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★**"Moonshine" of finite groups.**

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This is an almost verbatim reproduction of the author's lecture notes on "Moonshine of finite groups", written in 1983–84 at the Ohio State University, Columbus, Ohio, USA. A substantial update is given in the bibliography.

The original motivation of the author was to understand the moonshine phenomena of the Monster simple group $\mathbb{M}$. Over the last 20 plus years, there has been an energetic activity in the field of finite simple group theory related to the Monster $\mathbb{M}$. Most notably, influential works have been produced in the theory of vertex operator algebras whose research was stimulated by the moonshine of the finite groups.

However, a quarter of a century later, it does not seem that we understand these phenomena very well, although the main problem—the Conway-Norton conjecture—has been solved by Richard Borcherds. Still, we can ask the same questions now as we did some 30–40 years ago: What is the Monster simple group $\mathbb{M}$? Is it really related to the theory of the universe as it was vaguely so envisioned? What lies behind the moonshine phenomena of the Monster $\mathbb{M}$? It may appear that we have only scratched the surface.

On the other hand, the interest generated by researchers in the original mystery of moonshine of the Monster simple group seems to have faded away somewhat, due perhaps to the difficulty of solving it. This is one reason that the author has made the lecture notes available for a wider body of readers.

The author kept the notes essentially the same as in the original version. As one can see in the literature, for example T. Gannon's book [*Moonshine beyond the Monster*, Cambridge Monogr. Math. Phys., Cambridge Univ. Press, Cambridge, 2006; MR2257727] and its bibliography, the horizon encircling "moonshine" is vast, long and deep. It may even appear that we have yet to see a clear direction in which the truth of the moonshine will be found.

This book consists of four chapters. In the first chapter the author gives some background on modular functions and modular forms used in moonshine. Let $\mathbb{H}^*=\mathbb{H}\cup\mathbb{Q}\cup\{\infty\}$ and $\Gamma_0(N)\backslash\mathbb{H}^*$ be the corresponding Riemann surface. The genus of $\Gamma_0(N)\backslash\mathbb{H}^*$ is well known. In the moonshine phenomena of the Monster simple group $\mathbb{M}$, we are interested in the normalizer of $\Gamma_0(N)$. Let $\mathrm{GL}_2(\mathbb{R})^+=\{\alpha\in\mathrm{GL}_2(\mathbb{R})\colon \det(\alpha)>0\}$. The author defines $\Gamma_0(N)+$ to be the subgroup of $\mathrm{GL}_2(\mathbb{R})^+$ generated by $\Gamma_0(N)$ and all Atkin-Lehner involutions W_e of $\Gamma_0(N)$:

$$\Gamma_0(N)+=\langle\Gamma_0(N),W_e,W_f,\dots\rangle,\quad e\|N, f\|N,\dots.$$

Here $r\|s$ denotes $r|s$ and $(r,s/r)=1$. The author gives the genus of $\Gamma\backslash\mathbb{H}^*$ where $\Gamma=\Gamma_0(N)+$ and $N\le 300$.

In the second chapter, the author considers the Dedekind eta function $\eta(z)$ and the transformation formula for $\eta(z)$.

In the third chapter, the author discusses the moonshine of finite groups. Let $\mathfrak{F}$ be the set of functions $f(z)$ satisfying the following conditions:

(1) $f(z)$ is a modular function with respect to some discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$

that contains $\Gamma_0(N)$ for some N.

(2) The genus of $\Gamma\backslash\mathbb{H}^*$ is 0 and its function field is equal to $k(\Gamma\backslash\mathbb{H}^*)=\mathbb{C}(f)$.

(3) In a neighborhood of ∞, $f(z)$ is expressed in the form

$$f(z)=\frac{1}{q}+\sum_{n=0}^{\infty}a_nq^n,\quad q=e^{2\pi iz},\ z\in\mathbb{H},\ a_n\in\mathbb{C}.$$

We say a pair (G,ϕ) is a moonshine for a finite group G if ϕ is a function from G to $\mathfrak{F}$, and if, for $\sigma\in G$,

$$\phi_\sigma(z)=\frac{1}{q}+a_0(\sigma)+\sum_{n=1}^{\infty}a_n(\sigma)q^n,\quad q=e^{2\pi iz},$$

then the mapping $\sigma\mapsto a_n(\sigma)$ from G to $\mathbb{C}$ is a generalized character of G. In particular, ϕ is a class function of G.

Finding or constructing a moonshine (G,ϕ) for a given group G involves some nontrivial work. For each element σ of G, we have to find a natural number N_σ and a Fuchsian group Γ_σ containing $\Gamma_0(N_\sigma)$ in such a way that its function field $k(\Gamma_\sigma\backslash\mathbb{H}^*)$ is equal to $\mathbb{C}(\phi_\sigma)$ and that the coefficients $a_n(\sigma)$ of the expansion of $\phi_\sigma(z)$ at ∞ are generalized characters of the finite group G for all $n\geq 1$.

In the Appendix, the author provides a list of all Fuchsian groups Γ of genus 0 such that $\Gamma_0(N)\subseteq\Gamma\subseteq\Gamma_0(N)+$ for some N. There are exactly 123 possible Γ's. Some subgroups and some conjugates of those 123 Γ's are, in practice, the only Fuchsian groups that could be used for a moonshine of a finite group G.

Once Γ is chosen in some way, then we will need to find $f(z)\in\mathfrak{F}$ such that $k(\Gamma\backslash\mathbb{H}^*)=\mathbb{C}(f)$. In most known cases, $f(z)$ is expressed as a product of some Dedekind η functions. This leads to the following:

Definition. A symbol $\pi=\prod_t t^{r_t}$ is a generalized partition if $t\in\mathbb{N}$, $r_t\in\mathbb{Z}$ for all t and $\prod_t t^{r_t}$ is a product of finitely many t. Define $\deg(\pi)=\sum_t r_tt$, $\mathrm{wt}(\pi)=\frac{1}{2}\sum_t r_t$ and $\mathrm{sgn}(\pi)=\prod_t(-1)^{r_t-1}$. For a generalized partition $\pi=\prod_t t^{r_t}$, define

$$\eta_\pi(z)=\prod_t\eta(tz)^{r_t}.$$

The importance of the function $\eta_\pi(z)$ comes from the following fact:

Theorem 1. Let $\pi=\prod t^{r_t}$ be a generalized partition. Suppose that

(1) $\sum r_t=0$,

(2) $\sum tr_t\equiv 0\pmod{24}$,

(3) $\prod t^{|r_t|}$ is a square.

Choose a positive integer N satisfying

(4) $r_t=0$ if $t|N$,

(5) $\sum\frac{N}{t}\cdot r_t\equiv 0\pmod{24}$.

Then $\eta_\pi(z)$ is invariant under $\Gamma_0(N)$.

Theorem 1 is useful when we wish to show that $\eta_\pi(z)=\prod\eta(tz)^{r_t}$ is invariant under $\Gamma_0(N)$. However, there are not many N such that $\Gamma_0(N)\backslash\mathbb{H}^*$ is of genus 0. Therefore, we need to investigate a larger class of Fuchsian groups. Typically we deal with a group

$$\Gamma=\langle\Gamma_0(N),W_e,W_f,\dots\rangle,$$

where $W_e,W_f,\dots$ are Atkin-Lehner involutions of $\Gamma_0(N)$ (or conjugates of such Γ). The author gives a set of conditions on π in order for $\eta_\pi(z)$ to be invariant under $\Gamma_0(N)$ and W_e.

The author provides some examples for which $\eta_\pi(z)$ is invariant under $\Gamma_0(N)$, where $\{N,\pi\}$ is given, and other examples for which $\eta_\pi(z)$ is invariant under $\Gamma_0(N)+$, where

$\{N, \pi\}$ is given. He shows that the genera of the Riemann surfaces $\Gamma\backslash\mathbb{H}^*$ for the Fuchsian groups Γ appearing in these examples are all 0. Hence their function fields $k(\Gamma\backslash\mathbb{H}^*)$ are generalized by just one function: $k(\Gamma\backslash\mathbb{H}^*) = \mathbb{C}(f)$. The author proves that the $\eta_\pi(z)$ in these examples are the f for the corresponding Γ:

Theorem 2. Let π be a generalized partition. Assume that

(1) $\sum_t tr_t = -24$,
(2) $\eta_\pi(z)$ is invariant under the action of a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ containing $\Gamma_0(N)$ for some N,
(3) $\Gamma_\infty = \{\alpha \in \Gamma\colon \alpha(\infty) = \infty\}$ is equal to $\left\{\begin{pmatrix} \pm 1 & n \\ 0 & \pm 1 \end{pmatrix}\colon n \in \mathbb{Z}\right\}$,
(4) $z = \infty$ is the unique pole of $\eta_\pi(z)$ among all inequivalent cusps of Γ.

Then $\eta_\pi(z) \in \mathfrak{F}$ and the function field of $\Gamma\backslash\mathbb{H}^*$ is equal to $\mathbb{C}(\eta_\pi(z))$.

In the last chapter, the author gives McKay's conjecture on the multiplicative product of η functions and its proof. More precisely, let $\eta(z)$ be the Dedekind η function and $g = (k_1, k_2, k_3, \ldots, k_s)$ be a tuple of positive integers k_i, $k_1 \geq k_2 \geq \cdots \geq k_s \geq 1$, $1 \leq i \leq s$. Define

$$\eta_g(z) = \prod_{i=1}^{s} \eta(k_i z).$$

McKay conjectured that $\eta_g(z)$ is a primitive cusp form of some level N and some character χ if and only if the following conditions are satisfied:

(a) k_1 is a multiple of all k_i, $1 \leq i \leq s$,
(b) $k_1 k_s = k_i k_{s+1-i}$ for all $1 \leq i \leq s$,
(c) $\sum_{i=1}^{s} k_i = 24$, and
(d) s is even.

This conjecture was shown to hold by Dummit, Kisilevsky, and McKay, and an alternative proof was given by M. Koike.

Theorem 3. Let $g = (k_1, \ldots, k_s)$ be a tuple of positive integers satisfying the conditions (a)–(d) above. Then $\eta_g(z)$ is a primitive cusp form of weight $s/2$, of level $N = k_1 k_s$ and with some Dirichlet character χ. Conversely, if $\eta_g(z)$ is a primitive cusp form of some weight, level, and character, then g must satisfy (a)–(d).

This book is primarily for the benefit of readers who wish to start learning about modular functions used in moonshine.

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