Erratum:Uhlenbeck Compactness

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This note has some essential errata and a list of small corrections. Thanks to Malcolm Schreiber and Fabian Ziltener for meticulous reading!

Proof of theorem 2.2 :
The necessity of the condition $\int_M f = 0$ for the existence of a solution of the Neumann problem follows as in the case $p = 2$: If $u \in W^{k+2,p}(M)$ solves (NP) then by lemma N it also solves (wNP), which (tested with $\psi \equiv 1$) yields $\int_M f = 0$.

In order to prove the sufficiency of that condition let $f \in W^{k,p}(M)$ be given such that $\int_M f = 0$. Choose a sequence $\tilde{f}_i \in \mathcal{C}^\infty(M)$ that converges to f in the $W^{k,p}$-norm. Then also $\int_M \tilde{f}_i$ converges to $\int_M f = 0$ since M has finite volume. Thus

$$f_i \;:=\; \tilde{f}_i - \tfrac{1}{\mathrm{Vol}(M)} \int_M \tilde{f}_i \;\in \mathcal{C}^\infty(M)$$

is a sequence of functions with vanishing mean value that still converges to f in the $W^{k,p}$-norm. Then the L^2-theorems 1.5 and 1.3 provide solutions $u_i \in \mathcal{C}^\infty(M)$ of the Neumann problem (NP) with f replaced by f_i. We can choose the u_i to have vanishing mean value such that theorem 2.3 provides

$$\|u_i - u_j\|_{W^{k+2,p}} \;\leq\; C\|\Delta u_i - \Delta u_j\|_{W^{k,p}} \;=\; C\|f_i - f_j\|_{W^{k,p}} \underset{i,j\to\infty}{\longrightarrow} 0.$$

Thus these u_i converge to some $u \in W^{k+2,p}(M)$. The limit solves $\Delta u = f$ due to the continuity of $\Delta : W^{k+2,p}(M) \to W^{k,p}(M)$ and theorem B.10 implies that u also meets the Neumann boundary condition. Uniqueness follows from corollary 1.9. $\square$

Proof of theorem 2.1 :
Testing (wNP) with $\psi \equiv 1$ we see that $\int_M f = 0$ holds automatically. So from the already established theorem 2.2 we obtain a solution $\tilde{u} \in W^{k+2,p}(M)$ of the Neumann problem (NP) for the given $f \in W^{k,p}(M)$.
...

Theorem 3.1 :
Let $f \in W^{k,p}(M)$ and $g \in W^{k+1,p}_{\partial}(M)$. Then there exists a solution $u \in W^{k+2,p}(M)$ of (3.1) if and only if (3.2) holds. This solution is unique up to an additive constant.

Proof of theorem 3.1 :
The remark just before the theorem shows the necessity of (3.2) for the existence of a solution of (3.1).

For the sufficiency let functions $f \in W^{k,p}(M)$ and $g \in W^{k+1,p}_{\partial}(M)$ be given that satisfy (3.2). Choose some $G \in W^{k+1,p}(M)$ with $G|_{\partial M} = g$ then by theorem 3.4 there exists $v \in W^{k+2,p}(M)$ that solves the boundary condition $\frac{\partial v}{\partial \nu} = G|_{\partial M} = g$. Now we have by assumption

$$\int_M (f - \Delta v) \;=\; \int_M f + \int_{\partial M} \frac{\partial v}{\partial \nu} \;=\; \int_M f + \int_{\partial M} g \;=\; 0.$$

Thus theorem 2.2 asserts the existence of a solution $\tilde{u} \in W^{k+2,p}(M)$ of the Neumann problem (NP) with f replaced by $f - \Delta v$. The solution of the inhomogeneous problem (3.1) is then given by $u = \tilde{u} + v \in W^{k+2,p}(M)$. Uniqueness follows from corollary 1.9. □

Theorem 3.1, its proof, theorem 5.3, and proof of theorem 5.5:
One should replace $W^{k,p}_{\delta}$ *by* $W^{k,p}_{\partial}$.

Proof of lemma 5.6: To see (i) choose coordinates near a point in $N \subset \partial M$ such that $\nu = \frac{\partial}{\partial x^0}$ and $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ are orthonormal tangential directions.
...
For (iv) let $F = \langle\, \alpha\,,\, \beta\,\rangle$, then calculate in local geodesic coordinates

$$\begin{aligned}
\mathcal{L}_X F &= \sum_{i,j} X^j \partial_j \bigl(\alpha_i \beta_i\bigr) \\
&= \sum_{i,j} \bigl(X^j \partial_j \alpha_i + \alpha_j \partial_i X^j\bigr)\beta_i + \sum_{i,j} \alpha_i \bigl(X^j \partial_j \beta_i + \beta_j \partial_i X^j\bigr) \\
&\qquad - \sum_{i,j} \alpha_j \bigl(\partial_i X^j + \partial_j X^i\bigr)\beta_i \\
&= \langle\, \mathcal{L}_X \alpha\,,\, \beta\,\rangle + \langle\, \alpha\,,\, \mathcal{L}_X \beta\,\rangle - \langle\, \iota_{Y_\alpha}(\mathcal{L}_X g)\,,\, \beta\,\rangle\,.
\end{aligned}$$

Here we used the formulae $\mathcal{L}_X \alpha = X^j \partial_j \alpha_i + \alpha_j \partial_i X^j$ and $(\mathcal{L}_X g)_{ij} = \partial_i X^j + \partial_j X^i$ for the Lie derivatives in local geodesic coordinates, and $(Y_\alpha)^j = \alpha_j$ for the vector field Y_α that is dual to α. □

Proposition 7.6:
... Then there exists a subsequence $(\nu_i)_{i \in \mathbb{N}}$ and a sequence of gauge transformations $u^i \in \mathcal{G}^{2,p}_{loc}(P)$ such that

$$\limsup_{i \to \infty} \bigl\| u^{i\,*} A^{\nu_i} \bigr\|_{W^{\ell,p}(M_k)} < \infty \qquad \forall k \in \mathbb{N}, \ell \in I.$$

Proof of proposition 7.6 :
... Hence for every $\ell \in I$ and $k \in \mathbb{N}$

$$\begin{aligned}\limsup_{i\to\infty} \left\| u^{i\,*}A^{\nu_i} \right\|_{W^{\ell,p}(M_k)} &\leq \sup_{j>k} \left\| w(k,\mu_{j,j})^* A^{\mu_{j,j}} \right\|_{W^{\ell,p}(M_k)} \\ &\leq \sup_{i\in\mathbb{N}} \left\| w(k,\mu_{k,i})^* A^{\mu_{k,i}} \right\|_{W^{\ell,p}(M_k)} < \infty. \qquad \square\end{aligned}$$

Proof of theorem 7.5 :
... Now proposition 7.6 with $I = \{1\}$ provides a subsequence $(\nu_i)_{i\in\mathbb{N}}$ and a sequence of gauge transformations $u^i \in \mathcal{G}^{2,p}_{loc}(P)$ such that

$$\limsup_{i\to\infty} \left\| u^{i\,*}A^{\nu_i} \right\|_{W^{1,p}(M_k)} < \infty \qquad \forall k \in \mathbb{N}.$$

...

In the induction for the local slice theorem 8.1 the estimates on $A_1 - A_0$ are weaker than (8.13) and have to be established separately. The change of constants unfortunately affects the entire proof.

Proof of theorem 8.1 :
Fix a connection $\hat{A} \in \mathcal{A}^{1,p}(P)$ and a constant $c_0 > 0$ and consider a connection $A \in \mathcal{A}^{1,p}(P)$ that satisfies (8.1) for some $\delta > 0$. Again the idea of the proof is to use Newtons iteration method to solve the boundary value problem for u. One defines connections A_i and gauge transformations $u_i = \exp(\xi_1)\dots\exp(\xi_i)$ such that $u_i^* A = A_i$ and A_i converges to a connection A_∞ that is in relative Coulomb gauge with respect to $\hat{A}$. Then one proves that in fact $A_\infty = u^* A$ for some gauge transformation u.

In the case of varying metrics in remark 8.2 one chooses the $W^{1,\infty}$-neighbourhood of the given metric g as in lemma 8.5 (iii). Moreover, choose this neighbourhood, that is $\varepsilon > 0$, sufficiently small such that (8.6) holds with a uniform constant for all metrics g' that satisfy $\|g - g'\|_{W^{1,\infty}} \leq \varepsilon$. Then all constants in the following will be independent of the metric g' that is used in the boundary value problem. The constants in Sobolev inequalities are also independent of g' since they are defined with respect to g. That way the local slice theorem is proven with uniform constants for all metrics in the $W^{1,\infty}$-neighbourhood of g.

So we construct the sequences of gauge transformations $\exp(\xi_i) \in \mathcal{G}^{2,p}(P)$ and connections $A_i \in \mathcal{A}^{1,p}(P)$ by the following Newton iteration: $A_0 := A$ and $A_{i+1} := \exp(\xi_i)^* A_i$, where $\xi_i \in W^{2,p}(M, \mathfrak{g}_P)$ is provided by lemma 8.5 (ii). It is the solution of

$$\begin{cases} \mathrm{d}_{\hat{A}}^* \mathrm{d}_{\hat{A}} \xi_i = \mathrm{d}_{\hat{A}}^* (\hat{A} - A_i), \\ *\mathrm{d}_{\hat{A}} \xi_i|_{\partial M} = *(\hat{A} - A_i)|_{\partial M}, \end{cases}$$

with

$$\begin{aligned} &\|\xi_i\|_{W^{2,p}} \leq C_1 \bigl(\|\mathrm{d}_{\hat{A}}^* (A_i - \hat{A})\|_p + \| *(A_i - \hat{A})|_{\partial M} \|_{W^{1,p}_\partial} \bigr), \\ &\|\xi_i\|_{W^{1,q}} \leq C_1 \|A_i - \hat{A}\|_q. \end{aligned} \tag{8.11}$$

We claim that for sufficiently small $\delta > 0$ there exist constants C_0, C_I, C_{II} such that this sequence satisfies for all $i \in \mathbb{N}$

$$\|\mathrm{d}_{\hat{A}}^*(A_i - \hat{A})\|_p + \| *(A_i - \hat{A})|_{\partial M}\|_{W^{1,p}_\partial} \leq 2^{-i} C_I \|A - \hat{A}\|_q, \tag{8.12}$$

$$\|A_i - A_{i-1}\|_{W^{1,p}} \leq 2^{-i} C_{II} \|A - \hat{A}\|_q \quad \text{if } i \geq 2. \tag{8.13}$$

Let C_3 be the constant from (8.6) and let C_2 be the constant from lemma 8.6 (ii) for $c_2 = C_1 C_3 c_0$. The constants C_I and C_{II} will be determined from c_0, C_1, C_2, C_3, and some Sobolev constants. The induction step for (8.12) and (8.13) will require a sufficiently small choice of $\delta > 0$, depending on C_I and C_{II}. This is the same procedure as for theorem 8.3 – we first fix C_I and C_{II} and then determine a suitable $\delta > 0$, just that we do not give the more complicated formulae here.

Before starting the induction we note some estimates for $A_1 - A_0$. We choose $\delta \leq 1$, then lemma 8.6 and (8.11), (8.6) provide a constant $C_0 \geq 1$ such that

$$\|A_1 - A_0\|_q \leq C_2\bigl(1 + \|A - \hat{A}\|_q\bigr)\|\xi_0\|_{W^{1,q}} \leq C_1 C_2 (1+\delta)\|A - \hat{A}\|_q \leq C_0 \|A - \hat{A}\|_q,$$

$$\begin{aligned}
\|A_1 - A_0\|_{W^{1,p}} &\leq C_2\bigl(1 + \|A - \hat{A}\|_{W^{1,p}}\bigr)\|\xi_0\|_{W^{2,p}} \\
&\leq C_1 C_2 (1 + c_0)\bigl(\|\mathrm{d}_{\hat{A}}^*(A_j - \hat{A})\|_p + \| *(A_j - \hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\bigr) \\
&\leq C_1 C_2 C_3 (1 + 2c_0)\|A - \hat{A}\|_{W^{1,p}} \ \leq \ C_0 \|A - \hat{A}\|_{W^{1,p}}.
\end{aligned}$$

Now assume that (8.13) holds for all $i = 2, \ldots, j$ with some $j \geq 2$, then we have

$$\begin{aligned}
\|A_j - \hat{A}\|_{W^{1,p}} &\leq \|A_0 - \hat{A}\|_{W^{1,p}} + \|A_1 - A_0\|_{W^{1,p}} + \sum_{i=2}^{j} \|A_i - A_{i-1}\|_{W^{1,p}} \\
&\leq (1 + C_0)\|A - \hat{A}\|_{W^{1,p}} + \bigl(\textstyle\sum_{i=2}^{j} 2^{-i}\bigr) C_{II} \|A - \hat{A}\|_q \\
&\leq 2C_0 \|A - \hat{A}\|_{W^{1,p}} + C_{II}\|A - \hat{A}\|_q \\
&\leq 2C_0 c_0 + C_{II}\delta \ \leq \ 3C_0 c_0.
\end{aligned} \tag{8.14}$$

Here we choose $\delta \leq c_0 C_0 C_{II}^{-1}$. Moreover, (8.13) implies that with a Sobolev constant C and for $C_{II} \geq 2C^{-1}C_0$

$$\begin{aligned}
\|A_j - \hat{A}\|_q &\leq \|A_0 - \hat{A}\|_q + \|A_1 - A_0\|_q + \sum_{i=2}^{j} C\|A_i - A_{i-1}\|_{W^{1,p}} \\
&\leq (1 + C_0)\|A - \hat{A}\|_q + \bigl(\textstyle\sum_{i=2}^{j} 2^{-i}\bigr) C C_{II} \|A - \hat{A}\|_q \\
&\leq (2C_0 + CC_{II})\|A - \hat{A}\|_q \ \leq \ 2CC_{II}\delta.
\end{aligned} \tag{8.15}$$

Note that both (8.14) and (8.15) also hold for $j = 0$ and $j = 1$. That can be used as start of the induction. Then for the induction step suppose that (8.12) and (8.13) are true for all $i \leq j$ (so also (8.14) and (8.15) hold). In case $j = 1$ this means that we can only use (8.12), (8.14), and (8.15); in case $j = 0$ we will only use (8.14) and (8.15). Then we have to prove (8.12) and (8.13) for

$i=j+1$. Firstly, (8.11) provides the bound for lemma 8.6 (ii) that allows to use the estimates with the constant C_2 fixed above : In case $j=0$

$$\begin{aligned}\|\xi_0\|_{W^{2,p}} &\leq C_1\bigl(\|\mathrm{d}^*_{\hat{A}}(A_0-\hat{A})\|_p + \|*(A_0-\hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\bigr)\\ &\leq C_1C_3\|A-\hat{A}\|_{W^{1,p}}\\ &\leq C_1C_3c_0 \;=:\; c_2,\end{aligned}$$

and for the case $j\geq 1$ use (8.12) and choose $\delta\leq 2C_I^{-1}C_3c_0$ such that

$$\begin{aligned}\|\xi_j\|_{W^{2,p}} &\leq C_1\bigl(\|\mathrm{d}^*_{\hat{A}}(A_j-\hat{A})\|_p + \|*(A_j-\hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\bigr)\\ &\leq 2^{-j}C_1C_I\|A-\hat{A}\|_q\\ &\leq \tfrac{1}{2}C_1C_I\delta \;\leq\; c_2.\end{aligned}$$

Now since $\mathrm{d}^*_{\hat{A}}\hat{A}=\mathrm{d}^*_{\hat{A}}\bigl(\mathrm{d}_{\hat{A}}\xi_j+A_j\bigr)$ and $*\hat{A}|_{\partial M}=*(\mathrm{d}_{\hat{A}}\xi_j+A_j)|_{\partial M}$ we can rewrite

$$\begin{aligned}\mathrm{d}^*_{\hat{A}}(A_{j+1}-\hat{A}) &= \mathrm{d}^*_{\hat{A}}\bigl(\exp(\xi_j)^*A_j-A_j-\mathrm{d}_{A_j}\xi_j\bigr)+\mathrm{d}^*_{\hat{A}}\bigl[A_j-\hat{A},\xi_j\bigr], \qquad (8.16)\\ *\bigl(A_{j+1}-\hat{A}\bigr)|_{\partial M} &= *\bigl(\exp(\xi_j)^*A_j-A_j-\mathrm{d}_{A_j}\xi_j\bigr)|_{\partial M}+\bigl[*\bigl(A_j-\hat{A}\bigr)|_{\partial M},\xi_j\bigr].\end{aligned}$$

The first terms in both right hand side expressions are estimated by lemma 8.6 (ii) and with the help of (8.6), (8.11), and (8.14) :

$$\begin{aligned}&\bigl\|\mathrm{d}^*_{\hat{A}}\bigl(\exp(\xi_j)^*A_j-A_j-\mathrm{d}_{A_j}\xi_j\bigr)\bigr\|_p + \bigl\|*\bigl(\exp(\xi_j)^*A_j-A_j-\mathrm{d}_{A_j}\xi_j\bigr)|_{\partial M}\bigr\|_{W^{1,p}_\partial}\\ &\leq C_3\bigl\|\exp(\xi_j)^*A_j-A_j-\mathrm{d}_{A_j}\xi_j\bigr\|_{W^{1,p}}\\ &\leq C_2C_3\bigl(1+\|A_j-\hat{A}\|_{W^{1,p}}\bigr)\|\xi_j\|_{W^{1,q}}\|\xi_j\|_{W^{2,p}}\\ &\leq C_1^2C_2C_3(1+3C_0c_0)\|A_j-\hat{A}\|_q\bigl(\|\mathrm{d}^*_{\hat{A}}(A_j-\hat{A})\|_p+\|*(A_j-\hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\bigr)\end{aligned}$$

Now consider the upper second term in (8.16). Firstly, from the local formula (A.9) for $\mathrm{d}^*_{\hat{A}}$ and the Jacobi identity one obtains

$$\mathrm{d}^*_{\hat{A}}[A_j-\hat{A},\xi_j]=[\mathrm{d}^*_{\hat{A}}(A_j-\hat{A}),\xi_j]-\bigl\langle A_j-\hat{A}\,,\,\mathrm{d}_{\hat{A}}\xi_j\bigr\rangle.$$

As in the proof of lemma 8.6 let $\frac{1}{r}=\frac{1}{p}-\frac{1}{q}$, then the Sobolev inequality for $W^{2,p}\hookrightarrow W^{1,r}$ holds. Thus from (8.11) and with a finite constant C arising from several Sobolev constants one obtains

$$\begin{aligned}\bigl\|\mathrm{d}^*_{\hat{A}}[A_j-\hat{A},\xi_j]\bigr\|_p &\leq \|\mathrm{d}^*_{\hat{A}}(A_j-\hat{A})\|_p\|\xi_j\|_\infty+\|A_j-\hat{A}\|_q\|\mathrm{d}_{\hat{A}}\xi_j\|_r\\ &\leq C\bigl(\|\xi_j\|_{W^{1,q}}\|\mathrm{d}^*_{\hat{A}}(A_j-\hat{A})\|_p+\|A_j-\hat{A}\|_q\|\xi_j\|_{W^{2,p}}\bigr)\\ &\leq CC_1\|A_j-\hat{A}\|_q\bigl(\|\mathrm{d}^*_{\hat{A}}(A_j-\hat{A})\|_p+\|*(A_j-\hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\bigr)\end{aligned}$$

For the lower second term in (8.16) use (8.11) and lemma B.3 with $r=p$ and $s=q$ to obtain a constant C such that

$$\begin{aligned}\bigl\|[*(A_j-\hat{A})|_{\partial M},\xi_j]\bigr\|_{W^{1,p}_\partial} &\leq C\|*(A_j-\hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\|\xi_j\|_{W^{1,q}}\\ &\leq CC_1\|*(A_j-\hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\|A_j-\hat{A}\|_q.\end{aligned}$$

Now we have considered all terms in (8.16) and found a finite constant C_4 depending on c_0, C_0, C_1, C_2, C_3, and some Sobolev constants C such that

$$
\begin{aligned}
&\|\mathrm{d}^*_{\hat{A}}(A_{j+1}-\hat{A})\|_p + \|*(A_{j+1}-\hat{A})|_{\partial M}\|_{W^{1,p}_\partial} \\
&\qquad \le C_4\|A_j-\hat{A}\|_q\big(\|\mathrm{d}^*_{\hat{A}}(A_j-\hat{A})\|_p + \|*(A_j-\hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\big) \\
&\qquad \le C_4\cdot 2CC_{II}\delta\cdot 2^{-j}C_I\|A-\hat{A}\|_q \\
&\qquad \le 2^{-(j+1)}C_I\|A-\hat{A}\|_q
\end{aligned}
$$

In the above estimates we used (8.12) for $i=j$ and (8.15), and we made the possibly even smaller choice $\delta \le (4C_4CC_{II})^{-1}$. Since we used (8.12) this only holds for $j\ge 1$; in case $j=0$ one has to use (8.6) and (8.1) to estimate

$$
\begin{aligned}
&\|\mathrm{d}^*_{\hat{A}}(A_1-\hat{A})\|_p + \|*(A_1-\hat{A})|_{\partial M}\|_{W^{1,p}_\partial} \\
&\qquad \le C_4\|A_0-\hat{A}\|_q\big(\|\mathrm{d}^*_{\hat{A}}(A_0-\hat{A})\|_p + \|*(A_0-\hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\big) \\
&\qquad \le C_3C_4\|A-\hat{A}\|_q\|A-\hat{A}\|_{W^{1,p}} \\
&\qquad \le c_0C_3C_4\|A-\hat{A}\|_q.
\end{aligned}
$$

In both cases, this proves the induction step for (8.12); where in the step for $j=0$ the constant C_I is fixed as $C_I = 2c_0C_3C_4$.

Furthermore, (8.13) is shown in case $i=j+1\ge 2$ with the help of lemma 8.6 (ii), (8.14), (8.11), and again (8.12) for $i=j\ge 1$:

$$
\begin{aligned}
\|A_{j+1}-A_j\|_{W^{1,p}} &\le C_2\big(1+\|A_j-\hat{A}\|_{W^{1,p}}\big)\|\xi_j\|_{W^{2,p}} \\
&\le C_1C_2(1+3C_0c_0)\big(\|\mathrm{d}^*_{\hat{A}}(A_j-\hat{A})\|_p + \|*(A_j-\hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\big) \\
&\le 2^{-j}C_IC_1C_2(1+3C_0c_0)\|A-\hat{A}\|_q.
\end{aligned}
$$

This proves the induction step for (8.13) with $C_{II} = \frac{1}{2}C_IC_1C_2(1+3C_0c_0)$. So we have proved (8.12) and (8.13) by induction.

Now (8.13) shows that the A_i form a $W^{1,p}$-Cauchy sequence. Indeed, for all $k>j\ge 1$

$$
\|A_k-A_j\|_{W^{1,p}} \;\le\; \sum_{i=j+1}^{k}\|A_i-A_{i-1}\|_{W^{1,p}} \;\le\; \sum_{i=j+1}^{k}2^{-i}C_I\|A-\hat{A}\|_q \;\le\; 2^{-j}\delta C_I.
$$

Since $\mathcal{A}^{1,p}(P)$ is a Banach space this implies that the A_i converge in the $W^{1,p}$-norm to some $A_\infty \in \mathcal{A}^{1,p}(P)$. By continuity this limit connection also satisfies (8.14) and (8.15), hence one obtains a constant $C_{CG} = 2C_0 + CC_{II}$ (where C is the Sobolev constant for the embedding $W^{1,p}\hookrightarrow L^q$) such that

$$
\begin{aligned}
\|A_\infty-\hat{A}\|_{W^{1,p}} &\le C_{CG}\|A-\hat{A}\|_{W^{1,p}}, \\
\|A_\infty-\hat{A}\|_q &\le C_{CG}\|A-\hat{A}\|_q.
\end{aligned}
$$

From (8.12) one sees that

$$\mathrm{d}^*_{\hat A}(A_\infty - \hat A) = \lim_{i\to\infty} \mathrm{d}^*_{\hat A}(A_i - \hat A) \quad = 0,$$
$$*(A_\infty - \hat A)\big|_{\partial M} = \lim_{i\to\infty} *\big(A_i - \hat A\big)\big|_{\partial M} = 0.$$

So it remains to show that $A_\infty = u^*A$ for some $u \in \mathcal{G}^{2,p}(P)$. For that purpose consider the sequence $u_i = \exp(\xi_1)\ldots\exp(\xi_i)$. By lemma A.5 it lies in $\mathcal{G}^{2,p}(P)$, and it satisfies $u_i^*A = A_i$. Now lemma A.8 applies since A_i converges in the $W^{1,p}$-norm and A is uniformly $W^{1,p}$-bounded anyway. Thus there exists a subsequence of the u_i that converges in the $\mathcal{C}^0$-norm to some $u \in \mathcal{G}^{2,p}(P)$. For the same subsequence (again labelled by i) $u_i^{-1}\mathrm{d}u_i$ converges to $u^{-1}\mathrm{d}u$ in the L^{2p}-norm. Now $u^*A = A_\infty$ since this is the unique L^{2p}-limit of the sequence

$$u_i^{-1}Au_i + u_i^{-1}\mathrm{d}u_i \;=\; u_i^*A \;=\; A_i.$$

Thus u is the required gauge transformation that puts A in relative Coulomb gauge. □

Proof of theorem 8.3 :
... It remains to show that $A_\infty = u^*A$ for some $u \in \mathcal{G}^{1,r}(P)$. For that purpose consider the sequence $u_i = \exp(\xi_1)\ldots\exp(\xi_i)$. By lemma A.5 it lies in $\mathcal{G}^{1,r}(P)$, and it moreover satisfies $u_i^*A = A_i$. Now lemma A.8 applies (with $k = 1$ and $p = r$) since the A_i converge in the L^r-norm and A is uniformly L^r-bounded anyway. Thus there exists a subsequence of the u_i that converges in the $\mathcal{C}^0$-norm to some $u \in \mathcal{G}^{1,r}(P)$. For the same subsequence (again labelled by i) $u_i^{-1}\mathrm{d}u_i$ converges to $u^{-1}\mathrm{d}u$ in the weak L^r-topology. Now we obtain $u^*A = A_\infty$ since this is the unique weak L^r-limit of the sequence $u_i^{-1}Au_i + u_i^{-1}\mathrm{d}u_i \;=\; u_i^*A \;=\; A_i$. Thus u is the required gauge transformation that puts A in relative Coulomb gauge with respect to $\hat A$. □

Proposition 9.8 and Lemma 9.9 : Let $p > \frac{n}{2}$. **...**

Proof of theorem 10.3 :
... So proposition 7.6 with $I = \mathbb{N}$ provides a subsequence $(\nu_i)_{i\in\mathbb{N}}$ and a sequence of gauge transformations $u^i \in \mathcal{G}^{2,p}_{loc}(P)$ such that

$$\limsup_{i\to\infty} \big\|{u^i}^*\tilde A^{\nu_i} - \tilde A\big\|_{W^{\ell,p}(M_k)} < \infty \qquad \forall k,\ell \in \mathbb{N}.$$

...

Lemma A.8 : ...

(ii) There exists a subsequence of the u^ν that converges in the $\mathcal{C}^0$-topology to some $u^\infty \in \mathcal{G}^{k,p}(P)$ and for all trivializations $(u^\nu_\alpha)^{-1}\mathrm{d}u^\nu_\alpha \to (u^\infty_\alpha)^{-1}\mathrm{d}u^\infty_\alpha$ in the weak $W^{k-1,p}$-topology.

Corollary B.9 : Let U be a compact Riemannian n-manifold and let G be a compact Lie group. Let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $kp > n$. Then for every sequence $(u_i)_{i\in\mathbb{N}}$ in $\mathcal{G}^{k,p}(U)$ with a uniform bound on $\|u_i^{-1}\mathrm{d}u_i\|_{W^{k-1,p}}$ there exists a subsequence that converges in the $\mathcal{C}^0$-topology to a gauge transformation $u \in \mathcal{G}^{k,p}(U)$. Moreover, $u_i^{-1}\mathrm{d}u_i$ converges to $u^{-1}\mathrm{d}u$ in the weak $W^{k-1,p}$-topology.

...

Proof:

... Thus $u = \Phi^{-1} \circ v$ is well defined, lies in $\mathcal{G}^{k,p}(M,G)$, and is the $\mathcal{C}^0$-limit of the u_i. At the same time, $u^{-1}\mathrm{d}u$ is the weak $W^{k-1,p}$-limit of the $u_i^{-1}\mathrm{d}u_i$.

... (note that $\mathrm{d}\Phi$ is a bijection between $\mathrm{T}G$ and $\mathrm{T}(\Phi(G))$). □

various small corrections

p.17
We denote by $\langle\, u\,,\,\phi\,\rangle$ the pairing of a distribution $u\in\mathcal{D}(M)$ with $\phi\in\mathcal{C}^\infty(M)$.
p.34
In fact, theorem 2.3' is only used in chapter 5.
p.39
$\|\mathcal{L}_X u\|_{W^{\ell+2,p}} \le C\bigl(\|\Delta\mathcal{L}_X u\|_{W^{\ell,p}} + \|\mathcal{L}_X u\|_{W^{\ell+1,p}}\bigr)$
p.41
$c := (\mathrm{Vol}\,M)^{-1}\,\langle\, u, 1\,\rangle$
$\|u\|_{(W^{k,p^*}(M))^*} \le \ldots \le C\|f\|_{(W^{k,p^*}(M))^*} + C(\mathrm{Vol}\,M)^{-1}|\,\langle\, u\,,\,1\,\rangle\,|$
p.66
... ∇ is given by connection potentials $A_\alpha\in L^r(U_\alpha,\mathrm{T}^*U_\alpha\otimes\mathrm{End}\,V)$.
p.69
For the Sobolev embedding one checks that $1-\frac{n}{p} > -\frac{n}{2p}$ due to $p>\frac{n}{2}$.
p.78 within the boundary value problems
$\mathrm{d}^*\alpha\in W^{k,p}(M)$
$\alpha_0|_{\partial\mathbb{H}} = 0$
p.84 the estimates are meant for $\left|\int_M \alpha(X)\cdot\Delta\phi\right|$
p.88
If X is perpendicular to the boundary, then the estimate holds for all $\phi\in\mathcal{C}^\infty_\delta(M)$; if X is tangential, then it holds for $\phi\in\mathcal{C}^\infty_\nu(M)$.
p.96
... the Sobolev inequality for $W^{1,p}\hookrightarrow L^{2p}$
p.97
$\|F_{A_i}\|_p \;\le\; C\left(\|A_i\|_{W^{1,p}} + \|A_i\|^2_{W^{1,p}}\right)$
p.101
... the perturbation S also is a linear operator from $W^{2,p}_m(B,\mathfrak{g})$ to $\mathcal{Z}$
... we can use a property (A.6) of the norm on $\mathfrak{g}$
p.105
Now $W^{2,\frac{n}{2}}(M,G)$ can be defined as the closure of $\mathcal{C}^\infty(M,G)$ with respect to the $W^{2,\frac{n}{2}}$-norm on $\mathcal{C}^\infty(M,\mathbb{R}^m)$.
p.109
$(\tilde{u}^\nu_\alpha)^{-1}\phi_{\alpha\beta}\tilde{u}^\nu_\beta = g_\alpha(u^\nu_\alpha h^\nu_\alpha)^{-1}\phi_{\alpha\beta}(u^\nu_\beta h^\nu_\beta)g_\beta^{-1} = g_\alpha g_{\alpha\beta}g_\beta^{-1} = \phi_{\alpha\beta}$
p.120
This provides a subsequence $(\mu_{j+1,i})_{i\in\mathbb{N}}\subset(\nu_{j+3,i})_{i\in\mathbb{N}}$ and gauge transformations $w(j+1,\mu_{j+1,i})\in\mathcal{G}^{2,p}(P|_{M_{j+3}})$
p.128
$\mathrm{d}_{A^2}{}'(A^1-A^2)\,\eta = \ldots = -\mathrm{d}_{A^1}{}'(A^2-A^1)\,\eta\; - \int_M \langle\, *[A\wedge *A]\,,\,\eta\,\rangle$
$\bigl(\mathrm{d}^*_A(v^*\hat{A}-A)\bigr)_\alpha = -\bigl(\mathrm{d}^*_{v^*\hat{A}}(A-v^*\hat{A})\bigr)_\alpha = \ldots$
p.129
Newton iteration analogous to [CGMS, Thm.B.1]
p.132

$\alpha(t) = t \cdot \mathrm{d}_A\xi + \sum_{k=1}^{\infty} \frac{-(-t)^{k+1}}{(k+1)!}\mathrm{ad}_\xi^k(\mathrm{d}_A\xi)$

p.133

$$
\begin{aligned}
&\big|\nabla_{\hat{A}}\big(\exp(\xi)^*A - A - \mathrm{d}_A\xi\big)\big| \\
&\le \sum_{k=1}^{\infty} \tfrac{C^{k-1}}{(k+1)!}\big(k|\nabla_{\hat{A}}\xi| \cdot |\mathrm{d}_A\xi| + |\xi| \cdot |\nabla_{\hat{A}}\mathrm{d}_A\xi|\big) \\
&\le \tfrac{e^C-1}{C}\big(|\nabla_{\hat{A}}\xi|^2 + |\nabla_{\hat{A}}\xi| \cdot |A-\hat{A}| \cdot |\xi| + |\xi| \cdot |\nabla_{\hat{A}}^2\xi| + |\xi| \cdot |\nabla_{\hat{A}}[A-\hat{A},\xi]|\big)
\end{aligned}
$$

p.144 in (9.3)

$*F_A|_{\partial M} = 0$

p.146

Note that the assumptions on p in case $k=1$ of both the above proposition and corollary ensure $p \ge \frac{2n}{n+1}$

... then we can get to $W^{2,n}$ (and thus to $W^{2,p}$ if we started with $p < n$)

p.148

$W^{k,q} \hookrightarrow W^{k-1,p}$ (in case $q \ne p$ this is due to $\frac{1}{q} = \frac{2}{p} - \frac{1}{n} \le \frac{1}{p} + \frac{1}{n}$)

... with (k,p) replaced by $(k-1,q)$ and for $s = p$ and $p \le r < \infty$

p.152

Then corollary 9.6 (ii) with $M' = M_k^\ell$ and $M'' = M_k^{\ell+1}$

p.156

Indeed, $\mathrm{d}_{A^\nu}\beta$ converges in the L^{p^*}-norm to $\mathrm{d}_A\beta$ since $q \ge p^*$, and F_{A^ν} converges in the weak L^p-topology to F_A.

p.167

In the local trivialization a gauge transformation $u \in \mathcal{G}(P)$ is represented by $u_\alpha = \tilde{\phi}_\alpha \circ \bar{u} : U_\alpha \to G$

p.174

$\|F_A\|_{W^{k-1,q}} \le \|F_{\tilde{A}}\|_{W^{k-1,q}} + C\left(\|\alpha\|_{W^{k,q}} + \|\alpha\|_{W^{k,q}}^2\right).$

Here $|\mathrm{d}_{\tilde{A}}\alpha| \le 2\,|\nabla_{\tilde{A}}\alpha|$

p.175

In a trivialization over $U \subset M$ **...** with $s \in \mathcal{C}^\infty(U,G)$ and $\xi \in W^{k,p}(U,\mathfrak{g})$.

p.176

Thus in $u_i^*A_i = (u_i)^{-1}\mathrm{d}u_i + (u_i)^{-1}A_iu_i$ one immediately obtains the $W^{k-1,p}$-convergence of the first term and the L^p-convergence of the second term.

p.177

Indeed, for the first this is due to the Sobolev embedding $W^{\ell,p} \hookrightarrow W^{\ell-1,2p}$ and $\ell \le k-1$ (using the convergence criterion in lemma B.7 (iv)).

p.180

Note the following subtlety of the definition of $W^{k,p}(M,E)$:

If M is a compact manifold with boundary, then sections in $W^{k,p}(M,E)$ can be nonzero over ∂M. Sections in $W^{k,p}(M \setminus \partial M, E)$ however will be the limit of smooth sections with support in $M \setminus \partial M$. For $k = 0$ the completions are the same, but for $k \ge 1$ any section in $W^{k,p}(M \setminus \partial M, E)$ necessarily extends to zero over ∂M.

p.181
Here $V_i \subset \mathbb{R}^n$ is a compact coordinate chart of M and $\mathbb{R}^m$ is isomorphic to the fibres of E.
p.186
note that $k - m \geq \frac{n}{kp}(k - \frac{mk}{M})$ since $m \leq M \leq k$
p.188
Let $u = s \cdot \exp(\xi)$ with $s \in \mathcal{C}^\infty(M, G)$ and $\xi \in W^{k,p}(M, \mathfrak{g})$
... for some constant C_ξ $\|E(\xi) \circ \mathcal{L}_Y \xi\|_p \leq C_\xi \|\mathcal{L}_Y \xi\|_p$.
p.189
... for some constant C_ξ $\|\mathrm{d}_\xi E(\mathcal{L}_Z \xi) \circ \mathcal{L}_Y \xi\|_p \leq C_\xi \|\mathcal{L}_Z \xi\|_{2p} \|\mathcal{L}_Y \xi\|_{2p}$.
This proves that $u^{-1}\mathrm{d}u$ has finite $W^{1,p}$-norm.
Since G is compact $E(u)$ is bounded in the operator norm
p.196
Finally, if $k \neq l$ and both derivatives are included one checks

$$\left| x_l x_k x_{i_1} \ldots x_{i_s} \frac{\partial^{s+2} m}{\partial x_l \partial x_k \partial x_{i_1} \ldots \partial x_{i_s}} \right| \; = \; \ldots \; \leq \; 2 \cdot 2^{s+2} (s+2)! \; \leq \; 2^{n+1} n! \; .$$

Before one has $s \leq n$ or $s \leq n+1$ respectively, hence
... the criterion (C.1) is met with $A = 2^{n+1} n!$.
p.199

$$\int_{\mathbb{R}^n \setminus B_K} \rho_t \; \leq \; \tfrac{\varepsilon}{2\|f\|_p^p}$$

The second term is estimated as follows:

$$\ldots \leq \|f\|_p^p \int_{\mathbb{R}^n \setminus B_K} \rho_t(y) \, \mathrm{d}^n y \quad \leq \; \tfrac{\varepsilon}{2}$$

p.208
$W^{k+1,p}_\partial(M) = \left\{ G|_{\partial M} \;\middle|\; G \in W^{k+1,p}(M) \right\}$

p.212
The reference [W] split into

[W1] K.Wehrheim, Banach space valued Cauchy-Riemann equations with totally real boundary conditions, *Comm. Contemp. Math.* **6** (2004), no. 4, 601–635.

[W2]K.Wehrheim, Anti-self-dual instantons with Lagrangian boundary conditions I: Elliptic theory, *Comm. Math. Phys.* **254** (2005), no. 1, 45–89.