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★**Metric geometry of locally compact groups.**

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This outstanding book offers a thorough treatment of the foundations of what could be called the *geometric theory of locally compact groups*, namely, a unification of geometric group theory and locally compact group theory.

Geometric group theory emerged as an independent discipline under the influence of Gromov in the 1980's. Its main scope concerns the relation between the algebraic properties of a group and its geometric features, which are usually manifest through an action of the group on a metric/geometric space. That space can typically be the group itself, endowed with a Cayley graph structure associated with a generating set.

The origin of locally compact group theory goes back to Hilbert's fifth problem, which inspired a tremendous amount of research in the first half of the 20th century and culminated in the Gleason-Yamabe structure theorem in the early 1950's (see [T. C. Tao, *Hilbert's fifth problem and related topics*, Grad. Stud. Math., 153, Amer. Math. Soc., Providence, RI, 2014; MR3237440] for a recent account on those fundamental developments and their applications).

While locally compact groups often arise in geometric group theory as isometry groups of proper metric spaces, or enveloping groups of lattices, the metric/geometric approach to group theory has mostly focused on countable discrete groups. This book broadens the scope of that approach by providing a systematic treatment of the large scale metric geometry of locally compact groups in general. The emphasis is put on three fundamental finiteness conditions: σ-compactness, compact generation and compact presentation. These are generalizations of the familiar notions of countability, finite generation and finite presentation for abstract groups. A topological group is *σ-compact* if it has a countable cover by compact subsets; it is *compactly generated* if it has a compact generating set, and *compactly presented* if it has a presentation of the form $\langle S \mid R\rangle$ with a compact generating set S and a set R of relators whose word length is bounded. The notion of compact presentation was introduced and studied by the German school in the 1960's, but mainly disappeared from the landscape until recently. This book includes the first systematic treatment of that notion, and provides convincing evidence of its interest. While the properties of finitely presented groups usually constitute a good guide for the intuition in exploring compact presentability of locally compact groups, some surprises are also to be expected. For example, it is proved in the book that, for any non-discrete locally compact field $\mathbf{K}$, the locally compact group $\mathbf{K}^2 \rtimes \mathrm{SL}_2(\mathbf{K})$ is compactly generated, whereas it is compactly presented if and only if $\mathbf{K}$ is connected (hence isomorphic to $\mathbf{R}$ or $\mathbf{C}$).

Each of the three finiteness conditions is thoroughly discussed. For each of them, a fundamental result, which the authors call a *milestone*, provides an associated purely metric property:

Milestone 4.A.8. On every σ-compact locally compact group, there exists an adapted pseudo-metric that is well defined up to metric coarse equivalence.

Milestone 4.B.13. On every compactly generated locally compact group, there exists a geodesically adapted pseudo-metric that is well defined up to quasi-isometry.

Milestone 8.A.7. A compactly generated locally compact group G with geodesically adapted pseudo-metric d is compactly presented if and only if the pseudo-metric space (G, d) is coarsely simply connected.

The statement of Milestone 8.A.7 in the book provides, moreover, several other metric characterizations of compact presentability, notably in terms of the *Rips complex*, which is defined and studied in the context of non-discrete groups for the first time in this volume.

The book consists of eight chapters. The first is an introduction providing an overview of its content. The second introduces the basic notions used throughout the book, and constitutes an excellent, concise and efficient introduction to the theory of locally compact groups, including numerous well-chosen examples, as well as a brief account on the Gleason-Yamabe theorem mentioned above. The third chapter is purely metric. It introduces and develops two categories of pseudo-metric spaces that will turn out to be respectively relevant to σ-compact and compactly generated locally compact groups. The first is the *metric coarse category* (whose isomorphisms are the metric coarse equivalences), and the second is the *large-scale category* (whose isomorphisms are the quasi-isometries). The discussion of those notions in the context of pseudo-metric spaces, instead of metric spaces, is most natural in view of the authors' motivations: this is what allows them to treat locally compact groups without any hypothesis of metrizability. Chapter 4 is the first place where the metric notions developed thus far are applied to locally compact groups. It covers the first two milestones mentioned above. The extremely useful Chapter 5 is aimed at illustrating the concept of compact generation through the discussion of a huge variety of examples. These include connected groups, algebraic groups over non-discrete locally compact fields, and isometry groups of proper metric spaces. The last section collects a dozen families of examples of lattices in locally compact groups. Chapters 6 and 7 are technical preparations of the detailed treatment of compact presentability of locally compact groups, which is the ultimate goal of the book, achieved in Chapter 8.

The exposition is of stellar quality throughout. All concepts are systematically illustrated with numerous examples and non-examples. Moreover, although the emphasis is not put on history, the original references for most notions and results are cited, and the bibliography is extensive.

It should also be mentioned that several results covered by the book are new and have not been published elsewhere before. This is notably the case for the metric characterizations of locally elliptic groups in Section 4.D, and the non-discrete version of the Bieri-Strebel splitting theory in Section 8.C.

The authors have managed to achieve a perfect balance between erudition and readability. Their book provides an extensive and accessible reference that is both a friendly invitation to the subject for the novice and an invaluable tool for the expert. It is highly recommended to any group theorist. *Pierre-Emmanuel Caprace*