Chapter 1
Overview

Multivariate continuous problems are defined on spaces of functions of d variables, where d may be in the hundreds or even in the thousands. They occur in numerous applications including physics, chemistry, finance, economics, and the computational sciences. For path integration, which lies at the foundation of quantum mechanics, statistical mechanics and mathematical finance, the number of variables is infinite; approximations to path integrals result in arbitrarily large d.

Such problems can almost never be solved analytically. Since they must be solved numerically they can only be solved approximately to within a threshold ε. Algorithms for solving multivariate problems use n information operations given typically by either function values or linear functionals. Computational complexity is the study of the *minimal* resources needed to solve a problem. It is defined as the minimal number of information operations and combinatory operations needed to combine computed information operations in order to obtain the solution to within ε. The *information complexity* is defined as the *minimal* number $n(\varepsilon, d)$ of information operations needed to solve the d-variate problem to within ε. For all problems, the information complexity is a *lower* bound on the computational complexity. Surprisingly, for many computational problems, including in particular many linear multivariate problems, the information complexity is *proportional* to the computational complexity. For this reason, we concentrate on the information complexity in this book. It is studied in various settings including the worst case, average case, randomized and probabilistic settings for the absolute, normalized and relative error criteria. Depending on the setting and on the error criterion, ε has different meanings, but it always represents the error tolerance.

A central issue is the study of how the information complexity depends on ε^{-1} and d. If it depends *exponentially* on ε^{-1} or d, we say the problem is *intractable*. For many multivariate problems, we have exponential dependence on d, which is called after Bellman the *curse of dimensionality*. If the information complexity depends polynomially on ε^{-1} and d then we have *polynomial tractability*, if it depends only on a polynomial in ε^{-1} we have *strong polynomial tractability*. If the information complexity is *not* exponential in both ε^{-1} and d, then we have *weak tractability*. There are more types of tractability depending on how we measure the lack of exponential behavior.

There is a huge literature on the computational complexity of d-variate problems. Most of these papers and books study error bounds that lead to bounds on the information complexity. These bounds are usually sharp with respect to ε^{-1} but have, unfortunately, unknown dependence on d. But to determine if a problem is *tractable* we need to know the dependence on both ε^{-1} and d. Tractability requires new proof techniques to obtain sharp bounds on d. There is therefore a need to revisit even multivariate problems thoroughly studied in the past if we want to investigate their tractability.

Research on tractability of multivariate continuous problems started in 1994 and there are many surprising results. Today this subject is thoroughly studied by many people. This is the first book on tractability of multivariate continuous problems. We summarize the known results and present many new results. So far only *polynomial tractability* has been thoroughly studied in many papers. The study of more general tractability and, in particular, weak tractability has just begun. Therefore most of results on weak tractability are new. Weak tractability means that we allow a more general dependence on ε^{-1} and d, as long as it is non-exponential. This obviously enlarges the class of tractable problems.

Many multivariate problems suffer from the curse of dimensionality when they are defined over standard (unweighted) spaces. In this case, all variables and groups of variables play the same role and this causes the information complexity to be exponential in d.

But many practically important problems, such as problems in financial mathematics, are solved today for huge d in a reasonable time. One of the most intriguing challenges of the theory is to understand why this is possible. We believe the reason is that many practically important multivariate problems belong to *weighted* spaces. For weighted spaces, the dependence on the successive variables or groups of variables can be moderated by weights. We consider various weights such as product weights, order-dependent weights, finite-order and finite-diameter weights. For example, for *finite-order* weights, functions of d variables can be represented as sums of functions of ω variables, where ω is fixed and moderate, and d varies and can be arbitrarily large. For finite-order weights, most multivariate problems are polynomially tractable.

Multivariate problems may become weakly tractable, polynomially tractable or even strongly polynomially tractable if they are defined over *weighted* spaces with properly decaying weights. One of the main purposes of this book is to study weighted spaces and obtain necessary and sufficient conditions on weights for various notions of tractability.

The tractability results are illustrated for many specific multivariate problems. We consider general linear problems including multivariate integration, approximation, as well as a number of specific non-linear problems such as partial differential or integral equations, including the Schrödinger equation. Some of these applications will be presented in Volumes II and III.

The book contains a number of open problems, including the 15 open problems in Chapter 3, and the other 15 open problems in the remaining chapters. They should be of interest to a general audience of mathematicians. Volume I of the book contains a bibliography of over 290 papers and books, whereas Volume II will have additionally about 150 papers and books. We hope that the book will further intensify research on tractability of multivariate continuous problems.