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★**Foundations of Garside theory.**
With François Digne, Eddy Godelle, Daan Krammer and Jean Michel.
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The book under review describes in detail a new field of study in algebra that has emerged over the last few decades; its origins lie in the study of the braid group.

In the 1960s, Frank Garside introduced a new algebraic aspect to the study of the braid group B_n in his solution to the conjugacy problem in B_n [Quart. J. Math. Oxford Ser. (2) **20** (1969), 235–254; MR0248801]: Garside defined a submonoid B_n^+ of B_n and a distinguished element $\Delta_n \in B_n^+$ with the property that every element $g \in B_n$ can be written in the form $g = \Delta_n^m g'$ with $m \in \mathbb{Z}$ and $g' \in B_n^+$. He moreover proved that least common multiples exist in the monoid B_n^+. Building on Garside's work, Adyan, Elrifai and Morton and Thurston independently showed that every element of B_n has a distinguished representation as a product of divisors of Δ_n [S. I. Adyan, Mat. Zametki **36** (1984), no. 1, 25–34; MR0757642; E. A. Elrifai and H. R. Morton, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 479–497; MR1315459; D. B. A. Epstein et al., *Word processing in groups*, Jones and Bartlett, Boston, MA, 1992; MR1161694]; this normal form of elements has proved crucial to the study of both structural and algorithmic properties of braids.

Garside's approach was soon realised to be applicable to more general situations: Brieskorn and Saito and Deligne independently generalised the above methods to the braid groups associated to arbitrary finite Coxeter groups, i.e., Artin-Tits groups of spherical type; in this situation, the role that the symmetric group (the Coxeter group of type A) plays in the case of the braid group B_n is assumed by another finite Coxeter group [E. V. Brieskorn and K. Saito, Invent. Math. **17** (1972), 245–271; MR0323910; P. Deligne, Invent. Math. **17** (1972), 273–302; MR0422673].

In the 1990s, further generalisations and abstractions emerged, resulting in the definition of *Garside groups* and *Garside monoids* [P. Dehornoy and L. Paris, Proc. London Math. Soc. (3) **79** (1999), no. 3, 569–604; MR1710165]; the key concept is a monoid that is a combinatorial lattice with respect to left-divisibility and right-divisibility. The notions of Garside monoids and Garside groups have since become firmly established, and they have been instrumental in achieving many new results, both structural and algorithmic, for braids as well as in the general setting.

However, although the notions of Garside groups and Garside monoids have been useful, there are many instances in which they are not applicable although the "spirit" of Garside normal forms can be made to work: Specifically, Noetherianity conditions or requiring the non-existence of non-trivial invertible elements turn out to be dispensable. Moreover, a category theoretic setting is more useful in many cases.

The text under review presents a framework for a "maximal" generalisation of the ideas developed by Garside and others based on Garside's work.

The setting is a left-cancellative category $\mathcal{C}$ with a distinguished family $\mathcal{S} \subseteq \mathcal{C}$ of morphisms; we denote the set of invertible morphisms of $\mathcal{C}$ by $\mathcal{C}^\times$ and define $\mathcal{S}^\sharp = \mathcal{S}\mathcal{C}^\times \cup \mathcal{C}^\times$. Given two morphisms $g_1, g_2 \in \mathcal{C}$ that can be composed, we write their product in $\mathcal{C}$ as $g_1 g_2$, and distinguish it from the sequence (or path) of length 2 with the entries g_1

and g_2; the latter is written as $g_1|g_2$. Products and paths of length greater than 2 are defined analogously. For $f, h \in \mathcal{C}$ we write $f \preccurlyeq h$ if there exists $g \in \mathcal{C}$ such that $fg = h$ holds in $\mathcal{C}$.

Definition: Given $\mathcal{S} \subseteq \mathcal{C}$ and $g_1, g_2 \in \mathcal{C}$, the path $g_1|g_2$ is called $\mathcal{S}$-*greedy* if $s \preccurlyeq fg_1g_2$ for $s \in \mathcal{S}$ and $f \in \mathcal{C}$ implies $s \preccurlyeq fg_1$. A path $g_1|\cdots|g_k$ is called $\mathcal{S}$-greedy if $g_i|g_{i+1}$ is $\mathcal{S}$-greedy for every i.

A path is called $\mathcal{S}$-*normal* if it is $\mathcal{S}$-greedy and all its entries lie in $\mathcal{S}^\sharp$.

Definition: The family $\mathcal{S}$ is called a *Garside family* for $\mathcal{C}$ if for every element $f \in \mathcal{C}$ there exists an $\mathcal{S}$-normal path $g_1|\cdots|g_k$ such that $f = g_1 \cdots g_k$ holds.

Definition: A Garside family $\mathcal{S}$ for $\mathcal{C}$ is called *bounded* if there exists a map $\Delta\colon \mathrm{Obj}(\mathcal{C}) \to \mathcal{C}$ such that for all $x \in \mathrm{Obj}(\mathcal{C})$ one has the following:

(a) $\Delta(x)$ is an element of $\mathcal{S}$ with source x.

(b) For every $s \in \mathcal{S}$ with source x, one has $s \preccurlyeq \Delta(x)$.

(c) There exists $y \in \mathrm{Obj}(\mathcal{C})$ such that for every $s \in \mathcal{S}^\sharp$ with target x, there exists an $r \in \mathcal{S}^\sharp$ with source y such that $rs = \Delta(y)$ holds.

The connection with the notion of Garside monoids is as follows: If M is a Garside monoid with Garside element Δ, then the set of divisors of Δ forms a bounded Garside family for M.

Almost all known properties of Garside monoids extend to categories that admit a bounded Garside family. Moreover, in many cases where properties of Garside monoids don't readily extend to categories with a more general Garside family, addressing the questions in the more general setting leads to a deeper understanding and improved arguments.

The text is divided into two parts: The first part develops the general theory in detail; the second part consists of more specialised chapters, which are to some extent independent from each other and discuss special aspects or more intricate classes of examples.

Chapter I gives an introduction and a motivation by describing examples that illustrate the conventional theory of Garside monoids, as well as instances that cannot be captured by the conventional theory although normal forms with similar properties exist.

Chapter II describes preliminaries, including general results about the *reversing method*, a combinatorial tool of independent interest for dealing with the presented monoids or categories.

Chapter III develops the notions of normal decomposition and Garside families.

Chapter IV discusses properties that ensure that a certain subfamily of a given category is a Garside family.

Chapter V introduces the notion of bounded Garside families, connecting the new framework to the conventional theory of Garside monoids.

Chapter VI discusses the notion of *Garside germs*, dealing with the situation that a set $\mathcal{S}$ with a partial multiplication is given and the question is whether there is a category $\mathcal{C}$ into which $\mathcal{S}$ embeds as a Garside family. This approach is a rich source of examples, generalising the construction of Artin-Tits monoids from Coxeter groups.

Chapter VII investigates the connection between Garside families for a category $\mathcal{C}$ and Garside families for a subcategory of $\mathcal{C}$.

Chapter VIII addresses questions of conjugacy, including an extension of geometric methods for describing periodic elements.

Chapters IX to XIV form the second part of the text. Topics discussed include generalised braid groups (classical and dual Artin-Tits monoids), Deligne-Lusztig varieties, the geometry monoid of left self-distributivity, ordered groups, and set-theoretic solutions to the Yang-Baxter equation, as well as divided and decomposition categories, cyclic systems (a generalisation of RC-systems), an analogue of the braid group for $\mathbb{Z}^n$, and groupoids of cell decompositions. *Volker Gebhardt*