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★**Large scale geometry.**

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An area of mathematics is, to some extent, characterized by the types of embeddings which it studies. The most important for large scale geometry are the following two types of embeddings: Let $f\colon (X,d_X)\to(Y,d_Y)$ be a map between two metric spaces.

The map f is called a *quasi-isometric embedding* if there exist constants $0<L,C<\infty$ such that

$$\forall u,v\in X,\quad L^{-1}d_X(u,v)-C\le d_Y(f(u),f(v))\le Ld_X(u,v)+C.$$

The map f is called a *coarse embedding* if there exist non-decreasing functions $\rho_-,\rho_+\colon[0,\infty)\to[0,\infty)$ (observe that this condition implies that ρ_+ has finite values) such that $\lim_{t\to\infty}\rho_-(t)=\infty$ and

$$\forall u,v\in X,\quad \rho_-(d_X(u,v))\le d_Y(f(u),f(v))\le\rho_+(d_X(u,v)).$$

It is clear that these definitions impose no restrictions on bounded maps of bounded metric spaces; the definitions become useful only at a large scale.

The book of P. W. Nowak and G. L. Yu contains a nice overview of some of the most important notions and techniques of large scale geometry. We list the topics contained in the book by chapter.

Chapter 1: Basic definitions and results, including the Milnor-Schwarz lemma [A. S. Schwarz, Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 32–34; MR0075634; J. W. Milnor, J. Differential Geometry **2** (1968), 1–7; MR0232311] and a very short introduction to Gromov hyperbolic spaces [M. Gromov, in *Essays in group theory*, 75–263, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987; MR0919829].

Chapter 2: (1) Asymptotic dimension, introduced by Gromov [in *Geometric group theory, Vol. 2 (Sussex, 1991)*, 1–295, London Math. Soc. Lecture Note Ser., 182, Cambridge Univ. Press, Cambridge, 1993; MR1253544], and (2) the much more recent notion of decomposition complexity, introduced by E. Guentner, R. Tessera and Yu [Invent. Math. **189** (2012), no. 2, 315–357; MR2947546].

Chapter 3: Amenability. This is a classical notion introduced by J. von Neumann [Fund. Math. **13** (1929), 73–116; JFM 55.0151.01]. This notion is the subject of many surveys and monographs, such as [A. L. T. Paterson, *Amenability*, Math. Surveys Monogr., 29, Amer. Math. Soc., Providence, RI, 1988; MR0961261]. The authors provide a short introduction restricted to the case of finitely generated groups (the most important case for this book).

Chapter 4: Property A, introduced by Yu [Invent. Math. **139** (2000), no. 1, 201–240; MR1728880] in his work on the Baum-Connes conjecture; later it found many other applications. This metric property of a discrete space can be regarded as a weak version of amenability [N. Higson and J. Roe, J. Reine Angew. Math. **519** (2000), 143–153; MR1739727]. It is also shown (among other things) that finite asymptotic dimension implies Property A.

Chapter 5: Coarse embeddings, introduced by Gromov in [op. cit., MR1253544] under the name *uniform embeddings*; the name was later changed to avoid confusion with the standard notion in functional analysis. Here we find the following results (among others): (1) Property A implies coarse embeddability into Hilbert space [G. L. Yu, op. cit.]. (2) There exist locally finite metric spaces, which are coarsely embeddable into Hilbert space, but do not have Property A. This is a result of Nowak [J. Funct. Anal. **252** (2007), no. 1, 126–136; MR2357352]; a stronger recent example of G. N. Arzhantseva, Guentner and J. Špakula [Geom. Funct. Anal. **22** (2012), no. 1, 22–36; MR2899681], where the space has bounded geometry, is mentioned, but without proof. (3) Embeddability of bounded geometry metric spaces into strictly convex reflexive spaces ([N. P. Brown and E. Guentner, Proc. Amer. Math. Soc. **133** (2005), no. 7, 2045–2050; MR2137870]; see [M. I. Ostrovskii, Proc. Amer. Math. Soc. **140** (2012), no. 8, 2721–2730; MR2910760] and references therein for more general results). (4) Expanders: definition and coarse non-embeddability into Hilbert space (constructions are presented in Chapter 6). (5) Generalized expanders and coarse non-embeddability into Hilbert space. The presentation follows the work of Tessera [J. Topol. Anal. **1** (2009), no. 1, 87–100; MR2649350] (a slightly different result was independently proved by Ostrovskii [Topology Proc. **33** (2009), 163–183; MR2471569]). (6) Hilbert space compression, introduced by Guentner and J. Kaminker [J. London Math. Soc. (2) **70** (2004), no. 3, 703–718; MR2160829]. Here we find the result on Hilbert space compression of trees and (Compression $> \frac{1}{2}$) $\Rightarrow$ (Property A).

Chapter 6: Group actions on Banach spaces. Here we find an introduction to (1) affine isometric actions; (2) a-T-menability [M. Gromov, op. cit., MR1253544], also known as the Haagerup property [U. Haagerup, Invent. Math. **50** (1978/79), no. 3, 279–293; MR0520930], its relations with amenability and coarse embeddability into ℓ_2; (3) actions on ℓ_p-spaces; (4) Kazhdan's property (T) [D. A. Kazhdan, Funkcional. Anal. i Priložen. **1** (1967), 71–74; MR0209390] and a construction of expanders based on existence of finitely generated residually finite groups with property (T) [G. A. Margulis, Problemy Peredači Informacii **9** (1973), no. 4, 71–80; MR0484767]; and (5) Żuk's spectral criterion for property (T) [A. Żuk, Geom. Funct. Anal. **13** (2003), no. 3, 643–670; MR1995802].

Chapter 7: Coarse homology. This chapter contains an introduction to coarse locally finite homology introduced by Roe in [Mem. Amer. Math. Soc. **104** (1993), no. 497, x+90 pp.; MR1147350] and to uniformly finite homology introduced by J. L. Block and S. Weinberger [J. Amer. Math. Soc. **5** (1992), no. 4, 907–918; MR1145337], with an application to aperiodic tilings.

Chapter 8 contains a brief discussion of applications of large scale geometry to topology, geometry and index theory.

The book contains many exercises and open problems. Each chapter except Chapter 8 contains a Notes and Remarks section in which relevant results and references are mentioned.

This book will be very useful for people trying to enter this fascinating and currently very active research area, which has important connections with topology, group theory, and functional analysis.

There are a nontrivial number of misprints in this book; hopefully the authors will collect information about these misprints on their web pages. *Mikhail Ostrovskii*