Jahresber Dtsch Math-Ver (2016) 118:67–72
DOI 10.1365/s13291-015-0121-2

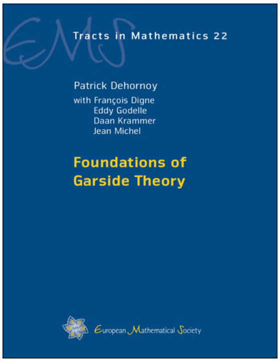
BOOK REVIEW

Patrick Dehornoy, François Digne, Eddy Godelle, Daan Krammer, Jean Michel: "Foundations of Garside Theory"

EMS Tracts in Mathematics, vol. 22, EMS, 2015, 710 pp.

Wolfgang Rump[1]

Published online: 19 August 2015



In 1925, Emil Artin invented the braid group to understand knots and links. As every conjugacy class in the braid group determines a link, the first challenge was to classify conjugacy classes in the braid group B_n on n strings. Forty years passed, until Frank Arnold Garside (1915–1988) in his 1965 thesis revealed the lattice structure of B_n, which enabled him to solve the conjugacy problem for all $n \geq 3$. In particular, he discovered a "fundamental element" $\Delta \in B_n$ which is *normal* in the sense that conjugation with Δ maps the cone of positive braid words onto itself. Thus any braid word can be put into a standard form $\Delta^m a$ with $m \in \mathbb{Z}$ where the positive braid a is taken from a finite set. With a little further effort, the conjugacy problem was solved. For $n \geq 3$, the centre of B_n is generated by Δ^2.

Artin's braid group became the prototype of what is now called a *Garside group*. Such groups are equipped with a lattice structure, every positive element is a product of irreducibles (atoms) such that the number of factors is bounded. The lattice order is given by left divisibility, and there is a *Garside element* Δ, a normal element for which the interval $[1, \Delta]$ is finite and generates the positive cone as a monoid. The concept is left-right symmetric, due to the fact that left and right divisibility are related by the inversion $a \mapsto a^{-1}$.

The braid group B_n maps onto the symmetric group S_n, a Coxeter group of type $\mathbb{A}_{n-1}$. So it was natural to ask whether Garside's method applies to generalized braid

✉ W. Rump
rump@mathematik.uni-stuttgart.de

1 Institute for Algebra and Number Theory, University of Stuttgart, Pfaffenwaldring 57, 70550 Stuttgart, Germany

groups with other Coxeter diagrams. For Artin groups of spherical type, this was done by Brieskorn and Saito [4], independently by Deligne [5], in two successive papers in the same journal. In 1998, Birman, Ko, and Lee gave a completely new approach to the word and conjugacy problem for B_n, related to a “dual” Garside structure on B_n. Here the set of atoms is enlarged to the set of all reflections, so that the Garside element becomes shorter and turns into a Coxeter element. Similar dual Garside structures were found (D. Bessis, 2003) for all Artin groups. The dual structure proved to be particularly successful, allowing a certain extension to Euclidean Garside groups (McCammond and Sulvay, 2014). Such groups are better called *quasi-Garside* groups, as their sets of atoms are no longer finite.

For a while, the Garside phenomenon may have been viewed as just another aspect of the inexhaustible richness of braid structure. So it took several decades until a concept of Garside group was fixed in papers of Dehornoy and Paris (1999) and Dehornoy (2002). Meanwhile, Garside-like structures occurred in connection with self-distributive systems. If the left multiplication of a left self-distributive set X is bijective, X is also called a *rack*, and it was known since Joyce’s 1982 and Matveev’s 1984 paper that knots are classified by their *fundamental rack*. To analyse the left self-distributive law (LD)

$$a \cdot (b \cdot c) = (a \cdot b) \cdot (a \cdot c),$$

its “geometry monoid” was introduced by Dehornoy: Note that the expressions which constitute a particular word in a free magma (set with a binary operation) can be arranged in a finite binary tree whose branches are associated to the letters (atoms) of the word. If each atom is split into a pair of atoms, the branching of the tree continues indefinitely: An infinite binary tree is obtained where the atoms have disappeared. So the self-distributive law (LD) essentially applies to each node α of an infinite tree, which yields a system of operators Σ_α, generating a monoid M_{LD}. Studying the operators Σ_α and their relations amounts to consider a “second homology” of (LD). Dehornoy found a single coherence relation among the Σ_α, similar to Mac Lane’s pentagon relation for braided categories. More precisely, the *geometry monoid* M_{LD} is defined by this “critical” and few other (obvious) relations, while it remains open whether they exhaust all the relations among the operators Σ_α. If the Σ_α are identified for all α of the same height n, one obtains operators Σ_n where Dehornoy’s critical relation specializes to the braid relation. In other words, M_{LD} projects onto the braid monoid B_∞^+ on infinitely many strings.

Dehornoy proved that M_{LD} has a *one-sided* Garside structure, with the further exception that instead of a single Garside element Δ, there are enough “local” Garside elements, that is, normal elements Δ_α such that the whole monoid is generated by the intervals $[1, \Delta_\alpha]$. The question whether the critical relation among the Σ_α’s (and the obvious ones) suffice can be rephrased as the question whether M_{LD} embeds into a group: the “Embedding Conjecture”. Examples like this indicate that “Garside Theory” is still in a state of flux, searching for unifying principles to define its limits properly.

The book under review is a timely account on various incarnations of Garside structure, to make this flourishing topic accessible to a wide audience. The book is divided into two parts, a first one on the general theory, a second one on specific examples. The general part starts with a chapter on examples of a prototypical nature, beginning with ordinary and dual braid monoids, to motivate the concept of Garside group. Another example is given by torus knot groups. Quasi-Garside groups are exemplified by D. Bessis' dual braid monoid for the free group F_n on n generators, based on the Hurwitz action of the braid group B_n. To show that Garside features exist beyond the limitations of quasi-Garside groups, infinite braids are mentioned (which have no Garside element), the fundamental group of the Klein bottle (where the noetherian property fails), and *ribbon categories* (where the group is replaced by a groupoid).

If a unifying principle is yet to be found, there is a golden thread that goes through the whole complex of Garside phenomena, namely, the *normal decomposition* of elements, preconceived in Garside's representation of braids as products $\Delta^m a$. Indeed, a main feature in that pre-normal decomposition is the *normal* element Δ, while a can be further normalized, using the lattice structure of the group. At the end of the first chapter, the reader is informed that the aim of the book is not to establish a theory of Garside groups in the strict sense, but rather to lay the foundation of an extended framework, to cover those examples where Garsideness appears in one or the other form. As a main structural result on Garside groups, almost the only one thus far, Picantin's decomposition theorem is just mentioned, which roughly states that every Garside group is an iterated bi-crossed product of Garside groups with cyclic centre. Here "iterated" means that after a first maximal bi-crossed product decomposition, the factors can be further decomposed, in general, so that a tree-like structure arises which eventually stops at groups with a trivial centre. It seems that this general theorem, indeed fundamental, has not yet found its full appreciation, probably by lack of a simple formulation, and because of its limited application to concrete cases considered thus far. For the pure braid group, the theorem suggests a natural quasi-Garside structure related to the *combing* of braids, used by E. Artin to obtain the historically first normal form for the braid group B_n [1].

After a chapter on preliminaries, the general part of the book takes off with an analysis of normal decompositions. For the beginner, it may take a little time to get used to the language, as the concepts are developed in the context of categories, with the usual order of composition changed – for practical reasons – to 'left-right'. Each chapter of the general part starts with a collection of main definitions and results. Combined with an efficient index, this greatly facilitates a quick orientation in the 700 pages treatise.

The chapter on normal decompositions is central. With respect to a class $\mathcal{S}$ of morphisms in a *left cancellative* category (i.e. all morphisms are epic), its existence is expressed by saying that $\mathcal{S}$ is a *Garside family*. Up to units, normal decompositions are unique. Unique prime factorization in an integral domain can be viewed as a special case. For Ore categories with left lcm's, existence and uniqueness are extended to the enveloping groupoid. The geodesic property of normal paths, the automatic structure of Garside groups, as well as the word problem, are discussed here. The focus upon normal forms entails that large parts of the book are occupied with algorithmic aspects, a feature that seems to be embraced by a majority of today's students. Here they are invited to their first light-bulb moments with word reversing.

Garside families are analysed in the next two chapters. It is shown that up to units, they are closed with respect to right divisor and satisfy a weak lcm existence condition. Simple characterizations of Garside families in terms of such closure properties are provided. The noetherian case is discussed separately. Here the original definition of a Garside element as a least common multiple of the atoms (Dehornoy and Paris, 1999) reappears in a statement asserting the existence of a smallest unit-closed Garside family. Without a noetherian hypothesis, even in the discrete case, a smallest Garside family need not exist, as the Klein bottle monoid shows. Garside families $\mathcal{S}$ naturally lead to presentations of the ambient category, with generators and relations taken from $\mathcal{S}$ (up to units). For such a presentation, an estimation of the length of transformations between equivalent paths is provided. Some results appear to be new, for example, a certain equivalence between "local" and "global" for the noetherian property. Under a suitable boundedness condition, a Garside family in a cancellative category gives rise to a functorial automorphism Φ which generalizes conjugation with the Garside element Δ in a Garside monoid, while Δ itself is replaced by a natural transformation $\Delta\colon 1 \to \Phi$. In the noetherian case, it is shown that the category admits lcm's (pushouts) and gcd's (pullbacks thereof). Several open questions are raised at the end of both chapters.

As Garside families give rise to a presentation of the ambient category, it is natural to characterize them by intrinsic properties. Endowed with a partial composition, they determine the ambient category (or monoid in case of a single object). Slight modifications of these *Garside germs* were introduced in the category context by Digne and Michel, and for Garside groups by Bessis, Digne, and Michel (2002). The idea of group germs, with or without lattice structure, is much older. It occurs in a paper of O. Wyler on Clans [8], and in R. Baer's paper on group amalgamations [2]. Several equivalent characterizations of Garside germs are given in the book. They are useful for the construction of particular Garside categories.

Two more chapters of the general part on subcategories (e.g. parabolic ones) and the conjugacy problem are more or less extensions of previous results to the category context. After all, the reader may ask why such an extension, and whether categories are really needed and useful for the analysis of Garside phenomena. Indeed, monoids and their groups of fractions are more tangible and mostly sufficient. On the other hand, examples of genuine Garside groupoids arose, e.g., in the study of pure braid groups [7] and complex reflection groups [3]. For example, the centralizer of an element in a Garside group need not be a Garside group, which led Bessis to introduce *weak Garside* groups, that is, groups equivalent to a Garside groupoid as a category. With this concept, the anomaly could be resolved.

The second part of the book starts with Artin groups and their generalization to braid groups of complex reflection groups. To any finite Coxeter group W, let $\mathcal{H}$ be the union of the complexified reflection hyperplanes. The pure Artin group can then be seen as the fundamental group of the complement $M(W)$ of $\mathcal{H}$, while the full Artin group G is $\pi_1(M(W)/W)$. Deligne has shown that $M(W)/W$ is a $K(G, 1)$-space, that is, its universal cover is contractible. Recently, a similar result was proved by Bessis [3] for well-generated finite complex reflection groups W. Ordinary and dual Garside structures are discussed for spherical Artin groups G. In the dual Garside structure, a Coxeter element c is Garside, and the interval $[1, c]$ is isomorphic to the lattice of (generalized) non-crossing partitions. The case of the braid group $G = B_n$ is explained in the book, with a brief sketch of Brady and Watt's 2008 treatment of the general (spherical) case. Some exotic Garside structures for the braid group B_3 are presented.

A chapter on Deligne-Lusztig varieties deals with geometric consequences of the abelian defect conjecture inasmuch as braid groups are involved. Using a theorem of Deligne [6], Broué and Michel (1996) associate a Deligne-Lusztig variety X_w to a positive braid w. If w is sufficiently regular (a root of the squared Garside element), one expects the endomorphism algebra of the ℓ-adic cohomology of X_w to be a specialization of a certain Hecke algebra. A brief introduction to Deligne-Lusztig varieties is given, and a geometric version of the abelian defect conjecture (for the principal block) is explained. In an appendix, Deligne's theorem [6] is extended to Garside families.

The next two chapters are on self-distributivity (as mentioned above) and right orders on Garside-like groups. The latter means that the partial order extends to a linear one which is invariant under right multiplication. Equivalently, this says that the group embeds into a lattice-ordered group. Introduced by Paul Conrad in 1959, right ordered groups always admit a left order, and vice versa. Certain mapping class groups, including braid groups as well as all knot groups, are (left) orderable, but it is widely open which fundamental groups of 3-manifolds are so. Pure braid groups are even bi-orderable (Kim and Rolfsen, 2003), and the same holds for the group G_{LD} of self-distributivity, given by the same relations as the geometry monoid M_{LD} (Dehornoy, 2001). For general Garside groups, the question of orderability is undecided. The set of all left orderings on G, viewed as a closed subset $\mathrm{LO}(G)$ of 2^G, is a *Stone space* (Sikora 2004) on which G acts by conjugation. The bi-orderings are the fixed points under this action. For a non-abelian free group G, the space $\mathrm{LO}(G)$ is homeomorphic to a Cantor set. Tararin has shown that $\mathrm{LO}(G)$ is finite if and only if G admits a unique normal series $1 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ such that each G_{i+1}/G_i is torsion-free abelian of rank one, and no quotient G_{i+2}/G_i is bi-orderable. Some results on isolated points of $\mathrm{LO}(G)$ and triangular presentations, are discussed in this chapter.

Most of the above mentioned Garside-like groups, with the exception of the Klein bottle group, have an underlying lattice which is far from distributive. The next chapter deals with the structure groups of a class of set-theoretic solutions of the quantum Yang-Baxter equation (QYBE). "Set-theoretic" means that for a suitable basis X of a K-vector space V, the operator $R \in \mathrm{End}_K(V \otimes_K V)$ providing the solution permutes the basis $X \times X$ of $V \otimes_K V$. The QYBE is a kind of braid relation, satisfied, e.g., for quantum groups based on V. The map $R\colon X \times X \to X \times X$ with $(x, y) \mapsto (x^y, {}^x y)$ is then given by two binary operations on X, and the *structure group* G of R is generated by X with quadratic relations $x \cdot y = ({}^x y) \cdot (x^y)$. Chouraqui observed in 2010 that for unitary non-degenerate solutions, G is always a Garside group. For such a group, the underlying lattice is distributive. Being connected with the QYBE, these Garside groups are linked to structures like regular affine groups, radical rings, Chevalley groups, Artin-Schelter regular rings, groups of I-type, dynamical systems, and triangular Hopf algebras.

Some additions and more examples are given in a final chapter. Among other things, it contains a section on divided categories (Bessis 2006), a new proof of Deligne's theorem [6] on braid actions on a monoidal category, Garside families and the QYBE, and Krammer's braid group of $\mathbb{Z}^n$, where the symmetric group S_n is replaced by $GL_n(\mathbb{Z})$.

Apart from the more advanced chapter on Deligne-Lusztig varieties which is more loosely connected to Garside structure, the book gives a self-contained, comprehensive, and up-to-date account on an exciting topic, addressed to a wide public to take part in the shaping and development of a rising theory around braiding and ordering. Equipped with numerous examples, 130 exercises, and 40 questions, the book can serve as an introduction as well as a source of inspiration for those who want to know more about Garside's thriving heritage.

References

1. Artin, E.: Theory of braids. Ann. Math. **48**, 101–126 (1947)
2. Baer, R.: Free sums of groups and their generalizations. An analysis of the associative law. Am. J. Math. **71**, 706–742 (1949)
3. Bessis, D.: Finite complex reflection arrangements are K(π, 1). Ann. Math. **181**(3), 809–904 (2015)
4. Brieskorn, E., Saito, K.: Artin-Gruppen und Coxeter-Gruppen. Invent. Math. **17**, 245–271 (1972)
5. Deligne, P.: Les immeubles des groupes de tresses généralisés. Invent. Math. **17**, 273–302 (1972)
6. Deligne, P.: Action du groupe des tresses sur une catégorie. Invent. Math. **128**, 159–175 (1997)
7. Krammer, D.: A class of Garside groupoid structures on the pure braid group. Trans. Am. Math. Soc. **360**(8), 4029–4061 (2008)
8. Wyler, O.: Clans. Compos. Math. **17**, 172–189 (1965)