial K = \partial G = \partial \overline{G}$. Since u and all u_n are harmonic functions, the maximum modulus principle for subharmonic functions (see, for instance, [67, Theorem 7.10]) yields that this convergence is uniform on $\overline{G}$.

Suppose now that the set $\mathbb{C} \setminus \overline{G}$ is not connected. By Proposition 1.5, part (a), each bounded component G_1 of the set $\mathbb{C} \setminus \overline{G}$ is simply connected and $\partial G_1 \subset \partial G$. Then, we can solve, for each G_1, the Dirichlet problem with boundary values $u|_{\partial G_1}$. Therefore, one can define a function $\tilde{u}\colon K \to \mathbb{R}$ as $u(z)$ for $z \in \overline{G}$ and $\tilde{u}(z) = (u|_{\partial G_1})^\wedge(z)$ given by (3.5). The key point is that $K^\circ = G \cup \bigcup_j G_j$. So, $\tilde{u} \in C(K) \cap \mathrm{Har}(K^\circ)$. Since $\mathbb{C} \setminus K$ is connected and $\partial G = \partial K$, then there exists the sequence (u_n) of harmonic polynomials that converges uniformly on K to u, then in particular on $\overline{G}$.

Let now there exists $w \in \partial G$ such that $w \notin \partial G_\infty$. Take $r > 0$ such that $\overline{D(w, 2r)} \cap \partial G_\infty = \emptyset$. Let $\rho\colon \mathbb{C} \to \mathbb{R}$ be a continuous function such that $\rho \equiv 1$ on $\overline{D(w, r)}$ and $\mathrm{Supp}(\rho) \subset D(w, 2r)$. Consider as before the solution $\widehat{\rho}$ of the Dirichlet problem in G with boundary function $\rho|_{\partial G}$. If there exists such a sequence (u_n) that $u_n \rightrightarrows \rho$ on $\overline{G}$ then $u_n \rightrightarrows 0$ on ∂G_∞. Then, $u_n \rightrightarrows 0$ on $G_\infty^\complement$. In particular, $u_n(w) \to \rho(w) = 1$, which gives a contradiction. ∎

As a consequence, of the previous result we have the next theorem, which was not explicitly stated in [133]. Occasionally it is also referred as Walsh–Lebesgue theorem (see, for instance, [99, Section 1]) and nowadays it is this statement that is perceived by experts in the theory of approximation by analytical functions as the most complete and general form of the Walsh–Lebesgue theorem.

Theorem 3.23 (Walsh–Lebesgue theorem; the third of such name). *Let $K \subset \mathbb{C}$ be a compact set. Then, each function from the space $C(K) \cap \mathrm{Har}(K^\circ)$ can be approximated uniformly on K by harmonic polynomials if and only if K is a Carathéodory compact set.*

Proof. Let K be a Carathéodory compact set. The keynote ingredient here is Proposition 1.8, because one has $\mathrm{Int}(\widehat{K}) = \mathrm{Int}(K) \cup \bigcup_j G_j$ and $\partial G_j \subset \partial K = \partial \Omega_\infty(K)$. If $g \in C(K) \cap \mathrm{Har}(K^\circ)$, then we define the function $\hat{g}$ on $\widehat{K}$ in such a way that $\hat{g}(z) = g(z)$ if $z \in K$, while $\hat{g}(z)$ is the solution of the Dirichlet problem with boundary function $g|_{\partial G_j}$ for $z \in G_j$. Then, $\hat{g} \in A_h(\widehat{K})$ then the proof is finished as before, applying Theorem 3.19.

Going further assume that $\partial K \neq \partial \widehat{K}$. Then, take a disk $D(a, r) \subset \mathrm{Int}(\widehat{K})$ with $a \in \partial K$. Next, taking $b \in D(a, r) \setminus K$, let us consider the function $g \in C(K)$ such that $g(z) = \log|z - b|$ for each $z \in K$. Then, there exists a sequence (q_n) of harmonic polynomials such that $\|g - q_n\|_K \to 0$ as $n \to \infty$. Then, by the maximus modulus principle

$$\|q_n - q_m\|_{\widehat{K}} = \|q_n - q_m\|_{\partial K} \to 0$$

as $n, m \to \infty$. Then, (q_n) is a Cauchy sequence on $\widehat{K}$, then it converges uniformly on $\widehat{K}$ to the function $g_1 \in C(\widehat{K}) \cap \mathrm{Har}(\mathrm{Int}(\widehat{K}))$. But $g_1(z) = \log|z-b|$ if $z \in D(a, r)$ which is a contradiction. ∎

Corollary 3.24. *Let K be a Carathéodory compact set. If $f \in C(K)$, then there exists a unique $u \in C(\widehat{K}) \cap \mathrm{Har}(\mathrm{Int}(\widehat{K}))$ such that $u(z) = f(z)$ for each $z \in K$.*

We must mention here that in [133] the condition in the part (b) of Theorem 3.21 stated by Walsh is different. He stated that "The compact K contains no region of infinite connectivity not included in a larger region of finite connectivity belonging to K. Then, in particular if K has no interior points, an arbitrary function $f(x, y)$ continuous on K can be so approximated". But the result with this formulation is not true, as one can see using the Deny's criterion for uniform approximation by functions harmonic in a neighborhood of K, see [36].

Ending our discussion on Walsh–Lebesgue theorem, let us mention the papers [95–97], where several interesting generalizations of this theorem were obtained in the situation when one deals with an approximation on boundaries ∂X of compact sets X in $\mathbb{C}$ with connected complement by functions of the form $P(\psi_1) + Q(\psi_2)$, where P and Q are polynomials in the complex variable, and ψ_1 and ψ_2 are two homeomorphisms of $\mathbb{C}$ to $\mathbb{C}$.

3.3 Pointwise polynomial approximation

Let us revert to the topic of approximation of functions by polynomials in the complex variable. We have seen in Theorem 3.2 that the locally uniform convergence of sequences of polynomials for each holomorphic function in a given open set implies certain topological restrictions on this set. But what can happen if we only suppose the pointwise convergence instead of the locally uniform one? Of course, the answer will depend on certain additional assumptions (such as, for instance, a boundedness of the corresponding sequence of approximating polynomials). In the most general case, when we did not demand anything else, the answer to this question was obtained by Montel: *For any open set $U \subset \mathbb{C}$ each function $f \in H(U)$ can be approximated by some sequence (P_n) of complex polynomials in such a way that $P_n(z) \to f(z)$ for every $z \in U$.* The proof of this fact may be obtained as follows. Let us take first some sequence (Y_n) of compact sets such that $\mathbb{C} \setminus Y_n$ is a connected set, $Y_n \subset U$, and $U = \liminf_{n\to\infty} Y_n$. A possible way to construct such sequence may be found in [85, Chaper IV, Section 2.3]. Next, Runge's theorem yields that for each n there exists a polynomial P_n such that $\|p_n - f\|_{Y_n} < 1/n$. Thus, the sequence (P_n) is as demanded. Notice that in such general setting we cannot conclude that (P_n) tends to f locally uniformly in U.

The most deep and important case of the aforesaid question arises when we assume that the function under approximation is bounded, and demand to approximate it by bounded sequence of polynomials. In this situation the picture changes completely. It became clear after works [45, 46] by O. J. Farrell in 1934–1935, where he proved the next Theorem 3.25. It is necessary to read simultaneously both papers to obtain the proof. However, these important papers are rarely mentioned in forthcoming works in the topic under consideration, so it causes errors in the attribution of who and what actually proved, see, for instance, [106, 112]. Farrell also mentioned that certain ideas of Carleman (see [23]) were of utility to prove both Theorems 3.25 and 4.1 below.

Theorem 3.25 (Farrell). *Let $G \neq \emptyset$ be a simply connected domain in $\mathbb{C}$. The following conditions are equivalent.*

(a) *For every function $f \in H^\infty(G)$ there exist a sequence of polynomials (P_n) such that $P_n(z) \to f(z)$ for each $z \in G$, and $\limsup_{n\to\infty} \|P_n\|_G \leqslant \|f\|_G$.*

(b) *G is a Carathéodory domain.*

Proof. Let G be a simply connected domain and put $T = \partial G_\infty$. Consider Q to be such component of $\mathbb{C} \setminus T$ that $G \subset Q$. Let f be some fixed conformal map from G onto $\mathbb{D}$. If the approximation properties stated in the part (a) holds, then there exists a sequence (P_n) of polynomials such that $P_n(z) \to f(z)$ and $|P_n(z)| \leqslant 2$ for each $z \in G$ and for each $n \in \mathbb{N}$ large enough. Then, $|P_m(z)| \leqslant 2$ for all $z \in \overline{G}$ and for some $m \in \mathbb{N}$, and $P_m \neq 0$, so that G needs to be bounded. Since $\partial Q \subset \overline{G}$, Montel's theorem (on the characterization of compact subsets of $H(G)$) shows that there exist a partial subsequence (P_{n_k}) and a holomorphic function $f_0\colon Q \to \mathbb{C}$ such that $P_{n_k}(z) \to f_0(z)$ for each $z \in Q$. Therefore, $f_0 = f$ in G, so that f_0 is non-constant. Moreover, since $Q \subset \widehat{G}$, then

$$|f_0(z)| = \limsup |P_{n_k}(z)| \leqslant \limsup \|P_{n_k}\|_Q \leqslant \limsup \|P_{n_k}\|_{\widehat{G}} \leqslant 1,$$

for each $z \in Q$. If we assume that G is not a Carathéodory domain, then $\partial G \setminus T \neq \emptyset$. Then, there exist $b \in \partial G \setminus T$ which is an accessible point from G by some end-cut $\mathcal{E}$ and there exists $\varepsilon > 0$ such that $D(b, \varepsilon) \subset Q$. Then,

$$|f_0(b)| = \lim_{\mathcal{E} \ni z \to b} |f_0(z)| = \lim_{z \to b} |f(z)| = 1,$$

but this is a contradiction since $|f_0|$ cannot achieve at the point b its maximum modulus over Q. Thus, the implication (a)$\Rightarrow$(b) is proved.

We are going now to prove the inverse implication. Take a function $f \in H^\infty(G)$. Fix $z_0 \in G$. Let us take, as usual, the sequence of Jordan curves (J_n) such that $D(J_n) \to G$ with respect to z_0 (in the sense of kernel convergence). Let $K = \widehat{G}$. Since $D(J_n)$ is a Jordan domain then $K \subset D(J_n)$ for all $n \geqslant 1$. Let us take the conformal maps φ_n from $D(J_n)$ onto $\mathbb{D}$ and the conformal map φ from $\mathbb{D}$ onto G such that $\varphi_n(z_0) = 0$ and $\varphi_n'(z_0) > 0$, while $\varphi(0) = z_0$ and $\varphi'(0) > 0$. Put $g_n = \varphi \circ \varphi_n$. Then, the function $f \circ g_n$ is holomorphic on $D(J_n)$, that is in an open neighborhood of K. Applying Runge's theorem, one can find a sequence of polynomials (P_n), such that

$$\|f \circ g_n - P_n\|_K < \frac{1}{n}. \tag{3.9}$$

From (3.9) it follows that $\|P_n\|_K \leqslant \frac{1}{n} + \|f\|_K = \frac{1}{n} + \|f\|_G$, which gives the conclusion of the theorem.

Notice that $g_n \rightrightarrows z$ in G. If $Y \subset G$ is a compact set then, for big enough n, one has

$$|f(g_n(z)) - f(z)| < \frac{1}{n}, \quad z \in Y. \tag{3.10}$$

From (3.9) and (3.10) we obtain that (P_n) converges uniformly on compact subsets of G to f. This implies the pointwise convergence in G. ■

Paying more attention into the proofs given above and doing a bit more, the following result can be obtained.

Corollary¶ 3.26. *Let G be a simply connected domain in $\mathbb{C}$.*

(a) *Let f map G conformally onto $\mathbb{D}$. Assume that there exists a sequence of polynomials (P_n) such that*

$$\sup_{n\in\mathbb{N}} \|P_n\|_G \leqslant C \quad \textit{and} \quad \lim_{n\to\infty} P_n(z) = f(z),\ z \in G, \tag{3.11}$$

for some constant C. Then, G is a Carathéodory domain.

(b) *Conversely, if G is a Carathéodory domain, then each function $h \in H^\infty(G)$ can be approximate by a sequence of polynomials (P_n) satisfying (3.11). In particular, if f is a conformal map from G onto $\mathbb{D}$, one can take the corresponding sequence in such a way that $C = 1$ in (3.11).*

Proof. Let us start with the part (a). We will use all notations introduced in the proof of the implication (a) ⇒ (b) in Theorem 3.25. So, we take the partial sequence (P_{n_k}) and the function f_0 such that $P_{n_k} \rightrightarrows f_0$ on Q. Assume that $\partial G \setminus T \neq \emptyset$, then for each accessible point $w \in \partial G \setminus T$ we know that $|f_0(w)| = 1$. By continuity this is true for all $w \in \partial G \setminus T$. Then, take $b_1 \in \partial G \setminus T$ such that $f_0'(b_1) \neq 0$. Therefore, is it to possible to find $\varepsilon > 0$ and a small closed disk W such that

$$W \subset D(b_1, \varepsilon) \cap G \subset \overline{D(b_1, \varepsilon)} \cap G \subset Q$$

and f_0 is a holomorphic homeomorphism from W onto its image, so that $1+\varepsilon < |f_0(z)|$ for all $z \in W$. Since the sequence (P_{n_k}) converges uniformly on W, then

$$1+\frac{\varepsilon}{2} \leqslant |P_{n_k}(z)|, \quad z \in W,\ k \geqslant k_0.$$

Taking limits when $k \to \infty$ the previous estimate yields that there exist many points z, where $|f(z)| \geqslant 1+\varepsilon/2$, which is a contradiction.

It remains to prove last assertion in the part (b). Let f be the conformal map from G onto $\mathbb{D}$. Since $f \circ g_n = \varphi_n$ is holomorphic in $D(J_n)$, then $\|\varphi_n\|_{\widehat{G}} \leqslant c_n < 1$. Then, take the corresponding polynomial P_n in such a way that

$$\|\varphi_n - P_n\|_{\widehat{G}} < 1 - c_n,$$

for every $N \geqslant 1$. Thus, $\|P_n\|_{\widehat{G}} \leqslant 1$. ■

This result for $C = 1$ is covered by the original proof in [45, 46], and it is [106, Theorem 2]. The author of the paper [106] and, highly likely, its referee were unaware that the respective result already has been proved 60 years prior to the publication of that paper.

Similar arguments can be used to prove the following result.

Proposition¶ 3.27. *Let G be a simply connected domain in $\mathbb{C}$. Assume that there is a subset $E \subset \partial G$ such that $\overline{E} = \partial G$ and for each point $a \in E$ the function $f(z) = \sqrt{z-a}$ can be boundedly approximated on G by a sequence of polynomials. Then, G is a Carathéodory domain.*

In [47] Farrell gave the estimate of the norm $\|f - p_n\|_G$ in terms of one special metrical concept. For a given domain G and a function $f \in H^\infty(G)$ let

$$D(f, \partial G) = \sup_{z \in \partial G} \operatorname{diam} C(f, z),$$

where $C(f, z)$ is the cluster set of f at the point z.

Theorem 3.28. *Let G be a Jordan domain and $f \in H^\infty(G)$. Then, there exists a sequence (P_n) of polynomials such that $P_n \rightrightarrows f$ in G and*

$$\limsup_{n\to\infty} \|f - P_n\|_G \leqslant D(f, \partial G). \tag{3.12}$$

Sketch of the proof. Fix $z_0 \in G$ and let (G_n) be the usual sequence of simply connected domains such that $G_n \to G$ with respect to z_0. Take the conformal map $\psi_n \colon G_n \to G$ such that $\psi(z_0) = z_0$ and $\psi'(z_0) > 0$. For the function f_n defined by the formula $f_n(z) = f(\psi_n(z))$ one can find an appropriate polynomial P_n in such a way that

$$|P_n(z) - f_n(z)| \leqslant \frac{1}{n}$$

for all $z \in \overline{G}$. Since $\psi_n \rightrightarrows z$ on $\overline{G}$ in view of Theorem 2.10, we have $P_n \rightrightarrows f$ locally in G.

The estimate (3.12) is obtained as a consequence of the following fact. If $w_0 \in \partial G$, if (z_n) is any sequence tending to w_0, and if (P_{k_n}) is a suitable subsequence of (P_n), then

$$\limsup_{n\to\infty} |f(z_n) - P_{k_n}(z_n)| \leqslant \operatorname{diam} C(f, w_0).$$

The omitted details may be found in [47]. ■

Question III. Whether it is true, that if G is a Carathéodory domain such that $\mathbb{C} \setminus \overline{G}$ is connected and $f \in H^\infty(G)$, then there exists a sequence (P_n) of polynomials such that $P_n \rightrightarrows f$ in G and

$$\limsup_{n\to\infty} \|f - P_n\|_G \leqslant D(f, \partial G).$$

It seems that the answer is affirmative, but the proof given in the case of Jordan domains cannot be adapted directly.

Continuing the analysis of Farrell's results let us observe that the Carathéodory hull U^* of an open set U can be defined as follows:

$$U^* = \operatorname{Int}\Big\{z_0 : |p(z_0)| \leqslant \sup_{z\in U} |p(z)|, \text{ for each } p \in \mathcal{P}\Big\}.$$

As far as we know, the first occurrence of a related notion to Carthéodory hull (without the corresponding name) was in Theorem D in Farrell's work [46]. In this paper it was considered the component of the Carathéodory hull of a given domain that contains this domain itself. The concept of a Carathéodory hull of a set has appeared with this name in [120]. In [31] the set U^* was called the outer envelope of U. In [111, 112] this concept also appeared without name. Perhaps the name of "extended Carathéodory–Farrell hull" of U will be more honest and appropriated because if G is a Carathéodory domain, then G is only a component of a (sometimes) bigger open set G^*. However, in order to avoid a new name creation, the name of a Carathéodory hull is enough good and has been adopted to denote this set. Let us also note that the notation U^* for the Carathéodory hull of U coincides with the notation of [98], although in that paper it is not given a special name for this object.

The following properties are interesting and easy to prove (recall Proposition 1.5, see also [112]).

Lemma 3.29. *Let G be a bounded open set in $\mathbb{C}$. Then, the following hold.*

(i) $G^* = \mathbb{C} \setminus \overline{\Omega_\infty(\overline{G})}$, $\mathbb{C} \setminus \widehat{G} = \Omega_\infty(\overline{G})$, *and* $\partial G^* = \partial\Omega_\infty(\overline{G})$.

(ii) G^* *is a Carathéodory open set and* $(G^*)^* = G^*$.

(iii) *The set* G^* *is simply connected.*

The next lemma clarifies the usefulness of the concept given in Definition 1.4.

Lemma 3.30. *Let G be a bounded open set, let $f \in H(G)$, and let (P_n) be such sequence of polynomials that*

$$\sup_{n\in\mathbb{N}} \|P_n\|_G \leqslant C \quad \text{and} \quad P_n(z) \to f(z), \text{ for all } z \in G \tag{3.13}$$

for some constant C. Then, the following hold.

(a) $P_n \rightrightarrows f$ *locally in G as $n \to \infty$.*

(b) *There exists a function $f^* \in H(G^*)$ such that $f^*|_G = f$, that is f^* extends f to G^*.*

Proof. (a) Take a partial sequence (P_{n_k}) of the sequence (P_n). By Montel's theorem there exists a new partial sequence $(P_{n'_k})$ of this subsequence (P_{n_k}) such that $P_{n'_k} \rightrightarrows g$ locally in G for some function $g \in H(G)$. But $g(z) = f(z)$ in each component of G, then $g = f$ on G, and so, $P_{n'_k} \rightrightarrows f$ on G. Since it is true for all partial sequences of (P_n), the proof is completed.

(b) Let us observe that (3.13) together with the maximum modulus principle implies that $\|P_n\|_{\widehat{G}} = \|P_n\|_{\overline{G}} \leqslant C$ for all n. Then, there exists a partial sequence (P_{n_k}) such that $P_{n_k} \rightrightarrows f^*$ locally in G^* for some function f^* holomorphic on G^*. Since $G \subset G^*$, then f^* is an extension of f. Notice, that such extension in not unique in a general case. ■

The final result by Farrell can be stated as follows.

Theorem 3.31. *Let $G \subset \mathbb{C}$ be a domain, and let $f \in H^\infty(G)$. The following conditions are equivalent.*

(a) *There exist a sequence of polynomials (P_n) such that (3.13) is satisfied.*

(b) *The function f is the restriction of some function belonging to $H^\infty(G^*)$.*

Near thirty years after publication of the above results, L. Rubel and A. Shields in [111, 112] obtained their generalization for a general bounded open sets. The following result is called nowadays *Farrell–Rubel–Shields theorem*.

Theorem 3.32. *Let $U \neq \emptyset$ be a bounded open subset of $\mathbb{C}$, and $f \in H^\infty(U)$. The following conditions are equivalent.*

(a) *There exists a sequence of (P_n), $P_n \in \mathcal{P}$, such that* $\sup_n \|P_n\|_U \leqslant \|f\|_U$ *and $P_n(z) \to f(z)$ for all $z \in U$.*

(b) *There exists such function $f^* \in H^\infty(U^*)$ that $f = f^*|_U$.*

The case that the set U is connected corresponds to the original Farrell's proof. There are two key points that distinguish the Rubel and Shields results from Farrell's ideas. The first one is the following thing. If U is an open set, then U^* is a Carathéodory open set, and hence each function $f \in H^\infty(U^*)$ can be bounded pointwise approximated by polynomials in U^*, but not only in U. The sequences of polynomials constructed in Farrell's proof cannot give directly the convergence in U^*. The second key point is related with the following observation. If U has infinitely many components G_j, then each $f|_{G_j}$ can be approximated by a sequence of polynomials $(P_{j,n})$. However, each of such sequence depends on j and it is not clear how to deal with all sequences. Rubel and Shields gave a clever idea how to avoid simultaneous work with several components of U.

Nowadays a proof of Theorem 3.32 using many important tools from the theory of uniform algebras consist in proving an abstract version of such theorem. From this abstract version the following result may be obtained which also gives Theorem 3.32 (the details of these proofs may be found in [56, pages 152–154]).

Theorem 3.33. *Let K be a finitely connected compact set in $\mathbb{C}$, and let $f \in H^\infty(K^\circ)$. Then, there is a sequence (f_n), $f_n \in R(K)$, such that $\sup_n \|f_n\|_K \leqslant \|f\|_\infty$ and $f_n(z) \to f(z)$ for all $z \in K^\circ$.*

Now, we will describe the pattern of the proof of Rubel–Shields theorem. We need to introduce yet one auxiliary construction.

Definition 3.34. Let U be a Carathéodory open set and let B be a component of U. The cluster $\mathcal{K}(B)$ is defined as the union of all components Q of U for which $Q \subset E_B$, where E_B is the component of $\overline{U}$ that contains B.

In order to illustrate this definition let us consider the outer snake (or cornucopia) Q_1 twisting around $\mathbb{D}$ with $Q_1 \subset D(0, 3/2)$ and another outer snake Q_2 with $Q_2 \subset D(3, 1)$; for example of the model for such Q_1 and Q_2 see G_1 on Figure 2. Take $U = \mathbb{D} \cup Q_1 \cup Q_2$. Then, $\mathcal{K}(\mathbb{D}) = \mathcal{K}(Q_1) = \mathbb{D} \cup Q_1$ and $\mathcal{K}(Q_2) = Q_2$.

The next result corresponds to Theorem 2.11 in the case of general Carathéodory open sets. It may be found in [112].

Theorem 3.35. *Suppose U be an open set.*

(a) *Let U be a Carathéodory open set. For each component B of U take a point $w_B \in B$. Then, there exists a sequence (U_n) of bounded simply connected open sets possessing the following properties:*

  (i) $\overline{U} \subset U_n \subset \overline{U}_n \subset U_{n-1}$, $n \geqslant 2$;

  (ii) *If B is any component of U and if B_n is the component of U_n containing B, then $\overline{B} \subset B_n \subset \overline{B}_n \subset B_{n-1}$, $n \geqslant 2$, and $B_n \to B$ with respect to w_B.*

(b) *If U is an open set such that there exists some sequence of open sets (U_n) satisfying the properties* (i) *and* (ii)*, then U is a Carathéodory open set.*

Notice, that in the frameworks of conditions of this theorem one has $\mathcal{K}(B) \subset B_n$ for every $n \in \mathbb{N}$.

The following lemma is one of key ingredients of the proof of Theorem 3.32.

Lemma 3.36. *Let E be a finite subset of U and let B be a component of U. Assume that $f = 1$ in all the other components of U and $\|f\|_U \leqslant 1$. Then, for each given $\varepsilon > 0$ there exists a polynomial P such that $|P(z)| \leqslant 1$ for each $z \in U$ and $|f(z) - P(z)| < \varepsilon$ for each $z \in E$.*

Sketched proof of Theorem 3.32. We need to prove that if U is a Carathéodory open set and $f \in H^\infty(U)$ with $\|f\|_U \leqslant 1$, then there exists a sequence of polynomials, uniformly bounded by 1 in U, and converging to f at each point of U. Let us assume that Lemma 3.36 is already proved.

Denote by $C_1, C_2, \dots$ some enumeration of all components of U, take a countable dense set $\{z_1, z_2, \dots\} \subset U$ and put $E_n = \{z_1, z_2, \dots, z_n\}$. Define the functions g_k, $k \in \mathbb{N}$, in such a way that $g_k(z) = f(z)$ for $z \in C_k$ and $g_k = 1$ in $U \setminus C_k$. Take (and fix) some $\varepsilon > 0$ and $n \in \mathbb{N}$. By Lemma 3.36 for every $k \in \mathbb{N}$ there exists a polynomial P_k such that $\|P_k\|_U \leqslant 1$ and $|g_k(z) - P_k(z)| < \varepsilon/n$ for each $z \in E_n$. Let now $f_n = g_1 g_2 \cdots g_n$ so that $f_n = f$ on $C_1 \cup C_2 \cup \dots \cup C_n$, while $f_n = 1$ on each C_k with $k > n$. For the polynomial $\widetilde{P}_n = P_1 P_2 \cdots P_n$ we have $\|\widetilde{P}_n\| \leqslant 1$ in U and

$$f_n - \widetilde{P}_n = \sum_{j=1}^{n} P_1 \cdots P_{j-1} \cdot (g_j - P_j) \cdot g_{j+1} \cdots g_n,$$

which gives

$$|f_n(z) - \widetilde{P}_n(z)| < \varepsilon \quad \text{for each } z \in E_n.$$

Then, by Montel's theorem, each partial subsequence of $(\widetilde{P}_n)$ converges to a function h such that $f = h$ on E, so $\widetilde{P}_n \rightrightarrows f$ in U. ∎

It remains now to prove Lemma 3.36. To do this it is sufficient (in view of Runge's theorem) to verify the next statement.

Lemma 3.37. *Let E a finite subset of U and let B be a component of U. Assume that $f = 1$ in all other components of U and $\|f\|_U \leqslant 1$. Then, for any $\varepsilon > 0$ there exists a simply connected domain Q with $\overline{U} \subset Q$, and a holomorphic function g in Q with $\|g\|_Q \leqslant 1$ such that $|f - g| \leqslant \varepsilon$ on E.*

Let B be the component mentioned in Lemma 3.36. According to Theorem 3.35 one can take (U_n), (B_n), (φ_n) and φ, where φ_n is the conformal map from B_n onto $\mathbb{D}$ normalized at some point $w_B \in B$ as $\varphi_n(w_B) = 0$, and φ is the conformal map from B onto $\mathbb{D}$ normalized by the same way. Passing to an appropriate subsequence of (φ_n), we obtain that $\varphi_n \rightrightarrows \psi$ in $\mathcal{K}(B)$, where $\psi = \varphi$ in B. Let now $\{Q_j\}$ be the collection of all components that formed $\mathcal{K}(B)$. It holds that $\psi = \zeta_j$ with $|\zeta_j| = 1$ in Q_j for all indices j. Since E is finite, it meets only finite number of components of $\mathcal{K}(B)$, says for definiteness, $Q_1, \dots, Q_n$. Put $E' = \varphi(E \cap B) \subset \mathbb{D}$, so that E' is a finite set. Consider the function $F = f \circ \varphi^{-1}$ such that $\|F\|_{\mathbb{D}} \leqslant 1$. Using [112, Lemma 3.13] one can find a new function F_1 which is close to F on E', while it is close to 1 near the points $\zeta_1, \dots, \zeta_n$. Finally, for sufficiently large n the function g defined in such a way that $g = F_1 \circ \varphi_n$ in B_n and $g = 1$ in $U_n \setminus B_n$ is the desired approximant for f in Lemma 3.37. All omitted technical details may be found in [112, Lemmas 3.11, 3.12, and 3.13].

Example 3.38. Let G be the outer cornucopia, and $U = G^* = G \cup \mathbb{D}$. Then, there exists a sequence (P_n) of polynomials, uniformly bounded by 1 such that $P_n(z) \to 0$ if $z \in G$ and $P_n(z) \to 1$ if $z \in \mathbb{D}$.

The next statement is an application of Rubel–Shields theorem. But we encourage the interested reader to find a proof using only Farrell's ideas, as well as the another one basing only on Runge's theorem.

Corollary 3.39. *Let G be a Carathéodory domain and let $f \in H^\infty(G)$. Then, there exists a sequence of polynomials (P_n) such that $P_n \rightrightarrows f$ locally in G and for each bounded component B of $\mathbb{C} \setminus \overline{G}$ one has $P_n'(z) \to 0$ for each $z \in B$.*

We end this section mentioning several interesting and important concepts related with the topic on bounded pointwise approximation. The first one is the concept of a Farrell set, which was introduced by Rubel and studied, for example, in [126]. Later on, O'Farrell and Perez–Gonzalez defined Farrell pairs for general open sets and the notion of a Farrell–Rubel–Shields set. Notice that the family of Farrell–Rubel–Shields sets includes the family of Carathéodory domains. The paper [98] gives a comprehensive theorem on pointwise bounded-on-a-subset approximation for Farrell–Rubel–Shields sets.

3.4 Uniform algebras on Carathéodory sets

We start this section by mentioning some connections between the Walsh–Lebesgue theorem and the theory of uniform algebras. We recall some notions of that theory, whose exhaustive exposition may be found in [18, 56, 69, 125].

A uniform algebra $\mathcal{A}$ on a compact Hausdorff space X is a uniformly closed (with respect to the norm $\|f\| = \sup\{|f(x)| : x \in X\}$) subalgebra of $C(X)$ which contains constants and separates points of X. A set $E \subset X$ is called a boundary for $\mathcal{A}$ if for each $f \in \mathcal{A}$ there exists $y \in E$ such that $|f(y)| = \|f\|$. The minimum closed boundary of $\mathcal{A}$ (which always exists) is called the Shilov boundary of $\mathcal{A}$. A subset $F \subset X$ is called a peak set for $\mathcal{A}$ if there exists a function $f \in \mathcal{A}$ such that $\|f\| = 1$ and $F = f^{-1}(1)$. A point $x \in X$ is a peak point of $\mathcal{A}$ if $\{x\}$ is a peak set.

If $\mathcal{A}$ is a uniform algebra on a compact space X, the maximal ideal space of $\mathcal{A}$ can be identified with the space of non-zero complex-valued homomorphism of $\mathcal{A}$, which will be denoted by $M_{\mathcal{A}}$. If $\Psi \in M_{\mathcal{A}}$, then Ψ is continuous and $\|\Psi\| = 1 = \Psi(1)$. Moreover, there exists a probability measure μ on X such that $\Psi(f) = \int_X f\, d\mu$ for each $f \in \mathcal{A}$. This measure is call a representing measure for Ψ. The set of such measures is convex and weak-star compact but, in general, is not a singleton. The Choquet boundary of $\mathcal{A}$ is the set of all those $x \in X$ for which the evaluation functional $\tau_x(f) = f(x)$ has a unique representing measure, of course it is needed to be the unit point mass δ_x supported at the point x. Moreover, if X is a metrizable space, the Choquet boundary of $\mathcal{A}$ is also the set of all peak points of $\mathcal{A}$. It can be proved that it is a boundary for $\mathcal{A}$ and its closure coincides with the Shilov boundary.

Recall that $\mathcal{A}$ is called a Dirichlet algebra on X, if $\operatorname{Re} \mathcal{A}$ is dense in $C(X, \mathbb{R})$, while $\mathcal{A}$ is called a logmodular algebra on X, if

$$\left\{\log|f| : f \text{ is an invertible element of } \mathcal{A}\right\}$$

is dense in $C(X, \mathbb{R})$.

Let K be a compact subset of $\mathbb{C}$. We are going to discuss here several results related to [39]. For better understanding of the matter we emphasize the following facts.

(1) Let $g \in P(K)$. Then, there exists a sequence of polynomials that converges uniformly to g. By the maximum modules theorem this sequence also converges uniformly on $\widehat{K}$ to an extension $\hat{g} \in P(\widehat{K})$ of g which has the same norm. The isometry $g \mapsto \hat{g}$ allow us to identify $P(K)$ with $P(\widehat{K})$ or even with $P(\partial K)$. These identifications will be used in what follows without explicit reference.

(2) Returning to the algebras appearing in (3.1) let us note that the maximal ideal spaces for all of them are identified with K. Moreover, the Shilov boundaries for $P(K)$, $R(K)$, $A(K)$ and $C(K)$ are $\partial\widehat{K}$, ∂K, ∂K and K, respectively. For $P(K)$ and $C(K)$ the Choquet boundaries coincide with their Shilov boundaries, but for $R(K)$ and $A(K)$ the Choquet boundaries are more involved (see [56, page 205]).

For better understanding the next Proposition, we prove that the Choquet boundary of $P(K)$ is $\partial\widehat{K}$. First note that if x is a peak point of $P(K)$, then $x \in \partial\widehat{K}$. Let $x \in \partial\widehat{K}$ and let μ be a representing measure of τ_x. Since μ is real, then $\operatorname{Re} g(x) = \int \operatorname{Re} g\, d\mu$ for each $g \in P(K)$. Because $P(K)$ is a Dirichlet algebra, then $r(x) = \int_{\widehat{K}} r(y)\, d\mu(y)$ for each continuous function $r \in C_{\mathbb{R}}(\partial\widehat{K})$. It means that μ is also a representing measure of τ_x for the algebra $C_{\mathbb{R}}(\partial\widehat{K})$, so $\mu = \delta_x$.

If $\mathcal{A}$ is a Dirichlet algebra on X, then $\mathcal{A}$ is also a logmodular algebra on X, and X is the Shilov boundary of $\mathcal{A}$.

Theorem 3.23 tell us that $P(K)$ is a Dirichlet algebra on $\partial\widehat{K}$.

In view of the aforesaid, all ingredients are readily available to obtain the following statement which is worth comparing with [39, Theorem 4].

Proposition 3.40. *Let K be a compact set in $\mathbb{C}$, and let $\Gamma = \partial K$. The following conditions are equivalent.*

(a) *K is a Carathéodory compact set.*

(b) *The Choquet boundary of $P(K)$ is Γ.*

(c) *The Shilov boundary of $P(K)$ is Γ.*

(d) *$P(K)$ is a Dirichlet algebra on Γ.*

(e) *$P(K)$ is a logmodular algebra on Γ.*

(f) *Each point of Γ is a peak point for $P(K)$.*

Now, we are interested in the question about maximal subalgebras. We recall the concept of maximality in the theory of uniform algebras. Let K be a compact subset of $\mathbb{C}$. A closed subalgebra $\mathcal{A}$ of the algebra $C(K)$ is called *maximal* if for each closed subalgebra $\mathcal{B}$ of $C(K)$ such that $\mathcal{A} \subset \mathcal{B}$ it holds either $\mathcal{B} = \mathcal{A}$ or $\mathcal{B} = C(K)$. In [56, page 38] it is assumed that $\mathcal{A} \neq C(K)$, but seems more appropriate not to use this convention.

The question on maximality of the algebra $P(K)$, where K is a compact subset of $\mathbb{C}$, was initiated by J. Wermer, who proved that $A(\overline{\mathbb{D}})$ is maximal, considering as a uniform algebra on its Shilov boundary, $\mathbb{T}$, or with more generality for every closed subalgebra of $C(\mathbb{T})$ that contains an injective function. This result is known as Wermer's maximality theorem, see the first proof of it in [135]. Later on E. Bishop [15] (see Theorem 6 of the cited work) established the following result.

Theorem 3.41. *Let K be a compact subset of $\mathbb{C}$ such that both sets K° and $\mathbb{C} \setminus K$ are connected. Then, $P(\partial K)$ is maximal on $C(\partial K)$.*

Proof. We follow the proof which was done by Bishop that used ideas due to Hoffman. Other proof may be found in [125] (see Theorem 25.12 in this book). Let $\mathcal{B}$ be some closed subalgebra of $C(\partial K)$ such that $P(\partial K) \subset \mathcal{B}$ and put $G = K^\circ$. Then, we need to prove that $\mathcal{B} = C(\partial K)$ or $\mathcal{B} = P(\partial K)$. We know that every function from $P(\partial K)$ can be extended to some function belonging to $A(K)$. Then, for every point $a \in G$ the mapping $\varphi_a \colon P(\partial K) \to \mathbb{C}$ defined by $\varphi_a(h) = h(a)$, is a homomorphism of the algebra $P(\partial K)$. Now, we distinguish two cases.

Case 1. Assume that φ_a can be extended to the algebra $\mathcal{B}$ for any $a \in G$. Then, $|\varphi_a(h)| \leqslant \|h\|_{\partial K}$ for all $h \in \mathcal{B}$. Therefore, φ_a can be extended to a bounded linear functional (with norm equals 1) on the space $C(\partial K)$. It means that there exists a measure μ_a (with $\|\mu_a\| = 1$) on ∂K such that $\int h(z)\mu_a(z) = \varphi_a(h)$ for every $h \in \mathcal{B}$. Since

$\mu_a(\partial K) = \varphi_a(1) = 1$, and since $\|\mu_a\| = 1$, we have that μ_a is a positive measure (see [18, page 80]). Therefore, for each polynomial P we have

$$\operatorname{Re} P(a) = \operatorname{Re} \varphi_a(P) = \operatorname{Re} \int P d\mu_a = \int \operatorname{Re} P \, d\mu_a. \tag{3.14}$$

Let us denote by $\hat{f}$ the harmonic complex extension of f given by Corollary 3.24 of f to K. Take $h \in \mathcal{B}$ and consider $\hat{h} \in C(K) \cap \operatorname{Har}(K^\circ)$. By Theorem 3.23 and (3.14) one has $\hat{h}(a) = \int \hat{h} d\mu_a = \int h \, d\mu_a = \varphi_a(h)$ and, moreover, since φ_a is a multiplicative functional, $\widehat{zh}(a) = a\hat{h}(a)$. Thus, $\hat{h}$ and $z\hat{h}$ are harmonic in G. Hence,

$$0 = \partial\overline{\partial}(\widehat{zh}) = \partial\overline{\partial}(z\hat{h}) = \partial(z\overline{\partial}\,\hat{h}) = \overline{\partial}\,\hat{h} + z\partial\overline{\partial}\,\hat{h} = \overline{\partial}\,\hat{h}.$$

In G, which yields that $\hat{h}$ is holomorphic in G. Since $\mathbb{C} \setminus K$ is connected, we conclude from Mergelyan's theorem that $\hat{h} \in P(K)$.

Case 2. Assume that there exists a point $a \in G$ such that the homomorphism φ_a cannot be extended to $\mathcal{B}$. Consider in such a case the principal ideal in $\mathcal{B}$

$$\mathcal{J} = \{h \in \mathcal{B} : h(z) = h_1(z)(z-a),\ z \in \partial K,\ h_1 \in \mathcal{B}\}.$$

Assume that $\mathcal{J} \neq \mathcal{B}$, then there exists a maximal ideal $\mathcal{M}$ such that $\mathcal{J} \subset \mathcal{M}$. Then, there exists such homomorphism $\Phi\colon \mathcal{B} \to \mathbb{C}$ that $\ker \Phi = \mathcal{M}$. Then, $\Phi(\jmath) = a$ (where, as before, $\jmath(z) = z$) and therefore $\Phi(P) = P(a)$ for each $P \in P(\partial K)$. It means that Φ is an extension of φ_a which contradicts our assumption in Case 2. Thus, $\mathcal{J} = \mathcal{B}$ and $1 \in \mathcal{J}$. It means that $1/(\jmath - a) \in \mathcal{B}$. In view of Mergelyan's theorem $C(\partial K)$ is the algebra generated by $\jmath$ and $1/(\jmath - a)$. Then, $\mathcal{B} = C(\partial K)$. ∎

In fact, the property that $P(K)$ is a maximal subalgebra of $C(K)$ imposes quite rigid topological restrictions on the compact set K. We prove now the converse statement for Wermer's maximality theorem, which was essentially obtained in [27].

Theorem¶ 3.42. *Let K be a compact subset of $\mathbb{C}$. If $P(K)$ is a maximal subalgebra of $C(K)$, then K is a Carathéodory compact set without interior. If, moreover, $K = \partial\Omega$, where Ω is a nonempty bounded open set in $\mathbb{C}$, then neither $\overline{\Omega}$ nor Ω does not separate the plane and both sets $\partial\Omega$ and Ω are connected.*

Proof. If $P(K) = C(K)$ then $K^\circ = \emptyset$ and $K = \widehat{K}$. Then, $\partial K = \partial\widehat{K}$.

Assume therefore that $P(K) \neq C(K)$. In such a case the set $\mathbb{C} \setminus \partial\widehat{K}$ has a bounded component. If $K^\circ = \emptyset$ this is a consequence of Lavrentiev's theorem. If $K^\circ \neq \emptyset$ we can choose a bounded component of K°. So, one has

$$P(\partial\widehat{K}) \neq C(\partial\widehat{K}). \tag{3.15}$$

By (3.15) there exists a measure μ on $\partial\widehat{K}$ such that $\mu \perp P(\partial\widehat{K})$ and $\mu \neq 0$ (the symbol $\perp$ expresses the fact of orthogonality of μ to the corresponding set of functions). Let us now assume that $\partial K \setminus \partial \widehat{K} \neq \emptyset$ or $K^\circ \neq \emptyset$ and let us take $a \in \partial K \setminus \partial \widehat{K}$ or $a \in K^\circ$. Then, there exists a function $f \in C(K)$ such that $f(a) = 1$ and $f|_{\partial \widehat{K}} = 0$.

Let now $\mathcal{B}$ be the closure of the set of functions having the form $\sum_{j=0}^{m} q_j f^j$, where $q_0, \dots, q_m$ are polynomials and $m \in \mathbb{N}$. Since $f \notin P(K)$, then $\mathcal{B} \neq P(K)$. Moreover, since $f|_{\partial \widehat{K}} = 0$, then

$$\int_K \left(\sum_{j=0}^{m} q_j f^j \right) d\mu = \int_{\partial \widehat{K}} q_0 \, d\mu + \sum_{j=1}^{m} \int_{\partial \widehat{K}} q_j f^j \, d\mu = 0.$$

Thus, $\mathcal{B} \neq C(K)$, and so, $P(K)$ is not maximal. Thus, $\partial K = \partial \widehat{K}$ and $K^\circ = \emptyset$.

Let now $K = \partial \Omega$, where $\Omega \neq \emptyset$ is a bounded open set.

Assume that $\overline{\Omega}$ separates the plane. Let G be a bounded component of the set $\mathbb{C} \setminus \overline{\Omega}$ and Ω_1 be some component of Ω. Take $z_1 \in \Omega_1$ and $z_2 \in G$. Consider the closed subalgebra $\mathcal{B}$ which is generated by $P(K)$ and by the function $g_1(z) = 1/(z - z_1)$, $z \in K$. Clearly $g_1 \notin P(K)$. Taking into account that $\partial G \subset K$, and $g_1|_{\overline{G}}$ is holomorphic in $\overline{G}$, an application of the Maximum modulus principle gives that the function $h(z) = 1/(z - z_2)$, $z \in K$, does not belong to $\mathcal{B}$. Therefore, $P(K)$ is not maximal, which gives a contradiction. Thus, $\overline{\Omega}$ does not separate the plane.

Let us assume that Ω separates the plane. In such a case $\mathbb{C} \setminus \Omega = F_\infty \cup F_1$, where F_∞ is a closed set such that $\overline{\Omega}_\infty \subset F_\infty$ and F_1 is a nonempty compact set such that $F_1 \cap F_\infty = \emptyset$. Since $F_1 \cap \partial \Omega$ is not empty, take a point $z \in F_1 \cap \partial \Omega$. Since $\partial \Omega$ is a Carathéodory compact set then $z \in \partial(\partial \Omega) = \partial \Omega = \partial(\widehat{\partial \Omega})$. Thus, there exists a sequence of points $\{z_n\}$ such that $z_n \notin \widehat{\partial \Omega}$ and $z_n \to z$ as $n \to \infty$. Since $z_n \in \Omega_\infty$, then $z \in \overline{\Omega}_\infty \cap F_1 = \emptyset$. Thus, a contradiction arises and therefore Ω does not separate the plane.

Going further let us assume that the set $\partial \Omega$ is not connected. Then, $\partial \Omega = F_1 \cup F_2$, where F_1 and F_2 are compacts sets and $F_1 \cap F_2 = \emptyset$. Then, $\Omega \cap \widehat{F_j} \neq \emptyset$ for $j = 1, 2$, because if $\Omega \cap \widehat{F_1} = \emptyset$, then $\mathbb{C} \setminus \Omega$ has a bounded component and Ω will separate the plane. Then, we consider the closed subalgebra

$$\mathcal{B} = \{ f \in C(\partial \Omega) : f|_{F_1} \in P(F_1) \}.$$

If we take the function $f(z) = 1/(z - a)$, where $a \in \Omega \cap \widehat{F_2}$, we can see that $\mathcal{B} \neq P(F_1 \cup F_2)$. Clearly, $\mathcal{B} \neq C(\partial \Omega)$, it may be readily verified by considering $g(z) = 1/(z - b)$, $b \in \Omega \cap \widehat{F_1}$. Thus, $P(\partial \Omega)$ would be not maximal. Therefore, the set $\partial \Omega$ is connected. The fact that the set Ω is connected may be proved by a similar way. ∎

Corollary¶ 3.43. *If $\Omega \neq \emptyset$ is a bounded open set, then $P(\partial \Omega)$ is maximal subalgebra of $C(\partial \Omega)$ if and only if Ω is a Carathéodory domain which does not separate the plane.*

Notice that a slightly weaker version of Corollary 3.43 (in the case when Ω is a priory assumed to be a simply connected domain) was obtained in [39].

Remark 3.44. In the proof of Theorem 3.42 it was shown that if Ω is a Carathéodory domain, and if $\overline{\Omega}$ does not separate the plane, then Ω itself does not separate the plane either. In the general case the properties "Ω does not separate the plane" and "$\overline{\Omega}$ does not separate the plane" are independent because all four possible situations can occur. The same can be said concerning connectivity properties of $\partial\Omega$ and Ω.

3.5 Orthogonal measures on Carathéodory sets

Many results in approximation theory were obtained in the frameworks of so-called dual approach, which is based on studies of linear functionals orthogonal to certain spaces of functions. In the case of uniform approximation on compact sets in $\mathbb{C}$ any linear functional on the space $C(X)$ has the form $f \mapsto \int f\,d\mu$, where μ is some complex-valued Borel measure with support on X. So that it is interesting and important to study properties of measures on X which are orthogonal to spaces of polynomials or rational functions, or to some other spaces of functions. One important and deep theorem in this theory which we will need in what follows is the F. and M. Riesz theorem (for the proof see, for instance, [115, Chapter 17] or [77, Chapter II]). For the reader's convenience we state it in such a way which makes evident the starting point of the research made by E. Bishop in his three papers that we will discuss in this section.

Theorem 3.45 (F. and M. Riesz). *Let ν be a complex measure on $\mathbb{T}$ which is orthogonal to all polynomials, that is $\int_{\mathbb{T}} P(\zeta)\,d\nu(\zeta) = 0$ for every $P \in \mathcal{P}$. Then, the following hold.*

(a) *The measure ν is absolutely continuous with respect to the measure $m_{\mathbb{T}}$, that is there exists a Borel measurable function u such that*

$$\nu(E) = \int_E u(\zeta)\,dm_{\mathbb{T}}(\zeta) = \frac{1}{2\pi i}\int_E \overline{\zeta}\,u(\zeta)\,d\zeta$$

for every Borel set $E \subset \mathbb{T}$.

(b) *Let the function f be defined in $\mathbb{D}$ by the formula*

$$f(z) = \frac{1}{2\pi i}\int \frac{d\nu(\zeta)}{\zeta - z},$$

and let $f_r(\zeta) = f(r\zeta)$ for $r > 0$ and $\zeta \in \mathbb{T}$. Then, $f_r \to u$ as $r \to 1$ in $L^1(\mathbb{T})$.

(c) *For a.a. points ζ on $\mathbb{T}$, one has that $f(z) \to u(\zeta)$ when $z \in \mathbb{D}$ tends to ζ non-tangentially.*

The aim of Bishop's research was to obtain a generalization of the F. and M. Riesz theorem for measures living on boundaries of general compact sets. The first problem arising in this connection is that if ∂K is not a rectifiable set, then it is not clear what is the absolute continuity property (with respect to what measure?) that needs to be used. In [13–15] E. Bishop has provided a fruitful investigation of the structure of measures orthogonal to rational functions on Carathéodory compacts sets. He used many tools in conformal mappings, in the theory of Hardy spaces, in measure theory. One key point he introduced is the concept of an analytic differential $g(z)\,dz$ that represents some complex measure μ. Let us briefly recall this concept. An analytic differential in a domain $\Omega \subset \mathbb{C}$ is a differential form $g(z)\,dz$, where $g \in H(\Omega)$. One says that the analytic differential $g(z)\,dz$ represents the measure μ on $\partial\Omega$ if the sequence of measures $\{g(z)\,dz|_{\Gamma_j}\}$ converges in the weak-star topology of the space of measures on $\overline{\Omega}$ to μ, where $\{\Gamma_j\}$ is some sequence of rectifiable contours such that $D(\Gamma_j) \subset D(\Gamma_{j+1}) \subset \Omega$ and $D(\Gamma_j) \uparrow \Omega$ as $j \to \infty$. Observe, that the analytic differential $g(z)\,dz$ in Ω is defined even in the case when $\partial\Omega$ is not a rectifiable set. This concept is not used nowadays and it has been only occasionally used in the mathematical literature.

To present the Bishop's results we need to recall some definitions and fix some notation. We will use notation from Section 3.2 concerning harmonic measure. Let now G be a simply connected domain in $\mathbb{C}$ and let f be some conformal map from $\mathbb{D}$ onto G. Assume for a moment, that ∂G is locally connected. Then, by Theorem 2.5, f has a continuous extension to $\overline{\mathbb{D}}$ onto $\overline{G}$. Moreover,

$$\omega(w, E, G) = \omega(f^{-1}(w), f^{-1}(E), \mathbb{D}), \tag{3.16}$$

for every point $w \in G$ and every Borel set $E \subset \partial G$. The equality (3.16) is called the invariance principle of the harmonic measure under conformal mapping. It can be readily proved by comparing both harmonic functions by its values on ∂G. The right-hand side of (3.16) can be readily calculated since

$$\omega(a, F, \mathbb{D}) = \int_F \frac{1-|a|^2}{|e^{it}-a|^2}\,\frac{dt}{2\pi}, \qquad F \subset \mathbb{T},\ a \in \mathbb{D},$$

and, moreover, this quantity may be represented in geometric terms. In the case that ∂G is not locally connected we have the following result.

Theorem 3.46. *Let $G \subset \mathbb{C}$ be a simply connected domain, and let f be a conformal map from $\mathbb{D}$ onto G. Then, $\omega(z, \partial_a G, G) = 1$ for every $z \in G$. Moreover, if $E \subset \partial_a G$ is a Borel set, then* (3.16) *holds. In particular, if $f(0) = z_0 \in G$, then*

$$\omega(z_0, E, G) = \omega(0, f^{-1}(E), \mathbb{D}) = m_{\mathbb{T}}(f^{-1}(E)). \tag{3.17}$$

For a proof of this theorem see [104, Section 6.2] and [59, page 206].

In the case of Carathéodory open sets the following useful property of a harmonic measure is satisfied.

Proposition 3.47. *Let U be a Carathéodory open set, and let W_1 and W_2 be two different components of U. Then, the measures $\omega(a,\cdot,U)$ and $\omega(b,\cdot,U)$ are mutually singular for all points $a \in W_1$ and $b \in W_2$.*

Proof. We know that it is enough to prove the desired assertion for two fixed points a and b belonging to the different components of U. Take $a \in W_1$. We have that W_1 is a Carathéodory domain, the measure $\omega(a,\cdot,U)$ is concentrated in $\partial_a W_1$, and W_2 is a component of $\mathbb{C} \setminus \overline{W_1}$. So, we can apply Proposition 1.15 to obtain the result. ∎

Notice that the result stated in Proposition 3.47 is clearly not true in the case, where the open set U is not assumed to be a Carathéodory open set. To better understand this curious behavior, the reader can remind the open set $U = \mathbb{D} \cup Q_1 \cup Q_2$ defined just after Definition 3.34. Another, slightly different, proof of Proposition 3.47 was given in [15, Lemma 10].

Let now G be a Carathéodory domain, let f be a conformal map from $\mathbb{D}$ onto G such that $f(0) = z_0 \in G$, and let $g = f^{-1}$ be the respective inverse mapping. In what follows we will (often implicitly) use all results about boundary behavior of f and g obtained in Chapter 2 (in particular, Theorem 2.24 and Corollary 2.25).

Take a function $h \in L^1(\mathbb{T})$ and consider the measure $h\,d\zeta$ on $\mathbb{T}$. Define the measure $f(h\,d\zeta)$ on ∂G by the formula

$$f(h\,d\zeta)(E) := \int_{g(E\cap\partial_a G)} h(\zeta)\,d\zeta = \int (\mathbf{1}_E \circ f)(\zeta) h(\zeta)\,d\zeta$$

for every Borel set $E \subset \partial G$ (where $\mathbf{1}_E$ stands of the characteristic function of E), or, equivalently,

$$\int \psi\, df(h\,d\zeta) = \int_{F(f)} \psi(f(\zeta))h(\zeta)\,d\zeta = \int_{\mathbb{T}} \psi(f(\zeta))h(\zeta)\,d\zeta \tag{3.18}$$

for every function $\psi \in C(\bar{G})$. Note that (3.17) implies that

$$\int_{\partial G} \psi(z)\,d\omega(z_0,z,G) = \int_{\mathbb{T}} (\psi \circ f)(\zeta)\,dm_{\mathbb{T}}(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} (\psi \circ f)(e^{i\vartheta})\,d\vartheta$$

in our situation.

We define the complex harmonic measure relative to G and z_0 as $\omega_c(z_0,\cdot,G) = f(d\zeta)$. Then,

$$\omega_c(z_0,\cdot,G) = 2\pi i g\, \omega(z_0,\cdot,G). \tag{3.19}$$

Moreover, if $h \in L^1(\mathbb{T})$ then

$$f(h\,d\zeta) = (h \circ f^{-1})\,\omega_c(z_0,\cdot,G) = (h_0 \circ f^{-1})\,\omega(z_0,\cdot,G), \tag{3.20}$$

where

$$h_0(z) := 2\pi i z h(z).$$

In view of (3.19), the properties that some measure μ on ∂G is absolutely continuous with respect to $\omega_c(z_0, \cdot, G)$ and $\omega(z_0, \cdot, G)$ are equivalent. For simplicity in what follows we will denote the complex measure $\omega_c(z_0, \cdot, G)$ just by ω assuming that the point z_0 is clear from the context and fixed.

The following results are essentially (but only implicitly) stated in the [13–15]. A proof of Theorem 3.48 below based on studies of analytic differentials representing measures can be extracted from the aforementioned papers of Bishop. We consider that it is interesting and in certain sense important to present a direct proof of this theorem which is free from the concept of analytic differentials. It was done in [26], but here we made some modifications. In [15] it was not mentioned that $\mu_G = \mu_{|\partial G}$ (in the second part of Theorem 3.48). This fact was proved in [26]. For an open set U we denote by $\mathcal{C}(U)$ the collection of all components of U.

Theorem¶ 3.48. *Let G be a Carathéodory domain, while X be a Carathéodory compact set in $\mathbb{C}$.*

(1) *Let μ be a measure on ∂G such that $\mu \perp R(\overline{G})$. Then, there exists a function $h \in H^1$ such that*

$$\mu = (h \circ g)\,\omega. \tag{3.21}$$

(2) *Let $X^\circ \neq \emptyset$, and let μ be a measure on ∂X such that $\mu \perp R(X)$. Then,*

$$\mu = \sum_{G \in \mathcal{C}(X^\circ)} \mu_G, \tag{3.22}$$

where

$$\mu_G = \mu|_{\partial G}, \quad \mu_G \perp R(\overline{G}),$$

and the series in (3.22) converges in the space of measures on ∂X.

(3) *Let μ be a measure on ∂X such that $\mu \perp R(X)$. Then, $\mu = 0$ on $X \setminus \overline{X^\circ}$ and $\mu \perp R(\overline{X^\circ})$.*

We recall that the Cauchy transform of a measure μ is the function

$$\widehat{\mu}(z) = \frac{1}{2\pi i} \int \frac{d\mu(w)}{w - z}$$

which is well defined for A-a.a. $z \in \mathbb{C}$. It is well known, that $\widehat{\mu}$ is holomorphic outside of $\mathrm{Supp}(\mu)$ and $\overline{\partial}\widehat{\mu} = \frac{i}{2}\mu$ in the sense of distributions.

We also recall that for a given class $\mathcal{F}$ of continuous functions and for a given measure μ the expression $\mu \perp \mathcal{F}$ means that μ is orthogonal to $\mathcal{F}$, i.e., $\int f\, d\mu = 0$ for each $f \in \mathcal{F}$.

Sketched proof of Theorem 3.48. Let us denote by G_j, where $j \in J$ and $J \subset \mathbb{N}_0$ is some finite or countable set of indexes, each element of the set $\mathcal{C}(X^\circ)$. We know that every G_j, $j \in J$, is a Carathéodory domain. In the case of the part (1), one has $J = \{0\}$ and $G_0 = G$.

For each $j \in J$ let f_j be some conformal mapping from $\mathbb{D}$ onto G_j, such that $f_j'(0) > 0$, let $\psi_j = f_j^{-1}$ be the inverse mapping and let $h_j := (\widehat{\mu} \circ f_j)\, f_j'$.

The proof will consist of several steps.

Step 1. $h_j \in H^1$ *for every* $j \in J$.

Proof. Take and fix $j \in J$. In view of Proposition 1.18 there exists a connected Carathéodory compact set Y such that $X \subset Y$ and $X^\circ = Y^\circ$. Choose some sequence (Γ_m) of rectifiable contours such that $Y \subset D(\Gamma_m) \subset D(\Gamma_{m-1})$ and $\overline{D(\Gamma_m)}$ converges to Y as $m \to \infty$. Notice that for any point $z_j \in G_j$ the kernel of the sequence $(D(\Gamma_m))$ with respect to z_j is exactly G_j.

Let $z_0 = f_j(0)$. Let g_m be the conformal mapping from $D(\Gamma_m)$ onto $\mathbb{D}$ such that $g_m(z_0) = 0$, $g_m'(z_0) > 0$. By Carathéodory kernel theorem the sequence (g_m) converges to $\psi_j = f_j^{-1}$ locally uniformly in G_j. Take a point $w \in \mathbb{D}$ and set $z_m = g_m^{-1}(w)$. Then, the function

$$\begin{cases} a(z) = \frac{1}{g_m(z) - g_m(z_m)} - \frac{1}{g_m'(z_m)(z - z_m)} & \text{for } z \neq z_m, \\ a(z_m) = -\frac{g_m''(z_m)}{2 g_m'(z_m)^2} \end{cases}$$

can be uniformly on X approximated by rational functions with poles lying outside X. Then, since $\mu \perp R(X)$, we have

$$\frac{1}{2\pi i} \int \frac{d\mu(z)}{g_m(z) - g_m(z_m)} = \frac{\widehat{\mu}(z_m)}{g_m'(z_m)}.$$

We define the measures ν_m supported on $\mathbb{D}$ by the formula $\nu_m(E) = \mu(g_m^{-1}(E \cap \mathbb{D}))$ for each Borel subsets E of $\mathbb{C}$. Taking into account the previous formula and the fact that $g_m(z_m) = w$, we have

$$\frac{1}{2\pi i} \int \frac{d\nu_m(\zeta)}{\zeta - w} = \widehat{\mu}(g_m^{-1}(w))(g_m^{-1})'(w). \tag{3.23}$$

Moreover, ν_m is orthogonal to polynomials and $\|\nu_m\| \leqslant \|\mu\|$.

Take now a weak-star limit point η of the sequence (ν_m). Then, $\operatorname{Supp}(\eta) \subset \mathbb{T}$ and η is orthogonal to polynomials. Thus, one can find a function $t_j \in H^1$ with the property $\eta = t_j\, d\zeta|_{\mathbb{T}}$. Passing to the limit in (3.23) we obtain

$$h_j(w) = \widehat{\mu}(f_j(w)) f_j'(w) = \widehat{\eta}(w) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{t_j(\zeta)\, d\zeta}{\zeta - w} = t_j(w)$$

for all $w \in \mathbb{D}$, so that $h_j \in H^1$. ∎

For each $j \in J$ we define the measures $\omega_j := f_j(d\zeta|_{\mathbb{T}})$ and $\mu_j := f_j(h_j d\zeta|_{\mathbb{T}}) = (h_j \circ \psi_j)\,\omega_j$.

Step 2. *One has*

(i) $\widehat{\mu}_j(z) = \widehat{\mu}(z)$ *for all* $z \in G_j$*;*

(ii) $\widehat{\mu}_j(z) = 0$ *for all* $z \notin \overline{G}_j$*, (that means that* $\mu_j \perp R(\overline{G}_j)$*).*

Proof. Take $z \notin \partial\overline{G}_j$. Then,

$$\widehat{\mu}_j(z) = \frac{1}{2\pi i}\int_{\partial G_j} \frac{h_j(\psi_j(\zeta))\,d\omega_j(\zeta)}{\zeta - z} = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{h_j(\zeta)\,d\zeta}{f_j(\zeta) - z} = 0,$$

because the function $w \mapsto h_j(w)/(f_j(w) - z)$ belongs to H^∞.

If $z \in G_j$ let us take $w_j = f_j^{-1}(z) \in \mathbb{D}$. Then, the function

$$\begin{cases} q(w) = \frac{w - w_j}{f_j(w) - f_j(w_j)}, & \text{for } w \neq w_j, \\ q(w_j) = \frac{1}{f_j'(w_j)} \end{cases}$$

belongs to H^∞. Therefore,

$$\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{h_j(\zeta)\,d\zeta}{f_j(\zeta) - f_j(w_j)} = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{h_j(\zeta)\,q(\zeta)\,d\zeta}{\zeta - w_j} = h_j(w_j)\,q(w_j).$$

It gives, that for $z \in G_j$ one has

$$\widehat{\mu}_j(z) = \frac{h_j(w_j)}{f_j'(w_j)} = \widehat{\mu}(f_j(w_j)) = \widehat{\mu}(z),$$

which ends the proof. ■

We are ready now to prove the first assertion of the theorem. Recall, that $G = G_0$ and $X = \overline{G}$ in this case. It follows from Step 2, that $\widehat{\mu}(z) = \widehat{\mu}_0(z)$ for all $z \notin G$, consequently $\mu - \mu_0 \perp R(\partial G)$. Since ∂G is a Carathéodory compact, in view of Theorem 3.10 we have $R(\partial G) = C(\partial G)$ and hence $\mu = \mu_0$. For each finite subset $I \subset J$ put $W_I := \bigcup_{j \in I} G_j$. The following assertion is the direct consequence of [15, Lemma 7].

Step 3. *There exists a sequence* (r_k) *of functions from* $R(X)$ *such that* $\|r_k\|_X \leqslant 1$, $r_k \rightrightarrows 1$ *locally in* W_I *and* $r_k \rightrightarrows 0$ *locally in* $X^\circ \setminus W_I$.

Let denote by $\mu_I \perp R(X)$ a weak-star limit in the space of measures on X of the sequence of measures $(r_k\mu)$.

Step 4. *One has*

(i) $\widehat{\mu}_I(z) = \widehat{\mu}(z)$ *for all* $z \in W_I$*;*

(ii) $\widehat{\mu}_I(z) = 0$ *for all* $z \in X^\circ \setminus W_I$.

Proof. Denote also by $(r_k\mu)$ the partial sequence that converges in the weak-star topology to μ_I. If $z \notin \partial X$, then

$$\widehat{\mu}_I(z) = \lim_{k\to\infty}\left(\frac{1}{2\pi i}\int \frac{(r_k(\zeta)-r_k(z))\,d\mu(\zeta)}{\zeta - z} + \frac{r_k(z)}{2\pi i}\int\frac{d\mu(\zeta)}{\zeta - z}\right) = \widehat{\mu}(z)\lim_{k\to\infty} r_k(z),$$

gives the desired assertion. ■

It follows from Steps 2 and 4, that

$$\widehat{\mu}_I(z) = \sum_{j\in I}\widehat{\mu}_j(z), \quad z \notin \partial X.$$

Since $R(\partial X) = C(\partial X)$, we conclude that

$$\mu_I = \sum_{j\in I}\mu_j. \tag{3.24}$$

Taking into account (3.24), Proposition 3.47 and the fact that $\mu_j \ll \omega_j$ we conclude, that $\mu_j \perp \mu_k$ for $j,k \in J$, $j \neq k$. Hence, we have

$$\sum_{j\in I}\|\mu_j\| = \Big\|\sum_{j\in I}\mu_j\Big\| = \|\mu_I\| \leqslant \|\mu\|,$$

which means that $\sum_{j\in J}\|\mu_j\| < \infty$. Let $\mu = \sum_{j\in J}\mu_j$. It is clear, that $\eta \perp R(X)$. For each $j \in J$ we have $\widehat{\eta}(z) = \widehat{\mu}_j(z)$ for all $z \in G_j$ and applying the result of Step 2 we conclude that $\widehat{\eta}(z) = \widehat{\mu}(z)$ on X. Then, $\eta = \mu$.

Take now $k \in J$. Since $\mu_j \perp \mu_k$ for $j \in J \setminus \{k\}$, then for every Borel set $E \subset \partial X$ we have

$$\mu_{|\partial G_k}(E) = \mu(E\cap\partial G_k) = \sum_{j\in J}\mu_j(E\cap\partial G_k) = \mu_k(E\cap\partial G_k) = \mu_k(E).$$

The remaining part (item (3)) follows from (3.22) if $X^\circ \neq \emptyset$, and from Theorem 3.10 otherwise.

Thus, the proof is finished. ■

Moreover, it is possible to find out in [15] certain additional facts concerning the objects that were introduced in the proof of Theorem 3.48. We present only two of them. In fact, one has

$$\sum_{j\in J}\int_{f_j(\rho\mathbb{T})}|\widehat{\mu}(\zeta)|\,d\zeta \leqslant C\|\mu\|,$$

for each $\rho \in (0,1)$, and

$$\sum_{j\in J}\|h_j\|_1 \leqslant C\|\mu\|,$$

where $C > 0$ is some absolute constant.

Remark 3.49. The part (2) of Theorem 3.48 remind us the Decomposition theorem for orthogonal measures, see [56, Theorem 7.11, Chapter II], see also [60]. We have not made the connection between both results, probably it can give another proof of Bishop result. Also it is curious to observe that Bishop's papers were not mentioned in Gamelin's book.

We have seen in Bishop's Theorem 3.48 that if G is a Carathéodory domain and μ is a measure on ∂G such that $\mu \perp R(\overline{G})$, then $\mu \ll \omega(a, \cdot, G)$ for every point $a \in G$. It turns out, that some converse result is also true. More precisely we have the following result, which must be compared with [38, Theorem 1].

Proposition¶ 3.50. *Let Ω be a non-empty bounded domain in $\mathbb{C}$, and let $a \in \Omega$.*

(a) *If Ω is a Carathéodory domain, and the set $\mathbb{C} \setminus \overline{\Omega}$ is not connected, then there exist $\mu \in P(\overline{\Omega})^{\perp}$ such that μ is not absolutely continuous with respect to $\omega(a, \cdot, \Omega)$.*

(b) *Assume that every measure which is orthogonal to $P(\overline{\Omega})$ is absolutely continuous with respect to $\omega(a, \cdot, \Omega)$. Then, Ω is a Carathéodory domain and the set $\mathbb{C} \setminus \overline{\Omega}$ is connected.*

Proof. (a) Assume that $\mathbb{C} \setminus \overline{\Omega}$ has a bounded component Ω_1. Then, Ω_1 is a Carathéodory domain and its complement is connected. This fact together with the part (1) of Theorem 3.48 yields that every measure of the form $(h \circ \rho)\, \omega$, where $h \in H^1$, ρ is some conformal map from Ω_1 onto $\mathbb{D}$ and ω is the complex harmonic measure on $\partial\Omega_1$ with respect some point $b \in \Omega_1$, is supported in $\partial\Omega_1 \subset \partial\Omega$, it is orthogonal to $\mathcal{P}$ and it is not absolutely continuous with respect to $\omega(a, \cdot, \Omega)$, since $\omega(a, \partial\Omega_1, \Omega) = 0$.

(b) Let $\Omega_0 = \mathbb{C} \setminus \overline{\Omega}_\infty$, i.e., Ω_0 is the interior of the complement of the unbounded component of the set $\mathbb{C} \setminus \overline{\Omega}$. If Ω is not a Carathéodory domain then there exists $z_0 \in \Omega_0 \cap \partial\Omega$. Consider now the measure

$$\mu_0 := \omega(z_0, \cdot, \Omega_0) - \delta_{z_0}.$$

Then, for every $P \in \mathcal{P}$, one has

$$\int P(\zeta)\, d\mu_0(\zeta) = \int_{\partial\Omega_0} P(\zeta)\, d\omega(z_0, \zeta, \Omega_0) - P(z_0) = 0,$$

because P is a harmonic function on $\overline{\Omega}_0$. Then, $\mu \perp P(\overline{\Omega})$ and it is not absolutely continuous with respect to $\omega(z_0, \cdot, \Omega)$. When we know that Ω is a Carathéodory domain, we apply the result of part (a) in order to complete the proof. ∎

Remark 3.51. The class of Carathéodory domains Ω for which $\overline{\Omega}$ does not separate the plane is (in view of Proposition 3.50) the largest class of domains for which the well-known F. and M. Riesz theorem may be extended from the unit disk preserving its formulation.

At the end of this section we present one refinement of Rudin's converse of the maximum modulus principle, where the concept of a Carathéodory set and Theorem 3.48 plays a crucial role.

Let us briefly recall the story of the aforementioned result. Let Ω be a bounded domain in $\mathbb{C}$, and let $f \in C(\overline{\Omega}) \cap H(\Omega)$. The classical maximum modulus principle states that for any $z \in \Omega$ the inequality $|f(z)| \leqslant \|f\|_{\partial\Omega}$ is satisfied. Moreover, if this inequality turns into equality at least at one point $z \in \Omega$, then the function f is constant. The question on whether it is possible to invert this principle arises quite naturally. In other words this is the question on whether it follows from the condition $|f(z)| \leqslant \|f\|_{\partial\Omega}$ (or from its certain weaker versions; see below) that the function $f \in C(\overline{\Omega})$ is holomorphic in Ω. One of the best known results of this kind is the following theorem due to W. Rudin (see [115, Theorem 12.13]). As before, $\mathfrak{j}$ stands for the function $\mathfrak{j}(z) = z$.

Theorem 3.52. *Let $\mathcal{F}$ be a subspace of the space $C(\overline{\mathbb{D}})$. Assume that $\mathcal{F}$ satisfies the following three conditions:* (i) $1 \in \mathcal{F}$; (ii) *for every function $f \in \mathcal{F}$ it holds $\mathfrak{j}f \in \mathcal{F}$; and* (iii) *the inequality*

$$|f(z)| \leqslant \|f\|_{\mathbb{T}} \tag{3.25}$$

is satisfied for every $f \in \mathcal{F}$ and $z \in \mathbb{D}$. Then, each function of $\mathcal{F}$ is holomorphic in $\mathbb{D}$.

Let $\overline{\mathcal{F}}$ be the closure of $\mathcal{F}$ in $C(\overline{\mathbb{D}})$. Since the conditions (i) and (ii) of Theorem 3.52 imply that $\mathcal{P} \subset \mathcal{F}$, then $A(\overline{\mathbb{D}}) = P(\overline{\mathbb{D}}) \subset \overline{\mathcal{F}} \subset A(\overline{\mathbb{D}})$. So that, if a given closed subspace $\mathcal{X} \subset C(\overline{\mathbb{D}})$ satisfies all conditions of Theorem 3.52, then $\mathcal{X} = A(\overline{\mathbb{D}}) = P(\overline{\mathbb{D}})$.

Rudin's theorem was a starting point for a number of further studies in the line of inversion of the maximum modulus principle. These studies were mainly related with consideration of certain weaker versions of the inequality (3.25) instead of the original one. Let us mention in this occasion the work by J. Anderson, J. Cima, N. Levenberg, and T. Ransford [4]. In this paper the inequality $|f(z)| \leqslant C_z\|f\|_{\mathbb{T}}$, where C_z is some positive number (which may depend on the point $z \in \mathbb{D}$), is considered in place of the inequality (3.25), and meromorphic functions in $\mathbb{D}$ are included into consideration. The result in question is formulated as follows.

Theorem 3.53 (Anderson, Cima, Levenberg, Ransford). *Let U be an open subset of $\mathbb{D}$ and let $g \in C(U \cup \mathbb{T})$. Assume that for any point $z \in U$ there exists a constant C_z such that the inequality*

$$|f_1(z) + g(z)f_2(z)| \leqslant C_z\|f_1 + gf_2\|_{\mathbb{T}}$$

is satisfied for all functions $f_1, f_2 \in A(\overline{\mathbb{D}})$. Then, there exist two functions $u, v \in H^\infty$ such that $g = u/v$ in U and for a.a. points $\zeta \in \mathbb{T}$ the equality of angular boundary *values* $g(\zeta) = u(\zeta)/v(\zeta)$ *holds. In particular, the function* g *is holomorphic in* U *and extends meromorphically to* $\mathbb{D}$.

It is also interesting to extend Rudin's theorem to domains which are different from the unit disk. However, this question is unstudied as yet. In [39] A. Dovgoshei considered it for the first time for Carathéodory domains G which do not separate the plane. He proved the following statement.

Theorem 3.54 (Dovgoshei). *Let* G *be a Carathéodory domain with the boundary* Γ, *and let* $\mathcal{A}$ *be a closed subalgebra of the algebra* $C(\overline{G})$ *such that* $1 \in \mathcal{A}$ *and* $\|f\|_{\overline{G}} = \|f\|_{\Gamma}$ *for any function* $f \in \mathcal{A}$. *The following two conditions are equivalent:*

(a) *if there exists a function* $g \in \mathcal{A}$ *such that* g *is injective on* $\overline{G}$ *and holomorphic in* G, *then* $\mathcal{A} = P(\overline{G})$;

(b) *the set* $\overline{G}$ *does not separate the plane.*

Notice that in this theorem one considers subspaces of the space $C(\overline{G})$ possessing certain additional (with respect to Rudin's theorem) conditions. Thus, as distinct from Rudin's theorem, we are dealing in that case with a closed subalgebra $\mathcal{A} \subset C(\overline{G})$, but not with a subspace $\mathcal{F} \subset C(\overline{G})$. Moreover, in Theorem 3.54 the condition of closedness of $\mathcal{A}$ with respect to multiplication by $\mathfrak{j}$ is replaced with the condition that $\mathcal{A}$ contains some univalent function. In fact, it was proved in [39] that for a Carathéodory domain G for which $\overline{G}$ does not separate the plane, the condition that a closed subalgebra $\mathcal{A} \subset C(\overline{G})$ contains some univalent function, yields that $\mathfrak{j} \in \mathcal{A}$. This result may be obtained as the consequence of Theorem 1.7 (more precisely, as the consequence of the weaker version of this theorem obtained in [39]). Let us also notice that the result of Theorem 3.54 in the case when G is a Jordan domain was previously obtained by Rudin in [113]. It is worth to observe that the assumptions which are imposed to $\mathcal{A}$ in Theorem 3.54 can be weakened and formulated as in Rudin's theorem. Indeed, the following result holds, see [51, Theorem 1].

Theorem¶ 3.55. *Let* G *be a Carathéodory domain.*

(a) *Let* G *be such that* $\overline{G}$ *does not separate the plane. If a subspace* $\mathcal{F}$ *of the space* $C(\overline{G})$ *satisfies the following three conditions:* (i) $1 \in \mathcal{F}$; (ii) *for every function* $f \in \mathcal{F}$ *it holds* $\mathfrak{j} f \in \mathcal{F}$; *and* (iii) *the inequality* $|f(z)| \leqslant \|f\|_{\partial G}$ *is satisfied for all* $f \in \mathcal{F}$ *and* $z \in G$; *then each function in* $\mathcal{F}$ *is holomorphic in* G.

(b) *A closed subspace* $\mathcal{X} \subset C(\overline{\Omega})$ *satisfying the conditions* (i)–(iii) *from the first part of the theorem (where* $\mathcal{F}$ *is replaced by* $\mathcal{X}$*) coincides with* $P(\overline{G})$ *if and only if* $\overline{G}$ *does not separate the plane.*

The proof of the direct statement in Theorem 3.55 is essentially based on the usage of Wermer's maximality theorem for Carathéodory domains that do not separate the plane. As it was shown previously in Theorem 3.42, the condition that $\overline{G}$ does not separate the plane cannot be dropped whenever we want to preserve the maximality theorem statement.

Since the notion of a Carathéodory domain has appeared in the same topics in complex analysis and theory of uniform algebras with theorems by Rudin, Wermer and Anderson–Cima–Levenberg–Ransford, it is quite natural to consider Carathéodory domains in the respective context. In fact, we have the following result, see [51, Theorem 2].

Theorem¶ 3.56. *Let G be a Carathéodory domain with the boundary Γ, and let U be an open subset of G. Let $g \in C(U \cup \Gamma)$. Assume that for any $z \in U$ there exists a constant C_z such that the inequality*

$$|f_1(z) + g(z) f_2(z)| \leqslant C_z \| f_1 + g f_2 \|_\Gamma \tag{3.26}$$

is satisfied for any function $f_1, f_2 \in A(\overline{G})$. Then, there exist two functions $u, v \in H^\infty(G)$ such that the equality

$$g(z) = \frac{u(z)}{v(z)} \tag{3.27}$$

holds everywhere in U and a.e. on Γ in the sense of conformal mappings. The latter means that for a.a. points $\zeta \in \mathbb{T}$ the following equality of angular boundary values holds $g(f(\zeta)) = u(f(\zeta))/v(f(\zeta))$, where f is some conformal map from the disk $\mathbb{D}$ onto G. In particular, the function g is holomorphic in U and extends meromorphically to G.

In the case, when $M \subset G$ is some finite set and $U = G \setminus M$, Theorem 3.56 gives the description of meromorphic functions in G with poles in M. In particular, if the set M is empty, then the respective description of holomorphic functions in G originates from this theorem.

Observe that in the case when G is a Jordan domain with rectifiable boundary, the equality (3.27) may be realized directly as the equality of angular boundary values a.e. on ∂G.

Corollary 3.57. *Let G be a Carathéodory domain for which $\overline{G}$ does not separate the plane. Assume that a function $g \in C(\overline{G})$ is such that for any functions $f_1, f_2 \in A(\overline{G})$ and for any point $z \in G$ the inequality*

$$|f_1(z) + g(z) f_2(z)| \leqslant \| f_1 + g f_2 \|_\Gamma \tag{3.28}$$

is satisfied. Then, the function g is holomorphic in G.

Notice, that the assertion of Rudin's theorem may be derived from this corollary, see [51, Section 3] for the details.

3.6 Approximation by polyanalytic functions

The topic on approximation of functions by polyanalytic polynomials and polyanalytic rational functions is the subject of active studying in contemporary complex analysis and approximation theory. The concept of a Carathéodory set appears in this topic very naturally. In this section let X be a compact set in the complex plane, and let $n \geqslant 1$ be an integer. We define

$$\mathcal{P}_n = \mathcal{P} + \bar{z}\,\mathcal{P} + \cdots + \bar{z}^{n-1}\,\mathcal{P},$$
$$\mathcal{R}_n = \mathcal{R} + \bar{z}\,\mathcal{R} + \cdots + \bar{z}^{n-1}\,\mathcal{R}.$$

The spaces $\mathcal{P}_n$ and $\mathcal{R}_n$ are modules of dimension n over $\mathcal{P}$ and $\mathcal{R}$, respectively, generated by the powers of the function $\bar{z}$. For a given integer $d \geqslant 1$ we will also consider modules $\mathcal{P}_{n,d}$ and $\mathcal{R}_{n,d}$ generated by powers of $\bar{z}^d$ instead of powers of $\bar{z}$. For instance,

$$\mathcal{P}_{2,d} = \mathcal{P} + \bar{z}^d\,\mathcal{P}, \quad \mathcal{R}_{2,d} = \mathcal{R} + \bar{z}^d\,\mathcal{R}, \ldots .$$

Let us recall, that a function f is said to be *polyanalytic of order* n (or, for the sake of brevity, n*-analytic*) in an open set $U \subset \mathbb{C}$, if it is of the form

$$f(z) = f_0(z) + \bar{z}\, f_1(z) + \cdots + \bar{z}^{n-1} f_{n-1}(z),$$

where $f_0, \ldots, f_{n-1} \in H(U)$. The functions $f_0, \ldots, f_{n-1}$ are usually called holomorphic components of f. As usual, n-analytic functions whose holomorphic components are polynomials and rational functions will be called polyanalytic polynomials and polyanalytic rational functions, respectively. In fact, a polyanalytic rational function is not, in the general case, a quotient of two polyanalytic polynomials. It can be readily verified that the set of all n-analytic function on an open set U coincides with the set of all functions $f \in C(U)$ each of which is satisfies in U (in the sense of distributions) the (elliptic) partial differential equation $\overline{\partial}^n f = 0$. One ought to notice right now, that elements of modules generated by $\bar{z}^d$ for every $d > 1$ no longer belong to the kernel of some elliptic differential operator with constant coefficients, but (under suitable additional assumptions) they belong to the kernel of the elliptic operator $f \mapsto \overline{\partial}(\bar{z}^{1-d}\,\overline{\partial} f)$.

Furthermore, for a closed set $E \subset \mathbb{C}$ we will denote by $\mathcal{R}_{n,d,E}$ the set of all functions $g \in \mathcal{R}_{n,d}$ such that all poles of all holomorphic components of g lies outside E. Finally, we put $\mathcal{R}_{n,E} = \mathcal{R}_{n,1,E}$ and define the space

$$A_n(X; \bar{z}^d) = C(X) \cap (H(X^\circ) + \bar{z}^d H(X^\circ) + \cdots + \bar{z}^{d(n-1)} H(X^\circ)),$$

and let $A_n(X) = A_n(X; \bar{z})$, so that $A_n(X)$ is the set of all functions which is continuous on X and n-analytic on its interior.

Let K be an arbitrary compact set in $\mathbb{C}$ containing X. It can be shown, that the uniform closures on X of the spaces $\mathcal{P}_{n,d}|_X$ and $\mathcal{R}_{n,d,K}|_X$ are contained in $A_n(X;\bar{z}^d)$.

Thus, the problem on to describe such compact sets X for which the set $\mathcal{P}_{n,d}|_X$ is dense in $A_n(X;\bar{z}^d)$ is of interest. We refer the reader to the recent survey paper [88], where the history of this problem and its state-of-the-art are established in details. Here, we only state two results, which highlight the role of Carathéodory sets in this topic. Before doing this let us present the following result which may be directly derived from the main results of [25] using the Runge's pole–shifting method.

Theorem 3.58. *Assume X to be such that the set $\mathbb{C}\setminus X$ is connected. Then, the following hold.*

(1) *For any integer $n \geqslant 1$ the space $\mathcal{P}_n|_X$ is dense in $A_n(X)$.*

(2) *For any integer $d \geqslant 2$ the space $\mathcal{P}_{2,d}|_X = (\mathcal{P} + \bar{z}^d\,\mathcal{P})|_X$ is dense in $A_2(X;\bar{z}^d)$.*

For formulation of next results we need the concept of a d-Nevanlinna domain. This is the special analytic characteristic of bounded simply connected domains in the complex plane which was originally introduced in the case $d = 1$ in [49] and [28], and later in [8] for $d > 1$. It will be clear from what follows, that this concept turned out to be crucial for the aforementioned problem.

Definition 3.59. Let $d \in \mathbb{N}$. A bounded simply connected domain $G \subset \mathbb{C}$ is called a d-Nevanlinna domain if there exists two functions $u, v \in H^\infty(G)$ such that the equality

$$\bar{z}^d = \frac{u(z)}{v(z)}$$

holds almost everywhere on ∂G in the sense of conformal mappings. The latter means, that the equality of boundary values $\overline{f(\zeta)}^d = (u \circ f)(\zeta)/(v \circ f)(\zeta)$ holds for almost all points $\zeta \in \mathbb{T}$, where f is some conformal mapping from $\mathbb{D}$ onto G.

The class of 1-Nevanlinna domains is just the class of Nevanlinna domains. Notice that properties of Nevanlinna domain and d-Nevanlinna domains has been studied in detail during the two last decades (see, for instance, [8–12, 50, 86, 87]).

Let us mentioned several simple examples. In fact, $\mathbb{D}$ is a d-Nevanlinna domain for all $d \geqslant 1$. At the same time, any domain bounded by an ellipse which is not a circle is not a d-Nevanlinna for any $d \geqslant 1$. Take any fixed $d > 1$. For a real $a > 1$ let g_a be the single valued branch of the function $\sqrt[d]{a-z}$ defined on $\mathbb{C}\setminus[a,+\infty)$ and such that $g_a(0) > 0$. Then, the domain $g_a(\mathbb{D})$ is a d-Nevanlinna, but not a Nevanlinna domain. At the first glance it seems that the concept of a Nevanlinna domain gives a slight refinement of the concept of a Schwarz function of an analytic arc (see, for instance, [35]), but it turns out that there exists Nevanlinna domains with not analytic, not smooth, not rectifiable boundaries and, moreover, Nevanlinna domains G such that the Hausdorff dimension of the set $\partial_a G$ could take any value in $[1, 2]$.

In the following statement we combine the results of [28, Theorem 2.2], [16, Theorem 4], and [26, Theorem 4]. Let $\mathcal{C}'_X$ be the set of all connected components of the set $\operatorname{Int}(\widehat{X})$ that are not contained in X, that is

$$\mathcal{C}'_X = \{\Omega \in \mathcal{C}(\operatorname{Int}(\widehat{X})) : \Omega \not\subset X\}.$$

Theorem¶ 3.60. *The following statements hold.*

(1) *Let X be a compact set in $\mathbb{C}$ such that the set $\mathcal{C}'_X$ is not empty. Then, the subspace $\mathcal{P}_n|_X$ is dense in $A_n(X)$ if and only if for every $\Omega \in \mathcal{C}'_X$ the space $\mathcal{R}_{n,\overline{\Omega}}|_{X\cap\overline{\Omega}}$ is dense in $A_n(X \cap \overline{\Omega})$.*

(2) *Let G be a bounded simply connected domain in $\mathbb{C}$. If G is a Nevanlinna domain, then $\mathcal{R}_{n,\overline{G}}|_{\partial G}$ is not dense in $C(\partial G)$ for any integer $n \geqslant 0$.*

(3) *Let G be a Carathéodory domain in $\mathbb{C}$. The subspace $\mathcal{R}_{n,\overline{G}}|_{\partial G}$ is dense in $C(\partial G)$ if and only if G is not a Nevanlinna domain.*

The same results hold in problem of approximating functions by elements of the space $\mathcal{P}_{2,d} = \mathcal{P} + \bar{z}^d\mathcal{P}$. In fact, we have (see [8, Theorems 1, 2, and Propositions 2, 3]).

Theorem¶ 3.61. *The following statements hold.*

(1) *Let X be a compact set in $\mathbb{C}$ such that the set $\mathcal{C}'_X$ is not empty. Then, the subspace $\mathcal{P}_{2,d}|_X$ is dense in $A_2(X; \bar{z}^d)$ if and only if for every $\Omega \in \mathcal{C}'_X$ the space $\mathcal{R}_{2,d,\overline{\Omega}}|_{X\cap\overline{\Omega}}$ is dense in $A_2(X \cap \overline{\Omega}; \bar{z}^d)$.*

(2) *Let G be a bounded simply connected domain in $\mathbb{C}$. If G is a d-Nevanlinna domain, then the space $\mathcal{R}_{2,d,\overline{G}}|_{\partial G}$ is not dense in $C(\partial G)$.*

(3) *Let G be a Carathéodory domain in $\mathbb{C}$. The subspace $\mathcal{R}_{2,d,\overline{G}}|_{\partial G}$ is dense in $C(\partial G)$ if and only if G is not a d-Nevanlinna domain.*

Notice that this result is established for modules of dimension 2 only. The general case remains open.

Remarks and hints concerning the proofs of Theorems 3.60 and 3.61. The first statements in Theorems 3.60 and 3.61 are proved using the following scheme consisting of two steps (see [16] and [8], respectively): at the first step it was proved that any measure on X which is orthogonal to $\mathcal{P}_n$ (respectively, to $\mathcal{P}_{2,d}$) is also orthogonal to $\mathcal{R}_{n,X}$ (respectively, to $\mathcal{R}_{2,d,X}$). The respective construction was essentially elaborated in [28] in the proof of Theorem 2.2 of this paper. At the second step, using the special refinement of the Vitushkin's localization technique, it was proved that the space $\mathcal{R}_{n,X}|_X$ is dense in $A_n(X)$ (respectively, the space $\mathcal{R}_{2,d,X}|_X$ is dense in $A_2(X; \bar{z}^d)$). The condition that $\mathcal{R}_{n,\overline{\Omega}}|_{X\cap\overline{\Omega}}$ is dense in $A_n(X \cap \overline{\Omega})$ (and the respective condition in the second case) allow us to construct the desired approximants.

In order to prove the second statements in both Theorems 3.60 and 3.61 it is sufficient to show, that if G is a d-Nevanlinna domain, then the function $(\bar{z}^d - \bar{a}^d)/(z - a)$, $a \in G$, cannot be approximated uniformly on ∂G by rational functions of the class $\mathcal{R}_{2,d,\bar{G}}$. The detailed exposition of this proof is in the proof of Theorem 4 in [26] and of the proof of Proposition 2 in [8].

Let us present the schematic exposition of the proof of the third statements of theorems under consideration, because in the respective constructions show the reasons why the Carathéodory domain and Nevanlinna domain concepts are important and crucial for the aforementioned topic.

Let f be a conformal mapping from $\mathbb{D}$ onto G. We recall, that Corollary 2.25 states that the functions f and f^{-1} can be extended to mutually inverse Borel measurable functions on $\mathbb{D} \cup F(f)$ and $G \cup \partial_a G$, respectively. Let $\omega = f(d\zeta)$ the complex harmonic measure with respect to $f(0)$, see (3.19). If the space $\mathcal{R}_{2,d,\bar{G}}$ is not dense in $C(\partial G)$, then there exists a non-zero measure μ on ∂G such that $\mu \perp \mathcal{R}_{1,\bar{G}}$ and $\bar{z}^d \mu \perp \mathcal{R}_{1,\bar{G}}$. In view of (3.21) there exists two functions $h_1, h_2 \in H^1$ such that $\mu = (h_1 \circ f^{-1})\,\omega$ and $\bar{z}^d \mu = (h_2 \circ f^{-1})\,\omega$. Therefore, for almost all $\zeta \in \mathbb{T}$ one has $\overline{f(\zeta)}^d h_1(\zeta) = h_2(\zeta)$. Going further, replacing the quotient h_2/h_1 by f_2/f_1 with $f_1, f_2 \in H^\infty$ and defining the functions u and v in G as follows: $u(z) = f_2(f^{-1}(z))$, $v(z) = f_1(f^{-1}(z))$ one obtains that $\bar{z}^d = u(z)/v(z)$ almost everywhere on ∂G in the sense of conformal mappings, as it is demanded. ∎

Finally, let X be a Carathéodory compact set. In such a case the set $\mathcal{C}'_X$ is exactly the set of all bounded connected components of the set $\mathbb{C} \setminus X$. Thus, the following statement is a direct corollary of Theorems 3.60 and 3.61:

Corollary¶ 3.62. *Let $X \subset \mathbb{C}$ be a Carathéodoty compact set.*

(1) *The space $\mathcal{P}_n|_X$ is dense in $A_n(X)$ if and only if each bounded connected component of the set $\mathbb{C} \setminus X$ is not a Nevanlinna domain.*

(2) *The space $\mathcal{P}_{2,d}|_X$ is dense in $A_2(X; \bar{z}^d)$ if and only if each bounded connected component of the set $\mathbb{C} \setminus X$ is not a d-Nevanlinna domain.*

Chapter 4

Approximation in L^p-norms on Carathéodory sets

In this chapter we consider the topic on approximation of functions on Carathéodory sets by rational functions or polynomials in L^p-norms for $0 < p < +\infty$.

For a bounded measurable set $E \subset \mathbb{C}$ let us denote by $L^p(E)$ the space of all measurable functions $f\colon E \to \mathbb{C}$ such that

$$\|f\|_{p,E} = \|f\|_{L^p(E)} = \left(\int_E |f(z)|^p \, dA(z)\right)^{\frac{1}{p}} < +\infty,$$

while by $A^p(E)$ we denote the space consisting of those functions in $L^p(E)$ that are holomorphic in the interior of E. In the case that E is a domain, the spaces $A^p(E)$ are usually called Bergman spaces. For $p \geqslant 1$ they are Banach spaces, but for $p \in (0, 1)$ the quantity $\|f\|_{p,E}$ is only a quasi-norm. The history and the state-of-the-art of the theory of Bergman spaces may be found in the books [41] and [66].

4.1 Approximation in Bergman spaces

Our first goal in this section is to prove and discuss the following result, which is due to O. J. Farrell [45, 46] and A. I. Markushevich [84], see also [85, Chapter v].

Theorem 4.1 (Farrell). *Let G be a Carathéodory domain and let $0 < p < +\infty$. For every function $f \in A^p(G)$ there exist a sequence (p_n) of polynomials such that*

$$\lim_{n\to\infty} \int_G |f(z) - p_n(z)|^p \, dA(z) = 0.$$

In order to find the original proof of this theorem given by Farrell, it is convenient to pass thought both his papers [45] and [46]. The case that $p = 2$ was considered independently by Markushevich, however, there are some evidences that he has proved the corresponding result in the general case too. Markushevich's proof given in his later book [85, Chapter v] uses some tools which are very useful in the case of Hilbert spaces.

Before proving Theorem 4.1 let us make some historical remarks concerning the matter. Let G be a bounded domain in the complex plane. As far as we know the first results on approximation of functions in the class $A^p(G)$, for a given domain $G \subset \mathbb{C}$ and a number $p \in (0, \infty)$, by polynomials were obtained in the beginning of the 1920s by T. Carleman [23]. He considered the case of Jordan domain starlike with respect to the origin. Since his result is completely covered by Theorem 4.1, we are not going to comment it or on the technique used in Carleman's proof. It is worth mentioning here, that in the case when $G = \mathbb{D}$, $p > 1$, and $f \in A^p(\mathbb{D})$, then one can take the sequence of Taylor polynomials of f (with the center at the origin) as the desired approximating sequence in Theorem 4.1. Notice also, that in the general case, if $f \in A^p(\mathbb{D})$ with $p \in (0, 1]$, the sequence of the Taylor polynomials of f does not converge to f. The details of these constructions may be found in [41, page 31]. Let us notice however, that the proof in the case $0 < p < \infty$ and $G = \mathbb{D}$ does show that f_ρ converges to f as $\rho \to 1$, where $f_\rho(z) = f(\rho z)$.

Proof of Theorem 4.1. Let the sequence (J_n) of Jordan curves such that $D(J_n)$ converges to G (in the sense of kernel convergence), the sequence (φ_n) of conformal maps from $D(J_n)$ onto $\mathbb{D}$, and the conformal mapping φ from $\mathbb{D}$ onto G be as in the proof of Theorem 3.25. Let g_n, $n \in \mathbb{N}$, be the function $g_n = \varphi \circ \varphi_n$ defined on $D(J_n)$. Then, $g_n(z) \to z$ and $g_n'(z) \to 1$ locally uniformly in G. Consider the function

$$f_n = (f \circ g_n)\,(g_n')^{2/p}$$

defined in $D(J_n)$, where the branch of ${g_n'}^{2/p}$ is taken in such a way that is positive at the point $z_0 = \varphi(0)$. Let

$$C_p = \max\{2^{p-1}, 1\},$$

so that $|a + b|^p \leqslant C_p(|a|^p + |b|^p)$ for every point $a, b \in \mathbb{C}$. Fix now $\varepsilon > 0$ and take $K \subset G$ to be the closure of some Jordan domain such that

$$C_p \int_{G \setminus K_1} |f(z)|^p \, dA(z) < \varepsilon/3, \tag{4.1}$$

where $K \subset K_1 \subset G$, and K_1 also is the closure of some Jordan domain. Choosing K_1 in such a way that $G \setminus g_n(K_1) \subset G \setminus K$ for all $n \geqslant n_0$ with some $n_0 \in \mathbb{N}$, one has

$$\begin{aligned} \int_G |f - f_n|^p \, dA &\leqslant \int_{K_1} |f - f_n|^p \, dA + \int_{G \setminus K_1} |f - f_n|^p \, dA \\ &\leqslant \int_{K_1} |f - f_n|^p \, dA + C_p \int_{G \setminus K_1} |f|^p \, dA + C_p \int_{G \setminus K_1} |f_n|^p \, dA. \end{aligned} \tag{4.2}$$

The last integral in (4.2) can be estimated using (4.1) and (2.1) as follows:

$$\begin{aligned} \int_{G \setminus K_1} |f_n|^p \, dA &= \int_{G \setminus K_1} |f(g_n(z))|^p \, |g_n'(z)|^2 \, dA(z) = \int_{g_n(G \setminus K_1)} |f|^p \, dA \\ &\leqslant \int_{G \setminus g_n(K_1)} |f|^p \, dA \leqslant \int_{G \setminus K} |f|^p \, dA \leqslant \frac{\varepsilon}{3C_p}. \end{aligned} \tag{4.3}$$

For $n \geqslant n_1$ with some $n_1 \in \mathbb{N}$. Since $f(z) - f(g_n(z))g_n'(z) \to 0$ uniformly on $z \in K_1$ we have

$$\int_{K_1} |f - f_n|^p \, dA(z) < \frac{\varepsilon}{3}, \tag{4.4}$$

for $n \geqslant n_2$ with some $n_2 \in \mathbb{N}$. Using (4.4), (4.3), and (4.2) we obtain

$$\int_G |f(z) - f_n(z)|^p \, dA(z) < \varepsilon, \quad n \geqslant n_3. \tag{4.5}$$

For $n \geqslant n_3 = \max\{n_0, n_1, n_2\}$.

Each function f_n is holomorphic in $D(J_n)$ which is an open simply connected neighborhood of $\widehat{G}$. Then, applying again the Runge's theorem, we conclude that there exists a polynomial P_n such that

$$\int_{\widehat{G}} |f_n(z) - P_n(z)|^p \, dA(z) < \varepsilon,$$

for $n \geqslant n_3$. Using this together with (4.5) we obtain

$$\int_G |f - P_n|^p \, dA \leqslant 2C_p\varepsilon,$$

for $n \geqslant n_3$, which completes the proof. ■

Let us observe that Theorem 3.25 can be regarded as the limiting case of Theorem 4.1 when $p = +\infty$.

It would be interesting to compare Theorems 3.25 and 4.1 with each other, as well as with Runge's theorem, at least in some simple cases. For instance, let us take $p = 1$ and put $G = \mathbb{D}$. In this case, the polynomials nz^n, $n \in \mathbb{N}$, converge to zero locally uniformly in $\mathbb{D}$, but they do not converge on $L^p(\mathbb{D})$ for any $p > 0$. On the other hand the polynomials z^n converge to zero locally uniformly in $\mathbb{D}$ and also converges in $L^p(\mathbb{D})$ for each $p > 0$. They do not, however, satisfy the estimate of Theorem 3.25. The polynomials z^n/n converge to zero locally uniformly in $\mathbb{D}$ and they satisfy the estimate of Theorem 3.25. Finally, the polynomials $p^n z^{nk^n}$ with $p > 0$ and $k \in \mathbb{N}$, converge in $L^p(\mathbb{D})$ when $p \leqslant k$ and diverge when $p > k$.

Since the convergence of some sequence of holomorphic functions defined on a given open set U in $L^p(U)$-norm (for some $p > 0$) implies the locally uniform convergence of this sequence in U, one can additionally conclude in Theorem 4.1 that the sequence (P_n) converges locally uniformly in G to f. The possibility of polynomial approximation in such theorem in the case when the set $\mathbb{C} \setminus \overline{G}$ has bounded components looks a bit surprising. For example, if we suppose G to be the left-hand-side domain in Figure 2 (the cornucopia), then the function $1/z$ can be approximated in $A^p(G)$ by a sequence of polynomials but, of course, it cannot be approximated by polynomials locally uniformly in G. Notice also, that in a given Carathéodory domain G there may exist a compact set $K \subset \overline{G}$ such that $\widehat{K}$ is not contained in $\overline{G}$ and the $L^p(G)$-convergence of some sequence of polynomials does not imply the uniform convergence of this sequence on $\widehat{K}$. It is worth comparing this observation with the next proposition.

Proposition¶ 4.2. *Let U be an open set in $\mathbb{C}$ and $K \subset U$ be a compact set. Let $p > 0$ and $f \in A^p(U)$. Assume that there exists a sequence (P_n) of polynomials such that*

$$\lim_{n\to\infty} \int_U |f - P_n|^p \, dA = 0. \tag{4.6}$$

Then, f has an analytic extension to $U \cup \widehat{K}$. Denoting again this extension by f one has

$$\lim_{n\to\infty} \int_{U\cup\widehat{K}} |f - P_n|^p \, dA = 0.$$

Proof. It is enough to consider the case $\widehat{K} \setminus U \neq \emptyset$. Take such $r > 0$ that $\operatorname{dist}(K, \partial U) > 2r$. We need the following statement asserting that functions in a Bergman space cannot grow too rapidly near the boundary (see [41, Theorem 1]).

Lemma 4.3. *Let $p > 0$. For each function $f \in A^p(U)$ and for each compact set $K \subset U$, we have*

$$\|f\|_K \leqslant \frac{\|f\|_{p,U}}{\pi^{1/p} \operatorname{dist}(K, \partial U)^{2/p}}. \tag{4.7}$$

In particular, if a sequence of functions (f_n), $f_n \in A^p(U)$, converges to f in $A^p(U)$, then $f_n \rightrightarrows f$ locally in U.

Consider the compact set

$$K_1 = K \cup \bigcup_{z\in K} \overline{D(z,r)} \subset U$$

and apply (4.7) to K_1. Then, one has

$$\|f - P_n\|_{K_1} = \sup_{z\in K_1} |f(z) - P_n(z)| \leqslant c_r \int_U |f - P_n|^p \, dA, \tag{4.8}$$

for some constant c_r depending on U, K, and p.

Then, (4.8) and (4.6) imply that (P_n) is a Cauchy sequence on K_1. By the maximum modulus principle the sequence (P_n) must converge uniformly to some holomorphic function g on $\operatorname{Int}(\widehat{K_1})$. Since $\widehat{K} \subset \operatorname{Int}(\widehat{K_1})$ then (P_n) converges uniformly on $\widehat{K}$, so it converges in the space $L^p(\widehat{K})$. By (4.8) it follows that $g(z) = f(z)$ on K, and the usage of the inequality

$$\int_{U\cup\widehat{K}} |f - P_n|^p \, dA \leqslant \int_U |f - P_n|^p \, dA + \int_{\widehat{K}} |g - P_n|^p \, dA$$

finishes the proof. ∎

We mention three simple examples showing the situation with polynomial approximation in L^p-norm in the case of non-Carathéodory domains.

Example 4.4. We mention three simple examples showing the situation with polynomial approximation in L^p-norm in the case of non-Carathéodory domains.

(i) Take $G_1 = \mathbb{D} \setminus (-1, 0]$ and $g(z) = 1/z$, $z \in G_1$. Then, $g \in A^p(G_1)$ for each p, $0 < p < 2$, but g cannot be approximated in the L^p-norm by polynomials for any $p \in (0, 2)$.

(ii) Let $g(z) = \log z$, $z \in G_1$ (where log stands for the principal branch of the log-function). Then, g cannot be approximated in the L^p-norm by polynomials for any $p > 0$.

(iii) Take $G_2 = \mathbb{D} \setminus (\overline{D(0, 1/2)} \cup [0, 1))$ and $g(z) = 1/z$, $z \in G_2$. Then, g cannot be approximated in the L^p-norm by polynomials for any $p > 0$.

The verification of all these statements may be done using (4.7).

The construction given in the first of the aforesaid examples may be refined and generalize by the following way.

Proposition¶ 4.5. *Let G be a Carathéodory domain and let $\mathcal{E}$ be some end-cut of G such that* $\text{Area}(\mathcal{E}) = 0$, *and let $G_{\mathcal{E}}$ be the corresponding slitted domain $G \setminus \mathcal{E}$. Then, the set of polynomials is not dense in $A^p(G_{\mathcal{E}})$ for any $p > 0$.*

Proof. Take a conformal map g from $G_{\mathcal{E}}$ to the unit disk. Then, $g \in A^p(G_{\mathcal{E}})$ for each $p > 0$. Assume that there exists a sequence of polynomials (P_n) that converges in $A^p(G_{\mathcal{E}})$ to the function g. Therefore, (P_n) is a Cauchy sequence in $A^p(G)$. So, it needs to converge uniformly on compact subsets of G to a function $\tilde{g} \in H(G)$ which coincides with g on $G \setminus \mathcal{E}$. But this is impossible since in each cut point $a \in \mathcal{E}$, the function g cannot be extended continuously to a neighborhood of a. ∎

In view of this proposition one can ask whether the condition that a given domain G is a Carathéodory domain, is necessary in order to have polynomial approximation in $A^p(G)$. The answer to this question is negative, as it may be observed from several constructions of so-called moon-shaped domains.

Recall, that a domain $M \subset \mathbb{C}$ is called a *moon-shaped* domain, if it has the form $M = D(J_1) \setminus \overline{D(J_2)}$, where $J_{1,2}$ are two Jordan curves such that $J_1 \cap J_2 = \{\xi\}$ and $J_2 \subset D(J_1) \cup \{\xi\}$. In what follows it would be appropriate to say that the domain M is determined by the curves $J_1 = J_1(M)$ and $J_2 = J_2(M)$. It will be also useful to write $D_1 = D_1(M) = D(J_1)$ and $D_2 = D_2(M) = D(J_2)$ in the situation under consideration.

The simplest example of a moon-shaped domain is the domain

$$M_r := \mathbb{D} \setminus \overline{D(r, 1 - r)}, \tag{4.9}$$

for $0 < r < 1$, see the left-hand side domain on Figure 9. In this situation $J_1 = \mathbb{T}$, while $J_2 = \{z : |z - r| = 1 - r\}$.

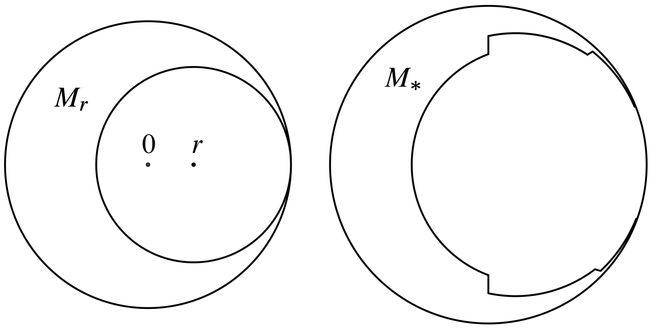

Figure 9. Two moon-shaped domains: M_r and M_*.

In the following two propositions we collect several results about moon-shaped domains, which are closely related with the topic on L^p-polynomial approximation being discussed.

Proposition¶ 4.6. *Let M be a moon-shaped domain, let $p \in (0, \infty)$ and $s = 2/(p + 2)$. The set of polynomials is dense in $A^p(M)$ if and only if there exists $b \in D_2(M)$ such that both functions $\varphi(z) = (z - b)^s$ and $\psi(z) = (z - b)^{-s}$ can be approximated by polynomials in $A^p(M)$.*

Sketch of the proof. The necessity of the stated condition is clear. For proving its sufficiency, let us note that the function φ maps conformally the given domain M onto some domain $W = \varphi(M)$. Since $s < 1$ then W is a Jordan domain. Put $w = \varphi(z)$, $z \in M$, then $z = \varphi^{-1}(w) = b + w^{1+p/2}$. Take $f \in A^p(M)$ and put $f_1(w) = f(\varphi^{-1}(w))$. Since $\varphi(z)^p(\varphi'(z))^2 = s^2$ for every $z \in M$, one has

$$\int_W |wf_1(w)|^p \, dA(w) = \int_M |\varphi(z)|^p |f(z)|^p |\varphi'(z)|^2 \, dA(z) = s^2 \int_M |f(z)|^p \, dA(z).$$

Then, $wf_1 \in A^p(W)$. Take any $\varepsilon > 0$. Since the set of polynomials is dense in $A^p(W)$ (because W is a Jordan domain), there exists a polynomial $P(w) = \sum_{k=0}^n a_k w^k$ such that

$$\int_W |wf_1(w) - P(w)|^p \, dA(w) = s^2 \int_M \left| f - a_0\psi - \sum_{k=1}^n a_k \varphi^{k-1} \right|^p dA < \varepsilon.$$

Thus, the possibility of approximation of both function φ and ψ by polynomials in $A^p(M)$ implies the possibility of approximation in the desired sense of any function $f \in A^p(M)$. ∎

Note that in the case $p = 2$ it can be shown that is it sufficient to approximate just ψ in order to have the conclusion in Proposition 4.6. This will be used in proving part (3) of the next proposition. At the same time the mentioned arguments cannot be used in the case $p \neq 2$. In view of this reason it is not clear how to modify the construction of the domain M_* to obtain a corresponding example that covers the case $p \neq 2$ in part (3) of Proposition 4.7.

Proposition 4.7. *The following approximation properties hold.*

(1) *Let M be a moon-shaped domain, while ξ be the common point of $J_1(M)$ and $J_2(M)$. If there exist a rectifiable Jordan curve Γ in $M \cup \{\xi\}$ such that $\xi \in \Gamma$, and a number $\alpha > 0$ such that*

$$\int_\Gamma \operatorname{dist}^{-\alpha}(z, \partial M)\, |dz| < +\infty,$$

then the set of polynomials is not dense in $A^p(M)$ for every $p > 0$.

(2) *The set of polynomials is not dense in $A^p(M_r)$ for any $p > 0$ and for any $r \in (0, 1)$.*

(3) *There exists a moon-shaped domain M_* such that the set of polynomials is dense in $A^2(M_*)$.*

Sketch of the proof. Part (1) The proof of this statement can be obtained following the pattern of the verification of Example 4.4, which is based on usage of (4.7); the case $p = 2$ may be found in [91, page 116].

Part (2) Using the notation of the previous part, let us take

$$\Gamma(t) = \frac{r}{2} + \Big(1 - \frac{r}{2}\Big)e^{it}, \quad t \in [0, 2\pi],\ r \in (0, 1),$$

It can be verified that this curve satisfies the conditions of the previous statement for every $\alpha < \frac{1}{2}$.

Part (3) Denote by arg the branch of the argument function defined on $\mathbb{C} \setminus \{0\}$ such that $\arg z \in (-\pi, \pi]$ for $z \neq 0$. Let us construct a sequence (α_n) with $0 < \alpha_n < 1$, a sequence of polynomials (P_n), and three sequences of sets (D_n), Δ_n and Ω_n as follows. Let $\alpha_0 = 1/4$ and

$$D_1 = \{z \in \mathbb{D} : |z - \alpha_0| > 1 - \alpha_0,\ |\arg z| > \pi/2\}.$$

According to Runge's theorem there exists a polynomial P_1 such that

$$\|z^{-1/2} - P_1\|_{2,D_1} < 1/\sqrt{2}.$$

Next there exists a sufficiently small $\alpha_1 \in (0, 1)$ and the domain

$$\Delta_1 = \{z \in \mathbb{D} : |z - \alpha_1| > 1 - \alpha_1,\ |\arg z| < \pi/2\}$$

such that

$$\|z^{-1/2} - P_1\|_{2,\Delta_1} < 1/\sqrt{2}.$$

Finally, let $\Omega_1 := \{z \in \mathbb{D} : |z - \alpha_1| > 1 - \alpha_1,\ |\arg z| > \pi/4\}$. Assume, that all desired objects are already constructed for $n = 1, \dots, N-1$, $N \geqslant 2$. Put $D_N = D_{N-1} \cup \Omega_{N-1}$. There exists a polynomial P_N such that

$$\|z^{-1/2} - P_N\|_{2,D_N} < 2^{-N/2},$$

and a (sufficiently small) number $\alpha_N \in (0,1)$ such that

$$\|z^{-1/2} - P_N\|_{2,\Delta_N} < 2^{-N/2},$$

for the domain $\Delta_N = \{z \in \mathbb{D} : |z - \alpha_N| > 1 - \alpha_N,\ |\arg z| < \pi/2^N\}$. Defining

$$\Omega_N = \left\{z \in \mathbb{D} : |z - \alpha_N| > 1 - \alpha_N, |\arg z| > 1/2^{N+1}\right\}$$

we finish the construction. Now, we are able to define the domain M_* as $\bigcup_{n=}^{\infty} D_n$. Since $M_* \subset D_n \cup \Delta_n$ for every integer $n > 0$ one has

$$\|z^{-1/2} - P_n\|_{2,M_*} < 2^{(1-n)/2}.$$

Then, $\psi(z) = z^{-1/2}$ belongs to $A^2(M_*)$, so the proof of (3) is completed. ■

It is also worth mentioning here yet another example given in [71] (see also [91]). Taking $\alpha > 4$ and $\lambda \in (0,1)$ let

$$\Upsilon_{\alpha,\lambda} := \left\{z = x + iy \in \mathbb{C} : y^2 = (\lambda + x)(1-x)^\alpha\right\}.$$

The moon-shaped domain M determined by $J_1 = \mathbb{T}$ and $J_2 = \Upsilon_{\alpha,\lambda}$ is homeomorphic to the domain M_* defined in the proof of the part (3) of Proposition 4.7. But it was proved in [71] that the set of polynomials is not dense in $A^2(M)$. So, the question on L^p-approximation by polynomials depends on certain metric properties of the domain under consideration.

At the end of this section we present yet another two proofs of Theorem 4.1 in the Hilbert space setting, namely, in the case that $p = 2$. We do it in order to highlight certain special properties of Carathéodory domains and their conformal maps on which these proofs are based. The first proof was given by A. I. Markushevich [83], and it is based on the following lemma.

Lemma 4.8. *Let G be a Carathéodory domain and $z_0 \in G$. Take a sequence (G_n) of Jordan domains such that $G_n \to G$ with respect to z_0, and let g_n be the conformal map from G_n onto $\mathbb{D}$ normalized by the conditions $g_n(z_0) = 0$ and $g_n'(z_0) > 0$ for every $n \in \mathbb{N}$. Moreover, let g be the conformal map from G onto $\mathbb{D}$ with the same normalization, so that $g_n \rightrightarrows g$ locally in G. Then,*

$$\lim_{n\to\infty} g_n^k g_n' = g^k g' \quad \text{in } A^2(G), \text{ for } k \in \mathbb{N}_0. \tag{4.10}$$

Proof. Taking $k \geqslant 1$, one has

$$\begin{aligned}|g_n^k g_n' - g^k g'|^2 &\leqslant \big(|g_n^k(g_n' - g')| + |(g_n^k - g^k)g'|\big)^2 \\ &\leqslant 2|g_n|^{2k}|g_n' - g'|^2 + 2|g'|^2|g_n^k - g^k|^2 \\ &\leqslant 2|g_n' - g'|^2 + 2k^2|g'|^2|g_n - g|^2.\end{aligned}$$

We have used here that $|g_n(z)| < 1$ and $|g(z)| < 1$ for each $z \in G$. In fact, these inequalities also hold for $k=0$. Since $g_n'(z) \to g'(z)$ for each $z \in G$ and $\int_G |g_n'|^2\, dA = \int_G |g'|^2\, dA = \pi$ we have that $\int_G |g_n' - g'|^2\, dA \to 0$. In order to verify this one can use, for instance, the following fact on convergence which may be found in [115, page 76].

Lemma 4.9. *Let μ be a positive measure on some set E, let $p \in (0, \infty)$, and let $f \in L^p(E, \mu)$, $f_n \in L^p(E, \mu)$, $n \geqslant 1$, and, finally let $f_n(x) \to f(x)$ for μ-a.a. $x \in E$ and $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$. Then, $\|f - f_n\|_p \to 0$ as $n \to \infty$.*

Thus, $\int_G |g_n - g|^2\, |g'|^2\, dA \to 0$, which is a consequence of Lebesgue dominated convergence theorem, since $|g'|^2 \in L^1(G, dA)$, and $|g_n - g| \leqslant 2$ and $|g_n(z) - g(z)| \to 0$ as $n \to \infty$ for each z. ∎

Sketched proof of Theorem 4.1 in the case $p = 2$. Take a function $h \in A^2(G)$. Using the conformal map $g\colon G \to \mathbb{D}$ let us "move" the function h to the unit disc. Namely we consider the function φ in $\mathbb{D}$ defined as follows:

$$\varphi(w) = (h \circ g^{-1})(w)(g^{-1})'(w), \quad |w| < 1.$$

Since

$$\int_{\mathbb{D}} |\varphi(w)|^2\, dA(w) = \int_G |h(z)|^2\, dA(z),$$

then $\varphi \in A^2(\mathbb{D})$. Take the Taylor expansion for φ at the origin $\varphi(w) = \sum_{k=0}^{\infty} a_k w^k$. Then, as it was mentioned above, the Taylor polynomials of φ converges in $A^2(\mathbb{D})$ to φ. Hence,

$$\int_{\mathbb{D}} \left|\varphi(w) - \sum_{k=0}^{N} a_k w^k\right|^2 dA(w) = \int_G \left|h(z) - \sum_{k=0}^{N} a_k g(z)^k g'(z)\right|^2 dA(z) \to 0.$$

Fixed N and using Lemma 4.8 each sum $\sum_{k=0}^{N} a_k g^k g'$ can be approximated in the space $A^2(G)$ by a function

$$h_n = \sum_{k=0}^{N} a_k g_n^k g_n'$$

for some value of n. Since the function g_n is defined on G_n and $\overline{G} \subset G_n$, one can use Runge's theorem to obtain a polynomial P_n which approximates the function h_n in $A^2(G)$. ∎

The next result is similar to Lemma 4.8. It was proved in [91]. We present here slightly different proof of this result working in the framework of more direct approach related with properties of Carathéodory domains.

Lemma 4.10. *Let G be a Carathéodory domain, let $z_0 \in G$, and let (G_n) be some sequence of Jordan domains such that $G_n \to G$ with respect to $z_0 \in G$. Let $\psi_n \colon G_n \to G$ be the conformal map normalized by conditions $\psi_n(z_0) = z_0$ and $\psi_n'(z_0) > 0$, $n \in \mathbb{N}$, and let h be some function of class $A^2(G)$. Then, $(h \circ \psi_n)\psi_n'$ converges to h in $A^2(G)$ as $n \to \infty$.*

Proof. Using the notations of Lemma 4.8 one can note that $\psi_n = g^{-1} \circ g_n$ for each $n \geqslant 1$. Then, $\psi_n(z) \to z$ for every $z \in G$, so that $h(\psi_n(z))\psi_n'(z) \to h(z)$ as $n \to \infty$ for every $z \in G$. Moreover, making the change of variables $w = \psi_n(z)$ we have

$$\int_G |h(\psi_n(z))\psi_n'(z)|^2 \, dA(z) = \int_{\psi_n(G)} |h(w)|^2 \, dA(w) \to \int_G |h(w)|^2 \, dA(w)$$

as $n \to \infty$. At that point we can finish the proof applying Lemma 4.9, as it was done in the proof of Lemma 4.8. ∎

The second alternative proof of Theorem 4.1 will be presented in Section 4.2, where we will deal with certain aspects of the subject under consideration related with the Hilbert space structure of $A^2(G)$.

Approximation on Carathéodory compact sets

The next contribution to the theory of L^p-polynomial approximation on Carathéodory sets was made by S. O. Sinanjan in [123]. He proved the two following theorems, and the first one is a generalization of Theorem 4.1 for the case that $1 \leqslant p < \infty$.

Theorem 4.11 (Sinanjan). *If $K \subset \mathbb{C}$ is a Carathéodory compact set, then the set of polynomials is dense in $A^p(K)$ for every $1 \leqslant p < \infty$.*

Scheme of the proof. The proof follows more or less directly the pattern of the original proof of Mergelyan's theorem. Without loss of generality it may be assumed, that K is a continuum. Let $R > 0$ be such that $K \subset D(0, R/2)$. Take a function $f \in A^p(K)$ and define it also for all points $z \notin K$ by setting $f(z) = 0$. Take and fix an arbitrary $\delta > 0$. Set

$$f_\delta(z) = \int_{\mathbb{C}} f(\zeta) \, K_\delta(|\zeta - z|) \, dA(\zeta),$$

where

$$K_\delta(r) = \begin{cases} \frac{3}{\pi\delta^2}\left(1 - \frac{r}{\delta}\right), & \text{if } 0 \leqslant r \leqslant \delta \\ 0, & \text{if } r > \delta. \end{cases}$$

The function f_δ possesses the following important properties:

(1) $\|f_\delta\|_p \leqslant 3\|f\|_p$;

(2) $\|f_\delta - f\|_p \leqslant 3\omega_p(f,\delta)$, where $\omega_p(f,\delta)$ is the L^p-modulus of continuity of f;

(3) $\omega_p(f_\delta, r) \leqslant 3\omega_p(f,r)$;

(4) for any function $\psi \in C(\mathbb{C})$ with $\psi(z) = 0$ for $z \in \mathbb{C} \setminus \overline{D(0,2R/3)}$ it holds

$$\|\overline{\partial}\psi_\delta\|_p \leqslant 6\frac{\omega_p(\psi,\delta)}{\delta};$$

(5) $f_\delta = f$ in $U = \{z : \operatorname{dist}(z, K^{\complement}) > \delta\}$, while $f_\delta = 0$ in $U \cap K^{\complement}$.

In contrast to the original Mergelyan proof in the case under consideration one needs to consider yet another convolution of the approximable function with the kernel K_δ defined above.

$$f_\delta^*(z) = \int_{\mathbb{C}} f_\delta(\zeta)\, K_\delta(|\zeta - z|)\, dA(\zeta).$$

It follows from the aforesaid properties of the function f_δ that

$$\|f - f_\delta^*\|_p \leqslant \|f - f_\delta\|_p + \|f_\delta - f_\delta^*\|_p \leqslant 12\omega_p(f,\delta).$$

Thus, it is enough to find a polynomial Q such that $\|f_\delta^* - Q\|_p \leqslant A_1\omega_p(f,\delta)$ for some absolute constant $A_1 > 0$.

Take a conformal map from $\Omega_\infty(K)^* = \Omega_\infty(K) \cup \{\infty\}$ to $\mathbb{D}$. Then, the preimages of the circles $|w| = 1 - 1/(n+1)$ under this transformation are denoted by Γ_n, they are analytic curves. Moreover, let $D_n = D(\Gamma_n)$. Now, the standard Cauchy–Green formula (see, for instance, [18, page 151]) gives

$$f_\delta^*(z) = \frac{1}{2\pi i}\int_{\Gamma_n} \frac{f_\delta^*(t)}{t - z}\, dt - \frac{1}{\pi}\int_{D_n} \frac{\overline{\partial} f_\delta^*(\zeta)}{\zeta - z}\, dA(\zeta), \quad z \in D_n. \tag{4.11}$$

Next one can choose a sufficiently large integer n in such a way that the following two conditions are fulfilled:

(a) $\operatorname{dist}(z, \Gamma_n) < \delta/2$ for each $z \in \partial K$;

(b) the following inequality holds

$$\int_K \left(\frac{1}{2\pi}\int_{D_n\setminus\widehat{K}} \left|\frac{\overline{\partial} f_\delta^*(\zeta)}{\zeta - z}\right| dA(\zeta)\right)^p dA(z) < \omega_p(f,\delta)^p. \tag{4.12}$$

Now, take $Y = \{z \in \widehat{K} : \operatorname{dist}(z, \partial K) < 2\delta\}$. At that point we need to recall the main [90, Lemma 2.2] (see also [115, Lemma 20.2]). The set Y can be covered by finitely many open discs $D(a_j, 2\delta)$, $1 \leqslant j \leqslant m$ with centers $a_j \in \Omega_\infty(K)$. Since

K is a Carathéodory compact set, then there exists a continuum (actually an arc) $L_j \subset D(a_j, 2\delta) \cap \Omega_\infty(K)$ such that diam L_j is comparable with δ. Using furthermore the conformal maps g_j, $j = 1, \ldots, m$, from $\mathbb{C}_\infty \setminus L_j$ to $\mathbb{D}$ such that $g_j(\infty) = 0$ and taking suitable linear combinations of g_j and g_j^2, one can find, for each point $\zeta \in Y$, a holomorphic (even rational) function R_ζ defined on the open set

$$\Omega := \mathbb{C}_\infty \Big\backslash \bigcup_j L_j \subset \mathbb{C}_\infty \backslash \widehat{K},$$

that R_ζ satisfies the following properties:

$$|R_\zeta(z)| \leqslant \frac{A_1}{\delta}, \quad \text{for } z \in \Omega, \tag{4.13}$$

$$\left| \frac{1}{\zeta - z} - R_\zeta(z) \right| \leqslant \frac{A_1 \delta^2}{|\zeta - z|^3}, \quad \text{for } z \in \Omega,\ |\zeta - z| \geqslant c_1 \delta, \tag{4.14}$$

where A_1 and c_1 are positive constants.

In view of (4.11) we define

$$Q_\delta(z) = -\int_Y \bar{\partial} f_\delta^*(\zeta)\, R_\zeta(z)\, dA(\zeta),$$
$$\varphi(z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{f_\delta^*(t)}{t - z}\, dt + Q_\delta(z),$$

where Q_δ is holomorphic in a neighborhood of $\widehat{K}$. Notice, that in order to prove the theorem it is sufficient to show that Q_δ is close to the function

$$\varphi_\delta(z) = -\frac{1}{\pi} \int_Y \frac{\bar{\partial} f_\delta^*(\zeta)}{\zeta - z}\, dA(\zeta),$$

namely, that

$$\|Q_\delta - \varphi_\delta\|_p \leqslant A_2 \omega_p(f, \delta) \tag{4.15}$$

for some constant $A_2 > 0$. Indeed, since φ is holomorphic in a neighborhood of $\widehat{K}$, Runge's theorem allow us to pick a polynomial P such that $\|\varphi - P\|_p \leqslant \omega_p(f, \delta)$. Therefore,

$$\|f - P\|_p \leqslant \|f - f_\delta^*\|_p + \|f_\delta^* - \varphi\|_p + \|\varphi - P\|_p \leqslant A_3 \omega_p(f, \delta),$$

for some positive constant A_3 because of (4.11), (4.12), (4.15), and the fact that $f_\delta^*(z) = 0$ for $z \notin Y$.

Thus, it remains to verify the estimate (4.15). In view of (4.13) and (4.14) we have

$$\|Q_\delta - \varphi_\delta\|_p \leqslant \|F_1\|_p + \|F_2\|_p + \|F_3\|_p \leqslant A_1(c_1 + 1)\|F_2\|_p + \|F_4\|_p, \tag{4.16}$$

where

$$F_1(z) = \int_{Y_1(z)} |\bar{\partial} f_\delta^*(\zeta)|\,|R_\zeta(z)|\,dA(\zeta),$$

$$F_2(z) = \int_{Y_1(z)} \left|\frac{\bar{\partial} f_\delta^*(\zeta)}{\zeta - z}\right| dA(\zeta),$$

$$F_3(z) = \int_{Y_2(z)} |\bar{\partial} f_\delta^*(\zeta)| \left|R_\zeta(z) - \frac{1}{\zeta - z}\right| dA(\zeta),$$

$$F_4(z) = \int_{Y_2(z)} \frac{|\bar{\partial} f_\delta^*(\zeta)|}{|\zeta - z|^3}\, dA(\zeta),$$

and $Y_1(z) = \{\zeta \in Y : |\zeta - z| < c_1\delta\}$, $Y_2(z) = \{\zeta \in Y : |\zeta - z| > c_1\delta\}$. The desired estimates of $F_2(z)$ and $F_4(z)$ was obtained in [123] as a result of using Hölder's inequality. Finally, the estimate (4.15) follows from (4.16), which finishes the proof. We skip here some details which can be found in [123]. ■

Corollary 4.12. *Let U be a Carathéodory open set, and let $1 \leqslant p \leqslant +\infty$. Then, for each $f \in A^p(\bar{U})$ there exists a sequence of polynomials (P_n) such that $P_n \to f$ in $A^p(\bar{U})$ as $n \to \infty$.*

This result is a consequence of Sininjan's theorem and the fact that $K = \bar{U}$ is a Carathéodory compact set in the case under consideration. Even in the case that U is a domain this result cannot be obtained as a consequence of Theorem 4.1. The difference can happen if ∂U has positive area.

In [123] the following conjecture was made: *for $p \in (1, 2)$ and for every compact set K, the set of functions holomorphic in a neighborhood of K is dense in $A^p(K)$.* V. P. Havin in [63] solved this problem by proving the fact that $R^p(K) = A^p(K)$ for each compact set K and $p \in (1, 2)$. Here, $R^p(K)$ stands for L^p-closure of rational functions with poles lying outside K. The problem when $R^p(K) = A^p(K)$ for $p \in [2, +\infty)$ has a long history, and finally this problem was solved in terms of certain capacity conditions (or, in other words, in terms of $(1, q)$-stability), see, for instance, [65]. Since these results do not concern the class of Carathéodory sets, we will not continue this line of exposition.

Let us also say a few words about the case of harmonic polynomials. If E is a measurable set in $\mathbb{C}$, let us denote by $A^p_{har}(E)$ the set of all harmonic in E° functions of the class $L^p(E, \mathbb{R})$ The first result that the authors are aware of in connection with L^p-approximation by harmonic polynomials were obtained by A. L. Šaginjan in [117]. He proved that every bounded harmonic function on a given domain G belongs to $A^p_{har}(G)$ if G satisfies either of the following two conditions:

(i) G is a Carathéodory domain;

(ii) G is a moon-shaped domain and the real harmonic polynomials are dense in $A^p(G)$.

The next contribution was made by Sinanjan in [123, pages 99–101]. The respective result states as follows.

Theorem 4.13. *Let K be a Carathéodory compact set. Then, the set of real harmonic polynomials is dense in the space $A^p_{\text{har}}(K)$ for every $p \geqslant 1$.*

Question IV. Whether the results stated in Theorems 4.11 and 4.13 hold true also for $p \in (0,1)$?

4.2 Some studies related with Hilbert space structure of $A^2(G)$

If $G \neq \emptyset$, then $A^2(G)$ is a separable Hilbert space with respect to the standard inner product $\langle f, g\rangle$ in $L^2(G)$, so that

$$\langle f, g\rangle = \int_G f\,\bar{g}\,dA.$$

We will use in this section all standard results from the Hilbert space theory without any special introduction and giving no references.

First of all let us give the second alternative proof of Theorem 4.1 in the case that $p = 2$ using some Hilbert space technique. In order to do that we need to show that the system of functions $\{1, z, z^2, \dots\}$ is complete in $A^2(G)$. The proof presented is due to A. L. Shaginyan, see [91].

Yet another proof of Theorem 4.1 for $p = 2$. Take a function $h \in A^2(G)$ and assume that $\langle h, z^m\rangle = 0$ for each $m \geqslant 0$. Then, it is enough to show that $h = 0$.

Given a big enough number $R > 0$, for w with $|w| > R$ we have

$$\int_G \frac{\overline{h(z)}}{z-w}\,dA(z) = -\sum_{n=0}^{\infty} \frac{1}{w^{n+1}} \int_G \overline{h(z)}\,z^n\,dA(z) = -\sum_{n=0}^{\infty} \frac{\langle z^n, h\rangle}{w^{n+1}} = 0.$$

Then,

$$\int_G \frac{\overline{h(z)}}{z-w}\,dA(z) = 0 \tag{4.17}$$

for every point $w \in G_\infty$, since G_∞ is a connected set.

Going further, take a sequence (Γ_n) of rectifiable Jordan curves such that the domains $G_n := D(\Gamma_n)$ converges to G with respect to some fixed point $z_0 \in G$ (in the sense of kernel convergence). Then, take as usual the sequence (ψ_n), $\psi_n\colon G_n \to G$, of conformal maps normalized by conditions $\psi_n(z_0) = z_0$ and $\psi_n'(z_0) > 0$. Multiplying (4.17) by $h(\psi_n(w))\psi_n'(w)$, integrating over Γ_n and applying Fubini's theorem and Cauchy integral formula we obtain

$$\int_G \overline{h(z)}\,h(\psi_n(z))\,\psi_n'(z)\,dA(z) = 0 \tag{4.18}$$

for all $n \geqslant 1$. By Lemma 4.10 we know that $(h \circ \psi_n)\psi_n' \to h$ as $n \to \infty$ in $A^2(G)$. Then, $\bar{h}(h \circ \psi_n)\psi_n' \to \bar{h}h$ as $n \to \infty$ in $A^1(G)$. By (4.18) one has $\int_G |h|^2 \, dA = 0$, which yields $h = 0$. ∎

The next result is standard but it shows the special role that Carathéodory domains play in the theory.

Proposition 4.14. *Let G be a bounded domain in $\mathbb{C}$. Then, the following hold.*

(1) *In the space $A^2(G)$ there exists an orthonormal sequence of polynomials (P_n) such that* $\deg P_n = n$ *for all $n \geqslant 0$.*

(2) *This sequence (P_n) is uniquely determined whenever one demands that the coefficient of P_n at z^n is positive.*

(3) *If G is a Carathéodory domain, then this sequence (P_n) is an orthonormal basis.*

Proof. The construction of the desired system (P_n) is nothing else, then the standard Gram–Schmidt orthogonalisation process applied to the sequence of functions $\{1, z, z^2, \dots\}$ in the space $A^2(G)$. So, we need to verify the part (3) only.

Let G be a Carathéodory domain. We need to prove that the orthonormal system (P_n) such that $\deg P_n = n, n \geqslant 0$, is a basis in $A^2(G)$. Take a function $h \in A^2(G)$ and assume that $\langle h, P_n \rangle = 0$ for every $n \geqslant 0$. Since any polynomial Q of degree m may be represented as a linear combination of $P_0, P_1, \dots, P_m$, then $\langle h, Q \rangle = 0$, but since h can be approximated by a sequence of polynomials, then $\langle h, h \rangle = 0$, and hence $h = 0$. Thus, (P_n) is complete and hence it is an orthonormal basis for $A^2(G)$. ∎

Going further let us observe that the space $A^2(G)$ has a reproducing kernel for every nonempty domain G. Recall, that the reproducing kernel for $A^2(G)$, which is usually called the *Bergman kernel* for G, is a function $K : \mathbb{C}^2 \to \mathbb{C}$ such that $K(\cdot, w) \in A^2(G)$ for every $w \in G$ and $h(w) = \langle h, K(\cdot, w) \rangle$ for every $h \in A^2(G)$ and $w \in G$. It is well-known, that if (v_n) is some orthonormal basis in $A^2(G)$, then $K(z, w) = \sum_{n=0}^{\infty} \overline{v_n(w)} v_n(z)$.

Let now G be a Carathéodory domain. According to Proposition 4.14 there exists the orthonormal basis (P_n) in $A^2(G)$ consisting of polynomials (with $\deg P_n = n$). In this case, we have

$$K(z, w) = \sum_{n=0}^{\infty} \overline{P_n(w)} P_n(z). \tag{4.19}$$

Using this representation of reproducing kernel we are able to obtain the explicit expression for the conformal radius of G and for the corresponding conformal map from G onto $D(0, R)$.

Take a point $a \in G$, let R be the conformal radius of G with respect to a, and let g_0 be the conformal map from G onto $D(0, R)$ with the standard normalization

$g_0(a) = 0$ and $g_0'(a) = 1$. It follows from Proposition 2.1 that

$$\inf\left\{\int_G |h'(z)|^2\,dA(z) : h \in A^2(G),\ h(a) = 1\right\} = \int_G |g_0'(z)|^2\,dA(z) = \pi R^2. \quad (4.20)$$

Let us define the functions

$$K_m(z, w) = \sum_{n=0}^{m} \overline{P_n(w)} P_n(z), \quad m \in \mathbb{N}_0. \quad (4.21)$$

Then, using (4.20) for some appropriate h and making a bit of computations, we have

$$\int_G \left|\frac{K_m(z,a)}{K_m(a,a)}\right|^2 dA(z) = \frac{1}{\sum_{n=0}^{m} |P_n(a)|^2} \geqslant \pi R^2.$$

It gives

$$M_a := \sum_{n=0}^{\infty} |P_n(a)|^2 \leqslant \frac{1}{\pi R^2}.$$

Now, since $g_0' \in A^2(G)$ we have, in particular, $g_0'(z) = \sum_{n=0}^{\infty} c_n P_n(z)$ for all $z \in G$, where $c_n = \langle g_0', P_n\rangle$. Therefore,

$$\pi R^2 \leqslant \int_G \left|\frac{K(z,a)}{M_a}\right|^2 dA(z) = \frac{1}{M_a} \leqslant \sum_{n=0}^{\infty} |c_n|^2 = \int_G |g_0'|^2\,dA(z) = \pi R^2.$$

Therefore, $M_a = 1/(\pi R^2)$ and $K(z, a) = M_a g_0'(z)$ for all $z \in G$. So that we have proved the following result.

Proposition 4.15. *Let G be a Carathéodory domain, let $a \in G$, let R be the conformal radius of G with respect to a, and let g_0 be the conformal map from G onto $D(0, R)$ with the standard normalization $g_0(a) = 0$ and $g_0'(a) = 1$. Then,*

$$R = \frac{1}{\sqrt{\pi M_a}}, \quad \textit{where } M_a = \sum_{k=0}^{\infty} |P_k(a)|^2;$$

$$K(z, a) = M_a g_0'(z) \quad \textit{for all } z \in G;$$

$$g(z) = \frac{1}{M_a} \sum_{k=0}^{\infty} \overline{P_k(a)} \int_a^z P_k(\zeta)\,d\zeta \quad \textit{for all } z \in G.$$

The representation of $K(z, w)$ in terms of conformal mapping (and vice-versa) given in this proposition may be adapted in a clear way for conformal mappings normalized by other ways. Thus, if g is the conformal map from G onto $\mathbb{D}$ such that $g(a) = 0$ and $g'(a) > 0$, then

$$g'(z) = \sqrt{\frac{\pi}{K(a,a)}}\, K(z, a), \quad K(z, a) = \frac{g'(a)}{\pi} g'(z). \quad (4.22)$$

In the case of a general conformal mapping g from G onto $\mathbb{D}$ (without special normalization) one has

$$K(z,a) = \frac{g'(z)\overline{g'(a)}}{\pi(1-\overline{g(a)}g(z))^2}. \tag{4.23}$$

In the simplest case that $G = D(0,R)$ for some $R > 0$ the corresponding function $K(\cdot,\cdot)$ and basis (P_n) in $A^2(G)$ may be easily computed:

$$P_n(z) = \frac{\sqrt{n+1}}{\sqrt{\pi}R^{n+1}}z^n, \quad n \geqslant 0 \quad \text{and} \quad K(z,w) = \frac{R^2}{\pi(R^2 - z\bar{w})^2}.$$

Remark 4.16. Going further we need to make the following observation.

(1) Let G_1 and G_2 be two simply connected domains, and let $\psi\colon G_2 \to G_1$ be a conformal map. Then, the map $f \mapsto (f \circ \psi)\,(\psi')^{2/p}$ provides an isometry of $A^p(G_1)$ onto $A^p(G_2)$ for each p, $0 < p < +\infty$.

(2) For instance, if $\{v_n : n \in \mathbb{N}\}$ is some orthonormal system in $A^2(G_1)$, then the system $\{(v_n \circ \psi)\,\psi' \colon n \in \mathbb{N}\}$ is an orthonormal system in $A^2(G_2)$.

(3) Let G be a simply connected domain, and let g be a conformal map from G onto $\mathbb{D}$ such that $g(a)=0$ and $g'(a)>0$ for some $a \in G$. Since $(\sqrt{(n+1)/\pi}\,z^n)$ is the orthonormal basis in $A^2(\mathbb{D})$, then the system of functions

$$\omega_n(z) = \sqrt{\frac{n+1}{\pi}}g(z)^n\, g'(z), \quad n \in \mathbb{N}_0$$

forms an orthonormal system in $A^2(G)$.

Example 4.17. In order to obtain yet another example of the orthonormal basis (P_n) in $A^2(G)$ for certain special domain G, let us consider the Cassini's oval $\{z : |z-1| \cdot |z+1| < \alpha\}$ with $\alpha \in (0,1]$. Let $G = O_\alpha$ be the component of this Cassini's oval lying in the right half-plane. The function $g\colon O_\alpha \to \mathbb{D}$ defined by $g(z) = (z^2-1)/\alpha$ gives the conformal map such that $g(1) = 0$ and $g'(1) > 0$. Then, according to the statement of the part (3) of Remark 4.16, an orthogonal basis in $A^2(O_\alpha)$ is formed from the polynomials

$$P_n(z) = \frac{2\sqrt{n+1}}{\alpha^{n+1}\sqrt{\pi}}\, z(z^2-1)^n, \quad n \geqslant 0.$$

The Bergman kernel for O_α may be also expresses explicitly:

$$K(z,w) = \frac{4\alpha^2\bar{w}z}{\pi(\alpha^2 - (\bar{w}^2-1)(z^2-1))^2}.$$

Let us now briefly describe the concept of a *Bieberbach polynomials* and their relations with Carathéodory domains. Let G be a domain in $\mathbb{C}$ and let $a \in G$. For each $n \geqslant 2$, let

$$\mathcal{P}_n(a) = \{P \in \mathcal{P} : \deg P = n,\ P(a) = 0,\ P'(a) = 1\}.$$

Definition 4.18. A polynomial $\pi_n \in \mathcal{P}_n(a)$ solving the following extremal problem

$$\int_G |\pi_n'(z)|^2 \, dA(z) = \inf\left\{\int_G |P'(z)|^2 \, dA(z) : P \in \mathcal{P}_n(a)\right\} \tag{4.24}$$

is called the nth Bieberbach polynomial (with respect to G and a).

The solution π_n of the extremal problem (4.24) always exists, because it is the primitive of a polynomial which is the orthogonal projection of 0 onto $\mathcal{P}_n'(a) = \{P' : P \in \mathcal{P}_n(a)\}$ in $A^2(G)$.

It turned out that in the case of Carathéodory domains the Bieberbach polynomials possess certain interesting and important properties, as it is shown in the following statement. For a given domain $G \subset \mathbb{C}$ let us recall that (P_n) is the orthonormal basis in $A^2(G)$ consisting of polynomials with $\deg P_n = n$ and that the function $K_n(z, w)$, $n \in \mathbb{N}_0$ is defined by (4.21).

Proposition 4.19. *Let G be a Carathéodory domain, $a \in G$ and let g_0 map G conformally onto $D(0, R)$, where R is the conformal radius of G with respect to a (so that $g_0(a) = 0$ and $g_0'(a) = 1$). Then,*

$$\pi_n(z) = \pi_n(z; G, a) = \sum_{j=0}^{n-1} \frac{\overline{P_j(a)}}{K_{n-1}(a,a)} \int_a^z P_j(\zeta)\, d\zeta.$$

Moreover, $\pi_n' \to g_0'$ in $A^2(G)$, and hence $\pi_n \rightrightarrows g_0$ locally in G.

The proof of this proposition may be found in several sources, for example in [61, Chapter iii, Section 1].

If G is a bounded domain in $\mathbb{C}$ such that the space $A^2(G)$ admits an orthonormal basis consisting of polynomials, one can prove the existence of Bieberbach polynomials for such a domain. So, Carathéodory domain is one of the most suitable class of domains when the aforementioned condition is always fulfilled.

Let us make one more remark about the conditions of Proposition 4.19. Let G be a Carathéodory domain and let $\mathcal{E}$ be some end-cut of G such that $\mathrm{Area}(\mathcal{E}) = 0$. Take $G_1 = G \setminus \mathcal{E}$. Then, the conditions determining the Bieberbach polynomials for G and for G_1 are the same (the corresponding extremal problem "does not see" $\mathcal{E}$), but it is clear that conformal maps from G onto $\mathbb{D}$ and from G_1 onto $\mathbb{D}$ differ significantly. So, certain condition that prevent "cuts" in domains under consideration is needed if we want to have results similar to Proposition 4.19, where the condition that G is a Carathéodory domain guaranties that G has no "cuts".

Let us give two examples showing how the Bieberbach polynomials look like.

(1) Let $G = \mathbb{D}$ and $a \in \mathbb{D}$, and let $b_n = \sum_{k=0}^{n-1}(k+1)|a|^{2k}$ for $n \in \mathbb{N}$, then

$$\pi_n(z; \mathbb{D}, a) = \frac{z-a}{b_n} + \frac{1}{b_n}\sum_{k=1}^{n-1} \bar{a}^k\left(z^{k+1} - a^{k+1}\right).$$

(2) Furthermore, the Bieberbach polynomials may be explicitly computed if the domain under consideration is $D(\Gamma_{\alpha,\beta})$, where $\Gamma_{\alpha,\beta}$ is the ellipse with semi-axes α and β for some $\alpha > \beta > 0$ having foci at the point ± 1 (so that $\alpha^2 - \beta^2 = 1$).
Let T_n and U_n, $n \geqslant 1$ stand for the Tchebyshev polynomials of the first and second kind, respectively. We recall, that $T_n(z) = \cos(n \arccos z)$ if $|\mathrm{Re} z| < 1$, and $U_n(z) = (n+1)^{-1} T'_{n+1}(z) = (1-z^2)^{-1/2} \sin((n+1) \arccos z)$.
It holds that the Bergman kernel for the domain $D(\Gamma_{\alpha,\beta})$ is

$$K(z,a) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{T'_{n+1}(z)\overline{U_n(a)}}{\rho^{n+1} - \rho^{-(n+1)}}, \quad \rho = (\alpha+\beta)^2,$$

while the respective Bierberbach polynomials have the form

$$\pi_n(z; D(\Gamma_{\alpha,\beta}), a) = \frac{1}{K_{n-1}(a,a)} \sum_{j=0}^{n-1} \frac{(T_{j+1}(z) - T_{j+1}(a))\overline{U_j(a)}}{\rho^{j+1} - \rho^{-(j+1)}}.$$

Moreover, if g maps $D(\Gamma_{\alpha,\beta})$ conformally onto $\mathbb{D}$ with $g(0) = 0$ and $g'(0) > 0$, then

$$g(\cos w) = \frac{\pi}{2\sqrt{d}} \sum_{n=0}^{\infty} \frac{(-1)^n \cos((2n+1)w)}{\rho^{2n+1} - \rho^{-(2n+1)}},$$

where w belongs to the rectangle $\{w : 0 < \mathrm{Re}\, w < \pi,\ |\mathrm{Im} w| < c\}$ such that $\cosh c = \alpha$, while

$$d = \sum_{n=0}^{\infty} \frac{2n+1}{\rho^{2n+1} - \rho^{-(2n+1)}}.$$

The proof of these statements uses the fact that the system $(c_n U_n)$, where

$$c_n = \frac{4\pi}{n+1}\left(\rho^{n+1} - \rho^{-(n+1)}\right)$$

forms a basis in the space $A^2(D(\Gamma_{\alpha,\beta}))$, see [93, page 258].

According to Farrell's theorem (see Theorem 3.4) in order to have uniform convergence of the sequence (π_n) on $\overline{G}$, it is necessary that G is a Carathéodory domain and all prime ends of G are simple. A natural question arises now: *Are these conditions sufficient to have uniform approximation of the corresponding conformal mapping by Bieberbach polynomials*? The answer to this question is negative. In [71] a starlike Jordan domain G was constructed whose boundary is analytic except at one point such that the corresponding sequence of Bierberbach polynomials diverges on some dense subset of ∂G. We refer the reader who is interested in more information about Bierberbach polynomials, to the book [127].

One moment problem in $A^2(G)$ and in $A^1(G)$

Observe that Proposition 4.14 yields immediately the following proposition.

Corollary 4.20. *Let G be a Carathéodory domain. Let $h \in A^2(G)$ be such that*

$$\int_G h(z)\,\bar{z}^m\,dA(z) = 0, \quad \text{for each } m = 0, 1, 2, \dots . \tag{4.25}$$

Then, $h = 0$ in G.

Indeed, let (P_n) be the orthogonal basis in $A^2(G)$ given by Proposition 4.14. Thus, (4.25) implies the property $\langle h, P_m\rangle = 0$ for each $m \in \mathbb{N}_0$. Then, $h = 0$ in $A^2(G)$.

It is natural to consider the following question.

Question V. Let G be a Carathéodory domain and let $h \in A^1(G)$. Is it true that the condition (4.25) implies that $h = 0$ in G?

We are able to give a partial answer to this question by proving the following statement. The proof presented below is based on some results about pointwise approximation and it is quite short and simple. A different proof without using these tools may be found in [124, page 261].

Theorem 4.21. *Let G be a Carathéodory domain and let φ be a conformal map from G onto $\mathbb{D}$ such that $\|\varphi'\|_G \leqslant C$ for some constant $C > 0$. If the function $h \in A^1(G)$ is such that (4.25) is fulfilled, then $h = 0$ in G.*

In order to prove this theorem we need the following lemma.

Lemma 4.22. *Let G be a simply connected domain and assume that there exists $a \in G$ such that the Bergman kernel $K(\cdot, a)$ is bounded. Then, for all $b \in G$ the function $K(\cdot, b)$ is also bounded. Moreover, if $h \in A^1(G)$ then*

$$h(a) = \int_G h(z)\overline{K(z,a)}\,dA(z), \quad a \in G.$$

Proof. Let g_a be the conformal map from G onto $D(0, R_a)$ with $g_a(a)=0$, $g_a'(a)=1$ (so that R_a is the conformal radius of G with respect to a). Taking into account (4.22) and the hypothesis that $K(\cdot, a)$ is bounded, we obtain $|g_a'(z)| = \pi R_a^2|K(z, a)| \leqslant C$ for each $z \in G$. Here, and in the sequel in this proof $C, C', \dots$ stand for some positive constants which may differ in different formulae. Take an arbitrary $b \in G$ and consider the analogous conformal map g_b constructed with respect to b. Then, $g_b \circ g_a^{-1}$ maps $D(0, R_a)$ onto $D(0, R_b)$, therefore this function is the restriction of a Möbius transformation, and hence,

$$|(g_b \circ g_a^{-1})'(w)| \leqslant C$$

for each $w \in D(0, R_a)$. Then, for every $w \in D(0, R_a)$ and $z = g_a^{-1}(w)$ we have (in view of (4.22)) that

$$|K(z,b)| = \frac{1}{\pi R_b^2}|g_b'(z)| \leqslant C|g_a'(z)| \leqslant C'.$$

So that $K(\cdot, b)$ is bounded.

Let now $h \in A^1(G)$. Put $g := g_a$ and $R := R_a$ and define $G^{(r)} := \{z \in G : |g(z)| < r\}$ for $0 < r < R$. Now, using (4.22) once again we obtain

$$\begin{aligned}
&\pi R^2 \int_G K(z,a)\, h(z)\, dA(z)\\
&\quad = \lim_{r\to R} \int_{G^{(r)}} \overline{g'(z)}\, h(z)\, dA(z)\\
&\quad = \lim_{r\to R} \int_{D(0,r)} \overline{g'(g^{-1}(w))}\, h(g^{-1}(w))|(g^{-1})'(w)|^2\, dA(w)\\
&\quad = \lim_{r\to R} \int_{D(0,r)} \frac{h(g^{-1}(w))}{g'(g^{-1}(w))}\, dA(w) = \pi R^2 h(a),
\end{aligned}$$

where we have used (2.1) and, further, the mean area value property in $D(0, r)$. ■

Proof¶ *of Theorem* 4.21. Put $z_1 = \varphi^{-1}(0)$. In view of (4.22) we have

$$|K(z, z_1)| \leqslant \frac{1}{\pi}|\varphi'(z)\varphi'(z_1)| \leqslant C_1,$$

where $C_1 = C^2/\pi$. By virtue of Lemma 4.22 just proved, $K(\cdot, w)$ is bounded for all $w \in G$. Fix $a \in G$ and take the conformal map g from G onto $D(0, R)$ such that $g(a) = 0$, $g'(a) = 1$. By Lemmas 4.22 and (4.22), one has

$$h(a) = \int_G h(z)\, \overline{K(z,a)}\, dA(z) = \frac{1}{\pi R^2} \int_G h(z)\overline{g'(z)}\, dA(z).$$

Since g' is bounded (because $K(z, a)$ is also bounded), Theorem 3.25 tells us that there exists a sequence of polynomials (P_n) such that $P_n(z) \to g'(z)$ and $|p_n(z)| \leqslant C'$ for each $z \in G$ and for some positive constant C'. Then,

$$h(a) = \int_G h(z)\, \overline{g'(z)}\, dA(z) = \lim_{n\to\infty} \int_G h(z)\, \overline{p_n(z)}\, dA(z) = 0.$$

Because this holds for each $a \in G$ we have $h = 0$. ■

One estimate for polynomials forming orthogonal basis in $A^2(G)$

The main aim of the subsection is to prove the following result.

Theorem 4.23. *Let G be a Carathéodory domain, and let (P_n) be the orthogonal basis consisting of polynomials such that each P_n has degree n and its coefficient at z^n is positive. Then, for each $h \in H(\mathbb{C})$ there exists a sequence (a_n) such that*

$$h(z) = \sum_{n=0}^{\infty} a_n P_n(z),$$

where the series converges locally uniformly in G. Moreover, for every $\rho > 1$ there exists $C > 0$ such that

$$|P_n(z)| \leqslant C\rho^n, \quad n = 0, 1, 2, \ldots, \; z \in \overline{G}.$$

Before proving this theorem we need to recall one construction related with the certain lemma due to Bernstein. Let K be a continuum and let $\Omega'_\infty(K) = \Omega_\infty(K) \cup \{\infty\}$. Then, there exists a unique conformal map Φ from $\Omega'_\infty(K)$ onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ such that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. For a given number $\rho > 1$ let us define the set

$$L_\rho = \{z \in \Omega_\infty(G) : |\Phi(z)| = \rho\}.$$

Lemma 4.24 (Bernstein). *Let K be a continuum and let $F\colon G_\infty(K) \to \mathbb{C}$ be holomorphic function having a pole of order $n \geqslant 1$ at infinity. Assume that*

$$\lim_{\rho \to 1+} \sup_{z \in L_\rho} |F(z)| = M < +\infty.$$

Then, for every $\rho > 1$ it holds that $|F(z)| \leqslant M\rho^n$ for each $z \in L_\rho$.

Proof. Put $f(w) = F(\Phi^{-1}(w))/(\Phi^{-1}(w))^n$ for $|w| > 1$. Then, f is bounded in $\mathbb{C}_\infty \setminus \mathbb{D}$ and $\limsup_{|w| \to 1+} |f(w)| = \lim_{\rho \to 1+} \sup_{z \in L_\rho} |F(z)| = M$. Finally, the maximum modulus theorem implies that $|F(\Phi^{-1}(w))| \leqslant M |\Phi^{-1}(w)|^n$ for each $|w| = \rho$ as desired. ∎

Proof of Theorem 4.23. Let $z^n = \sum_{k=0}^{n} b_k^n P_k$ for each n, and let (a'_n) be the Taylor coefficients of h. Then, $a_k = \sum_{n \geqslant k} a'_n b_k^n$. Let us prove the growth estimate for P_n. Let g be some fixed conformal map from G onto $\mathbb{D}$. Put $G^{(r)} := g^{-1}(D(0, r))$ for $0 < r < 1$. For $r \in (0, 1)$ take some conformal map Φ_r from $G_\infty^{(r)} \cup \{\infty\}$ onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$, and take some conformal map Φ from $G_\infty \cup \{\infty\}$ onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$. Take an arbitrary $\rho > 1$ and define $L_{r,\rho} := \{z : |\Phi_r(z)| = \rho\}$ and $L_\rho = \{z : |\Phi(z)| = \rho\}$. Furthermore, let $G_{r,\rho} := D(L_{r,\rho})$ and $G_\rho := D(L_\rho)$. Then, $\overline{G} \subset G_{r,\rho}$ for some r sufficiently close to 1. Note that from (4.19) we have $|P_n(z)|^2 \leqslant K(z, z)$ and (4.23) implies that

$$\lim_{\rho \to 1} \sup_{z \in L_{r,\rho}} |P_n(z)| \leqslant \sup_{z \in \overline{G^{(r)}}} \frac{|g'(z)|}{\sqrt{\pi}(1 - |g(z)|^2)} = C_r < +\infty.$$

Therefore, one can apply Lemma 4.24 to $K = \overline{G^{(r)}}$ in order to obtain that $|P_n(z)| \leqslant C_r \rho^n$ for each $z \in G_{r,\rho}$ and for each $n \geqslant 0$. Since $\overline{G} \subset G_{r,\rho}$, the proof is finished. ∎

4.3 Topics on weighted Bergman spaces

Let U be an open set, and let $\mathbf{w}\colon U \to [0,\infty)$ be a measurable function (a weight). For p, $1 \leqslant p < +\infty$, the weighted space $A^p(U,\mathbf{w})$ is defined as follows:

$$A^p(U,\mathbf{w}) = \left\{ f \in H(U) : \|f\|_{p,\mathbf{w}} = \left(\int_U |f(z)|^p \, \mathbf{w}(z)\, dA(z) \right)^{1/p} < +\infty \right\}.$$

In order that the space $A^p(U,\mathbf{w})$ to be complete with respect to the norm $\|\cdot\|_{p,\mathbf{w}}$ one needs to assume that $\mathbf{w}$ satisfies the condition that for each compact set $K \subset U$ there exists a constant $c_K > 0$ such that

$$c_K |f(a)|^p \leqslant \int_U |f(z)|^p \, \mathbf{w}(z)\, dA(z) \quad \text{for each } a \in K \text{ and } f \in A^p(U,\mathbf{w}). \tag{4.26}$$

This inequality may be regarded as an analogue of the estimate (4.7). It yields, in particular, that the convergence in $A^p(U,\mathbf{w})$ implies the locally uniform convergence in U. This fact shows that the space $A^p(U,\mathbf{w})$ is a Banach space for all p under consideration, while for $p = 2$ it is also a Hilbert space with respect to the inner product $\langle f, g\rangle = \int_U f\, \bar{g}\, \mathbf{w}\, dA$.

One important family of weights is the family $\{\mathbf{w} = |h| : h \in A^1(U) \text{ and } h \not\equiv 0\}$. In this case, (4.26) holds, and its proof is analogous to the proof of (4.7).

In the case, where $p = 2$ it is convenient to consider the weighted Bergman spaces $A^2(U,\mathbf{w})$ with respect to the weight $\mathbf{w} = |h|^2$, where $h \in A^2(U)$ and $h \not\equiv 0$. Such weights are called an analytic weights.

Denote by $P^p(U,\mathbf{w})$ the closure of the set $\mathcal{P}|_U$ in $A^p(U,\mathbf{w})$. The following question arises in a natural way: to describe U, p and $\mathbf{w}$ such that

$$P^p(U,\mathbf{w}) = A^p(U,\mathbf{w}). \tag{4.27}$$

Note that for $\mathbf{w} = 1$, Theorem 4.1 gives a sufficient condition for (4.27), which is that U needs to be a Carathéodory domain. But the problem just stated is very far from completely solved. In this section we are going to present some results concerning the matter which have certain connections with Carathéodory sets.

The equality (4.27) implies some restrictions on $\mathbf{w}$ and to U.

Proposition 4.25. *The following statements hold.*

(a) *Let $\mathbf{w} = |h|^2$ with $h \in A^2(U)$. If h has zeros in U, then $P^2(U,\mathbf{w}) \neq A^2(U,\mathbf{w})$.*

(b) *If U is not simply connected, then (4.27) does not hold for any $p \geqslant 1$ and for any weight $\mathbf{w}$.*

From now on we will assume that the open set U is simply connected, while $\mathbf{w}\colon U \to (0,+\infty)$. Let now G be a Carathéodory domain and $\mathcal{E}$ be some end-cut in G with $\mathrm{Area}(\mathcal{E}) = 0$. Then, in view of Proposition 4.5, we have $P^2(\Omega) \neq A^2(\Omega)$ for

$\Omega = G \setminus \mathcal{E}$. Thus, the hypothesis that G is a Carathéodory domain plays some role in the theory. However, for a $\mathbf{w} \neq 1$ the situation is more complicated, as it may be seen from the following example.

Example 4.26. Take $t \in (0,1)$ and consider the function

$$h_t(z) = \exp\left(\frac{z+1}{z-1}\right)^t \quad \text{for } z \in \mathbb{D}.$$

If $0 < t < 1$ then $P^p(\mathbb{D}, |h_t|) = A^p(\mathbb{D}, |h_t|)$ and $P^p(\mathbb{D}, |h_1|) \neq A^p(\mathbb{D}, |h_1|)$ for each p with $1 \leqslant p < +\infty$.

The proof of the fact that $P^2(\mathbb{D}, |h_1|) \neq A^2(\mathbb{D}, |h_1|)$ is given in [73], where it was proved that the function

$$f(z) = \exp((1+z)/(2(1-z)))$$

does not belong to $P^2(\mathbb{D}, |h_1|)$. This proof may be also extended to all values of p under consideration. The first assertion follows from one result of Hedberg, that we will see in Example 4.37.

Let us discuss the case $p = 2$ and $\mathbf{w} = |h|^2$, $h \in A^2(U)$, in more detail. In this case, the map $f \in A^2(U, |h|^2) \mapsto fh \in A^2(U)$ is an isometry between the respective Hilbert spaces. This fact allow us to use the general Hilbert space tools for study the approximation problem under consideration. In particular, the equality (4.27) can be verified using the construction of orthogonal basis, or Bessel's inequality, or Parseval's formulae in $A^2(G)$ or in $A^2(U, \mathbf{w})$. The following lemma shows how conformal maps may be used in the theory.

Lemma 4.27 (Keldysh). *Let G be a simply connected domain, let f maps $\mathbb{D}$ conformally onto G, and let $\mathbf{w}$ be defined on G. Put $g = f^{-1}$. If $P^2(\mathbb{D}, \mathbf{w} \circ f) = A^2(\mathbb{D}, \mathbf{w} \circ f)$ and if $g^m g' \in P^2(G, \mathbf{w})$ for each $m \in \mathbb{N}_0$, then $P^2(G, \mathbf{w}) = A^2(G, \mathbf{w})$.*

Outline of the proof. Let $F \in A^2(G, \mathbf{w})$. Using (2.1) we have

$$\int_G |F(z)|^2 \,\mathbf{w}(z)\, dA(z) = \int_{\mathbb{D}} |F(f(v))|^2 \,\mathbf{w}(f(v)) |f'(v)|^2 \, dA(v) < +\infty,$$

so $(F \circ f) f' \in A^2(\mathbb{D}, \mathbf{w} \circ f)$. Then, given $\varepsilon > 0$ one can find a polynomial Q such that

$$\int_{\mathbb{D}} |F(f(v)) f'(v) - Q(v)|^2 \,\mathbf{w}(f(v))\, dA(v) =$$

$$= \int_G |F(z) - Q(g(z)) g'(z)|^2 \,\mathbf{w}(z)\, dA(z) < \varepsilon.$$

Since $(Q \circ g) g'$ is a sum of functions of the type $g^m g' \in P^2(G, \mathbf{w})$, $m \in \mathbb{N}_0$, we conclude that $F \in P^2(G, \mathbf{w})$. ∎

Going further and working with space $A^2(U, |h|^2)$, let us consider the sequence (ψ_k), $\psi_k(z) = h(z)z^k$, $k \in \mathbb{N}_0$. Then, the Gram–Schmidt procedure in $A^2(U)$ applied to this sequence gives a new sequence (φ_k), $\varphi_k(z) = h(z)q_k(z)$, $q_k \in \mathcal{P}$, $\deg q_k = k$, and we can assume that the coefficient at z^k of q_k is positive. In [124, Section 3.1.8] one can find all details of this procedure. But for general simply connected open set U the sequence (φ_k) is not a basis for $A^2(U)$. Let us formulate the criterion in order that the equality (4.27) holds for $U = \mathbb{D}$ and $p = 2$.

Theorem 4.28. *Let $h \in A^2(\mathbb{D})$ and let h have no zeros in $\mathbb{D}$. The equality*

$$P^2(\mathbb{D}, |h|^2) = A^2(\mathbb{D}, |h|^2)$$

holds if and only if one of the following conditions is fulfilled:

(1) *There exists a sequence (u_n), $u_n \in \mathcal{P}$, such that* $\lim_{n\to\infty} \int_{\mathbb{D}} |1 - hu_n|^2 \, dA = 0$.

(2) *The sequence (φ_n), $\varphi_n = hq_n$, defined above is a basis for $A^2(\mathbb{D})$.*

(3) *It holds that* $\sum_{k=0}^{\infty} |h(0)|^2 |q_k(0)|^2 = 1/\pi$.

A few remarks about the proof of Theorem 4.28. Assume that the weight function h is such that $P^2(\mathbb{D}, |h|^2) = A^2(\mathbb{D}, |h|^2)$. Since $1/h \in A^2(\mathbb{D}, |h|^2)$ then there is a sequence of polynomials (u_n) such that

$$\int_{\mathbb{D}} \left| \frac{1}{h} - u_n \right|^2 |h|^2 \, dA = \int_{\mathbb{D}} |1 - hu_n|^2 \, dA \to 0.$$

Since hq_n, $n \in \mathbb{N}_0$, form an orthonormal system in $A^2(\mathbb{D})$, then q_n form an orthonormal system in $A^2(\mathbb{D}, |h|^2)$. Moreover, the construction of q_n implies that the closed linear span of (q_n) in $A^2(\mathbb{D}, |h|^2)$ coincides with the closure of $\mathcal{P}$ in this space. Therefore, (q_n) is a basis for $A^2(\mathbb{D}, |h|^2)$ and hence (φ_n) is a basis for $A^2(\mathbb{D})$. If (hq_k) is a basis for $A^2(\mathbb{D})$, then the Parseval's equality for $f = 1$ gives $\pi = \sum_{n=1}^{\infty} |\langle 1, hq_k \rangle|^2$. The fact that $\langle 1, hq_k \rangle = \pi h(0) q_k(0)$ implies (3).

Let us check the sufficiency of the conditions stated. Assume that (φ_n), $\varphi_n = hq_n$, is a basis for $A^2(\mathbb{D})$. Take $g \in A^2(\mathbb{D}, |h|^2)$ and observe that gh is the sum of its Fourier series $\sum c_n h q_n$ in $A^2(\mathbb{D})$. Since the partial sums of this series converges to gh in $A^2(\mathbb{D})$, then the partial sums $\sum_{n=1}^{m} c_n q_n$ converges to g in $A^2(\mathbb{D}, |h|^2)$. Thus, $g \in P^2(\mathbb{D}, |h|^2)$.

Assume now that (1) is satisfied. If $\lim_{n\to\infty} \int_{\mathbb{D}} |1 - h(z)u_n(z)|^2 \, dA(z) = 0$, then

$$\lim_{n\to\infty} \int_{\mathbb{D}} |z^m - h(z) z^m u_n(z)|^2 \, dA(z) = 0.$$

So, each polynomial is the limit in $A^2(\mathbb{D})$ of functions hp, where $p \in \mathcal{P}$. Since $A^2(\mathbb{D}) = P^2(\mathbb{D})$ then the set of hp, $p \in \mathcal{P}$, are dense in $A^2(\mathbb{D})$. It means that (hq_n) is a basis for $A^2(\mathbb{D})$.

If (3) holds, then the conclusion follows from the fact that if Bessel's inequality becomes the equality for a certain orthonormal system, then this system is a basis. ∎

For a general simply connected domain we have the following sufficient condition.

Theorem 4.29. *Let G be a simply connected domain, and let h and g be as before. Assume that $g^m g' \in P^2(G)$ for each $m \in \mathbb{N}_0$. Then, $P^2(G, |h|^2) = A^2(G, |h|^2)$ if and only if there exists a sequence (u_n), $u_n \in \mathcal{P}$, such that*

$$\lim_{n\to\infty} \int_G |1 - hu_n|^2 \, dA = 0. \tag{4.28}$$

In view of (2.1) we have $\int_G |g^m(z)g'(z)|^2 \, dA(z) = \int_{\mathbb{D}} |z^m| \, dA(z) < +\infty$, so that $g^m g' \in A^2(G)$. Therefore, $g^m g' \in P^2(G)$ by Theorem 4.1. Then, Theorem 4.29 implies the following consequence.

Theorem 4.30. *Let G be a Carathéodory domain and let h be as before. Then, the equality $P^2(G, |h|^2) = A^2(G, |h|^2)$ holds if and only if there exists a sequence of polynomials (u_n) such that (4.28) is satisfied.*

Sketch of the proof. The proof of Theorem 4.29 is based on the following observation which is also a consequence of (2.1).

Let G_1 and G_2 be two simply connected domains, let f be some conformal map from G_1 onto G_2, and let $g = f^{-1}$. Take $h \in A^2(G_2)$. Then, the spaces $A^2(G_2, |h|^2)$ and $A^2(G_1, |h \circ f|^2)$ are isometric by means of the map $F \to (F \circ f) f'$. Its inverse is $R_1 \to (R_1 \circ g) g'$.

The fact that $g^m g' \in P^2(G)$ means that given $\varepsilon > 0$ there exists $q \in \mathcal{P}$ such that $\|g^m g' - q\|_{2,G} < \varepsilon$. In view of (2.1) one has

$$\int_G |g^m g' - q|^2 \, dA = \int_{\mathbb{D}} |z^m - q(f(z)) f'(z)|^2 \, dA(z).$$

That means that the closed subspace generated in $L^2(\mathbb{D})$ by $\{(q \circ f) f : q \in \mathcal{P}\}$ is the same as the one generated by $\mathcal{P}$. The fact that $P^2(G, |h|^2) = A^2(G, |h|^2)$ is equivalent (in view of the isometry described above) to the fact that $P_1^2(\mathbb{D}, |h \circ f|^2) = A^2(\mathbb{D}, |h \circ f|^2)$, where $P_1^2(\mathbb{D}, |h \circ f|^2)$ is the closure of the set $\{(q \circ f) f' : q \in \mathcal{P}\}$ which is dense. Then, one can find a basis in $P_1^2(\mathbb{D}, |h \circ f|^2)$ and after that the proof may be finished using ideas from the proof of Theorem 4.28. ∎

Let us now present two examples of situation when the condition (4.28) is fulfilled for a general Carathéodory domain G.

Example 4.31. Let G be a Carathéodory domain, and let $\alpha_1, \alpha_2, \ldots$ be real numbers such that $\alpha_k > -1$, $k \in \mathbb{N}$, among which there is only a finite number of negative ones, and $\sum_{k=1}^{\infty} \alpha_k < +\infty$. Take a sequence of points (z_k), $z_k \notin G$, in such a way that α_k needs to be integer if $z_k \in \partial_a G$. Finally, fix a point $a \in G$. Then, the function

$$\Phi(z) = \sum_{k=1}^{\infty} \alpha_k \log\left(1 - \frac{z-a}{z_k - a}\right), \quad z \in G,$$

where the branch of logarithm is defined on G and equal to zero at a, is well defined, and the weight

$$h(z) = C \prod_{k=1}^{\infty} \left(1 - \frac{z-a}{z_k - a}\right)^{\alpha_k}$$

is such that (4.28) is satisfied, and, therefore, the equality $A^2(G, |h|^2) = P^2(G, |h|^2)$ holds.

The proof of the fact that this function h satisfies (4.28) is rather involved. All details may be found in [124, Section 3.2.3]. The starting step of this proof is to show that $h \in A^2(G)$ which is not difficult. Later on it is needed to consider three consecutive cases. The first one is related with the simplest possible function $\Phi(z) = \alpha \log(1 - (z-a)/(z_1 - a))$ constructed by one point z_1. This case is analyzed with the help of the special analogue of Mergleyan's key lemma. The second case is related with the finite set of points $\{z_1, \dots, z_n\}$. The important ingredient of the proof in the general case is the fact that the product of two functions h_1 and h_2 satisfying (4.28) is again the function satisfying this property.

Example¶ 4.32. Let G be a Carathéodory domain. If $h \in H(\overline{G^*})$, and $h(z) \neq 0$ for each $z \in G$, then the property (4.28) holds for h, and hence $A^2(G, |h|^2) = P^2(G, |h|^2)$.

It turns out that for general weights $\mathbf{w}$, the assumption that G is a Carathéodory domain is not necessary in order to have (4.27) because the following result holds.

Proposition 4.33. *Let $G \neq \emptyset$ be a simply connected domain in $\mathbb{C}$. Then, there exists a weight $\mathbf{w}$ such that $P^2(G, \mathbf{w}) = A^2(G, \mathbf{w})$.*

Outline of the proof. Let g be (as before) a conformal mapping from G onto $\mathbb{D}$, and take an increasing sequence (ρ_n) of positive real numbers. Define $G_n = \{z \in G : |g(z)| < \rho_n\}$. For each k the function $g^k g'$ is holomorphic in $\overline{G_n}$, so there exist polynomials $Q_{n,k}$ such that

$$\int_{G_n} |g^k g' - Q_{n,k}|^2 \, dA < \frac{1}{2^n}.$$

Also it is possible to find $\alpha_k > 0$ such that

$$\alpha_k \int_{G \setminus G_n} |g^k g' - Q_{n,k}|^2 \, dA < \frac{1}{2^n}.$$

Then, the function h may be defined on $G_{n+1} \setminus G_n$ as $h(z) = \min\{1, \alpha_1, \alpha_2, \dots, \alpha_n\}$, the details of the proof may be found in [73]. ■

Now, we will mention some general results for $1 \leqslant q < +\infty$ that were proved in [64]. Let G be a Carathéodory domain, let $\mathbf{w}: G \to (0, +\infty)$ be continuous functions such that $\mathbf{w} \in L^1(G, dA)$. Define

$$M(\mathbf{w}, z, r) = \frac{1}{r^2} \int_{|w| \leqslant r} \mathbf{w}(z + \zeta)\, dA(\zeta), \quad z \in G,\ r > 0,$$

where $\mathbf{w}(z + \zeta) = 0$ if $z + \zeta \notin G$.

Theorem 4.34. *Let G, $\mathbf{w}$, M be as before, and let $g \in A^q(G, \mathbf{w})$.*

(1) *Let $q > 1$. If $\sup_{r>0} \int_G |g(z)|^q\, M(\mathbf{w}, z, r)\, dA(z) < +\infty$, then $g \in P^q(G, \mathbf{w})$.*

(2) *Let $q = 1$. If $\int_G |g(z)| \sup_{r>0} M(\mathbf{w}, z, r)\, dA(z) < +\infty$, then $g \in P^1(G, \mathbf{w})$.*

One of the crucial ingredient of the proof of this theorem is the fundamental Mergelyan's lemma. The proof is obtained as an appropriate combination of this lemma, duality arguments and standard L^p-estimates.

The next result gives yet another generalization of Theorem 4.1.

Corollary 4.35. *Let G and $\mathbf{w}$ be as before, let $1 \leqslant p < +\infty$, and assume that $\mathbf{w} \in L^s(G)$ for some s, $1 < s \leqslant +\infty$. If $v \in A^p(G, \mathbf{w}) \cap L^{pt}(G)$, where $1/s + 1/t = 1$, then $v \in P^p(G, \mathbf{w})$.*

Notice that Theorem 4.34 give a sufficient approximation condition for individual functions. For the special classes of weights it is also possible to find sufficient approximation conditions for classes of functions.

Theorem 4.36. *Let G be a Carathéodory domain, and let $\mathbf{w} = |h|$, where $h \in A^1(G)$ and $|h(z)| > 0$ for all $z \in G$.*

(1) *If there exist $\varepsilon > 0$ such that*

$$\sup_{r>0} \int_G |h(z)|^{-\varepsilon}\, M(|h|, z, r)\, dA(z) < +\infty,$$

then $P^q(G, |h|) = A^q(G, |h|)$ for all $q \in (1, +\infty)$.

(2) *If there exist $\varepsilon > 0$ such that*

$$\int_G |h(z)|^{-\varepsilon} \sup_{r>0} M(|h|, z, r)\, dA(z) < +\infty,$$

then $P^1(G, |h|) = A^1(G, |h|)$.

Let us also present some class of weights $\mathbf{w} = |h|$ constructed in [64] such that $A^q(G, |h|) = P^q(G, |h|)$ for all $q \in [1, \infty)$ for a given general Carathéodory domain G.

Example 4.37. Let G be a Carathéodory domain, $q \in [1, \infty)$, and let $v \in H(G)$ be such that $\operatorname{Re} v > 0$ in G. Put $h(z) = e^{-v(z)}$ and assume that there exist two positive constants, says c_1 and c_2, such that $|\operatorname{Im} v| \leqslant c_1 \operatorname{Re} v + c_2$ in G. Then, $P^q(G, |h|) = A^q(G, |h|)$.

In order to prove that $P^q(G, |h|) = A^q(G, |h|)$ we need firstly to show that $v \in P^q(G, |h|)$. Notice, that the function $v_1 = (v-1)/(v+1)$ is bounded by 1 in G. Then, $v_1^m \in P^q(G, |h|)$ for all $m \in \mathbb{N}$. It gives that

$$v = \frac{1+v_1}{1-v_1} = 1 + 2\sum_{n=1}^{\infty} v_1^n \in P^q(G, |h|).$$

By induction one can prove that $v^n \in P^q(G, |h|)$. Now, for each $z \in G$, we have

$$\sum_{n=0}^{N} \frac{t^n v(z)^n}{n!} \to e^{tv(z)} = \frac{1}{|h(z)|^t}.$$

This convergence holds in $L^q(G, |h|)$ for each t such that $0 \leqslant t \leqslant t_h = (qc_1 + q)^{-1}$, it can be proved using Lebesgue's dominated convergence theorem. Thus, $|h|^{-t_h} \in P^q(G, |h|)$. Some more argument is needed to conclude the desired approximation result from Theorem 4.36 with $\varepsilon = 1/(c_1 + 1)$.

4.4 Topics on Hardy spaces

We start by recalling some basic facts about Hardy spaces in general domains in the complex plane. An appropriate reference for the next statements is [54]. During this section p will denote a number belonging to $(0, \infty)$ (we will mention below only the special restrictions on p, if needed). Let $G \subset \mathbb{C}$ be a bounded domain. *By definition* $H^p(G)$ is the space consisting of all functions $f \in H(G)$ for which there exists a positive harmonic function u in G such that $|f(z)|^p \leqslant u(z)$ for each $z \in G$. Such function u is called a harmonic majorant of $|f|^p$ in G. If $f \in H^p(G)$, then there exists a unique least harmonic majorant u_f such that $|f|^p \leqslant u_f$ in G. Then, we put $\|f\|_{H^p(G)} = (u_f(z_0))^{1/p}$, where $z_0 \in G$ is some fixed point. In the case that $p \geqslant 1$ this quantity is a norm in $H^p(G)$ and the resulting topology is independent on the choice of z_0.

Lemma 4.38. *Let K be a compact subset of G. Then, there exist a constant $C = C(K, p)$ such that for every $f \in H^p(G)$ and $z \in K$ one has $\|f\|_K \leqslant C\|f\|_{H^p(G)}$.*

Using this lemma it may be readily obtained that $H^p(G)$ is a Banach space for $p \geqslant 1$. Also it is well-known that $H^p(G)$ is conformally invariant, that is if φ is a conformal map from some domain G_1 onto another domain G_2, then $H^p(G_1) = \{g \circ \varphi : g \in H^p(G_2)\}$ and both these spaces are isometric.

Lemma 4.39. *Let $f \in H(G)$. Then, $f \in H^p(G)$ if and only if for each C^1-exhaustion (Ω_n) of G there exists a constant C such that for some point $a \in G$ one has*

$$\int_{\partial\Omega_n} |f|^p(\zeta)\, d\omega(a,\zeta,\Omega_n) \leqslant C.$$

In the case when $G = \mathbb{D}$ the spaces $H^p(\mathbb{D})$ are the classical Hardy spaces in the unit disk, and they are denoted usually by H^p.

Lemma 4.40. *If $f \in H^p$ then there exists a sequence of polynomials (P_n) which converges to f in norm in H^p.*

The principal part of the proof of this lemma is to prove the fact that $f_r \to f$ in H^p as $r \to 1$, where $f_r(z) = f(rz)$, see, for instance, [77, page 71]. After that it remains to observe that the Taylor series of f_r converges uniformly on $\overline{\mathbb{D}}$, and hence the Taylor polynomials of f give the desired approximation.

Let us also recall, that the Hardy spaces $H^p(\mathbb{T})$ on the unit circle are the spaces consisting of all functions $h \in L^p(\mathbb{T})$ such that $\int_{\mathbb{T}} h(\zeta)\overline{\zeta}^n\, dm_{\mathbb{T}}(\zeta) = 0$ for every integer $n < 0$. According to Fatou's theorem, every function $f \in H^p$ has a.e. on $\mathbb{T}$ angular boundary values, which determine a function in the class $H^p(\mathbb{T})$. The mapping which maps a function $f \in H^p$ to its boundary function is an isometric isomorphism between the spaces H^p and $H^p(\mathbb{T})$. When $p = \infty$, this mapping is also a weak-star homeomorphism. In what follows, functions in H^p and their boundary functions will be denoted by the same symbols.

Weak-star generators in H^∞

The space H^∞ is isometric to $H^\infty(\mathbb{T})$, while this space is a subset of $L^\infty(\mathbb{T})$ which is isometric to the dual space of $L^1(\mathbb{T})$. Then, we can consider the weak-star topology in $L^\infty(\mathbb{T})$, with a basis of neighborhoods of zero formed by sets

$$\left\{ f \in L^\infty(\mathbb{T}) : \left| \int_0^{2\pi} f(e^{it}) g_j(e^{it})\, dt \right| < r_j,\ j = 1,\dots,n \right\}$$

for all possible choice of numbers $r_1,\dots,r_n > 0$ and functions $g_1, g_2, \dots, g_n \in L^1(\mathbb{T})$. The weak-star topology in H^∞ is that induced by the isometry. For some mote detailed explanation of weak topologies see, for instance, [114, Chapter 3]. Notice also, that the aforesaid week topologies are not metrizable in the general case. Then, the sequences are not enough to manage with this topology. For sequences in H^∞ the convergence in the weak-star topology can be easily characterized.

Lemma 4.41. *The following statements hold.*

(a) *Let $f_n \in H^\infty$. Then, the sequence (f_n) converges in the weak-star topology to f if and only if this sequence is uniformly bounded in $\mathbb{D}$ and $f_n(z) \to f(z)$ for each $z \in \mathbb{D}$.*

(b) *Let $f_\alpha \in H^\infty$ be a net. Then, if (f_α) converges in the weak-star topology to some f, then $\sup_{\alpha, z\in\mathbb{D}} |f_\alpha(z)|$ is finite and $f_\alpha(z) \to f(z)$ for each point $z \in \mathbb{D}$.*

This is a well-known result. The proof of the part (a) may be found in [120]. The proof of the part (b) is similar.

The next definition was given by D. Sarason in [119].

Definition 4.42. Let $\varphi \in H^\infty$. Then, the following hold.

(a) φ is a weak-star generator if the set $\{P \circ \varphi : P \in \mathcal{P}\}$ is weak-star dense in H^∞.

(b) φ is a (weak-star) sequential generator if every function in H^∞ is the weak-star limit of a sequence of polynomials in φ.

It is clear that a sequential generator is a weak-star generator, but the converse is very far to being true, as it will be shown later. The main reason to introduce the concept of a weak-star generator was because of its relations with the theory of invariant subspaces for certain multiplication operators. Let us recall, that for a given function $\varphi \in H^\infty$ the operator $S_\varphi \colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ acts as follows: $S_\varphi \colon h \mapsto \varphi h$, while the Toeplitz operator $T_\varphi \colon H^2 \to H^2$ is defined as follows: $T_\varphi \colon h \mapsto P_+(\varphi h)$, where P_+ stands for the orthogonal projection from $L^2(\mathbb{T})$ to H^2. In the special case when $\varphi = \mathrm{j}$ the operator S_z is called the *bilateral shift*, while the operator T_z is called the *unilateral shift*. The study of shift-invariant subspaces in H^2 was initiated by Beurling, Helson–Lowdenslager and Halmos. The following simple fact whose proof may be found in [69, page 106] shows the specific role of the unilateral shift in the topic under consideration.

Lemma 4.43. *Let E a closed subspaces of H^2. Then, E is T_z-invariant if and only if it is T_φ-invariant for all $\varphi \in H^\infty$.*

The descriptions of shift-invariant (closed) subspaces of H^2 and $L^2(\mathbb{T})$ are well-known; they are given by the following nowadays become classical results whose proofs may be found in [69, Chapter 7].

Theorem 4.44 (Beurling). *Let E be a non-empty closed subspace of H^2. Then, E is T_z-invariant if and only if $E = K_\Theta = \Theta H^2$, where Θ is an inner function.*

We recall, that a function $\Theta \in H^\infty$ is said to be *inner*, if $|\Theta(\zeta)| = 1$ for a.a. $\zeta \in \mathbb{T}$.

Theorem 4.45. *Let W be a closed S_z-invariant subspace of $L^2(\mathbb{T})$.*

(a) *If $zW = W$, then $W = \{f \in L^2(\mathbb{T}) : f|_B = 0\}$, where $B \subset \mathbb{T}$ is some Borel set.*

(b) *If $zW \neq W$, then $W = FH^2$, where F is a measurable function on $\mathbb{T}$ of modulus 1.*

For non closed subspaces the situation is fairly different. Let us mention in this connection the next result obtained in [119].

Proposition 4.46. *Let $\varphi \in H^\infty$. Then, the following are equivalent.*

(a) *φ is a weak-star generator of H^∞.*

(b) *The operator S_φ has the same invariant subspaces as S_z.*

(c) *The operator T_φ has the same invariant subspaces as T_z.*

Two simple necessary conditions for a function φ to be a weak-star generator of H^∞ were obtained in [119].

Proposition 4.47. *Let $\varphi \in H^\infty$ be a weak-star generator of H^∞. Then, the following hold.*

(i) *φ in univalent on $\mathbb{D}$.*

(ii) *There exists a set $I \subset \mathbb{T}$ such that $m_{\mathbb{T}}(I) = 0$ and $\varphi|_{\mathbb{T}\setminus I}$ is injective.*

If some function φ satisfies the second condition of this proposition we will call it *univalent almost everywhere on* $\mathbb{T}$.

Sketch of the proof of Proposition 4.47. Let φ be a weak-star generator of H^∞ and assume that $\varphi(a) = \varphi(b)$ for some $a, b \in \mathbb{D}$ with $a \neq b$. Then, there exist a family $\{P_\alpha\}$ of polynomials such that the net $P_\alpha(\varphi)$ converging to z in the weak-star topology. Fix $a \in \mathbb{D}$. Then, the point evaluation functional $f \mapsto f(a)$, defined for each $f \in H^\infty$, is weak-star continuous because $f(a)$ is obtained via the standard Poisson formula and the Poisson kernel belongs to $L^1(\mathbb{T})$. This continuity implies $a = \lim P_\alpha(\varphi(a)) = \lim P_\alpha(\varphi(b)) = b$, which gives a contradiction.

Because the evaluation at an arbitrary point e^{it} is not defined for $f \in H^\infty$ in the general case, the proof of the second condition needs to be different from the previous one. Let E be the closed span of the elements $\{1, \varphi, \varphi^2, \dots\}$ in $L^2(\mathbb{T})$. So that $S_\varphi E \subset E$. If φ is a weak-star generator then, by Proposition 4.46, the space E is also S_z-invariant, and hence $z \in E$. Then, there exists a sequence (P_n) of polynomials that converges to z in $L^2(\mathbb{T})$. So, there is a partial subsequence (that will be denoted by the same symbol) such that

$$P_n(\varphi(e^{it})) \to e^{it} \quad \text{for a.e } e^{it} \in \mathbb{T}. \tag{4.29}$$

If we assume that for each measurable set $M \subset \mathbb{T}$ of positive measure there exist two points $e^{it} \in M$ and $e^{is} \in M$ with $e^{it} \neq e^{is}$ and $\varphi(e^{it}) = \varphi(e^{is})$ we arrive to a contradiction with (4.29). So, the second property also holds. ■

If φ a weak-star generator of H^∞, then $G = \varphi(\mathbb{D})$ is a simply connected domain and ∂G cannot have a lot of cut points. For example, the set of all cut points of ∂G should have harmonic measure zero with respect to $\varphi(0)$. It implies that each conformal map from $\mathbb{D}$ onto $\mathbb{D} \setminus [0, 1)$ is not a weak-star generator. The statement of the part (ii) of Proposition 4.47 was improved in [108] in the way shown in the next theorem.

Theorem 4.48. *The following statements hold.*

(a) *If φ is a weak star generator of H^∞ then the boundary function defined on $F(\varphi)$ is one to one.*

(b) *There exists a bounded univalent function φ in $\mathbb{D}$ such that $F(\varphi) = \mathbb{T}$ and φ is injective on $\overline{\mathbb{D}}$, but it is not a weak-star generator of H^∞.*

The proof of the part (a) is a bit technically involved since it uses classical results about conformal maps together with certain tools from ordinal number theory. To verify the statement (b) it is enough to take the conformal map from $\mathbb{D}$ onto the domain G_2 in Figure 1, but some work is needed in order to show that it is not a weak-star generator.

Following Sarason let us pay attention to the sequential generators of H^∞ because it admits certain characterizations in topological terms.

Proposition 4.49. *Let φ be a conformal map from $\mathbb{D}$ onto a simply connected domain $G \subset \mathbb{C}$. Then, φ is a sequential generator of H^∞ if and only if G has the following property: for every $h \in H^\infty(G)$, there is a sequence of polynomials which is uniformly bounded on G and converges to f at every point of G.*

Proof. Assume that φ is a sequential generator and let $h \in H^\infty(G)$. Then, $h \circ \varphi \in H^\infty$ and let (P_n) be such sequence of polynomials that $P_n(\varphi(z)) \to h(\varphi(z))$ for every $z \in \mathbb{D}$, and $p_n \circ \varphi$ is uniformly bounded on $\mathbb{D}$. Then, $P_n(w) \to h(w)$ for every $w \in G$ and P_n is uniformly bounded in G. For the converse it enough to consider $g \circ \varphi^{-1}$ for $g \in H^\infty$. ∎

Corollary 4.50. *A conformal map onto a Jordan domain is always a weak-star generator. A conformal map onto a moon-shaped domain is never a weak-star generator.*

Now, combining Proposition 4.49 with Theorem 3.31 we arrive at the following result (see [120, Proposition 2]).

Proposition 4.51. *Let G be a bounded simply connected domain and let φ a conformal map from $\mathbb{D}$ onto G. Then, φ is a sequential generator of H^∞ if and only if G is a component of the set G^* (the latter property exactly means that G is a Carathéodory domain).*

We will denote by $\widetilde{G}$ the component of G^* that contains G. With the same notations as in Proposition 4.49 the following result holds.

Proposition 4.52. *Let $h \in H^\infty$. Then, h is the weak star limit of a sequence of polynomials on φ if and only if $h \circ \varphi^{-1}$ is the restriction of a function belonging to $H^\infty(\widetilde{G})$.*

Sarason in [120] has obtained a characterization of weak generators, adapting the statement of Farrell's theorem to a certain more general setting. This is a reason to give here a simple overview of his results. Also we believe that the notion of order of a simply connected domain introduced by Sarason may be regarded as a further generalization of the concept of a Carathéodory domain.

Take $\varphi \in H^\infty$. Denote by M^0 the set of polynomials in φ, that is $M^0 = \{P \circ \varphi : \varphi \in \mathcal{P}\}$. Furthermore, let M^1 denote the set of all weak-star limits of sequences of functions in M^0. If α is a countable ordinal we define M^α inductively to be the linear manifold of H^∞ consisting of all functions which are weak-star limits of sequences of functions on $\bigcup_{\beta<\alpha} M^\beta$. By a property of weak topologies, see [7, pages 124, 213], there exists a least countable ordinal α' such that $M^{\alpha'} = M^\beta$ if $\beta > \alpha'$. Moreover, $M^{\alpha'}$ is the weak closure of M^0. We say that φ is a generator of H^∞ of order α' if $M^{\alpha'} = H^\infty$.

In order to understand the definition of the order of a simply connected domain we need the following definition.

Definition 4.53. Let G be a bounded domain in $\mathbb{C}$, and let Ω be a simply connected domain such that $G \subset \Omega$. The relative hull of G in Ω (or, for brevity, Ω-hull) is the set

$$\operatorname{Int}\left\{w \in \Omega : |f(w)| \leqslant \sup_{z\in G} |f(z)| \text{ for every } f \in H^\infty(\Omega)\right\}.$$

One crucial step in Sarason's papers is to show that if $G \subset \mathbb{D}$, then the $\mathbb{D}$-hull of G is G^*. Also a geometric description of the Ω-hull of G is given by the next result.

Proposition 4.54. *Let Ω and G be as before and let Y be the closure in Ω of the Ω-hull of G. Then, $\Omega \setminus Y$ consists of those points of Ω that can be separated from G by a cross cut of Ω. Moreover, the Ω-hull of G is the interior of Y.*

With these tools the following generalization of Farrell's result is readily followed.

Theorem 4.55. *Let Ω be a domain and let G be a bounded simply connected domain such that $G \subset \Omega$. Denote by G_Ω the component of the Ω-hull of G that contains G. Let $f \in H^\infty(G)$. Then, a sequence of bounded holomorphic functions in Ω which is uniformly bounded in G and converges to f at the every point of G exists if and only if f is the restriction of some function $f_1 \in H^\infty(G_\Omega)$.*

Let G be a simply connected domain. For every countable ordinal α let us define inductively a domain G^α containing G as follows. For $\alpha = 1$ we put G^1 as the component of G^* that contains G. If α has an immediate predecessor we define G^α as the component of the $G^{\alpha-1}$-hull of G that contains G. If α has no immediate predecessor we define G^α to be the component of the interior of $\bigcap_{\beta<\alpha} G^\beta$ that contains G. Then, G^α is simply connected and, moreover, there exists a least countable ordinal γ such that $G^\gamma = G^{\gamma+1}$. So, $G^\gamma = G^\omega$ for $\omega > \gamma$. This γ is called the order of G. For better understanding this notion the reader can see that both domains in Figure 3 have order 1. Next result generalizes Proposition 4.52.

Proposition 4.56. $M^\alpha = \{h \in H^\infty : h \circ \varphi^{-1} = F|_G$ *for some* $F \in H^\infty(G^\alpha)\}$.

Corollary 4.57. *Now, the characterization of generators of* H^∞ *obtained by Sarason can be stated in the following form.*

(1) *If* φ *is a generator of* H^∞ *of order* γ *then the domain* $G = \varphi(\mathbb{D})$ *has order* γ *and* $G^\gamma = G$. *Conversely, if a given domain* G *has order* γ *and* $G^\gamma = G$, *then every conformal mapping* φ *from* $\mathbb{D}$ *onto* G *is a generator of* H^∞ *of order* γ.

(2) *The function* $\varphi \in H^\infty$ *fails to be a generator of* H^∞ *if and only if there exists a domain* Ω *which properly contains* G *and is such that* $\|f\|_\Omega = \|f\|_G$ *for every* $f \in H^\infty(\Omega)$.

(3) *If* φ *is a generator of* H^∞, *then* $G = \operatorname{Int}(\overline{G})$.

We refer to [120, Figures 1 and 2] to see domains which are the images of $\mathbb{D}$ under mapping by weak-star generators of order 2 and 3, respectively. The orders of these domains can be computed by using Proposition 4.54. In [121] the author was able to construct domains of arbitrary order using the fact that every countable well-ordered set can be realized as a subset of $\mathbb{R}$. We do not know whether it is possible to obtain some other type of characterization of weak-star generators avoiding, in particular, the usage of ordinals.

Finally, let us notice that the concept of domains of order 2 is underlying the result of [112, Theorem 4.1], so this theorem was the precursor of Sarason's studies.

Density of polynomials in $H^p(G)$

Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be a non-constant holomorphic function. The composition operator $C_\varphi\colon H(\mathbb{D}) \to H(\mathbb{D})$ is defined by the setting $C_\varphi(f) = f \circ \varphi$. If $\varphi(0) = 0$ then the Littlewood subordination theorem (see, for example, [42, Theorem 1.7]) implies

$$\|f \circ \varphi\|_{H^p} \leqslant \|f\|_{H^p}$$

for each $p \in (0, \infty)$. If $\varphi(0) \neq 0$ then $\|f \circ \varphi\|_{H^p} \leqslant M\|f\|_{H^p}$, where M is some constant depending only on $|\varphi(0)|$. Thus, $C_\varphi\colon H^p \to H^p$ is a bounded operator for each p, $1 \leqslant p \leqslant \infty$. A lot of efforts were applied for studying of such operators. In particular, in [31] the problem when $C_\varphi(H^p)$ is dense in H^p were considered.

We have the following clear facts.

Lemma 4.58. *If* $C_\varphi(H^p)$ *is dense in* H^p *for some* $0 < p < \infty$, *then* φ *is univalent in* $\mathbb{D}$.

Proof. Indeed, if $\varphi(z) = \varphi(w)$ for some $z \neq w$ in $\mathbb{D}$, then $f(\varphi(z)) = f(\varphi(w))$ for each $f \in H^p$. So, the function j does not belong to the closure of $C_\varphi(H^p)$ in H^p. ∎

Lemma 4.59. *Let φ maps conformally $\mathbb{D}$ onto some domain $G \subset \mathbb{D}$, while $0 < p < \infty$. Then, $C_\varphi(H^p)$ is dense in H^p if and only if the set of polynomials is dense in $H^p(G)$.*

Proof. If $f \in H^p(G)$, then $f \circ \varphi \in H^p$. Therefore, if $C_\varphi(H^p)$ is dense in H^p, then Lemma 4.40 yields that there exists a sequence (P_n) of polynomials such that $P_n \circ \varphi \to f \circ \varphi$ in H^p, which imply that $P_n \to f$ in H^p. The converse is clear. ∎

The following theorem was proved in [31].

Theorem 4.60 (Caughran). *Let p be such that $1 \leqslant p \leqslant \infty$. If G is a Carathéodory domain, then the set of polynomials is dense in $H^p(G)$. Conversely, if polynomials are dense in $H^p(G)$ and $\varphi \in C(\overline{\mathbb{D}})$, where φ is some conformal map from $\mathbb{D}$ onto G, then G is a Jordan domain.*

The Caughran's original proof, was made for $p = 2$ and it used the ideas of proving the sufficiency in Theorem 3.25. J. Caughran has mentioned that the given proof is valid, if interpreted properly, for H^p with $1 \leqslant p < \infty$. The following result is an immediate corollary of Caughran's theorem.

Corollary 4.61. *If φ maps $\mathbb{D}$ conformally onto a Carathéodory domain, $\|\varphi\|_{H^\infty} \leqslant 1$, then $C_\varphi(H^p)$ is dense in H^p for each $1 \leqslant p < \infty$.*

Later on in [109] the next generalization of the results under consideration was obtained.

Theorem 4.62 (Roan). *Let φ a weak star generator of H^∞, then the set of polynomials is dense in $H^p(G)$, where $G = \varphi(\mathbb{D})$ and $0 < p < \infty$.*

Proof. Assume that φ is a weak-star generator of H^∞. Denote by M the subspace $\{P \circ \varphi : P \in \mathcal{P}\}$, by M^1 the subspace of all functions in H^∞ which are weak-star limits of sequences of functions in M. Let $h \in M^1$, then there exists a sequence (P_n) of polynomials which are uniformly bounded and $P_n(\varphi(z)) \to h(z)$ for each $z \in \mathbb{D}$. We need the following lemma which corresponds to Lemma 4.41 for H^p.

Lemma 4.63. *Let $0 < p < \infty$, and let (f_n) be a bounded sequence in H^p. Assume that $f_n(z) \to f(z)$ for each $z \in \mathbb{D}$. Then, $f_n \to f$ in the weak topology in H^p.*

Notice that for $p \geqslant 1$ the proof of this lemma is essentially the same as it was done in [120, Lemma 1]. For the case $0 < p < 1$, it follows from [40], where it was proved that the point evaluation belongs to $(H^p)^*$ and the principle of uniform boundedness and the closed graph theorem remain valid for H^p.

By the lemma just mentioned we know that $P_n \circ \varphi \to h$ weakly in H^p. So, $h \in \mathrm{Clos}_{w;H^p}(M)$, the weak closure of M in H^p. Then, $M^1 \subset \mathrm{Clos}_{w;H^p}(M)$. One has

$$\mathrm{Clos}_{w;H^p}(M) = \mathrm{Clos}_{H^p}(M), \tag{4.30}$$

where the right-hand side of (4.30) is the closure of M in the original topology of H^p. Equality (4.30) follows from [114, Theorem 3.12] in the case when $1 \leqslant p < \infty$. In the case $p < 1$, (4.30) follows from [40, Lemma 8]. Then, $M^1 \subset \mathrm{Clos}_{H^p}(M)$. Now, inductively $M^\sigma \subset \mathrm{Clos}_{H^p}(M)$ for every countable ordinal number σ. Since φ is a weak-star generator of H^∞ there exists a countable ordinal τ such that $M^\tau = H^\infty$. Then, $H^\infty = \mathrm{Clos}_{H^p}(M)$, and hence M is dense in H^p. But the density of M in H^p exactly means the density of polynomials in $H^p(G)$. ∎

The proof of above theorem is quite simple. The crucial reason why this theorem implies Theorem 4.60 is the fact, given by Proposition 4.51, that a Carathéodory domain is the image of some sequential generator of H^∞.

In view of the Lemma 4.59, Theorem 4.62 can be reformulated as follows.

Theorem 4.64 (Roan). *Let $0 < p < \infty$, and let φ be a weak-star generator of H^∞ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then, the range of C_φ is dense in H^p.*

We end this section by mentioning some results obtained in [17] and related with Bergman spaces.

Theorem 4.65 (Bourdon). *Let φ be a weak-star generator of H^∞ and let $G = \varphi(\mathbb{D})$. Then, the polynomials are dense in $A^2(G)$.*

Because there are many weak-star generators of H^∞ which map $\mathbb{D}$ onto non-Carathéodory domains, this result is more general (for $p = 2$) than Theorem 4.1.

The proof of Theorem 4.65 use a theorem of Hedberg that says that if G is a simply connected domain of finite area, then $H^\infty(G)$ is dense in $A^2(G)$ and certain properties of cyclic vectors of multiplication operators acting in $A^2(G)$ and $H^2(G)$, see [17] for the detailed explanation. Then, one has yet another proof of Theorem 4.60 for $p = 2$.

A key idea of work [17] is to relate the approximation by polynomials in $H^2(G)$ with the approximation also by polynomials in some weighted Bergman spaces.

Proposition 4.66. *Let φ map $\mathbb{D}$ conformably onto G. Then, the polynomials are dense in $A^2(G, (1 - |\varphi^{-1}(w)|^2)\, dA)$ if and only if the polynomials in φ are dense in H^2.*

Sketched proof. If $f \in H^2$ and $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$, then

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2.$$

But this norm is equivalent to

$$\|f\|^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(w)|^2 (1-|w|^2)\, dA(w),$$

as it can directly verified by the considering of the corresponding Taylor expansion. From now on, one can proceed as follows. If the polynomials are dense in $A^2(G, (1-\varphi^{-1}(w))\, dA)$, then the set $\{(P \circ \varphi)\varphi' : P \in \mathcal{P}\}$ is dense in $A^2(\mathbb{D}, (1-|z|^2)\, dA)$, but this implies (via the integration) that the set $\{P \circ \varphi : P \in \mathcal{P}\}$ is dense in $H^2(\mathbb{D})$. The converse may be verified by differentiation. ■

Corollary 4.67. *Let φ map $\mathbb{D}$ conformally onto G.*

(1) *The density of polynomials in $A^2(G)$ or in $A^2(\mathbb{D}, (1-|\varphi^{-1}(w)|^2) dA)$ implies the density of polynomials in $H^2(G)$.*

(2) *If polynomials are dense in $A^2(G)$ or in $A^2(\mathbb{D}, (1-|\varphi^{-1}(w)|^2)\, dA)$, then φ is univalent almost everywhere on $\mathbb{T}$.*

Note that the part (1) of Corollary 4.67 says that Theorem 4.1 for $p = 2$ together with Proposition 4.66 give a direct proof of Theorem 4.60 in the case $p = 2$. Note also that the part (2) of Corollary 4.67 is the analogue for $A_a(G)$ of Proposition 4.47. Also it seems that Bourdon's techniques are only appropriate for $H^2(G)$ and not for $p \neq 2$.

Let us mention the paper [37], where another proof of Theorem 4.62 was given. It seems the author was unaware of Roan's, Caughran's and Bourdon's papers.

4.5 Approximation by polynomials on boundaries of domains

Let A be an uniform algebra on some compact Hausdorff space X, let $\phi \in M_A$ and assume that there exist a unique representative positive measure σ for ϕ (recall that this assumption is needed here, because, in general, such measure is not unique). Put $M = \ker \phi$ and denote by $M^+(X)$ the set of finite positive Borel measures on X. Let us recall the following result.

Theorem 4.68 (Szegö, Kolmogorov, Krein). *Let $\mu \in M^+(X)$, and suppose that $\mu = w \cdot \sigma + \nu$ is the Lebesgue decomposition of μ with respect to σ, where $w = d\mu/d\sigma \in L^1(\sigma)$ is the Radon–Nikodym derivative of μ and ν is singular with respect to σ. Let $0 < q < +\infty$. Then,*

$$\inf_{f \in M} \int |1-f|^q\, d\mu = \inf_{f \in M} \int |1-f|^q w\, d\sigma = \exp \int \log w\, d\sigma.$$

Szegö has proved this theorem when $A = A(\overline{\mathbb{D}})$, $M = \{P \in \mathcal{P} : P(0) = 0\}$, $\mu \ll dt$, $q = 2$ and $\phi(f) = f(0)$, $f \in A$. Later Kolmogorov and Krein showed that the infimum depends only on the absolutely continuous part of μ. A complete proof in the case $A(\overline{\mathbb{D}})$, $1 \leqslant q < +\infty$ and $M = \{P \in \mathcal{P} : P(0) = 0\}$ is given in [77, Chapter vii]. The proof of the general version may be found in [56, Chapter v] or in [18, page 236]. Observe that, by Jensen inequality, one has

$$\exp \int \log w \, d\sigma \leqslant \int w \, d\sigma < +\infty.$$

So, always,

$$-\infty \leqslant \int \log w \, d\sigma \leqslant \text{Const}\,.$$

Now, we will use the notation and result from Sections 3.2 and 3.4. Let G be a Carathéodory domain with the boundary Γ, then $P(\Gamma)$ is a Dirichlet algebra, and for each point $a \in G^*$ (the Carathéodory hull of G) the measure $\omega(a, \cdot, G)$ is the unique representative measure on the Shilov boundary Γ of the element of the spectrum of $P(\Gamma)$ defined by $P \mapsto P(a)$. In this context, given $\mu \in M^+(\Gamma)$, Theorem 4.68 can be applied, and this is the most general setting (in some sense) that the previous theorem can be applied. For example, if $G = \mathbb{D}$ we have the following.

Corollary 4.69. *Let be $\mu \in M^+(\mathbb{T})$ and let $0 < q < +\infty$. The set $\{P \in \mathcal{P} : P(0) = 0\}$ is dense in $L^q(\mu)$ if and only if*

$$\int_{\mathbb{T}} \log\left(\frac{d\mu}{dt}\right) dt = -\infty.$$

Proof. First note that

$$\inf_{p \in \mathcal{P}} \int \left| \frac{1}{z} - p \right|^q d\mu = \inf_{P : P(0)=0} \int_{\mathbb{T}} |1 - P|^q \, d\mu = 0,$$

where the equality to zero is obtained applying Theorem 4.68. Then, z and $1/z$ are limits in $L^q(\mu)$ of polynomials. But each $f \in C(\mathbb{T})$ can be uniform approximated by a sequence of polynomials in z and $\bar{z}$ and $C(\mathbb{T})$ is dense in $L^q(\mu)$ for each $0 < q < +\infty$. ∎

Abdullaev and Dovgoshei in [3] have provided an interesting study of the question on how to generalize Corollary 4.69 for other domains. It turns out that the notion of Carathéodory domains plays a central role in this question. Before discussing their results let us fix yet more notation. Let $z_0, z_1, z_2, \ldots$ be a collection of points such that it contains only one point from each component of G^*. Moreover, we assume that $z_0 \in G_0 = G$ and denote by G_j, $j \geqslant 1$, the other bounded components of $\mathbb{C} \setminus \overline{G}$ (if they exist). Let $\omega_j = \omega(z_j, \cdot, G)$, for each $j \geqslant 0$. We know that each ω_j is supported on ∂G_j. Given $\mu \in M^+(\Gamma)$ let us denote by $P^q(\mu)$ the closure in $L^q(\mu)$ of the set of polynomials, and by $P^q(\mu, z_0)$ the closure in $L^q(\mu)$ of the set of polynomials that vanishes at z_0. With this notation we can state the result.

Theorem 4.70. *Assume that G is a bounded simply connected domain, $z_0 \in G$, and $0 < q < +\infty$. Then, the following assertions hold.*

(1) *Let G be a Carathéodory domain. Then,*

$$\begin{aligned}\{\mu \in M^+(\Gamma) : P^q(\mu, z_0) = P^q(\Gamma)\} \\ = \left\{\mu \in M^+(\Gamma) : \int_\Gamma \log\left(\frac{d\mu}{d\omega_0}\right) d\omega_0 = -\infty\right\}.\end{aligned} \tag{4.31}$$

(2) *Conversely, if the sets defined on (1) are equal, then G is a Carathédodory domain.*

(3) *Let G be a Carathéodory domain. Then,*

$$\begin{aligned}\{\mu \in M^+(\Gamma) : L^q(\Gamma) = P^q(\Gamma)\} \\ = \left\{\mu \in M^+(\Gamma) : \int \log\left(\frac{d\mu}{d\omega_j}\right) d\omega_j = -\infty \textit{ for all } j\right\}.\end{aligned}$$

(4) *In order to have that G is a Carathéodory domain that does not separate the plane it is necessary and sufficient that*

$$\begin{aligned}\{\mu \in M^+(\Gamma) : L^q(\Gamma) = P^q(\Gamma)\} = \{\mu \in M^+(\Gamma) : L^q(\mu) = P^q(\mu, z_0)\} \\ = \left\{\mu \in M^+(\Gamma) : \int_\Gamma \log\left(\frac{d\mu}{d\omega_0}\right) d\omega_0 = -\infty\right\}.\end{aligned}$$

Sketch of the proof. (1) Always $P^q(\mu, z_0) \subset P^q(\mu)$. Theorem 4.68 may be applied in our case to give that $\int \log(d\mu/d\omega_0)\, d\omega_0 = -\infty$ if and only if there exists a sequence of polynomials (P_n) such that $P_n \to 1$ in $L^q(\mu)$. Then, if $h \in \mathcal{P}$ then $hP_n \to h$ in $L^q(\mu)$.

(2) Assume that G is not a Carathéodory domain. Then, we need to find a measure that shows that both sets in (4.31) are different. Let Ω be the component of G^* that contains z_0 and let take $\widetilde{\omega} = \omega(z_0, \cdot, \Omega)$. Then, $\Omega \supset G$ and $\widetilde{\omega}$ is a positive measure on Γ but it is supported on $\partial\Omega$. Since G is not a Carathéodory domain, we know that $L := \Gamma \setminus \partial\Omega \neq \emptyset$, and even more, $\omega_0(L) > 0$. Since $\widetilde{\omega}$ vanishes on L, one has $\frac{d\widetilde{\omega}}{d\omega_0}(z) = 0$ for almost all points z in L. Then, $\int \log(d\widetilde{\omega}/\omega_0)\, d\omega_0 = -\infty$, so $\widetilde{\omega}$ belongs to the set in the right-hand side of (4.31). Because Ω is a Carathéodory domain, we can apply the result just proved in (1). Since $\int \log(d\widetilde{\omega}/d\widetilde{\omega})\, d\widetilde{\omega} = 0$, we know that $P^q(\partial\Omega, z_0) \neq P^q(\partial\Omega)$. So, $P^q(\Gamma, z_0) \neq P^q(\Gamma)$.

(3) Let $\mu \in M^+(\Gamma)$. If $L^q(\Gamma) = P^q(\Gamma)$ then, for each z_j, $j \geqslant 0$, one has $L^q(\mu|_{\partial G_j}) = P^q(\mu|_{\partial G_j})$. Since each G_j is a Carathéodory domain, the result of part (1) may be used to obtain that $\int_{\partial G_j} \log \frac{d\mu}{d\omega_j}\, d\omega_j = -\infty$ for each j. Assume now

$$\int_{\partial G_j} \log \frac{d\mu}{d\omega_j}\, d\omega_j = -\infty \tag{4.32}$$

for all $j \geqslant 0$. The important fact now is the assumptions in (4.32) do not depend on the point selected in each component. In other words, if $\omega'_j = \omega(b_j, \cdot, G)$ for other points $b_j \in G_j$, $j \geqslant 0$, then, by (3.4) and (4.32) remains true if we replace ω_j with ω'_j. By Corollary 3.11 one has $R(\Gamma) = C(\Gamma)$. We know that $C(\Gamma)$ is dense in $L^q(\Gamma)$ for each $0 < q < +\infty$. Then, it is enough to prove that for each fixed $b \in G^* \setminus G$ the function $z \to (z-b)^{-1}$ can be approximated in $L^q(d\mu)$ by polynomials. Applying Theorem 4.68 to $P(\Gamma)$ and to $M = \{P : P(b) = 0\}$ we obtain $\inf\{\int |1-P|^q \, d\mu = 0 : P \in \mathcal{P},\ P(b) = 0\} = 0$. If $P(b) = 0$ then $P(z) = (z-b)P_1(z)$ and hence one has

$$\left|\frac{1}{z-b} - P_1(z)\right| \asymp |1-(z-b)P_1(z)|$$

for each $z \in \Gamma$ and $P_1 \in \mathcal{P}$. Then,

$$\inf\left\{\int\left|\frac{1}{z-b} - P_1\right|^q d\mu = 0 : P_1 \in \mathcal{P}\right\} = 0.$$

So, $1/(z-b) \in P^q(\mu)$.

This is a consequence of (1) and (3). In particular, if the set $\mathbb{C} \setminus \overline{G}$ has some bounded component G_j, then the assumption that $\int_{\partial G_j} \log \frac{d\mu}{d\omega_j} \, d\omega_j = -\infty$ cannot be dropped.

■

The next result gives a sufficient condition for approximation.

Corollary 4.71. *Let G be a Carathéodory domain such that $\overline{G}$ does not separate the plane, and let $0 < p < +\infty$. Assume that $\mu \in M^+(\partial G)$ is such that* Supp $\mu \neq \partial G$. *Then, $P^q(\mu) = L^q(\mu)$.*

Theorem 2 in [3] gives yet other characterization of Carathéodory domains, however it is a bit technical in a nature and hence we do not state it here, but only mentioned for the interested reader.

Chapter 5

Miscellaneous results about Carathéodory sets

In this chapter we briefly mention some results, where the concept of a Carathéodory sets plays a certain role, but which cannot be placed appropriately into any of the above chapters and sections.

Approximation by polynomials of controlled degree

In this section we present one result which is formally related with Carathéodory sets (at least the corresponding assumption was made in it formulation), but actually it is independent on this concept.

Let K be a continuum and let Φ be the conformal map from $\Omega'_\infty(K)$ onto $\{w \in \mathbb{C}_\infty : |z| > \rho\}$ with the normalization $\Phi(\infty) = \infty$ and $\Phi'(\infty) = 1$, where $\rho > 0$ is determined uniquely by this normalization of Φ. Recall that the Taylor series of Φ at ∞ has the form

$$\Phi(z) = z + a_0 + \frac{a_1}{z} + \cdots, \quad |z| > R_1$$

for some $R_1 > 0$. Then, for each $n \geqslant 1$, one has

$$\Phi^n(z) = z^n + a_{n-1}^{(n)} z^{n-1} + \cdots + a_0^{(n)} + \frac{a_{-1}^{(n)}}{z} + \cdots, \quad |z| > R_1.$$

The polynomial

$$\Phi_n(z) = z^n + a_{n-1}^{(n)} z^{n-1} + \cdots + a_0^{(n)}$$

is called the *n-Faber polynomial* with respect to K. For each $r > 1$ let

$$C_r = \Phi^{-1}(\{z : |z| = r\}).$$

The question on studies of approximation of functions by polynomials of degree at most n was posed already in the thesis of S. N. Bernstein. Here, we mention two results.

Theorem 5.1 (Bernstein theorem). *Let K be a continuum and let $f \in C(K)$. Then, for every $\varepsilon > 0$ and $0 < q < 1$ there exists a sequence of polynomials (P_n) such that* $\deg P_n \leqslant n$ *and*

$$|f(z) - P_n(z)| \leqslant C(\varepsilon)(q + \varepsilon)^n, \quad n = 0, 1, 2, \ldots, \ z \in K,$$

if and only if f has an analytic extension $\tilde{f}$ to $D(C_{\rho/q})$. In the case that there exists such extension, the sequence (P_n) converges to $\tilde{f}$ locally uniformly in $D(C_{\rho/q})$.

Now, we will use the following notation. Let $1 \leqslant p \leqslant \infty$. If $f \in A^p(G)$ then define

$$E^p_{n,G}(f) = \inf\{\|f - P\|_{p,G} : P \in \mathcal{P},\ \deg P \leqslant n\}.$$

Theorem 5.2 (Bernstéin–Walsh theorem). *Let $K \subset \mathbb{C}$ be a compact set such that $\mathbb{C} \setminus K$ is connected. If $f \in H(K)$, then*

$$\limsup_{n\to\infty} E^\infty_{n,K}(f)^{1/n} \leqslant \theta < 1,$$

where $\theta = 0$ if K has logarithmic capacity zero, while θ is a positive number (related with the Green function of $\mathbb{C} \setminus K$) if capacity of K is positive.

Note that previous result is a quantitative version of Runge's theorem. A proof can found in [107, page 170]. Let us revert to Theorem 5.1. A proof can be seen in [85]. In the case that f has the continuous extension, the key point is to show that

$$f(z) = \sum_{n=0}^{\infty} a_n \Phi_n(z)$$

uniformly on K and one can take $P_m = \sum_{n=0}^{m} a_n \Phi_n$. In the proof the following estimate is obtained

$$|f(z) - P_n(z)| \leqslant \frac{3}{2}\bar{M}(f, r)\frac{(r'/r)^{n+1}}{1 - (r'/r)}, \tag{5.1}$$

for each $z \in K$, where $r > r' > \rho$ and $\bar{M}(f, r) = \sup\{|f(z)| : z \in C_r\}$.

In a sequence of papers, see the references in [70], the following problem was studied: *whether the condition* $\lim(E^p_{n,G}(f))^{1/n} = 0$ *does imply that f is an entire function*. In [70] two results of such kind were obtained under the assumption that the domain G under consideration is a Carathéodory domain. However, it seems that this assumption is not relevant to the problem under consideration and it is not needed in the first of the aforementioned results. Let us reformulate and proof the corresponding statement.

Theorem¶ 5.3. *Let $1 \leqslant p \leqslant \infty$.*

(a) *Let $f \in H(\mathbb{C})$. Then, for each bounded domain G it holds*

$$\lim_{n\to\infty} E^p_{n,G}(f)^{1/n} = 0.$$

(b) *Let U be an open set in $\mathbb{C}$ and let $f \in H(U)$. If there exists an open disc D with $\bar{D} \subset U$, such that*

$$\lim_{n\to\infty} E^p_{n,D}(f)^{1/n} = 0,$$

then there exists $F \in H(\mathbb{C})$ such that $F|_U = f$.

Sketch of the proof. Put $K := \widehat{G}$, take the corresponding Φ for such K (see the beginning of this subsection), and put $r' = 2\rho$. If P_n is a polynomial which satisfies (5.1), then

$$E^p_{n,G}(f) \leqslant \|f - P_{n-1}\|_{p,K} \leqslant \sqrt[p]{\mathrm{Area}(K)}\, \|f - P_n\|_K.$$

Using now the estimate (5.1) we obtain

$$E^p_{n,G}(f) \leqslant \mathrm{Const}\, \overline{M}(f, r) \frac{(\rho/r)^{n+1}}{1 - (\rho/r)}$$

whenever $r > r' = 2\rho$. Then, $\limsup_{n\to\infty} E^p_{n,G}(f)^{1/n} \leqslant \rho/r$ and letting $r \to \infty$, the conclusion follows.

Let us prove the statement (b). Let $D = D(z_0, R)$. We know $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ uniformly on $\bar{D}$. Assume that P_{n-1} is a polynomial of degree at most $n-1$, then

$$\frac{\pi R^{2n+2}}{n+1} a_n = \int_D f(z)\,(\bar{z} - \overline{z_0})^n \,dA(z) = \int_D (f(z) - P_{n-1}(z))\,(\bar{z} - \overline{z_0})^n \,dA(z),$$

which in ones turn gives that

$$\frac{\pi R^{2n+2}}{n+1} |a_n| \leqslant R^n \|f - p_{n-1}\|_{1,D} \leqslant (\pi R^2)^{\frac{1}{q}} R^n \|f - P_{n-1}\|_{p,D},$$

where q is the conjugate exponent for p if $p > 1$. Then,

$$|a_n|^{1/n} \leqslant \mathrm{C}(n+1)^{1/n} E^p_{D,n-1}(f)^{1/n} \to 0$$

by the initial assumptions. So, $f \in H(\mathbb{C})$.

In the case of $p = \infty$ the theorem is essentially due to Winiarski, [137]. ∎

Dualities between $\mathcal{A}^{-\infty}(G)$ and $\mathcal{A}^{\infty}(\mathbb{C}_\infty \setminus G)$

The problem of characterization of the dual of the Fréchet space $H(G)$ for an open set G is a classical problem studied in several papers in the 1950s. It comes that there is an isomorphism from $H(G)^*$ onto $H_0(\mathbb{C}_\infty \setminus G)$. This is called the Main duality theorem. We recommend to the interested reader to look at the proof of this result and some related topics in [81, Chapter 8]. With this background he will understand perfectly the germinal ideas on this short section on dualities between spaces of analytic functions.

Let B be a bounded domain in $\mathbb{C}$. Consider the space $\mathcal{A}^{-\infty}(B)$ consisting of all holomorphic functions in B with polynomial growth near ∂B, so that

$$\mathcal{A}^{-\infty}(B) = \bigcup_{k=0}^{\infty} \{f \in H(B) : \|f\|_{(k)} = \sup_{z\in B} |f(z)|\, \mathrm{dist}(z, \partial B)^k < \infty\},$$

and the space $\mathcal{A}^{\infty}(\mathbb{C}_{\infty} \setminus B)$ consisting of all C^{∞}-functions defined on $\mathbb{C}_{\infty} \setminus B$ vanishing at ∞ and holomorphic in the interior of $\mathbb{C}_{\infty} \setminus B$. We are going to mention the recent result of the paper [2], where it was show that the Cauchy transformation of functionals establish a mutual duality between these introduced spaces in the case when B is a Carathéodory domain. Let us recall, that the Cauchy transformation of functionals is the mapping

$$L \mapsto L\left(\frac{1}{a - \jmath}\right),$$

where $a \in B$ is some point and $\jmath$ stands for the identity function $\jmath(z) = z$.

Let us denote by S^* the dual space of a locally convex topological space S endowed with the strong topology. The aforementioned result of [2] is as follows (see Theorems 4.3 and 4.5 in this paper).

Theorem 5.4. *The following statements hold.*

(1) *Let G be a Carathéodory domain. Then, the Cauchy transformation of functionals is an isomorphism from $\mathcal{A}^{-\infty}(G)^*$ onto $\mathcal{A}^{\infty}(\mathbb{C}_{\infty} \setminus G)$.*

(2) *Let B be a bounded domain in $\mathbb{C}$ with rectifiable boundary possessing the property $B = \operatorname{Int}(\overline{B})$, then the Cauchy transformation of measures is an isomorphism from $\mathcal{A}^{\infty}(\mathbb{C}_{\infty} \setminus B)^*$ onto $\mathcal{A}^{-\infty}(B)$.*

Let us give an outline of the proof of Theorem 5.4, part (1). In order to prove the theorem it is enough (in view of [2, Proposition 4.1]) to verify that the system of Cauchy kernels $\{\frac{1}{z-a} : a \in \mathbb{C}_{\infty} \setminus G\}$ is complete in $\mathcal{A}^{-\infty}(G)$. In order to prove this completeness property we need to prove first that the set of all polynomials is dense in $\mathcal{A}^{-\infty}$. This fact is the consequence of Hedberg's theorem (see Theorem 4.34 above) applied for the weights $\mathbf{w} = \operatorname{dist}(\cdot, \partial\Omega)^k$, $k \in \mathbb{N}_0$. The final step is to approximate each polynomial by respective Cauchy kernels in the topology of the space $\mathcal{A}^{-\infty}(G)$. ■

As it was mentioned above (see Propositions 1.5 and 1.6) any Carathéodory domain G is simply connected and possesses the property $G = \operatorname{Int}(\overline{G})$. Moreover, the latter condition is equivalent to the Carathéodory one whenever G is a bounded simply connected domain whose closure does not separate the plane.

As a corollary of Theorem 5.4 in [2] (see Corollary 4.6 of the paper cited) it was stated the following result: If G is a Carathéodory domain with rectifiable boundary, then the Cauchy transformation of measures establishes a mutual duality between the spaces $\mathcal{A}^{-\infty}(G)$ and $\mathcal{A}^{\infty}(\mathbb{C}_{\infty} \setminus G)$.

In this connection it is worth to recall Corollary 2.13 which says that any Carathéodory domain with rectifiable boundary is a Jordan domain. Thus, the mutual duality between the spaces $\mathcal{A}^{-\infty}(G)$ and $\mathcal{A}^{\infty}(\mathbb{C}_{\infty} \setminus G)$ is actually established only for the class of Jordan domains.

The same remark holds for the result of [2, Theorem 5.7] which we do not state explicitly because it goes too far from our main line of considerations.

Analytic balayage of measures supported in Carathéodory domains

Let us briefly discuss one topic concerning the structure of measures that are orthogonal to rational functions, in which the concept of a Carathéodory domain plays a certain role.

Let G be a Jordan domain with rectifiable boundary, and let μ be a measure with $\mathrm{Supp}(\mu) \subset G$. Then, by [28, Lemma 4.1], the measure $\mu + \widehat{\mu}\, d\zeta|_{\partial G}$ is orthogonal to $\mathcal{P}$ (as before, the symbol $\widehat{\mu}$ denotes the Cauchy transform of μ). In view of the term $\widehat{\mu}\, d\zeta|_{\partial G}$ it is not clear how to extend this observation to a wider class of domains. The following result was proved in [26, Proposition 3].

Proposition¶ 5.5. *Let G be a Carathéodory domain in $\mathbb{C}$, let f be a conformal map from $\mathbb{D}$ onto G, and ω be the corresponding complex harmonic measure on ∂G.*

(1) *Let μ be a measure with* $\mathrm{Supp}(\mu) \subset G$. *Then, the measure*

$$\mu^* = \mu + (\widehat{\eta} \circ f^{-1})\, \omega, \quad \text{where } \eta = f^{-1}(\mu),$$

is orthogonal to $A(\overline{G})$.

(2) *Let $K \subset G$ be a compact set, and σ be a measure on $K \cup \partial G$ with $\sigma \perp R(\overline{G})$. Then, there exists a function $h \in H^1$ such that $\sigma = (\sigma|_K)^* + (h \circ f^{-1})\, \omega$.*

Proof. We start with the proof of the first assertion. Put $E := \mathrm{Supp}(\eta)$. Since $\widehat{\eta}$ is holomorphic outside E, then $\widehat{\eta}\, d\zeta$ is a well-defined measure on $\mathbb{T}$ and $\nu = f(\widehat{\eta}\, d\zeta)$ is a measure on ∂G. Take a function $g \in A(\overline{G})$, so that $g \circ f \in H^\infty$. Using Fubini and Cauchy theorems and the definition of $\widehat{\eta}$ we have

$$\int g\, d\mu^* = \int g\, d\mu + \int g\, d\nu = \int g\, d\mu + \int_{\mathbb{T}} g(f(\zeta))\, \widehat{\eta}(\zeta)\, d\zeta$$
$$= \int g d\mu + \int_E \left[\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g(f(\zeta)) d\zeta}{w - \zeta}\right] d\eta(w) = \int g d\mu - \int_E g(f(w)) d\eta(w) = 0.$$

In order to prove the second assertion we need to observe that $\sigma - (\sigma|_K)^*$ is a measure on ∂G orthogonal to $R(\overline{G})$. It remains to apply (3.21). ∎

The representation of orthogonal measures obtained in this proposition has an interesting connection with the notion of an analytic balayage of measures, which was introduced by D. Khavinson [74], and which turned out to be a useful tool in approximations by analytic functions.

Definition 5.6. Let X be a compact set in $\mathbb{C}$, and let μ be a measure such that $\mathrm{Supp}(\mu) \subset X^\circ$. The measure ν on ∂X is called an *analytic balayage* of μ if $\mu - \nu \perp R(X)$, and for any measure $\widetilde{\nu}$ on ∂X such that $\mu - \widetilde{\nu} \perp R(X)$, the inequality $\|\widetilde{\nu}\| \geqslant \|\nu\|$ holds.

In all cases considered below, the analytic balayage of a given measure is uniquely determined. Having this remark in mind, we will denote the analytic balayage of μ by $\alpha(\mu) = \alpha(\mu, \partial X)$.

The presented definition of an analytic balayage of measures was given in [74, Definition 2] for finitely connected compact sets with piecewise analytic boundaries, but it also makes sense for general compact sets. For measures μ supported on X (but not only on X°), the analytic balayage was defined in another way by means of a special implicit construction, namely, a weak-star limit of analytic balayages of the initial measure to (piecewise analytic) boundaries of certain finitely connected compact sets approaching X (see [74, Definition 3]).

Let us see what an analytic balayage looks like in a simple case. Let G be a Jordan domain with piecewise analytic boundary Γ, and let μ be a measure such that $\operatorname{Supp}(\mu) \subset G$. As was shown in [74, Proposition 2]

$$\alpha(\mu) = g^* \, dz|_\Gamma - \widehat{\mu} \, dz|_\Gamma, \tag{5.2}$$

where $g^* \in R(\overline{G})$ is such that

$$\|\widehat{\mu} - g^*\|_{L^1(\Gamma)} = \inf \|\widehat{\mu} - g\|_{L^1(\Gamma)},$$

the infimum being taken over all functions $g \in R(\overline{G})$, and the Lebesgue space $L_1(\Gamma)$ is considered with respect to the measure $|dz|$ on Γ. The fact that the analytic balayage of μ is uniquely determined in this case is the consequence of [74, Proposition 3]. The formula (5.2) highlights the role of the term $\widehat{\mu} \, dz|_\Gamma$ which have appeared also in the part (1) of Proposition 5.5.

Since the explicit expression for analytic balayage is known only for finitely connected compact sets with piecewise analytic boundaries, it would be interesting to find such formulae for a wider class of compact sets. The class of Carathéodory compact sets fits this problem most naturally. This is mainly due to the structural properties of orthogonal measures stated in Proposition 5.5 which hold for the class of Carathéodory domains but not for any other known wider class of domains in $\mathbb{C}$.

The next result which was obtained in [52], see also [1], gives the desired expression for analytic balayage in the case of Carathéodory domains. We recall, that $H^1 = H^1(\mathbb{D})$ and the space $L^1 = L^1(\mathbb{T})$ is considered with respect to the measure $m_{\mathbb{T}}$.

Theorem¶ 5.7. *Let G be a Carathéodory domain and μ be a measure with* $\operatorname{Supp}(\mu) \subset G$. *Then, $\alpha(\mu, \partial G)$ is concentrated on $\partial_a G$ and has the form*

$$\alpha(\mu, \partial G) = (h^* \circ f^{-1}) \, \omega - (\widehat{\eta} \circ f^{-1}) \, \omega,$$

where f is a conformal map from $\mathbb{D}$ onto G, the measure η is defined as $\eta = f^{-1}(\mu)$, and the function $h^* \in H^1$ is the solution of the extremal problem

$$\|\widehat{\eta} - h^*\|_{L^1} = \inf_{h \in H^1} \|\widehat{\eta} - h\|_{L^1}. \tag{5.3}$$

It follows from this theorem that the analytic balayage of μ in the case under consideration is uniquely determined (notice that the extremal problem (5.3) has a unique solution, see [58, Chapter iv, Section 1.2]).

Sketch of the proof of Theorem 5.7. By part (1) of Proposition 5.5 one has

$$\mu^* = \mu + (\widehat{\eta} \circ f^{-1})\, \omega \perp R(\overline{G}).$$

Let now $\widetilde{\nu}$ be an arbitrary measure on ∂G such that $\mu - \widetilde{\nu} \perp R(\overline{G})$. Then,

$$\widetilde{\nu} + (\widehat{\eta} \circ f^{-1})\, \omega = \widetilde{\nu} + (\mu^* - \mu) = \mu^* - (\mu - \widetilde{\nu}) \perp R(\overline{G}).$$

Since $\widetilde{\nu} + (\widehat{\eta} \circ f^{-1})\, \omega$ is a measure on ∂G, there exists some function $h \in H^1$ such that

$$\widetilde{\nu} + (\widehat{\eta} \circ f^{-1})\, \omega = (h \circ f^{-1})\, \omega,$$

and hence

$$\widetilde{\nu} = (h \circ f^{-1})\, \omega - (\widehat{\eta} \circ f^{-1})\, \omega.$$

It remains to observe that the measure $\nu = \alpha(\mu, \partial G)$ is the measure among $\widetilde{\nu}$ that has the minimum norm. ■

The formula for analytic balayage given in Theorem 5.7, has a simpler form in the case when the solution h^* of the extremal problem (5.3) is zero. Let us describe the measures for which it is the case. The following result was proved in [1].

Proposition 5.8. *Let G be a Carathéodory domain in $\mathbb{C}$, and let f, μ and η be as in Theorem 5.7. Then, $\alpha(\mu, \partial G) = -(\widehat{\eta} \circ f^{-1})\, \omega$ if and only if μ is a finite sum of point-mass measures, one of which is supported at the point $f(0)$.*

In order to verify this result we need to use the concept of badly approximable functions in L^1. The function $\varphi \in L^1$ is called *badly approximable*, if only the function $g^* \equiv 0$ solves the extremal problem

$$\|\varphi - g^*\|_{L^1} = \inf_{g \in H^1} \|\varphi - g\|_{L^1}.$$

It follows from [58, Theorem 1.2, Chapter iv] that the solution of this extremal problem is unique. The class of badly approximable functions admits the following description.

Proposition 5.9. *A function $\varphi \in L^1$ is badly approximable if and only if it has the form $\varphi = \overline{\Theta}\, \Phi$, where Θ is an inner function (i.e., $\Theta \in H^\infty$ and $|\Theta(\zeta)| = 1$ for a.a. $\zeta \in \mathbb{T}$), $\Theta(0) = 0$, and $\Phi \in L^1$ is such that $\Phi \geqslant 0$.*

This result may be found in [75, Theorem 1], where it was obtained in a slightly different terms, and the proof provided was lengthly and technically involved. A new readable proof of this fact was given in [1].

Outline of the proof of Proposition 5.8. Let K be a compact subset of $\mathbb{D}$, and let $\varphi \in H(\mathbb{C}_\infty \setminus K)$. It follows from Proposition 5.9 that φ is badly approximable if and only if $\varphi = c\overline{B}$ on $\mathbb{T}$, where c is a positive constant, and B is a finite Blaschke product with $B(0) = 0$.

This fact yields that the solution h^* of the extremal problem (5.3) is zero if and only if the function $\widehat{\eta}$ coincides on $\mathbb{T}$ with the conjugation of some finite Blaschke product vanishing at 0. It means that η (and hence μ) is a finite sum of point-mass measures, one of which is supported at the origin (at the point $f(0)$, respectively). ■

Harmonic reflection over boundaries of Carathéodory domains

Recently, interest has intensified in the problems on reflection of harmonic functions over boundaries of domains in the plane and in space and in the problems on preservation of the smoothness properties of functions under such reflection. It is known several different approaches to define the harmonic reflection. Many of them are based on constructions of point-to-point reflection related with different variations of the symmetry principle for harmonic functions. At the same time in [53, 100–102] the construction was studied that was based on usage of the Dirichlet problems for harmonic functions in a given domain and in its complement. This construction is closely connected with the notion of a Carathéodory domain. In the rest of this section let $k \in \mathbb{N}$ and $k > 1$.

Definition 5.10. A nonempty bounded domain $G \subset \mathbb{R}^k$ is called a simple Carathéodory domain if it possesses the following properties:

(1) the set $\Omega = \mathbb{R}^k \setminus \overline{G}$ is a domain;

(2) $\partial G = \partial \Omega$;

(3) if $k \geqslant 3$ then both domains G and Ω are regular with respect to the classical Dirichlet problem for harmonic functions.

In fact, a simple Carathéodory domain in $\mathbb{R}^2$ is a Carathéodory domain, whose closure does not separate the plane. Notice that the third property in Definition 5.10 is assumed only for $N \geqslant 3$, since any Carathéodory domain in $\mathbb{R}^2$ is simply connected (see Proposition 1.5), and hence it is regular with respect to the Dirichlet problem for harmonic functions (see, e.g., Section 3.2).

Recall that for $m \in (0, 1]$ and for a closed set $X \subset \mathbb{R}^k$ (containing at least two points) the Lipschitz–Hölder space of order m is defined as follows:

$$\mathrm{Lip}^m(X) = \left\{ h \in C(X) : \|h\|'_{X,m} := \sup \frac{|h(\boldsymbol{x}) - h(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^m} < +\infty \right\},$$

where sup is taken over all couples of points $\boldsymbol{x}, \boldsymbol{y} \in X$ with $\boldsymbol{x} \neq \boldsymbol{y}$. The norm of a function $h \in \mathrm{Lip}^m(X)$ is defined as follows: $\|h\|_{X,m} := \max\{\|h\|'_{X,m}, \|h\|_X\}$.

Furthermore, for $m \in (0,1)$ we put

$$C^m(X) = \left\{ h \in \operatorname{Lip}^m(X) : \lim_{\delta\to 0} \sup_{0<|\boldsymbol{x}-\boldsymbol{y}|<\delta} \frac{|h(\boldsymbol{x}) - h(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^m} = 0 \right\}.$$

Notice that using the Whitney extension theorem and a regularization operator, it can be readily verified that for compact sets $X \subset \mathbb{R}^k$ the space $C^m(X)$ for $m \in (0,1)$ coincides with the closure in $\operatorname{Lip}^m(X)$ of the subspace $C^\infty(\mathbb{R}^k)|_X$.

Denote by $C_{\mathbb{R}}(X)$ the space of all real-valued continuous functions on a given closed set $X \subset \mathbb{R}^k$. Put

$$C_H(X) := C_{\mathbb{R}}(X) \cap \operatorname{Har}(\operatorname{Int}(X))$$

if X contains no punctured neighborhood of ∞, or, otherwise,

$$C_H(X) := \{h \in C_{\mathbb{R}}(X) \cap \operatorname{Har}(\operatorname{Int}(X)) : h(\boldsymbol{x}) = O_{|\boldsymbol{x}|\to\infty}(|\boldsymbol{x}|^{2-k})\}.$$

Take a simple Carathéodory domain $G \subset \mathbb{R}^k$, and let $\Omega = \mathbb{R}^k \setminus \overline{G}$, so that Ω is a domain and $\partial\Omega = \partial G$.

Let us define two operators, related with the Dirichlet problem for harmonic functions in G and in Ω. The first one is the *Poisson operator* $\boldsymbol{P}_G$ which maps a given function $\varphi \in C_{\mathbb{R}}(\partial G)$ to the function $f \in C_H(\overline{G})$ such that $f|_{\partial G} = \varphi$. The Poission operator $\boldsymbol{P}_\Omega$ is defined by the same way. The second one is the *harmonic reflection operator* $\boldsymbol{R}_G$ that acting from the space $C_H(\overline{G})$ to the space $C_H(\overline{\Omega})$ and that maps a given function $f \in C_H(\overline{G})$ to the function $g \in C_H(\overline{\Omega})$ such that $g|_{\partial\Omega} = f|_{\partial G}$.

Let us consider the question what conditions on G are necessary and sufficient in order that the operators $\boldsymbol{P}_G$ or $\boldsymbol{R}_G$ preserve smoothness properties of functions, when smoothness is understood in the sense of Lip^m-spaces for $0 < m \leqslant 1$ and C^m-spaces for $0 < m < 1$. This question is interesting both in its own, and in connection with problems on C^m-extension and C^m-approximation for harmonic and subharmonic functions.

Let now m and m' be such that $0 < m' \leqslant m \leqslant 1$. One says that the operator $\boldsymbol{P}_G$ is (m,m')-continuous, if it is continuous as an operator from the space $\operatorname{Lip}^m(\partial G)$ to the space $\operatorname{Lip}^{m'}(\overline{G}) \cap \operatorname{Har}(G)$. Respectively, one says that the operator $\boldsymbol{R}_G$ is (m,m')-continuous, if it is continuous as an operator from $\operatorname{Lip}^m(\overline{G}) \cap \operatorname{Har}(G)$ to $\operatorname{Lip}^{m'}(\overline{\Omega}) \cap C_H(\overline{\Omega})$.

Similarly, for m and m' such that $0 < m' \leqslant m < 1$, the operator $\boldsymbol{P}_G$ is called $C(m,m')$-continuous, if it is continuous operator from $C^m(\partial G)$ to $C^{m'}(\overline{G}) \cap \operatorname{Har}(G)$, while the operator $\boldsymbol{R}_G$ is called $C(m,m')$-continuous if it is continuous operator from $C^m(\overline{G}) \cap \operatorname{Har}(G)$ to $C^{m'}(\overline{\Omega}) \cap C_H(\overline{\Omega})$.

Finally, one says that $\boldsymbol{P}_\Omega$ is (m,m')-continuous, if it is continuous operator from $\operatorname{Lip}^m(\partial\Omega)$ to $\operatorname{Lip}^{m'}(\overline{\Omega}) \cap C_H(\overline{\Omega})$. Respectively, $\boldsymbol{P}_\Omega$ is $C(m,m')$-continuous, if it is continuous from $C^m(\partial\Omega)$ to $C^{m'}(\overline{\Omega}) \cap C_H(\overline{\Omega})$.

The next proposition combines the results obtained in [100] and [53].

Theorem 5.11. *The following holds.*

(1) *For any Jordan Lyapunov–Dini domain G in $\mathbb{R}^k$ the operator $\boldsymbol{R}_D$ is $(1,1)$-continuous; but there exist Jordan domains G with C^1-smooth boundaries for which it is not the case.*

(2) *For every simple Carathéodory domain $D \subset \mathbb{R}^k$ both operators $\boldsymbol{P}_D$ and $\boldsymbol{P}_\Omega$ are not $(1,1)$-continuous.*

We are not going here to give a precise definitions of a Jordan Lyapunov–Dini domain, but we mention that it is a Jordan domain with C^1-smooth boundary, whose boundary satisfies additional Dini-type continuity condition on inner normal vector.

Theorem 5.11 shows that the problem on (m,m')-continuity for operators $\boldsymbol{P}_G$ and $\boldsymbol{R}_G$ are independent in the general case. At the same time, in many instances the problems on (m,m')- and $C(m,m')$-continuity of the operator $\boldsymbol{R}_G$ can be reduced to the corresponding problems for the operator $\boldsymbol{P}_\Omega$. Notice that the domain Ω is unbounded, and assume, without loss of generality, that the initial domain G contains the origin. Using the classical Kelvin transform we can further reduce the problems on (m,m')- and $C(m,m')$-continuity of the operator $\boldsymbol{P}_\Omega$ to the corresponding problems for the operator $\boldsymbol{P}_{B(\Omega)}$, where $B(\Omega) = \{\boldsymbol{x} \in \mathbb{R}^k : \boldsymbol{x}/|\boldsymbol{x}|^2 \in \Omega\}$. Let us recall that the Kelvin transform maps a given function $h(\boldsymbol{x})$ to the function $|\boldsymbol{x}|^{2-k} h(\boldsymbol{x}/|\boldsymbol{x}|^2)$; this mapping is an isomorphism of the spaces $Q^m(\overline{B(\Omega)}) \cap \mathrm{Har}(B(\Omega))$ and $Q^m(\overline{\Omega}) \cap C_H(\overline{\Omega})$, where $Q^m(\cdot)$ stands for both $\mathrm{Lip}^m(\cdot)$ and $C^m(\cdot)$.

Theorem 2 in [53] gives the following criterion for Lip^m-continuity of the Poisson operator.

Theorem 5.12. *Let G, with $\mathrm{diam}(G) \leqslant 1$, be a simple Carathéodory domain in $\mathbb{R}^k$, and let $0 < m' \leqslant m \leqslant 1$. The following conditions are equivalent:*

(a) *the operator $\boldsymbol{P}_G$ is (m,m')-continuous;*

(b) *there exists $A > 0$ such that for each point $\boldsymbol{b} \in \partial D$ and for $\varphi(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{b}|^m$ one has*
$$\boldsymbol{P}_G(\varphi) \in \mathrm{Lip}^{m'}(\overline{G}) \quad \text{and} \quad \|\boldsymbol{P}_G(\varphi)\|_{\overline{D},m'} \leqslant A;$$

(c) *there exists $A > 0$ such that for every point $\boldsymbol{a} \in G$ and for each point $\boldsymbol{a}' \in \partial G$ with the condition $\delta = |\boldsymbol{a} - \boldsymbol{a}'| = \mathrm{dist}(\boldsymbol{a}, \partial G)$ the following estimate is satisfied:*
$$\sum_{n=1}^{N} (n\delta)^m \omega(\boldsymbol{a}, E_n, G) \leqslant A\delta^{m'},$$
where we set
$$E_0 = \{\boldsymbol{x} \in \partial G : |\boldsymbol{x} - \boldsymbol{a}'| \leqslant \delta\},$$
$$E_n = \{\boldsymbol{x} \in \partial G : n\delta < |\boldsymbol{x} - \boldsymbol{a}'| \leqslant (n+1)\delta\}, \quad n \geqslant 1,$$
and, where N is the maximal integer such that $E_N \neq \emptyset$.

Using this theorem one can show that for every simple Carathéodory domain G in $\mathbb{R}^k$ there exists a number $m_G \in [0,1]$ possessing the following properties: the operator $\boldsymbol{P}_G$ is (m,m)-continuous for all $m \in (0, m_G)$, but it is not the case for all $m \in (m_G, 1]$. Moreover, the operator $\boldsymbol{P}_G$ is (m,m')-continuous for all (m,m') such that $0 < m' < m_G$ and $m' \leqslant m \leqslant 1$.

Theorem 1 in [102] says that a similar picture holds in the case of C^m-continuity of the operator $\boldsymbol{P}_G$.

Theorem 5.13. *Let G be a simple Carathéodory domain in $\mathbb{R}^k$, and let m_G be the number defined in the previous statement. The operator $\boldsymbol{P}_G$ is $C(m,m)$-continuous for all $m \in (0, m_G)$, but it is not the case for all $m \in (m_G, 1)$. Moreover, the operator $\boldsymbol{P}_G$ is $C(m,m')$-continuous for all (m,m') such that* $0 < m' < m_G$ *and* $m' \leqslant m < 1$.

It follows from [53, Corollaries 3, 8, and 9] that for any simple Carathéodory domain $G \subset \mathbb{R}^2$ we have $m_D \in [1/2, 1]$, while in the case that $k \geqslant 3$ the number m_D may take any value from the segment $[0,1]$ in the general case. Let us now clarify what the numbers m_G and m_Ω are equal to in the case when G and Ω satisfy certain special geometrical conditions.

Given $\nu \in (0,1)$ and $r \geqslant 0$ let us define (closed spherical) *truncated cone* (closed sector in the two-dimensional case) $K(\nu, r)$ in $\mathbb{R}^k$ as follows:

$$K(\nu, r) = \{\boldsymbol{x} \in \mathbb{R}^k : 0 < |\boldsymbol{x}| \leqslant r,\ \theta_{\boldsymbol{x}} \leqslant \nu\pi\} \cup \{0\},$$

where

$$\theta_{\boldsymbol{x}} = \arccos(x_1/|\boldsymbol{x}|)$$

stands for the angle between the vector $\boldsymbol{x} = (x_1, \dots, x_k)$ and the direction of the axes Ox_1.

One says that a simple Carathéodory domain $G \subset \mathbb{R}^k$ satisfies the *external truncated cone condition* with parameters (α, r), where $\alpha \in (0,1]$ and $r > 0$, if for every point $\boldsymbol{a} \in \partial G$ there exists a truncated cone $K_{\boldsymbol{a}}$ congruent to $K(\alpha/2, r)$ with the vertex $\boldsymbol{a}$, and such that $K_{\boldsymbol{a}} \cap G = \emptyset$. The *internal truncated cone condition* is defined by the same way.

For further considerations we need one auxiliary construction, see [82, Section 1] and [92, Section 2]. For every $k \geqslant 2$ and $\lambda > 0$ there exists a unique function $g_{k,\lambda} \in C^2([0,\pi))$ such that

$$g''_{k,\lambda}(t) + (k-2)\cot(t) g'_{k,\lambda}(t) + \lambda(\lambda + k - 2) g_{k,\lambda}(t) = 0, \quad t \in (0,\pi),$$

with $g_{k,\lambda}(0) = 1$ and $g'_{k,\lambda}(0) = 0$. Moreover, the function $g_{k,\lambda}$ has its first (with respect to the increasing order) positive zero $\theta_k(\lambda)$ in the interval $(0,\pi)$; the function $\theta_k(\cdot)\colon (0, +\infty) \to (0,\pi)$ is continuous and strictly decreasing; the corresponding inverse function $\lambda_k(\cdot)\colon (0,\pi) \to (0,+\infty)$ is also continuous and injective.

Both functions $\theta_k(\cdot)$ and $\lambda_k(\cdot)$ may be found in an explicit form for $k = 2, 4$. Thus, in the case $k = 2$ the corresponding equation for the function $g_{k,\lambda}$ has a very simple form $g''_{2,\lambda} + \lambda^2 g_{2,\lambda} = 0$, so that $g_{2,\lambda}(t) = \cos(\lambda t)$, $\theta_2(\lambda) = \pi/(2\lambda)$ and $\lambda_2(\theta) = \pi/(2\theta)$. For $k = 4$ it can be shown, that $\theta_4(\lambda) = \pi/(\lambda + 1)$ and $\lambda_4(\theta) = -1 + \pi/\theta$, respectively.

For $\alpha \in (0, 1]$ we can define the number $m_{k,\alpha} := \lambda_k(\pi - \alpha\pi/2)$ so that, in particular, $m_{2,\alpha} = 1/(2 - \alpha)$. It was shown in [53] that for a simple Carathéodory domain $G \subset \mathbb{R}^k$ satisfying the external truncated cone condition with parameters (α, r) for some $\alpha \in (0, 1)$ and $r > 0$, it holds $m_G \geqslant m_{k,\alpha}$.

In several cases when a given simple Carathéodory domain G satisfies certain additional conditions stated in terms of external (or internal) truncated cone conditions (with some α), we have that $m_G = m_{k,\alpha}$ (or, respectively, $m_\Omega = m_{k,\alpha}$). In the latter case the operator $\boldsymbol{R}_G$ is (m, m')-continuous for $0 < m' < m_\Omega$ with $m' \leqslant m \leqslant 1$, and it is not the case for all m, m' such that $m_\Omega < m' \leqslant m \leqslant 1$. Moreover, the operator $\boldsymbol{R}_G$ in this case is $C(m, m')$-continuous for all m and m' such that $0 < m' < m_\Omega$ and $m' \leqslant m < 1$, but it is not the case for all m and m' with $m_\Omega < m' \leqslant m < 1$. These results and related discussions may be found in [53, Section 3] and [102, Section 2].

Carathéodory domains and invariant subspace problem

We end this chapter and the whole survey by stating one result showing the application of Carathéodory domains to the invariant subspace problem. The respective result was recently obtained in [76]. It states as follows.

Theorem 5.14. *Let T be a bounded linear operator on a separable infinite-dimensional Hilbert space $\mathcal{H}$ with the spectrum $\sigma(T)$. Assume that*

(i) *T is such that $\|P(T)\| \leqslant \|P\|_{\sigma(T)}$ for every $P \in \mathcal{P}$, and*

(ii) *$\widehat{\sigma(T)}$ is the closure of a Carathéodory domain such that for every $\zeta \in \partial_a G$ there exists a rectifiable arc $\Upsilon \subset \partial G$ containing ζ.*

Then, T has a nontrivial invariant subspace $\mathcal{H}_0$ (so that $T\mathcal{H}_0 \subset \mathcal{H}_0$).

As the corollary of this theorem, in [76] the existence of nontrivial invariant subspace for a certain subclass of hyponormal operators was proved.

References

[1] E. Abakumov and K. Y. Fedorovskiy, Analytic balayage of measures, Carathéodory domains, and badly approximable functions in L^p. *C. R. Math. Acad. Sci. Paris* **356** (2018), no. 8, 870–874

[2] A. V. Abanin and L. H. Khoi, Cauchy transformation and mutual dualities between $A^{-\infty}(\Omega)$ and $A^{\infty}(\Omega^{\complement})$ for Carathéodory domains. *Bull. Belg. Math. Soc. Simon Stevin* **23** (2016), no. 1, 87–102

[3] F. G. Abdullaev and A. A. Dovgoshei, Szegö theorem, Carathéodory domains and boundedness of calculating functionals. *Math. Notes* **77** (2005), no. 1, 3–14

[4] J. T. Anderson, J. A. Cima, N. Levenberg, and T. J. Ransford, Projective hulls and characterizations of meromorphic functions. *Indiana Univ. Math. J.* **61** (2012), no. 6, 2111–2122

[5] N. U. Arakeljan, Uniform approximation on closed sets by entire functions. *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 1187–1206

[6] S. Axler, Harmonic functions from a complex analysis viewpoint. *Amer. Math. Monthly* **93** (1986), no. 4, 246–258

[7] S. Banach, *Théorie des opérations linéaires*. Chelsea, New York, 1955

[8] A. D. Baranov, J. J. Carmona, and K. Y. Fedorovskiy, Density of certain polynomial modules. *J. Approx. Theory* **206** (2016), 1–16

[9] A. D. Baranov and K. Y. Fedorovskiy, Boundary regularity of Nevanlinna domains and univalent functions in model subspaces. *Sb. Math.* **202** (2011), no. 12, 1723–1740

[10] A. D. Baranov and K. Y. Fedorovskiy, On L^1-estimates of derivatives of univalent rational functions. *J. Anal. Math.* **132** (2017), 63–80

[11] S. Y. Belov and K. Y. Fedorovskiy, Model spaces containing univalent functions. *Russ. Math. Surverys* **73** (2018), no. 1, 172–174

[12] Y. Belov, A. Borichev, and K. Y. Fedorovskiy, Nevanlinna domains with large boundaries. *J. Funct. Anal.* **277** (2019), no. 8, 2617–2643

[13] E. Bishop, Measures orthogonal to polynomials. *Proc. Natl. Acad. Sci. USA* **44** (1958), no. 3, 278–280

[14] E. Bishop, The structure of certain measures. *Duke Math. J.* **25** (1958), 283–289

[15] E. Bishop, Boundary measures of analytic differentials. *Duke Math. J.* **27** (1960), 331–340

[16] A. Boivin, P. M. Gauthier, and P. V. Paramonov, On uniform approximation by n-analytic functions on closed sets in $\mathbb{C}$. *Izv. Math.* **68** (2004), no. 3, 447–459

[17] P. S. Bourdon, Density of the polynomials in Bergman spaces. *Pacific J. Math.* **130** (1987), no. 2, 215–221

[18] S. Browder, *Introduction to function algebras*. W. A. Benjamin, New York-Amsterdam, 1969

[19] R. B. Burckel, *An introduction to classical complex analysis. Vol. 1*. Pure Appl. Math. 82, Academic Press, Harcourt Brace Jovanovich, New York-London, 1979

[20] C. Carathéodory, Untersuchungen über die konformen Abbildungen von festen und veränderlichen Gebieten. *Math. Ann.* **72** (1912), no. 1, 107–144

[21] C. Carathéodory, Über die Begrenzung einfach zusammenhängender Gebiete. *Math. Ann.* **73** (1913), no. 3, 323–370

[22] C. Carathéodory, Über die gegenseitige Beziehung der Ränder bei der konformen Abbildung des Inneren einer Jordanschen Kurve auf einen Kreis. *Math. Ann.* **73** (1913), no. 2, 305–320

[23] T. Carleman, Über die Approximation analytischer Funktionen durch lineare Aggregate von vorgegebenen Potenzen. *Arkiv. Mat. Astron. Fys.* **17** (1922), no. 9, 1–30

[24] L. Carleson, Mergelyan's theorem on uniform polynomial approximation. *Math. Scand.* **15** (1964), 167–175

[25] J. J. Carmona, Mergelyan's approximation theorem for rational modules. *J. Approx. Theory* **44** (1985), no. 2, 113–126

[26] J. J. Carmona and K. Y. Fedorovskiy, Conformal maps and uniform approximation by polyanalytic functions. In *Selected topics in complex analysis*, pp. 109–130, Oper. Theory Adv. Appl. 158, Birkhäuser, Basel, 2005

[27] J. J. Carmona and K. Y. Fedorovskiy, New conditions for uniform approximation by polyanalytic polynomials. *Proc. Stelklov Inst. Math.* **279** (2012), 215–229

[28] J. J. Carmona, K. Y. Fedorovskiy, and P. V. Paramonov, On uniform approximation by polyanalytic polynomials and the Dirichlet problem for bianalytic functions. *Sb. Math.* **193** (2002), no. 10, 1469–1492

[29] J. J. Carmona and C. Pommerenke, On the argument oscillation of conformal maps. *Michigan Math. J.* **46** (1999), no. 2, 297–312

[30] J. J. Carmona and C. Pommerenke, Decomposition of continua and prime ends. *Comput. Methods Funct. Theory* **3** (2003), no. 1-2, 253–272

[31] J. G. Caughran, Polynomial approximation and spectral properties of composition operators on H^2. *Indiana Univ. Math. J.* **21** (1971), 81–84

[32] E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets*. Cambridge Tracts in Math. Phys. 56, Cambridge University Press, Cambridge, 1966

[33] J. B. Conway, *Functions of one complex variable. I*. 2nd edn., Grad. Texts in Math. 11, Springer, New York-Berlin, 1978

[34] J. B. Conway, *The theory of subnormal operators*. Math. Surveys Monogr. 36, American Mathematical Society, Providence, RI, 1991

[35] P. Davis, *The Schwarz function and its applications*. Carus Math. Monogr. 17, Mathematical Association of America, Buffalo, NY, 1974

[36] J. Deny, Sur l'approximation des fonctions harmoniques. *Bull. Soc. Math. France* **73** (1945), 71–73

[37] A. A. Dovgoshei, Polynomial approximation of functions from Hardy classes in simply-connected domains in the plane. *Ukrainian Math. J.* **42** (1990), no. 9, 1126–1130

[38] A. A. Dovgoshei, The F. and M. Riesz' theorem and Carathéodory domains. *Anal. Math.* **21** (1995), no. 3, 165–175

[39] O. Dovgoshey, Certain characterizations of Carathéodory domains. *Comput. Methods Funct. Theory* **5** (2005), no. 2, 489–503

[40] P. Duren, B. W. Romberg, and A. Shields, Linear functionals on H^p spaces with $0 < p < 1$. *J. Reine Angew. Math.* **238** (1969), 32–60

[41] P. Duren and A. Schuster, *Bergman spaces*. Math. Surveys Monogr. 100, American Mathematical Society, Providence, RI, 2004

[42] P. L. Duren, *Theory of H^p spaces*. Pure Appl. Math. 38, Academic Press, New York-London, 1970

[43] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*. Stud. Adv. Math., CRC Press, Boca Raton, FL, 1992

[44] O. J. Farrell, On approximation to a mapping function by polynomials. *Amer. J. Math.* **54** (1932), no. 3, 571–578

[45] O. J. Farrell, On approximation to an analytic function by polynomials. *Bull. Amer. Math. Soc.* **40** (1934), no. 12, 908–914

[46] O. J. Farrell, On approximation by polynomials to a function analytic in a simply connected region. *Bull. Amer. Math. Soc.* **41** (1935), no. 10, 707–711

[47] O. J. Farrell, On the representation of bounded analytic functions by sequences of polynomials. *Amer. J. Math.* **60** (1938), no. 3, 573–576

[48] O. J. Farrell, On approximation measured by a surface integral. *SIAM J. Numer. Anal.* **3** (1966), 236–247

[49] K. Y. Fedorovskiy, Uniform n-analytic polynomial approximation of functions on rectifiable contours in $\mathbb{C}$. *Math. Notes* **59** (1996), no. 4, 435–439

[50] K. Y. Fedorovskiy, On some properties and examples of Nevanlinna domains. *Proc. Steklov Inst. Math.* **253** (2006), 186–194

[51] K. Y. Fedorovskiy, Carathéodory domains and Rudin's converse of the maximum modulus principle. *Sb. Math.* **206** (2015), no. 1, 161–174

[52] K. Y. Fedorovskiy, Carathéodory sets and analytic balayage of measures. *Sb. Math.* **209** (2018), no. 9, 1376–1389

[53] K. Y. Fedorovskiy and P. Paramonov, On Lip^m-reflection of harmonic functions over boundaries of simple Carathéodory domains. *Anal. Math. Phys.* **9** (2019), no. 3, 1031–1042

[54] S. D. Fisher, *Function theory on planar domains*. Pure Math. Appl. (New York), John Wiley, New York, 1983

[55] D. Gaier, *Lectures on complex approximation*. Birkhäuser, Boston, MA, 1987

[56] T. W. Gamelin, *Uniform algebras*. Chelsea, New York, 1984

[57] S. Gardiner, *Harmonic approximation*. London Math. Soc. Lecture Note Ser. 221, Cambridge University Press, Cambridge, 1995

[58] J. Garnett, *Bounded analytic functions*. Pure Appl. Math. 96, Academic Press, Harcourt Brace Jovanovich, New York-London, 1981

[59] J. B. Garnett and D. E. Marshall, *Harmonic measure*. New Math. Monogr. 2, Cambridge University Press, Cambridge, 2005

[60] I. Glicksberg, The abstract F. and M. Riesz theorem. *J. Functional Analysis* **1** (1967), 109–122

[61] G. M. Goluzin, *Geometric theory of functions of a complex variable*. Transl. Math. Monogr. 26, American Mathematical Society, Providence, RI, 1969

[62] F. Hartogs and A. Rosenthal, Über Folgen analytischer Funktionen. *Math. Ann.* **104** (1931), no. 1, 606–610

[63] V. P. Havin, Approximation in the mean by analytic functions. *Dokl. Akad. Nauk SSSR* **178** (1968), 1025–1028

[64] L. I. Hedberg, Weighted mean approximation in Carathéodory regions. *Math. Scand.* **23** (1968), 113–122

[65] L. I. Hedberg and T. H. Wolff, Thin sets in nonlinear potential theory. *Ann. Inst. Fourier (Grenoble)* **33** (1983), no. 4, 161–187

[66] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman spaces*. Grad. Texts in Math. 199, Springer, New York, 2000

[67] L. L. Helms, *Introduction to potential theory*. Pure Appl. Math. XXII, Wiley-Interscience A division of John Wiley, New York-London-Sydney, 1969

[68] J. G. Hocking and G. S. Young, *Topology*. Addison-Wesley, Reading, Mass.-London, 1961

[69] K. Hoffman, *Banach spaces of analytic functions*. Dover, New York, 1988

[70] G. P. Kapoor and A. Nautiyal, Approximation of entire functions over Carathéodory domains. *Bull. Austral. Math. Soc.* **25** (1982), no. 2, 221–229

[71] M. V. Keldych, Sur l'approximation en moyenne quadratique des fonctions analytiques. *Rec. Math. [Mat. Sbornik] N.S.* **5** (1939), no. 47, 391–401

[72] M. V. Keldych, Sur la represénotations par des séries de polynomes des functions d'una variable complex dans de domains fermés. *Rec. Math. [Mat. Sbornik] N.S.* **8** (1940), no. 50, 137–148

[73] M. V. Keldych, Sur l'approximation en moyenne par polynomes des fonctions d'une variable complexe. *Rec. Math. [Mat. Sbornik] N.S.* **16** (1945), no. 58, 1–20

[74] D. Khavinson, F. and M. Riesz theorem, analytic balayage, and problems in rational approximation. *Constr. Approx.* **4** (1988), no. 4, 341–356

[75] S. Ya. Khavinson, On some extremal problems of the theory of analytic functions. *Moskov. Gos. Univ. Učenye Zapiski Matematika* **148** (1951), no. 4, 133–143. English translation: Amer. Math. Soc. Transl. (2) **32** (1963), 139–154

[76] J. Kim and W. Y. Lee, Invariant subspaces for operators whose spectra are Carathéodory regions. *J. Math. Anal. Appl.* **371** (2010), no. 1, 184–189

[77] P. Koosis, *Introduction to H_p spaces*. 2nd edn., Cambridge Tracts in Math. 115, Cambridge University Press, Cambridge, 1998

[78] K. Kuratowski, *Topology. Vol. II*. Academic Press, New York-London, 1968

[79] M. A. Lavrentiev, *Sur les fonctions d'une variable complex représentables par des séries de polynômes*. Hermann, Paris, 1936

[80] H. Lebesgue, Sur le problème de Dirichlet. *Rend. Circ. Mat. di Palermo* **29** (1907), no. 1, 371–402

[81] D. H. Luecking and L. A. Rubel, *Complex analysis*. Universitext, Springer, New York, 1984

[82] F.-Y. Maeda and N. Suzuki, The integrability of superharmonic functions on Lipschitz domains. *Bull. London Math. Soc.* **21** (1989), no. 3, 270–278

[83] A. I. Markouchevitch, Sur la représentation conforme des domaines à frontières variables. *Rec. Math. [Mat. Sbornik] N.S.* **1(43)** (1936), no. 6, 863–886

[84] A. I. Markushevich, *Conformal mapping of regions with variable boundary and application to the approximation of analytic functions by polynomials*. Ph.D. thesis, Moscow, 1934

[85] A. I. Markushevich, Theory of analytic functions. Vols. 1 and 2. *Prentice-Hall, Englewood Cliffs, NJ* (1965)

[86] M. Y. Mazalov, Example of a nonrectifiable Nevanlinna contour. *St. Petersburg Math. J.* **27** (2016), no. 4, 625–630

[87] M. Y. Mazalov, On Nevanlinna domains with fractal boundaries. *St. Petersburg Math. J.* **29** (2018), no. 5, 777–791

[88] M. Y. Mazalov, P. V. Paramonov, and K. Y. Fedorovskiy, Conditions for the C^m-approximability of functions by solutions of elliptic equations. *Russ. Math. Surveys* **67** (2012), no. 6, 1023–1068

[89] S. Mazurkiewicz, Uber erreichbare Punkte. *Fund. Math.* **26** (1936), 150–155

[90] S. N. Mergelyan, Uniform approximations of functions of a complex variable. *Uspehi Matem. Nauk (N.S.)* **7** (1952), no. 2, 31–122. English translation: Amer. Math. Soc. Transl. (2) **101** (1954)

[91] S. N. Mergelyan, On completeness of systems of analytic functions. *Uspehi Matem. Nauk (N.S.)* **8** (1953), no. 4, 3–63. English translation: Amer. Math. Soc. Transl. (2) **19** (1962), 109–166

[92] K. Miller, Extremal barriers on cones with Phragmén–Lindelöf theorems and other applications. *Ann. Mat. Pura Appl. (4)* **90** (1971), 297–329

[93] Z. Nehari, *Conformal mapping*. Dover, New York, 1975

[94] M. H. Newman, *Elements of the topology of plane sets of points*. Cambridge University Press, New York, 1964

[95] A. G. O'Farrell, A generalised Walsh–Lebesgue theorem. *Proc. Roy. Soc. Edinburgh Sect. A* **73** (1975), 231–234

[96] A. G. O'Farrell, Theorems of Walsh–Lebesgue type. In *Aspects of contemporary complex analysis (Proc. NATO Adv. Study Inst., Univ. Durham, Durham, 1979)*, edited by D. A. Brannan and J. G. Clunie, pp. 461–467, Academic Press, London-New York, 1980

[97] A. G. O'Farrell, Five generalisations of the Weierstrass approximation theorem. *Proc. Roy. Irish Acad. Sect. A* **81** (1981), no. 1, 65–69

[98] A. G. O'Farrell and F. Pérez-González, Pointwise bounded approximation by polynomials. *Math. Proc. Cambridge Philos. Soc.* **112** (1992), no. 1, 147–155

[99] P. V. Paramonov, C^m-approximations by harmonic polynomials on compact sets in $\mathbb{R}^n$. *Russian Acad. Sci. Sb. Math.* **78** (1994), no. 1, 231–251

[100] P. V. Paramonov, C^1-extension and C^1-reflection of subharmonic functions from Lyapunov–Dini domains into $\mathbb{R}^N$. *Sb. Math.* **199** (2008), no. 12, 1809–1846

[101] P. V. Paramonov, On Lip^m and C^m-reflection of harmonic functions with respect to boundaries of Carathéodory domains in $\mathbb{R}^2$. *Vestn. Mosk. Gos. Tekh. Univ. im. N.E. Baumana, Estestv. Nauki [Herald of the Bauman Moscow State Tech. Univ., Nat. Sci.]* (2018), no. 4, 36–45

[102] P. V. Paramonov and K. Y. Fedorovskiy, On C^m-reflection of harmonic functions and C^m-approximation by harmonic polynomials. *Sb. Math.* **211** (2020), no. 8, 1159–1170

[103] C. Pommerenke, *Univalent functions*. Studia Math./Mathematische Lehrbücher XXV, Vandenhoeck & Ruprecht, Göttingen, 1973

[104] C. Pommerenke, *Boundary behaviour of conformal maps*. Grundlehren Math. Wiss. 299, Springer, Berlin, 1992

[105] I. I. Privalov, *Boundary properties of analytic functions*. 2nd edn., GITTL, Moscow, 1950. German translation: *Randeigenschaften analytischer Funktionen*, Hochschulbücher für Mathematik, Bd. 25. VEB Deutscher Verlag der Wissenschaften, Berlin, 1956

[106] J. Z. Qiu, Density of polynomials. *Houston J. Math.* **21** (1995), no. 1, 109–118

[107] T. Ransford, *Potential theory in the complex plane*. London Math. Soc. Stud. Texts 28, Cambridge University Press, Cambridge, 1995

[108] R. C. Roan, Weak* generators of H^∞ and l^1. *Pacific J. Math.* **71** (1977), no. 2, 537–544

[109] R. C. Roan, Composition operators on H^p with dense range. *Indiana Univ. Math. J.* **27** (1978), no. 1, 159–162

[110] A. Roth, Approximationseigenschaften und Strahlengrenzwerte meromorpher und ganzer Funktionen. *Comment. Math. Helv.* **11** (1938), no. 1, 77–125

[111] L. A. Rubel and A. L. Shields, Bounded approximation by polynomials. *Bull. Amer. Math. Soc.* **69** (1963), 591–593

[112] L. A. Rubel and A. L. Shields, Bounded approximation by polynomials. *Acta Math.* **112** (1964), 145–162

[113] W. Rudin, Analyticity, and the maximum modulus principle. *Duke Math. J.* **20** (1953), no. 3, 449–457

[114] W. Rudin, *Functional analysis*. Tata McGraw-Hill, New Delhi, 1977

[115] W. Rudin, *Real and complex analysis*. 3rd edn., McGraw-Hill, New York, 1987

[116] C. Runge, Zur Theorie der Eindeutigen Analytischen Functionen. *Acta Math.* **6** (1885), no. 1, 229–244

[117] A. L. Šaginyan, On approximation in the mean by harmonic polynomials. *Akad. Nauk Armyan. SSR Dokl.* **19** (1954), 97–103

[118] S. Saks and A. Zygmund, *Fonctions analytiques*. Masson et Cie, Éditeurs, Paris, 1970

[119] D. Sarason, Invariant subspaces and unstarred operator algebras. *Pacific J. Math.* **17** (1966), 511–517

[120] D. Sarason, Weak-star generators of H^{∞}. *Pacific J. Math.* **17** (1966), 519–528

[121] D. Sarason, On the order of a simply connected domain. *Michigan Math. J.* **15** (1968), 129–133

[122] A. Shields, Carathéodory and conformal mapping. *Math. Intelligencer* **10** (1988), no. 1, 18–22

[123] S. O. Sinanjan, Approximation by analytical functions and polynomials in the mean with respect to the area. *Mat. Sb. (N.S.)* **69** (1966), 546–578. English translation: Amer. Math. Soc. Transl. (2) **74** (1968), 91–124

[124] V. I. Smirnov and N. A. Lebedev, *Functions of a complex variable: Constructive theory*. The MIT Press, Cambridge, MA, 1968

[125] E. L. Stout, *The theory of uniform algebras*. Bogden & Quigley, Tarrytown-on-Hudson, NY, 1971

[126] A. Stray, Pointwise bounded approximation by functions satisfying a side condition. *Pacific J. Math.* **51** (1974), 301–305

[127] P. K. Suetin, Polynomials orthogonal over a region and Bieberbach polynomials. *Proc. Steklov Inst. Math.* **100** (1974), 1–91

[128] A. G. Vitushkin, The analytic capacity of sets in problems of approximation theory. *Russian Math. Surveys* **22** (1967), no. 6, 139–200

[129] J. L. Walsh, Über die Entwicklung einer analytischen Funktion nach Polynomen. *Math. Ann.* **96** (1927), no. 1, 430–436

[130] J. L. Walsh, Über die Entwicklung einer Funktion einer komplexen Veränderlichen nach Polynomen. *Math. Ann.* **96** (1927), no. 1, 437–450

[131] J. L. Walsh, On approximation to an arbitrary function of a complex variable by polynomials. *Trans. Amer. Math. Soc.* **30** (1928), no. 3, 472–482

[132] J. L. Walsh, Über die Entwicklung einer harmonischen Funktion nach harmonischen Polynomen. *J. Reine Angew. Math.* **159** (1928), 197–209

[133] J. L. Walsh, The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions. *Bull. Amer. Math. Soc.* **35** (1929), no. 4, 499–544

[134] J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*. Amer. Math. Soc. Colloq. Publ. XX, American Mathematical Society, Providence, RI, 1969

[135] J. Wermer, On algebras of continuous functions. *Proc. Amer. Math. Soc.* **4** (1953), 866–869

[136] G. T. Whyburn, *Analytic topology*. Amer. Math. Soc. Colloq. Publ. 28, American Mathematical Society, New York, 1971

[137] T. Winiarski, Approximation and interpolation of entire functions. *Ann. Polon. Math.* **23** (1970), 259–273

MEMOIRS OF THE EUROPEAN MATHEMATICAL SOCIETY

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Carathéodory Sets in the Plane

This work is devoted to the class of sets in the complex plane which nowadays are known as Carathéodory sets, more precisely speaking, as Carathéodory domains and Carathéodory compact sets. These sets naturally arose many times in various research areas in Real, Complex and Functional Analysis and in the Theory of Partial Differential Equations. For instance, the concept of a Carathéodory set plays a significant role in such topical themes as approximation in the complex plane, the theory of conformal mappings, boundary value problems for elliptic partial differential equations, etc. The first appearance of Carathéodory domains in the mathematical literature (of course, without the special name at that moment) was at the beginning of the 20th century, when C. Carathéodory published his famous series of papers about boundary behavior of conformal mappings. The next breakthrough result which was obtained with the essential help of this concept is the Walsh–Lebesgue criterion for uniform approximation of functions by harmonic polynomials on plane compacta (1929). Up to now the studies of Carathéodory domains and Carathéodory compact sets remains a topical field of contemporary analysis and a number of important results were recently obtained in this direction. Among them one ought to mention the results about polyanalytic polynomial approximation, where the class of Carathéodory compact sets was one of the crucial tools, and the results about boundary behavior of conformal mappings from the unit disk onto Carathéodory domains. Our aim in the present paper is to give a survey on known results related with Carathéodory sets and to present several new results concerning the matter. Starting with the classical works of Carathéodory, Farrell, Walsh, and passing through the history of Complex Analysis of the 20th century, we come to recently obtained results, and to our contribution to the theory.

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