Dehornoy, Patrick; Digne, François; Godelle, Eddy; Kramer, Daan; Michel, Jean* 1370.20001
Foundations of Garside theory.
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Artin's braid group is the first example of a *Garside group*, a group G with a lattice structure, so that positive elements are products of irreducibles (atoms) and the number of factors is bounded. The lattice order is given by left divisibility, and there is a *Garside element* Δ, an element for which the interval $[1, \Delta]$ is finite and generates the positive cone G^+ as a monoid, and $\Delta G^+ = G^+\Delta$. The braid group B_n maps onto the symmetric group S_n, a Coxeter group of type $\mathbb{A}_{n-1}$.

E. Brieskorn and *K. Saito* [Invent. Math. 17, 245–271 (1972; Zbl 243.20037)] proved that all Artin groups related to finite Coxeter groups are Garside. In [Adv. Math. 139, No. 2, 322–353 (1998; Zbl 937.20016)], *J. Birman* et al. found a "dual" Garside structure of B_n, where the set of atoms is enlarged to the set of all reflections, so that the Garside element becomes shorter and turns into a Coxeter element. Similar dual Garside structures were found for all Artin groups [*D. Bessis*, Ann. Sci. Éc. Norm. Supér. (4) 36, No. 5, 647–683 (2003; Zbl 1064.-20039)]. Garside structures also occurred in low-dimensional topology, and in connection with self-distributive systems.

The book under review is a timely account on various incarnations of Garside phenomena, to make this topic accessible to a wide audience. The book is divided into two parts. The general part starts with a chapter on examples of a prototypical nature, beginning with ordinary and dual braid monoids, to motivate the concept of Garside group. Another example is given by torus knot groups. Quasi-Garside groups are exemplified by D. Bessis' dual braid monoid for the free group F_n on n generators, based on the Hurwitz action of the braid group B_n. To show that Garside features exist beyond the limitations of quasi-Garside groups, infinite braids (which have no Garside element), and the fundamental group of the Klein bottle (where the Noetherian property fails), are mentioned. The second part deals with more specific examples. As a golden thread through the whole complex of Garside phenomena, the *normal decomposition* of elements, preconceived in Garside's representation of braids as products $\Delta^m a$, is studied extensively. At the end of the first chapter, the reader is informed that the aim of the book is not to establish a theory of Garside groups in the strict sense, but rather to lay the foundation of an extended framework, to cover those examples where Garsideness appears in one or the other form. As a main structural result on Garside groups, almost the only one thus far, Picantin's iterated decomposition theorem of Garside groups is mentioned without proof.

After a chapter on preliminaries, the general part of the book takes off with an analysis of normal decompositions. For the beginner, it may take a little time to get used to the language, as the concepts are developed in the context of categories, with the usual order of composition changed – for practical reasons – to 'left-right'. Each chapter of the general part starts with a collection of main definitions and results. Combined with an efficient index, this greatly facilitates a quick orientation in the 700 pages treatise.

Garside families are analysed in the next two chapters. It is shown that up to units, they are closed with respect to right divisor and satisfy a weak lcm existence condition. Simple characterizations of Garside families in terms of such closure properties are provided. The Noetherian case is discussed separately. Here, the original definition of a Garside element as a least common multiple of the atoms [*P. Dehornoy* and *L. Paris*, Proc. Lond. Math. Soc. (3) 79, No. 3, 569–604 (1999; Zbl 1030.20021)] reappears in a statement asserting the existence of a smallest unit-closed Garside family. Without a Noetherian hypothesis, even in the discrete case, a smallest Garside family need not exist, as the Klein bottle monoid shows.

As Garside families give rise to a presentation of the ambient category, it is natural to characterize them by intrinsic properties. Endowed with a partial composition, they determine the ambient category (or monoid in case of a single object). Slight modifications of these *Garside germs* were introduced in the category context by Digne and Michel, and for Garside groups by *D. Bessis* et al. [Pac. J. Math. 205, No. 2, 287–309 (2002; Zbl 1056.20023)].

The second part of the book starts with Artin groups and their realization as complex reflection groups. To any finite Coxeter group W, let $\mathcal{H}$ be the union of the complexified reflection hyperplanes. The pure Artin group can then be seen as the fundamental group of the complement $M(W)$ of $\mathcal{H}$, while the full Artin group G is $\pi_1(M(W)/W)$. Deligne has shown that $M(W)/W$ is a $K(G, 1)$-space, that is, its universal cover is contractible. Ordinary and dual Garside structures are discussed for spherical Artin groups G. In the dual Garside structure, a Coxeter element c is Garside, and the interval $[1, c]$ is isomorphic to the lattice of (generalized) non-crossing partitions.

A chapter on Deligne-Lusztig varieties deals with geometric consequences of the abelian defect conjecture inasmuch as braid groups are involved. The next two chapters are on self-distributivity (as mentioned above) and right orders on Garside-like groups. The latter means that the partial order extends to a right invariant linear order. Introduced by Paul Conrad in 1959, right ordered groups always admit a left order, and vice versa. Certain mapping class groups, including braid groups and all knot groups, are (left) orderable, but it is widely open which fundamental groups of 3-manifolds are so.
Most of the above mentioned Garside-like groups, with the exception of the Klein bottle group, have an underlying lattice which is far from distributive. The next chapter deals with the structure groups of a class of set-theoretic solutions to the Yang-Baxter equation (YBE). Chouraqui observed in 2010 that the structure group associated to a finite involutive non-degenerate solution is a Garside group. The underlying lattice of such a group is distributive. Some additions and more examples are given in a final chapter. Apart from the more advanced chapter on Deligne-Lusztig varieties which is more loosely connected to Garside structure, the book gives a self-contained, comprehensive, and up-to-date account on an exciting topic, addressed to a wide public to take part in the shaping and development of a rising theory around braiding and ordering. Equipped with numerous examples, 130 exercises, and 40 questions, the book can serve as an introduction as well as a source of inspiration for those who want to know more about Garside's thriving heritage. *Wolfgang Rump (Stuttgart)*