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Christodoulou, Demetrios (CH-ETHZ)

★**The formation of black holes in general relativity.**
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In Minkowski space-time, outgoing light fronts emanating from a round sphere increase in area, while ingoing ones decrease in area. Clearly something similar will hold for weak gravitational fields. A trapped surface, as defined by R. Penrose, will have radically different properties: roughly speaking, it is a surface from which both ingoing and outgoing light fronts decrease in area.

Penrose has shown [Phys. Rev. Lett. **14** (1965), 57–59; MR0172678] that, under natural hypotheses, existence of trapped surfaces leads to geodesic incompleteness. Hence the interest in understanding the formation of such surfaces. Furthermore, existence of a trapped surface will sometimes signal the presence of a black hole, but this requires many supplementary conditions.

The whole book under review is devoted to the proof of the theorem, that sufficiently focused "short pulse" initial data on a light-cone lead to the formation of a trapped surface within their Cauchy development.

The title of the book is rather misleading, as there are major steps missing for inferring formation of black holes out of the initial data considered.

In the prologue of the book the following version of the theorem is announced: Let k, l be positive constants, $k > 1$, $l < 1$. Let us be given smooth asymptotic initial data at past null infinity which is trivial for advanced time $\underline{u} \leq 0$. Suppose that the incoming energy per unit solid angle in each direction in the advanced time interval $[0, \delta]$ is not less than $k/8\pi$. Then if δ is suitably small, the maximal development of the data contains a closed trapped surface S with area larger than or equal to $4\pi l^2$.

Unfortunately the proof of the above is not to be found in the book, though one can imagine a limiting argument, based on the a priori estimates established in the book, that would prove the result. Here many details would have been welcome.

What can really be found in the book is the following: The author considers initial data on a light-cone. The data are governed by a small parameter δ, which can be thought of as the duration of a burst of intense gravitational waves. The intensity of the waves increases, in a specific way, as δ goes to zero. This set-up is referred to as the "short pulse method". Most of the book consists in a formidable argument showing that a certain set of inequalities can be bootstrapped when the vacuum Einstein equations hold. It follows that any solution of the vacuum Einstein equations with initial data satisfying the author's conditions will also satisfy the bootstrap inequalities provided that the solution exists long enough. One trusts the author when he asserts that this implies existence of the solution on the region where the bootstrap inequalities have been established, though a more detailed justification would have been welcome. (Here it is hard to avoid worrying about the fact that existence theorems for characteristic Cauchy problems only provide solutions defined near the intersection of the characteristic hypersurfaces. Things are not made easier by the need of imposing coordinate conditions in existence proofs for Einstein equations.) The bootstrap estimates are shown to imply existence of trapped surfaces in the region where they hold.

In spite of the above reservations, most likely originating in this reviewer's ignorance, this is a visionary proof, of terrifying complexity, which opens new avenues in our understanding of mathematics of the Einstein equations. *Piotr T. Chruściel*