Thomas, Anne

Geometric and topological aspects of Coxeter groups and buildings. (English) Zbl 06891810

Zurich Lectures in Advanced Mathematics. Zürich: European Mathematical Society (EMS) (ISBN 978-3-03719-189-7/pbk; 978-3-03719-689-2/ebook). xii, 147 p. (2018).

This book originated from lectures for graduate students given by the author at the ETH Zurich and the Institute for Mathematical Research (FIM). It has the liveliness and occasionally the informality of live lectures. This is a text directed to graduate students, who have already a broad knowledge of mathematics.

The author gives an extensive survey of Coxeter groups and of buildings, drawing from the contents of many books. She gives frequent and detailed references to books on her list of references, making them a lively and essential part of her presentation. She emphasizes [16] and [23] for Coxeter groups and [10] and [23] for buildings. The author sets the stage in Chapter 1 "Examples and motivation". Here she presents examples of geometric reflection groups such as dihedral groups, triangle groups and isometry groups of classical polytopes accompanied by colourful enlightening figures. Triangle groups and isometry groups of classical polytopes are studied. Colourful figures, display barycentric subdivisions of the cube and the dodecahedron. She introduces symmetric groups, tiling of Euclidean space by cubes, tilings of the hyperbolic plane by right-angled polygons and tiling of hyperbolic 3-space by right-angled dodecahedra. There are colourful figures of tessellations, barycentric subdivision of the cube and the dodecahedron.

Let I be a finite indexing set and let $S = \{s_i\}_{i\in I}$. Let $M = (m_{ij})_{i,j\in I}$ be a matrix such that $m_{ii} = 1$ for all $i \in I$; $m_{ij} = m_{ji}$ for all $i, j \in I$; and $m_{ij} \in \{2, 3, 4, \dots\} \cup \{\infty\}$ for all distinct $i, j \in I$. The group $W = \langle S \mid (s_i s_j)^{m_{ij}} = 1\ \forall\, i, j \in I\rangle$ is called Coxeter group and (W, S) is called a Coxeter system. Weyl groups are introduced and certain finite Coxeter groups are seen to be Weyl groups.

In Chapter 2 the author investigates Coxeter systems, the Cayley graphs of Coxeter systems, reflection systems, the deletion condition and the exchange condition.

In Chapter 3 the author introduces the Tits representation $\rho : W \to \mathrm{GL}_n(\mathbb{R})$ of a Coxeter group W. She investigates its construction and faithfulness as well as the geometric realization of finite and affine Coxeter groups. She gives the motivation for other geometric realizationes.

In Chapter 4 "The basic construction of a geometric realization" the author first investigates simplicial complexes, followed by the definition of the mirror structure on X over S for a Coxeter system (W, S) and a connected, Hausdorff topological space X. She defines the basic construction as the quotient $\mathcal{U}(W, X) = W \times X/\sim$ equipped with the quotient topology. An investigation of its properties follows. She studies the action of W on the basic construction, the universal property of the basic construction, and the basic construction and geometric reflection groups.

In Chapter 5 the author defines: The Davis complex $\Sigma = \Sigma(W, S)$ is the basic construction $\Sigma = \mathcal{U}(W, K) = W \times K/\sim$, where K is a certain chamber with mirror structure $(K_s)_{s\in S}$. She investigates spherical special subgroups and the nerve as well as the Davis complex as the geometric realization of a poset. She proves the following assertions. The Davis complex $\Sigma = \Sigma(W, S)$ is contractable; We can identify the Davis complex Σ with the geometric realization of the poset $\{wW_T \mid w \in W, T \subseteq S \text{ and } W_T \text{ is finite}\}$, ordered by inclusion; We can identify the Coxeter complex with the geometric realization of the poset $\{wW_T \mid w \in W, T \subseteq S\}$, ordered by inclusion. Then she considers the Davis complex as a CW complex; and shows that the Davis complex is CAT(0); she investigates the cohomology of Coxeter groups and gives applications.

In Chapter 6 "Buildings as unions of apartments" the author gives the Definition: Let (W, S) be a Coxeter system. A building of type (W, S) is a simplicial complex Δ which is a union of subcomplexes called apartments with each apartment being a copy of the Coxeter complex for (W, S). The maximal simplices in Δ are called chambers and (1) Any two chambers are contained in a common apartment. (2) If A and A' are two apartments, there is an isomorphism $A \to A'$ which fixes $A \cap A'$ pointwise. First examples of buildings follow: a 'thin building', a Coxeter complex, a tessellation of $\mathbb{X}^n$, a Davis complex, a collections of points, complete bipartite graph $W \cong C_2 \times C_2$, (there are colourful pictures), trees, products of trees, products of buildings, Bourdon's building. There is also an extended example: the building for $GL_3(q)$. The Heawood graph and also the Bruhat decomposition of G are introduced.

Chapter 7 "Buildings as chamber systems" contains: Definition: Let (W, S) be a Coxeter system with $S = \{s_i \mid i \in I\}$. A building of type (W, S) is a chamber system $\triangle$ over I which is equipped with a W-valued distance function, and is such that each panel has at least two chambers. The author shows the equivalence of the two definitions of a building; and she compares the definitions. The main advantage of the first definition of a building is that it has good geometric and topological properties as shown in Theorem 7.14. Let Δ be a building (as in 6) so that its apartments are copies of some basic construction $\mathcal{U}(W, X)$. (1) If Δ is a spherical building, then Δ is a CAT(1) space. (2) If Δ is a Euclidean or affine building, then Δ is a CAT(0) space. (3) If Δ is a hyperbolic building, then Δ is a CAT(-1) space.

Chapter 8 "Retractions" Let Δ be a building of type (W, S). A retraction is a mapping from Δ to any of its apartments. Examples : $\Delta = T_3$ being the 3-regular tree or the product of two 3-regular trees, and also for the case that Δ is the building $\mathrm{GL}_3(q)$.

In Chapter 9 "BN-pairs" the author defines: Let G be a group. A *BN-pair* or a *Tits system* in G is a pair of subgroups (B, N) such that (0) G is generated by B and N; (1) $T = B \cap N$ is normal in N, and $W = N/T$ is a Coxeter group with distinguished generating set $S = \{s_i | i \in I\}$; (2) for all $w \in W$ and all $s_i \in S$, $BwB \cdot Bs_iB = BwBs_iB \subseteq BwB \cup Bws_iB$, (3) for all $i \in I, s_iBs_i^{-1} = s_iBs_i \neq B$. The author gives $G = \mathrm{GL}_3(q)$ as an example. She shows the following. If the group G has a *BN-pair* then $G = \bigsqcup_{w \in W} BwB$. If $i \in I$ then $P_i = \sqcup Bs_iB$ is a subgroup of G. Let $J \subseteq I$ then $P_J = \bigsqcup_{w \in W_J} BwB$ is a subgroup of G. In 9.5, "Spherical, affine and Kac-Moody BN-pairs" the author quotes several results of Tits.

In Chapter 10 the author reports on "Exotic buildings".

The author gives a lively introduction to Coxeter groups and buildings. She covers a large number of topics and draws on many of the available sources in the literature. As has to be expected, several results are only stated or just indicated. On the other hand, this presentation contains many illuminating examples, often worked out in detail. There are also many colourful diagrams facilitating the understanding. This book may serve to introduce graduate student to the intricacies of this field and prepare them for research in this topic.

Reviewer: Erich W. Ellers (Toronto)

MSC:

20-02	Research monographs (group theory)
20E42	Groups with a BN-pair; buildings
20F55	Reflection groups; Coxeter groups
51E24	Buildings and the geometry of diagrams
57M07	Topological methods in group theory

Keywords:

Coxeter group; reflection group; Tits representation; basic construction; simplicial complex; Davis complex; building; chamber system; BN-pair

Full Text: DOI