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★**Bases in function spaces, sampling, discrepancy, numerical integration.**
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The title is very descriptive of the content of the book (it basically covers the titles of the chapters).

The book starts with an introductory chapter on function spaces which aims to provide the necessary background to read what follows. As with the rest of the book, this first chapter is very carefully written. And, even though it consists basically of review material, we can find some formulations which are more general than can be seen elsewhere. This is, for example, the case with the atomic representations and the characterizations by local means.

Section 1.1 deals with so-called isotropic spaces, both on $\mathbb{R}^n$ and on domains. Not only Besov and Triebel-Lizorkin spaces in the context of tempered distributions are considered here: the author also considers spaces of this type in the framework of measurable functions.

Section 1.2 deals with spaces of dominating mixed smoothness and Section 1.3 deals with so-called logarithmic spaces, which are special cases of what are generally known as spaces of generalized smoothness. Subsection 1.3.3 is quite curious in an introduction, as it actually contains some research proposals. The last subsection of the introductory chapter mixes the idea of logarithmic with dominating mixed smoothness, being inspired by some of the estimates to be obtained later.

Chapter 2 deals with Haar bases in the previously introduced spaces, but starts with the introductory Section 2.1 dealing with the classical theory both for Haar and Faber systems and corresponding historical remarks. Then the following two sections deal first with Haar bases on $\mathbb{R}$ and intervals and afterwards with Haar bases on $\mathbb{R}^n$ and cubes, in both cases in isotropic spaces. The corresponding bases for spaces of dominating mixed smoothness, the so-called Haar tensor bases, are dealt with in Section 2.4. The chapter ends with a section on spline bases, but this only plays a side role as far as the main aims of the book are concerned.

Chapter 3 deals with Faber bases, first for isotropic spaces on intervals and then with Faber (tensor) bases in spaces of dominating mixed smoothness on cubes, though the picture for spaces of type F is far from complete. Faber bases in spaces of dominating mixed smoothness in the framework of measurable functions are also considered. In Section 3.4, Haar and Faber bases for isotropic spaces in one dimension and for spaces of dominating mixed smoothness in dimension 2 are obtained. The final section of this chapter deals, as in Chapter 2, with spline bases, now of Faber type. The results do not have the same final character as in the previous sections, but they show the potential to overcome some difficulties with the consideration of higher smoothness parameters in the spaces. Actually, the author ends up making some research proposals in this respect.

Though Haar bases are of interest by themselves, in this book they are also used as a means to get to Faber bases, as the functions in this system can, in one dimension, basically be obtained by integration of the functions of the former system, this being the starting point to step from results with Haar bases to results with Faber bases. On the other hand, the interest in the latter in the context of this book stems from the fact that the coefficients in the representation of a function in a Faber basis are calculated by means of the values of the function at some points. That is, the knowledge of a function at some points allows us to recover the whole function by means of such a representation. Such a procedure is at the core of sampling, one of the main themes of this book and the subject of Chapter 4, where the results previously obtained on Faber bases are put to work, not only to derive theoretical results but also to construct specific sampling methods.

Section 4.1 describes the basic ingredients of sampling methods and defines the so-called sampling numbers and linear sampling numbers of a natural embedding involving the function spaces considered in this book. It also describes sampling in isotropic spaces and compares corresponding sampling numbers with entropy and s-numbers, as far as the concept of information uncertainty is concerned. Section 4.2 studies sampling in isotropic spaces on intervals and Section 4.3 does the same for spaces of dominating mixed smoothness on squares. At the end of this section it becomes apparent that in higher dimensions it is more convenient to use this type of spaces instead of isotropic counterparts, since, for the *same* smoothness, the main order of decay is the same either when using isotropic spaces in one dimension or when using spaces of dominating mixed smoothness in two dimensions. That is, the negative influence of dimension exhibited by the sampling numbers when one steps to higher dimensions in the context of isotropic spaces does not show up in the framework of spaces of dominating mixed smoothness (at least in two dimensions), though some unavoidable logarithmic perturbation shows up in the latter case. Afterwards, in Section 4.4, mixing logarithmic with dominating mixed smoothness in the spaces to be considered allows one, at least in some cases, to remove the logarithmic dependence in the sampling results, though, for best results, a modification of the previously defined logarithmic spaces with dominating mixed smoothness had to be introduced. The chapter ends with comments on how to extend the results to the context of higher dimensions, even formulating some concrete research proposals.

Chapter 5 deals with another one of the major themes of the book, namely numerical integration. Section 5.1 gives the necessary definitions, including for the so-called integral numbers, relates them with sampling numbers and sets the goals for the chapter, with a special emphasis on getting explicit quadrature and cubature formulae based on Faber expansions. Then Section 5.2 studies numerical integration in isotropic spaces on intervals and Section 5.3 extends such considerations to spaces of dominating mixed smoothness in squares and cubes in higher dimensions, ending also with a discussion on integration based on logarithmic spaces. The line of thought here parallels the one in the preceding chapter: consider spaces of dominating mixed smoothness in higher dimensions, instead of corresponding isotropic spaces, in order to keep the main order of decay in the estimates (now of the integral numbers) exhibited in the context of isotropic spaces on intervals; since this introduces some logarithmic perturbations, try afterwards to remove them by considering also logarithmic perturbations in the definitions of the spaces. In the final Section 5.4 the author returns to planar domains, but this time not restricting to squares.

The book ends with Chapter 6, where the third major theme, namely discrepancy, is exploited. Section 6.1 gives the necessary definitions, including the clue that discrepancy measures the deviation of sets of points from uniformity. It also includes an elementary proof of the estimates for the so-called discrepancy numbers for Lebesgue spaces on intervals. Section 6.2 relates integral and discrepancy numbers in the framework of spaces of dominating mixed smoothness and then this relation is exploited in Section 6.3 in order to give corresponding estimates for discrepancy numbers. Afterwards the author returns to one dimension (so, to isotropic spaces), where he is able to give more precise results, and ends the section with comments, problems and proposals.

The book ends with a bibliography (including reference to the pages where the works are cited), a list of figures, a list of symbols and an index.

One idea that is stressed throughout the book is that some difficulties in getting a perfect match between lower and upper estimates come from the method, which uses a distribution of points which is too regular.

One distinctive feature of this book, as becomes apparent from what has been written above, is that it contains some research proposals in several places, so that it should be of interest to the active researcher in the area. One warning, though: in order not to take some conjecture as a proven result, one should carefully read the surrounding text in those parts where such research proposals are made.

Another feature of this book is that the author makes a systematic effort to render the text intelligible for the occasional reader by recalling at crucial points where relevant definitions have been given and important comments have been made. As a result, the book can be used as a reference text. This is reinforced by the fact that, though the results presented come in many variants and there could be the temptation to just write that some result or definition can be analogously phrased in a different context, quite on the contrary the author very often writes down explicitly what he means on such occasions. The intelligibility of the text is also a consequence of the care put into providing some historical perspective for the subjects under study, as well as plans for what follows at convenient places.

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