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★3-manifold groups.

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The book under review gives a detailed survey of certain facets of the very extensive theory of 3-manifold groups. The authors concentrate on results around and following from Perel'man's proof of the full Geometrization Conjecture, Kahn and Marković's proof of the Surface Subgroup Conjecture, Wise's theory of groups with quasi-convex hierarchies, and Agol's proof of Thurston's Virtual Haken Conjecture and Virtual Fibered Conjecture.

The book is organized by topics in logical progression, as summarized below. The purpose of the book is to give as detailed an account of the theory as possible, emphasizing the historical and conceptual development of the theory. A relatively limited number of proofs are given. The authors give extensive commentaries on results surrounding the main theorems, and they discuss at length the contributions of the multitude of authors to the discussion at hand.

Decomposition theorems. The authors summarize some classical results on decomposing 3-manifolds into canonical pieces, up through Perel'man's proof of the full Geometrization Conjecture. Topics include the topological vs. the PL vs. the smooth category, the Prime Decomposition Theorem, the Loop and Sphere Theorems, Seifert fibered manifolds, the JSJ Decomposition Theorem, geometrization à la Thurston and Perel'man, geometrization of fibered 3-manifolds à la Thurston, and the classification of abelian, nilpotent, and solvable groups appearing as fundamental groups of 3-manifolds.

Classification of 3-manifolds by their fundamental groups. The authors systematically investigate how the algebraic structure of the fundamental group of a 3-manifold can determine the topological properties of the manifold. Topics include Kneser's Conjecture, Waldhausen's Theorem, peripheral structures on 3-manifolds with boundary, conjugacy classification of compact submanifold groups, and topological consequences of noncyclic centralizers of elements in 3-manifold groups.

3-manifold groups after geometrization. This chapter gives a detailed description of the various group-theoretic properties enjoyed by the fundamental group of a compact, orientable, irreducible 3-manifold with empty or toroidal boundary, concentrating on the case of infinite, non-virtually solvable groups.

The work of Agol, Kahn-Marković, and Wise. This chapter summarizes the monumental progress made in the last 30 years towards understanding hyperbolic 3-manifold groups and 3-manifold groups with nontrivial JSJ decomposition, concentrating on the contributions of Agol, Kahn and Marković, and Wise, and on the centrality of right-angled Artin groups and their subgroups to the theory. Topics include the Tameness Theorem of Agol and of Calegari and Gabai, the Virtually Compact Special Theorem, the theory of special cube complexes, Wise's theorem on groups with quasi-convex hierarchies, Kahn and Marković's proof of the Surface Subgroup Conjecture, Agol's proof of the Virtual Haken Conjecture, and the virtual speciality of 3-manifold groups with nontrivial JSJ decomposition.

Consequences of the Virtually Compact Special Theorem. This chapter discusses the very large number of consequences of the Virtually Compact Special Theorem. A useful

summary is given by a flowchart on page 94. This chapter differs significantly from the previous ones in that more proofs of the various claims are given.

Subgroups of 3-manifold groups. This chapter discusses the structure of nontrivial finitely generated subgroups of the fundamental group of a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. By Scott's Core Theorem, any such subgroup is finitely presented. The specific types of subgroups which are considered include abelian, finite, and normal subgroups. The case where the ambient group is the fundamental group of a hyperbolic manifold is also considered in detail. In many cases, the membership problem for these subgroups is solvable.

Open questions. The book closes with a discussion of central open problems in the theory. These problems include Wall's Conjecture on Poincaré duality groups, Cannon's Conjecture, and the Simple Loop Conjecture. Other open problems concern a group-theoretic characterization of knot groups, non-LERF of 3-manifold groups with nontrivial JSJ decomposition, linearity of 3-manifold groups, stable commutator length and the PQL property, random models for 3-manifolds, and finite covers of 3-manifolds.

This book performs an invaluable service to the mathematical community by carefully documenting the topics in 3-manifold theory in a relatively concise and digestible manner. This service is important in light of the stupendous speed at which the theory has been developing.

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