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★**The formation of shocks in 3-dimensional fluids. (English summary)**

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In this monograph the author studies the maximally defined, smooth solutions to the relativistic Euler equations of motion for a perfect fluid in Minkowski spacetime M^{3+1}.

The discussion begins with a review of earlier works, including pioneering work on shock formation by Riemann on isentropic fluid flows with plane symmetry and, more generally, on nonlinear hyperbolic systems of two conservation laws in one space variable: smooth solutions develop singularities in finite time. The formulation of the physically correct jump relations was later found by Rankine and Hugoniot. Further fundamental work was done by Friedrichs and Lax, and the general problem of shock formation for hyperbolic systems of conservation laws in one space dimension was solved by Lax in 1964 (for genuinely nonlinear systems) and John in 1974 (for general systems).

The strategy in the above works was to deduce an ordinary differential inequality for a quantity constructed from the first-order derivatives of the solution, and to show that this quantity must blow-up in finite time, at least under certain assumptions on the structure of the hyperbolic system.

More recently, for the Euler equations of perfect compressible fluids, an entirely different approach was introduced by Sideris in 1985 which, instead, used integral quantities associated with the solution. The main drawback of this method is that it tells us nothing about the nature of the breakdown. Moreover, it requires the pressure of the fluid to be strictly convex in terms of the density. In another direction, in 1983, Majda began an ambitious program on the stability of shock fronts for nonlinear hyperbolic systems in several space dimensions; this was continued and expanded by Gues, Metivier, and followers.

In the present work, the author considers the relativistic Euler equations for a perfect fluid with an arbitrary equation of state. Initial data are imposed on a given spacelike hyperplane and are constant outside a compact set. Attention is restricted to the evolution of the solution within a region limited by two concentric spheres. Given a smooth solution to the Euler equations, the main objective of the author is to investigate the geometry of the boundary of its domain of definition, that is, the locus where shock waves may form.

At the end of this book, under certain smallness assumptions on the size of the initial data, a remarkable and complete picture of the formation of shock waves in three dimensions is obtained. In addition, sharp sufficient conditions on the initial data for the formation of shocks in the evolution are established, and sharp lower and upper bounds for the time of existence of a smooth solution are derived.

The main strategy proposed in this book is as follows. Given an arbitrary initial data that is constant outside a sphere and under suitable smallness conditions on this initial data, the author controls first the solution for a time interval of order $1/\eta$, where η is a reference sound speed. He shows that at the end T of this time interval the flow is irrotational and isentropic within an annular region limited by two concentric spheres. Then, he proceeds by studying the maximal development of the restriction of the data at the time T to the exterior of the inner sphere. Next, he relies on the property that for irrotational and isentropic flows, there exists a function ϕ which suffices to characterize the fluid and satisfies a wave equation which is the main equation studied here.

This book represents an amazing "tour de force" by the author. The analysis relies heavily on differential geometric concepts and methods; one key unknown of the Euler equations is the one-form velocity field, denoted here by β, suitably multiplied by the relativistic enthalpy of the fluid; in the irrotational case, β coincides with the exterior differential of the potential ϕ mentioned earlier. The estimates derived in this work are based on the natural action principle associated with the fluid equations and on the construction of vector fields adapted to the geometry of the solution. High-order energy-type estimates are derived which yield a sharp control of a geometric foliation of the solution.

The book is structured as follows. The first four chapters provide notation and set up the general framework. Chapters 5 through 13 restrict attention to irrotational and isentropic fluids and culminate with the shock formation result in Theorem 13.1. Chapter 5 contains the fundamental energy estimate. Chapter 6 contains a discussion of the properties of several vector fields of interest. Chapter 7 deals with source-terms arising in higher-order estimates and presents a recursion formula for these terms. Chapters 8 and 9 contain the crucial technical part of the present work, and establish the higher-order estimates. In particular, in Chapter 9 the evolution of the second fundamental form of the leaves of the foliation is discussed and the key structure of the problem (elliptic equations on two-dimensional submanifolds of the foliation, ordinary differential equations along the generators of the foliation) is uncovered.

Theorem 14.1 provides sharp sufficient conditions on the initial data for shock waves to form during the evolution. The rest of Chapter 14 is devoted to the general problem of shock formation for flow that needs be irrotational and isentropic. The connection is made here with Theorem 14.1, since the author finds conditions at the beginning of the time interval of definition of the solution which guarantee that the solution at the end of the time interval will satisfy the assumptions of Theorem 14.1. This analysis leads to the main result stated in Theorem 14.2.

Next, Chapter 15 is devoted to the investigation of the geometry of the boundary of the domain of definition of the solution. Another main result of this book is that the boundary of the domain of definition of a solution consists of a singular part $S = \partial_{-}UH \cup H$ and a regular part C. Here, $\partial_{-}H$ denotes the past boundary of H. Each component of C is an incoming characteristic hypersurface having a singular past boundary, while S is the locus where the inverse density vanishes.

In so-called acoustical coordinates (associated with a metric taking into account the acoustic part of the relativistic Euler equations), the solution extends smoothly up to the boundary, but a particular function associated with the solution and denoted by μ vanishes on the singular part S. On the other hand, the function μ is positive on the regular part of the boundary, and the solution is smooth in this part even in the original coordinates. In addition, the author shows that each connected component of the boundary $\partial_{-}H$ is a smooth two-dimensional embedded submanifold in Minkowski spacetime which is spacelike with respect to the acoustical metric. On the other hand, the corresponding component of H is a smooth embedded three-dimensional submanifold ruled by invariant curves of vanishing arc length with respect to the acoustical metric, having past end points on the component of $\partial_{-}H$. The corresponding component of C is precisely the incoming null hypersurface associated with the component of $\partial_{-}H$. It is ruled by incoming null geodesics of the acoustical metric with past end points on the component of $\partial_{-}H$.

The author also points out that the limit toward the non-relativistic Euler equation does not involve any singular behavior and, therefore, from his results one can deduce similar results about shock formation in non-relativistic fluids.

Finally, the author discusses the physical continuation of the solution. He observes that the standard notion of maximal development is not appropriate up to H. In order to determine the physically correct hypersurface of discontinuity (denoted by K below), the author then defines a shock development problem, as follows.

Given a component A of $\partial_{-}H$ and the corresponding components C^A and H^A of C and H, respectively, the physically relevant problem associated with A is as follows: Find a hypersurface $K \subset M^{3+1}$ lying in the past of H^A and with the same past boundary (namely A) and the same tangent hyperplane at each point along A, together with a solution of the Euler equations defined on the domain of M^{3+1} bounded in the past by C and K and such that on the regular part C^A the solution coincides with the data induced by the maximal solution; on the other hand, across K, the solution must suffer a jump with respect to the prescribed data (given the maximal solution).

Of course, the jump relations under consideration are those associated with the integral form of the Euler equations. Moreover, the hypersurface K must be spacelike for the acoustical metric induced by the maximal solution (in the past of K), but timelike with respect to the new solution (in the future of K). The author then proceeds with a rigorous derivation of the jump conditions, justifies that his construction is consistent with the second law of thermodynamics and provides the physically relevant discontinuous solution. *Philippe G. LeFloch*