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★**Cohomological theory of crystals over function fields.**
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Let k be a finite field with q elements, and let A be a finitely generated Dedekind domain over k with A^* finite. At first glance, this book appears to be concerned with so-called A-motives, which are generalizations of Drinfeld modules, and later Anderson's t-motives. In fact the authors go much further and study a more flexible notion, which they call τ-sheaves. This generalization is crucial as it enables the authors to introduce certain triangulated categories of τ-sheaves that behave much as the bounded derived category of constructible sheaves in the ℓ-adic setting.

The definition of a τ-sheaf is as follows: Let X/k be a finite type k-scheme, and let

$$\sigma_X\colon X\to X$$

be the q-power Frobenius morphism on X. Then a coherent τ-sheaf over A on X is a pair $\underline{\mathcal{F}}=(\mathcal{F},\tau_{\mathcal{F}})$, where $\mathcal{F}$ is a coherent sheaf on $X\times C$ and $\tau_{\mathcal{F}}$ is a morphism of coherent sheaves on $X\times X$,

$$(\sigma_X\times\mathrm{id})^*\mathcal{F}\to\mathcal{F}.$$

Here C denotes the spectrum of A. Morphisms of τ-sheaves are defined in the natural way.

It turns out that in order to get a good theory for compactly supported cohomology, one must invert a certain subcategory of the category of coherent τ-sheaves. Namely, a τ-sheaf $(\mathcal{F},\tau_{\mathcal{F}})$ is called nilpotent if some iterate $\tau_{\mathcal{F}}^n$ of $\tau_{\mathcal{F}}$ is zero. Inverting morphisms that are nilpotent, the authors obtain a new category $\mathrm{Crys}(X,A)$, which they call the category of A-crystals. One main point of the book is the development of suitable derived categories of A-crystals and the construction of functors f^*, $\otimes^{\mathbb{L}}$, and $Rf_!$ on these derived categories that behave analogously to the corresponding operations in ℓ-adic theory.

Now the main reason for studying these τ-sheaves is that there are natural L-functions attached to them. Namely, if $(\mathcal{F},\tau_{\mathcal{F}})$ is such a sheaf over a scheme X, then for any k-point $x\in X(k)$ we can consider the A-module M_x obtained by restricting $\mathcal{F}$ to $\{x\}\times C$. The map $\tau_{\mathcal{F}}$ defines an A-module homomorphism

$$\tau_x\colon M_x\to M_x$$

and, assuming some flatness hypothesis (for example that $\mathcal{F}$ is pulled back from a sheaf on X), one can consider the characteristic polynomial of this endomorphism. This can also be defined for points over finite extensions of k by weighting the characteristic polynomial by the degree of the field extension. Multiplying over all closed points of X one obtains the naive L-function

$$L^{\mathrm{naive}}(X,\underline{\mathcal{F}},t)\in 1+tA[[t]].$$

Perhaps the main theorem of the book is a trace formula relating this L-function to the L-function attached to the compactly supported cohomology of the τ-sheaf $(\mathcal{F},\tau_{\mathcal{F}})$.

The spirit is very much as in the étale setting. As in the étale setting, the heart of the matter is to lay the correct foundations for the derived categories and the functors between them, and much of the book is devoted to these foundations.

Finally, it should be mentioned that the introduction is very clear and a pleasure to read. *Martin C. Olsson*