Fujiwara, Kazuhiro; Kato, Fumiharu * **1400.14001**
Foundations of rigid geometry I.
EMS Monographs in Mathematics. Zürich: European Mathematical Society (EMS) (ISBN 978-3-03719-135-4/hbk; 978-3-03719-635-9/ebook). xxxiv, 829 p. (2018).

This is the first volume of the authors ambitious book project aiming at laying the foundations for an approach to rigid geometry via formal schemes, following Raynaud's by now classical idea.
Besides performing the first and most foundational steps towards this goal – which the authors interprete in its most extensive manner conceivable – the present book provides an enormous wealth of foundational material on formal schemes, formal algebraic spaces, complete rings and related topics, the main point and novelty being here that rings are not assumed to be Noetherian.
The avowed intention is to treat the material in an *encyclopedic* matter throughout. The present reviewer has no doubt that, indeed, this will become the normative book (series) for rigid geometry.
The introduction begins with a historical review of rigid geometry: "In the early stage of its history, *rigid geometry* has been first envisaged in an attempt to construct a *non-Archimedean analytic geometry*, an analogue over non-Archimedean valued fields, such as p-adic fields, of complex analytic geometry. Later, in the course of the development, rigid geometry has acquired several rich structures, considerably richer than being merely 'copies' of complex analytic geometry, which endowed the theory with a great potential of applications. This theory is nowadays recognized by many mathematicians in various research fields to be an important and independent discipline in arithmetic and algebraic geometry. This book is the first volume of our prospective book project, which aims to discuss the rich overall structures of rigid geometry, and to explore its applictions."
Historically, the principle obstacle in developing a reasonable theory of non-Archimedean analytic geometry was the total disconnectedness of non-Archimedean fields and the ensuing difficulty to find reasonable links between local and global notions of e.g. analytic functions: the problem of analytic continuation, or, as it is called in the present book, the *globalization problem.* The first and fundamental solution to the globalization problem was given by John Tate who came up with what he called *rigid analytic geometry.* Its main feature consists in endowing the maximal spectra of the so called affinoid algebras (i.e. quotients of completed polynomial rings over the fixed non-archimedean base field) with a certain Grothendieck topology of *admissible open* subsets and *admissible coverings.*
As efficient as rigid analytic geometry turned out to be for its first purposes, there also emerged essential theoretical difficulties, e.g. the *functoriality problem* – a general extension of the non-archimedean base field does not come along with a base change map from a based changed rigid space to the original one – and problems related to the failure of an analogue of the Gelfand-Mazur theorem. These problems arise because the concept of *points* (maximal ideals in affinoid algebras) in Tate's rigid analytic geometry is simply too narrow. The spectrum of an affinoid algebra must be increased. One way of doing this is to consider all height-one valuations (seminorms) on affinoid algebras; it leads to Berkovich's analytic geometry.
Another way of increasing the spectrum is taken up in this book: all valuations (not only those of height one) are considered to be points of the spectrum of an affinoid algebra, they constitute its *Stone-Zariski style spectrum.* In contrast to Berkovich's analytic geometry (which has other significant benefits, e.g. it typically leads to Hausdorff topological spaces), the natural topology now is good enough for solving the globalization problem, without recourse to the said Grothendieck topology. To deal with these Stone-Zariski style spectra adequately one is led to systematically work with integral structures in affinoid algebras. The approach is then further divided into the one advocated by R. Huber (*adic spaces*) on the one hand, and M. Raynaud's viewpoint via formal geometry as a *model geometry* on the other hand. It is this second one which is adopted in this book. Raynaud had discovered that the category of (Tate's) rigid analytic spaces (with some finiteness conditions) over a non-Archimedean field K is equivalent to a quotient category of the category of finite type formal schemes over the valuation ring of K; more precisely, the quotient is the one obtained by inverting all modifications (in particular, blow-ups) that are isomorphisms over K. One huge conceptual gain is that in this way classical tools from algebraic geometry can be brought to bear on the study of rigid analytic spaces.
From the introduction: "For us, rigid geometry is a geometry obtained from birational geometry of model geometries. This being so, the main purpose of this book project is to develop such a theory of formal geometry, thus generalizing Tate's rigid analytic geometry and building a more general analytic geometry. Thus to each formal scheme X we associate an object of a

resulting category, denote by X^{rig}, which itself should already be regarded as a rigid space. Then we define general rigid spaces by patching these objects. Note that, here, the rigid spaces are introduced as an 'absolute' object, without reference to a base space."

"Among what Raynaud's theory suggests, the most inspiring is, we think, the idea that rigid geometry should be a birational geometry of formal schemes. We would like to adopt this perspective as one of the core ideas of our theory."

This project is pursued in a generality which far exceeds previous literature on the subject. Although there is singled out a specific class of formal schemes to be considered – *locally universally rigid-Noetherian formal schemes* – these formal schemes are not themselves locally Noetherian. This in particular exceeds what is covered by EGA, hence the need to work out from scratch a theory of non-Noetherian adic rings of a fairly general kind. Besides this there is much more additional and helpful foundational material gathered in the preliminary chapter 0 – it takes up more than 250 pages. It is followed by chapter I on formal schemes which is almost just as long. In fact, here even a full fledged theory of formal algebraic spaces is worked out. As a result, the definition of a rigid space in chapter II is to be found not earlier than on page 473. For the development of their general theory some of the key words are: morphisms of various types, Zariski-Riemann spaces, tubes, coherent sheaves, affinoids, Stein affinoids, GAGA, dimension, maximum modulus principle. The appendices of chapter II in particular include the thorough comparisons with the other theories mentioned earlier (that of Tate, that of Berkovich, that of Huber), culminating in theorems stating equivalences between suitable subcategories of those studied in these varied theories.

It may be instructive to list the titles of the various sections together with their starting pages:

Elmar Große-Klönne (Berlin)