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Punctured Logarithmic Maps

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Dan Abramovich
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Bernd Siebert

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Authors

Dan Abramovich
Department of Mathematics
Brown University
151 Thayer Street
Providence, RI 02912, USA

Email: dan_abramovich@brown.edu

Qile Chen
Department of Mathematics
Boston College
140 Commonwealth Avenue
Chestnut Hill, MA 02467-3806, USA

Email: qile.chen@bc.edu

Mark Gross
DPMMS, Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge, CB3 0WB, UK

Email: mgross@dpmms.cam.ac.uk

Bernd Siebert
Department of Mathematics
The University of Texas at Austin
2515 Speedway, Stop C1200
Austin, TX 78712, USA

Email: siebert@math.utexas.edu

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Abstract

We introduce a variant of stable logarithmic maps, which we call *punctured logarithmic maps*. They allow an extension of logarithmic Gromov–Witten theory in which marked points have a negative order of tangency with boundary divisors.

As a main application we develop a gluing formalism which reconstructs stable logarithmic maps and their virtual cycles without expansions of the target, with tropical geometry providing the underlying combinatorics.

Punctured Gromov–Witten invariants also play a pivotal role in the intrinsic construction of mirror partners by the last two authors, conjecturally relating to symplectic cohomology, and in the logarithmic gauged linear sigma model in work of Qile Chen, Felix Janda and Yongbin Ruan.

Keywords. logarithmic Gromov–Witten invariant, punctured Gromov–Witten invariant, punctured map, punctured curve, puncturing, functorial tropicalization, tropical punctured map, tropical moduli, tropical type, marking by tropical type, basic monoid, basic logarithmic structure, negative contact order, gluing formula, splitting, Artin fan, perfect obstruction theory, virtual fundamental class, logarithmic geometry, logarithmic stack

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Contents

Chapter 1

Introduction

Logarithmic Gromov–Witten theory, developed by the authors in [2, 15, 30], has proved a successful generalization of the notion of relative Gromov–Witten invariants developed in [47–49]. Relative Gromov–Witten invariants are invariants of pairs (X, D) where X is a non-singular variety and D is a smooth divisor on X; these invariants count curves with imposed contact orders with D at marked points. Logarithmic Gromov–Witten theory allows D instead to be normal crossings, or more generally, allows (X, D) to be a toroidal crossings variety.

1.1 Scope and motivation

The purpose of the present work is to extend logarithmic Gromov–Witten theory to admit *negative contact orders*. Working over a field $\Bbbk$, an example for how negative contact orders arise naturally is by restricting a normal crossings degeneration, such as

$$\pi : \mathbb{A}^2 = \operatorname{Spec} \Bbbk[z, w] \to \mathbb{A}^1 = \operatorname{Spec} \Bbbk[t], \quad \pi^\sharp(t) = zw,$$

to the irreducible component $C = V(w)$ of the central fiber $\pi^{-1}(0)$. Viewing π as a morphism of log spaces for the toric log structures on $\mathbb{A}^2$ and $\mathbb{A}^1$, denote by s_z, s_w, s_t the global sections of the log structure $\mathcal{M}_{\mathbb{A}^2}$ induced by z, w and $\pi^\sharp(t)$, respectively. The induced log structure $\mathcal{M}_C = \mathcal{M}_{\mathbb{A}^2}|_C$ on $C = \operatorname{Spec} \Bbbk[z]$ is generated by the restrictions of s_z, s_w, denoted by the same symbols. Note that the structure morphism $\mathcal{M}_C \to \mathcal{O}_C$ maps s_z to z, which has a first order zero at the origin $0 \in \mathbb{A}^1$ as given by a marked point, while s_w and s_t map to 0. The point is that viewed as a log space over $\mathbb{A}^1$, the equation $s_z s_w = s_t$ implies that away from $0 \in \mathbb{A}^1$, we have

$$s_w = z^{-1} \cdot s_t.$$

Such sections do not exist on log smooth curves over the standard log point. The power of z occurring in this equation reflects the negative contact order. Since z^{-1} is defined on the punctured curve $\mathbb{A}^1 \setminus \{0\}$, we call the resulting extension of log smooth curves, stable logarithmic maps and logarithmic Gromov–Witten theory *punctured curves*, *punctured (logarithmic) maps* and *punctured Gromov–Witten theory*.

Our motivation for studying punctured Gromov–Witten theory comes from three sources. First, as illustrated in the example, negative contact orders arise naturally when gluing a logarithmic stable map from its restrictions to closed subcurves, as desired in degeneration situations [3]. Note that in transverse situations, as achieved by the expanded degeneration technique in [48], negative contact orders can be avoided by turning a punctured map over a standard log point into a stable logarithmic map to an irreducible component of the target over the trivial log point; see [26, Sections 6, 7] for details. This simplification is not possible when an irreducible component of the curve maps into a deeper stratum. See [3, Section 5.2.4] for an example where no decomposition of the target splits any of the nodes into a pair of marked points with non-negative contact order. A treatment of gluing situations based on punctured maps is contained in Chapter 5.

The second motivation comes from mirror constructions and their link to symplectic cohomology, relating to the program on mirror symmetry of Gross and Siebert via toric degenerations. It turns out that the algorithmic construction of mirrors via wall structures in [29] admits a vast, intrinsic generalization by using punctured invariants [33]. Punctured invariants are used in this context to define the structure coefficients of the coordinate ring of the mirror degeneration, with the space of non-negative contact orders representing generators. The structure coefficients require punctured invariants with two positive and one negative contact order. The gluing techniques developed in Chapter 5 are the crucial ingredient in proving associativity of the resulting multiplicative structure. In [32], the gluing techniques for punctured invariants are also crucial in constructing a consistent wall structure in the intrinsic mirror symmetry setup, thus linking the mirror constructions in [29, 33] via [28]. Further, building on [8, 32] gives an algorithmic method of calculating certain one-pointed punctured invariants on blow-ups of toric varieties.

Another interesting related fact is the interpretation of punctured invariants as structure coefficients in some versions of symplectic cohomology. Thus punctured invariants provide an algebraic-geometric path to computing otherwise hard to compute symplectic invariants. See [7, 23, 24, 63, 66] for some steps in this direction.

The third motivation is from work of the second author on the logarithmic gauged linear sigma model. In the papers [17, 19], punctured maps are shown to be a key for computing the invariants of the logarithmic gauged linear sigma model of [18].[1] This provides the geometric foundation for calculating higher genus invariants of quintic 3-folds [34, 35], and for proving [55, Conjecture A.1] on the cycle of holomorphic differentials [16].

[1] In [19], punctured maps to a smooth boundary divisor with extra structure called R-maps, are studied. The moduli of punctured maps provide different virtually birational models over which effective formulas for computing higher genus Gromov–Witten invariants hold [17]. These crucial virtually birational models do not exist as moduli of rubber maps with expansions [25, 48].

1.2 Main features of punctured Gromov–Witten theory

Several aspects of the theory of punctured Gromov–Witten invariants appear to be straightforward generalizations from ordinary logarithmic Gromov–Witten theory. The formal similarity can, however, be quite misleading. In fact, finding the right setup and point of view took a very long time, and was only made possible by developing the theory simultaneously with the mentioned applications.

One major difference is the more singular and more interesting nature of the base space for moduli spaces of punctured maps. In ordinary Gromov–Witten theory, the natural base space is the Artin stack $\mathbf{M}$ of nodal curves. While non-separated, $\mathbf{M}$ is smooth, hence is locally pure dimensional. The relative obstruction theory of the moduli space of stable maps over $\mathbf{M}$ thus produces a virtual fundamental cycle by virtual pullback of the fundamental class $[\mathbf{M}]$. The picture in logarithmic Gromov–Witten theory is much the same, with $\mathbf{M}$ now replaced by the stack $\mathfrak{M} = \mathrm{Log}_{\mathbf{M}}$ of log smooth curves of the given genus and numbers of marked points over fine saturated (fs) log schemes. This stack is log smooth over the base field, hence is also locally pure-dimensional.

For punctured invariants, the analogue of $\mathfrak{M}$ is the stack $\breve{\mathfrak{M}}$ of logarithmic curves with punctures. One crucial feature of the deformation theory of punctured curves is that $\breve{\mathfrak{M}}$ is typically not pure-dimensional. In fact, the map $\breve{\mathfrak{M}} \to \mathbf{M}$ forgetting the log structure turns out to be only idealized logarithmically étale (Proposition 3.3). This means that locally in the smooth topology $\breve{\mathfrak{M}} \to \mathbf{M}$ is isomorphic to the composition of a closed embedding defined by a monomial ideal followed by a toric morphism of affine toric varieties with associated lattice homomorphism an isomorphism over $\mathbb{Q}$.

The induced stratified structure of punctured maps turns out to be captured by tropical geometry. The second main feature of punctured Gromov–Witten theory is thus the central role of tropical geometry, exceeding by far its increasingly recognized role in logarithmic Gromov–Witten theory. Working over a base space B, we first factor the log smooth target $X \to B$ over the *relative Artin fan* $\mathcal{X} \to B$ from [5, Corollary 3.3.5], an algebraic stack glued from quotients of toric charts for $X \to B$ by the fiberwise acting torus, see [3, Section 2.2]. Working with $\mathcal{X}$ as a target amounts to working with nodal curves and compatible families of tropical maps, thus making the theory of such punctured maps a combinatorially enriched version of the theory of stable curves. A better base space than $\breve{\mathfrak{M}}$ to work with is then the algebraic stack $\mathfrak{M}(\mathcal{X}/B)$ of punctured maps to $\mathcal{X}/B$. Indeed, the forgetful map

$$\mathfrak{M}(\mathcal{X}/B) \to \mathbf{M} \times B$$

is also idealized logarithmically étale (Theorem 3.25), but now with idealized structure easy to extract from the tropical geometry of the situation. In particular, Remark 3.31 gives a complete characterization of the strata of $\mathfrak{M}(\mathcal{X}/B)$ in terms of types of tropical maps. Due to its fundamental nature for punctured Gromov–Witten theory, we emphasize the role of tropical geometry throughout, including adapted presentations of material from [30] in Chapters 2, 3 and Appendix C.

A third feature of punctured Gromov–Witten theory developed here, but already relevant to ordinary logarithmic Gromov–Witten theory, is the introduction of *evaluation stacks* for imposing point conditions compatible with the virtual formalism. Since $X \to \mathcal{X}$ is smooth in the ordinary sense, we can choose a lift to X of the image of each marked point in $\mathcal{X}$ to arrive at an algebraic stack $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B)$ smooth over $\mathfrak{M}(\mathcal{X}/B)$ and such that the relative obstruction theory over $\mathfrak{M}(\mathcal{X}/B)$ arises from a relative obstruction theory over $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B)$. It is this *evaluation stack* $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B)$ that one needs to work with to impose conditions on the evaluations at the marked points rather than the product $X \times_B \cdots \times_B X$ in ordinary Gromov–Witten theory. Evaluation stacks also play a crucial role in our gluing formalism, see Section 5.3.

1.3 Statements of main results

For simplicity of the presentation of the main results we now assume $X \to B$ to be a projective log smooth morphism of log schemes and B the standard log point $\mathrm{Spec}(\mathbb{N} \to \Bbbk)$ or B log smooth over the trivial log point $\mathrm{Spec}\,\Bbbk$, where $\Bbbk$ is a field of characteristic 0. For more detailed statements we refer to the main body of the text.

Similar to logarithmic Gromov–Witten theory, as presented in [3, Section 2.5], we introduce (decorated) types of punctured maps and of (families of) tropical maps. Types restrict the combinatorics of punctured maps over geometric points as seen by their tropicalizations, such as the dual intersection graph, the contact orders of punctured and nodal points and the genera and the curve classes of its irreducible components (Definition 2.24). An appropriate global version of contact orders developed in Section 2.4 leads to the notion of global decorated type $\boldsymbol{\tau}$ that can be used to define moduli spaces of punctured maps *marked by* $\boldsymbol{\tau}$ (Definition 3.8). Theorem 3.10, Corollary 3.19 and Theorem 3.25 establish the basic properties of these moduli spaces, which can be summarized as follows.

Theorem A. *Let $\boldsymbol{\tau}$ be a decorated global type. Then the stacks $\mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau})$ and $\mathcal{M}(X/B, \boldsymbol{\tau})$ of $\boldsymbol{\tau}$-decorated basic stable punctured maps (Definition 3.8) to $\mathcal{X} \to B$ and to $X \to B$, respectively, are logarithmic algebraic stacks. Moreover, $\mathcal{M}(X/B, \boldsymbol{\tau})$ is Deligne–Mumford and proper over B.*[2] *If in addition X is simple* [3, Definition 2.1]*, $\mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau})$ is idealized smooth over B.*

For the definition of punctured Gromov–Witten invariants we work over the evaluation stacks $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau)$, which lift the evaluations at a set $\mathbf{S}$ of punctured and

[2]We prove properness under the technical assumption that the log structure on X is Zariski and $\bar{\mathcal{M}}_X^{\mathrm{gp}}$ is globally generated. The latter assumption has meanwhile been removed in [39].

nodal points from $\mathcal{X}$ to X (Definition 5.14). We suppress $\mathbf{S}$ in the notation. The following theorem summarizes Propositions 4.2 and 4.5.

Theorem B. *For $\boldsymbol{\tau}$ a decorated global type let $\pi : \mathcal{C}(X/B,\boldsymbol{\tau}) \to \mathcal{M}(X/B,\boldsymbol{\tau})$ and $f : \mathcal{C}(X/B,\boldsymbol{\tau}) \to X$ be the universal curve and universal punctured map over the moduli space $\mathcal{M}(X/B,\boldsymbol{\tau})$ of $\boldsymbol{\tau}$-marked basic punctured maps to $X \to B$. Let $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})$ be the corresponding evaluation stack for a subset $\mathbf{S}$ of punctured sections, and $Z \subset \underline{\mathcal{C}}(X/B,\boldsymbol{\tau})$ the closed substack defined by the union of the images of these sections. Then there is a canonical perfect relative obstruction theory*

$$\mathbb{G} \simeq (R\pi_* f^*\Theta_{X/B}(-Z))^\vee \to \mathbb{L}_{\mathcal{M}(X/B,\boldsymbol{\tau})/\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})}$$

for the natural morphism $\varepsilon : \mathcal{M}(X/B,\boldsymbol{\tau}) \to \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})$. Here $\Theta_{X/B}$ denotes the logarithmic tangent bundle of X over B.

A similar statement holds if $\mathbf{S}$ also contains nodal sections, see Proposition 4.5. Virtual pullback [50] now provides punctured Gromov–Witten invariants, with the basic correspondence the homomorphism

$$(\underline{\mathrm{ev}} \times p)_* \varepsilon^!_{\mathbb{G}} : A_*(\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})) \to A_{*+d(g,k,A,n)}\Big(\prod_L \underline{Z}_L \times \mathcal{M}_{g,k}\Big)$$

on rational Chow groups defined in Definition 4.6. Here each $\underline{Z}_L$ is an evaluation stratum, the closed stratum of $\underline{X}$ that evaluation at a punctured section maps to by the choice of decorated global type $\boldsymbol{\tau}$, and $\mathcal{M}_{g,k}$ is the Deligne–Mumford stack of k-marked stable curves of genus g. A formula for the relative virtual dimension $d(g,k,A,n)$ is stated in (4.18).

The most challenging part of this paper was an efficient and practically useable treatment of gluing. While the final results may look straightforward, they rely on a number of careful choices and subtle points which became clear to us only after a long series of futile attempts.[3] Here we only summarize the results and refer to Remark 5.22 for some further discussion.

The formal setup for gluing takes a decorated global type $\boldsymbol{\tau}$ and splits the graph underlying $\boldsymbol{\tau}$ at a subset of edges, leading to a set $\{\boldsymbol{\tau}_1,\ldots,\boldsymbol{\tau}_r\}$ of decorated global types, with each split edge now producing a pair of legs in the graphs for the $\boldsymbol{\tau}_i$. Our first result on gluing reduces all gluing questions to the unobstructed evaluation stacks, as proved in Proposition 5.17 and Theorem 5.19.

[3] For some time our formalism only worked for gluing problems appearing in certain mirror constructions. We emphasize that the final results below are general and practical, as demonstrated in [71].

Theorem C. *There is a cartesian diagram*

$$\begin{tikzcd} \mathcal{M}(X/B,\boldsymbol{\tau}) \arrow[r, "\delta"] \arrow[d, "\varepsilon"'] & \prod_{i=1}^r \mathcal{M}(X/B,\boldsymbol{\tau}_i) \arrow[d, "\widehat{\varepsilon}=\prod_i \varepsilon_i"] \\ \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau) \arrow[r, "\delta^{\mathrm{ev}}"] & \prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau_i) \end{tikzcd}$$

with horizontal arrows the splitting maps from Proposition 5.4, finite and representable by Corollary 5.15, and the vertical arrows the canonical strict morphisms.[4]

Moreover, if $\widehat{\varepsilon}^!$ and $\varepsilon^!$ denote Manolache's virtual pullback defined using the two given obstruction theories for the vertical arrows, then for $\alpha \in A_*(\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau))$, we have the identity

$$\widehat{\varepsilon}^! \delta_*^{\mathrm{ev}}(\alpha) = \delta_* \varepsilon^!(\alpha).$$

A numerical gluing formula follows from Theorem C for those Chow classes α on $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})$ such that δ_*^{ev} can be written as a sum of products, see Corollary 5.20.[5] A straightforward consequence of Theorem C is a gluing formula for the degeneration situation from [3], see Corollary 5.26.

Our second result on gluing provides a description of the splitting map δ^{ev} in terms of an fs-fiber diagram, which apart from proving the properties stated in Theorem C, provides a route to using Theorem C in explicit computations. For the following statement the log stacks $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})$, $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau}_i)$ have to be replaced by closely related log stacks $\widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})$ and $\widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau}_i)$, which however have the same underlying reduced stacks, hence have identical Chow theories (Proposition 5.7).[6] These stacks come with logarithmic evaluation morphisms such as

$$\mathrm{ev}_{\mathbf{E}} : \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau}) \to \prod_{E\in\mathbf{E}} X,$$

where $\mathbf{E}$ is the set of nodal sections to split. The following is Corollary 5.15 to which we refer for more details.

[4]There is an entirely equivalent formalism allowing for disconnected punctured curves and disconnected types, in which case the products on the right form a single moduli stack corresponding to a disconnected decorated global type $\boldsymbol{\tau}$.

[5]Conversely, if there is no such decomposition, a numerical gluing formula cannot be achieved within Chow theory—a phenomenon already present in the classical case of a smooth gluing locus. A generally applicable gluing formula should therefore require working with a homology theory with a Künneth decomposition. It is possible that the formalism of virtual pullback in Borel–Moore homology developed in [40] may be useful.

[6]These stacks are closely related to Parker's moduli of cut curves introduced in [57].

Theorem D. *The splitting morphism* $\delta^{\mathrm{ev}} : \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau}) \to \prod_i \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau}_i)$ *fits into the cartesian diagram*

$$\begin{tikzcd}
\widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau}) \arrow[r, "\delta^{\mathrm{ev}}"] \arrow[d, "\mathrm{ev}_{\mathbf{E}}"'] & \prod_{i=1}^{r} \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau}_i) \arrow[d, "\mathrm{ev}_{\mathbf{L}}"] \\
\prod_{E \in \mathbf{E}} X \arrow[r, "\Delta"] & \prod_{E \in \mathbf{E}} X \times_B X
\end{tikzcd}$$

of fs log stacks. Here Δ *restricts to the diagonal morphism* $X \to X \times_B X$ *on each factor.*

We emphasize that the diagram in Theorem D is typically not cartesian on the level of underlying stacks due to the more subtle nature of fs fiber products. Theorem D is nevertheless a powerful tool for explicit computations. For example, under the assumption of toric gluing strata, Yixian Wu in [71] derives from Theorem D an explicit decomposition of the term $\delta^{\mathrm{ev}}_*[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau})]$ appearing in Theorem C in terms of the strata in $\prod_i \mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau}_i)$.

1.4 Organization of the paper

Chapter 2 contains the basic definitions and related concepts concerning punctured curves and punctured maps, with Section 2.1 introducing pre-stable and stable punctured maps, and Section 2.2 along with Appendix C the tropical language, including the definition of types and the modified balancing condition. The subject of Section 2.4 is the discussion of contact orders in a simplified version sufficient for most applications, and the associated notion of global types. The more involved general picture concerning contact orders is discussed in Appendix A. Section 2.3 presents the concept of basicness for punctured maps, which while largely the same as for logarithmic stable maps, emphasizes the tropical point of view, and hence might be of some independent interest. Section 2.5 introduces the new phenomenon of puncturing log ideals that each family of punctured curves or punctured maps comes with. Section 2.6 discusses the generalization to targets with monodromy.

Chapter 3 deals with the moduli theory of punctured maps, proving Theorem A among other things. Section 3.1 introduces stacks of punctured curves, with the main result the idealized smoothness statement in Proposition 3.3, followed in Section 3.2 by definitions of various stacks of punctured maps marked by types. Sections 3.3 and 3.4 establish properness of the moduli spaces to projective targets by adapting the boundedness and stable reduction theorems from [30]. The topic of Section 3.5 is the idealized smoothness of the spaces $\mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau})$ and the induced stratified structure, all characterized in terms of tropical geometry.

Chapter 4 deals with obstruction theories, using the approach of [13]. Section 4.1 gives a careful treatment of functoriality as well as of compatibility of obstruction theories for maps of pairs. Section 4.2 applies this discussion to construct the desired relative obstruction theory for $\mathcal{M}(X/B, \boldsymbol{\tau}) \to \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau})$, in particular proving Theorem B. Another application of this discussion is to the virtual treatment of gluing, presented in Section 5.3. The section ends with the definition of punctured Gromov–Witten invariants in Section 4.3.

The last Chapter 5 contains our results on gluing. Section 5.1 introduces the splitting morphism, while Section 5.2 treats the converse operation of gluing, essentially proving Theorem D, maybe the hardest single result in the paper with a very long genesis. The virtual treatment of gluing, proving Theorem C, is the objective of Section 5.3. The last Section 5.4 applies our results to the degeneration situation of [3].

1.5 Relation to other work

We end this introduction by discussing related work. First, our approach owes a great deal to Brett Parker's program of exploded manifolds, [56, 58–62]. We have often found ourselves trying to translate Parker's results in the category of exploded manifolds into the category of log schemes. Indeed, some of the original versions of the definition of punctured invariants, as well as the approach to gluing, arose after discussions with Parker.

A gluing formula for logarithmic Gromov–Witten invariants without expansions in the case of a degeneration X_0 with smooth singular locus is due to Kim, Lho and Ruddat [44]. This case does not require punctured invariants or evaluation spaces, but is otherwise close in spirit to the present treatment. A gluing formula in a special case for certain rigid analytic Gromov–Witten invariants has been proved by Tony Yu [72].

After the earlier manuscript version of this paper was distributed, Mohammed Tehrani [22], in developing a symplectic analogue of stable logarithmic maps, found that punctures were naturally described in the theory. Even more recently, [21, 68] defined negative contact order Gromov–Witten invariants by a limiting version of orbifold Gromov–Witten invariants. However, as observed by Dhruv Ranganathan, the invariants as currently defined cannot coincide with logarithmic invariants as they do not satisfy the correct invariance properties under log étale modifications. Work of Battistella, Nabijou and Ranganathan [11] takes this into account and shows how genus zero logarithmic invariants can be recovered from the orbifold invariants after sufficient blowing up. Their work [12] considers the case of negative contact orders in the orbifold theory, and makes a somewhat more subtle comparison which involves the puncturing ideal defined in Section 2.5. We send the reader to those papers for more details.

Besides the immediate applications of punctures already mentioned above, punctures also have been used by Hülya Argüz in [7] to build a logarithmic analogue of certain tropical objects in the Tate elliptic curve related to Floer theory.

Finally, we also mention recent work of Dhruv Ranganathan [65] taking a different point of view on gluing in log Gromov–Witten theory using an approach closer in spirit to the expanded degeneration picture of Jun Li.

1.6 Conventions

All schemes and stacks are defined over an algebraically closed field $\Bbbk$ of characteristic 0. By a logarithmic algebraic stack, we mean an algebraic stack equipped with a log structure. We follow the convention that if X is a log scheme or stack, then $\underline{X}$ is the underlying scheme or stack. To unclutter notation, we nevertheless often write $\mathcal{O}_X$ instead of $\mathcal{O}_{\underline{X}}$, and f^* instead of $\underline{f}^*$ for the pullback by the schematic morphism underlying a log morphism $f : X \to Y$. Unless stated otherwise, $\mathcal{M}_X$ denotes the sheaf of monoids on X, and $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ the structure map. The affine log scheme with a global chart defined by a homomorphism $Q \to R$ from a monoid Q to a ring R is denoted $\operatorname{Spec}(Q \to R)$. We use the notations $X \times_Z Y$, $X \times^{\mathrm{f}}_Z Y$, $X \times^{\mathrm{fs}}_Z Y$ to distinguish the fiber products in the category of all log schemes, and in the fine or the fine and saturated categories, respectively.

Throughout B denotes either a log point $\operatorname{Spec}(Q \to \Bbbk)$ with Q a toric monoid with $Q^\times = 0$, or an fs log scheme log smooth over $\operatorname{Spec} \Bbbk$.[7]

A *nodal curve* over a scheme $\underline{W}$ is a proper flat morphism $\underline{C} \to \underline{W}$ with all geometric fibers reduced of dimension one and with at worst nodes as singularities. A *pre-stable curve* is a nodal curve with all geometric fibers connected.

The category of rational polyhedral cones from [3, Section 2.1] is denoted **Cones**. An object σ of **Cones** comes with a lattice N_σ with $\sigma \subseteq (N_\sigma)_\mathbb{R} = N_\sigma \otimes_\mathbb{Z} \mathbb{R}$, and we denote by $\sigma_\mathbb{Z} = \sigma \cap N_\sigma$ the submonoid of integral points of σ. If P is a monoid, we write $P^\vee := \operatorname{Hom}(P, \mathbb{N})$, $P^* = \operatorname{Hom}(P, \mathbb{Z})$, and $P^\vee_\mathbb{R} = \operatorname{Hom}(P, \mathbb{R}_{\geq 0})$. We write $P^\times$ for the group of invertible elements of P. We write $\Bbbk[P]$ for the monoid ring of P with coefficients in the field $\Bbbk$, with $\Bbbk$-basis consisting of symbols z^p for $p \in P$.

[7] We only use these assumptions in the proof of Theorem 3.25 to assure that the reduced logarithmic strata are defined by logarithmic ideals. This theorem is at the heart of everything we do involving moduli spaces of punctured maps marked by a type.

Chapter 2

Punctured maps

2.1 Definitions

2.1.1 Puncturing

Definition 2.1. Let $Y = (\underline{Y}, \mathcal{M}_Y)$ be a fine and saturated logarithmic scheme with a decomposition $\mathcal{M}_Y = \mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}$. A *puncturing* of Y along $\mathcal{P} \subset \mathcal{M}_Y$ is a fine sub-sheaf of monoids

$$\mathcal{M}_{Y^\circ} \subset \mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}^{\mathrm{gp}}$$

containing $\mathcal{M}_Y$ with a structure map $\alpha_{Y^\circ} : \mathcal{M}_{Y^\circ} \to \mathcal{O}_Y$ such that

(1) The inclusion $\mathrm{p}^\flat : \mathcal{M}_Y \to \mathcal{M}_{Y^\circ}$ is a morphism of logarithmic structures on $\underline{Y}$.

(2) For any geometric point $\bar{x}$ of $\underline{Y}$ let $s_{\bar{x}} \in \mathcal{M}_{Y^\circ,\bar{x}}$ be such that $s_{\bar{x}} \notin \mathcal{M}_{\bar{x}} \oplus_{\mathcal{O}^\times} \mathcal{P}_{\bar{x}}$. Representing $s_{\bar{x}} = (m_{\bar{x}}, p_{\bar{x}}) \in \mathcal{M}_{\bar{x}} \oplus_{\mathcal{O}^\times} \mathcal{P}^{\mathrm{gp}}_{\bar{x}}$, we have $\alpha_{Y^\circ}(s_{\bar{x}}) = \alpha_{\mathcal{M}}(m_{\bar{x}}) = 0$ in $\mathcal{O}_{Y,\bar{x}}$.

Denote by $Y^\circ = (\underline{Y}, \mathcal{M}_{Y^\circ})$. We will also call the induced morphism of logarithmic schemes $\mathrm{p} : Y^\circ \to Y$ a *puncturing* of Y along $\mathcal{P}$, or call Y° a *puncturing* of Y along $\mathcal{P}$. We refer to Figure 2.1 for illustration.

We say the puncturing is *trivial* if p is an isomorphism.

Remark 2.2. In all examples in this paper, $\mathcal{P}$ is a $DF(1)$ log structure, that is, there is a surjective sheaf homomorphism $\underline{\mathbb{N}} \to \overline{\mathcal{P}}$. In this case the condition $\alpha_{\mathcal{M}}(m_{\bar{x}}) = 0$ is redundant. Indeed, for $s_{\bar{x}} = (m_{\bar{x}}, p_{\bar{x}}) \notin \mathcal{M}_{\bar{x}} \oplus_{\mathcal{O}^\times} \mathcal{P}$, suppose $\alpha_{Y^\circ}(s_{\bar{x}}) = 0$. Note that the $DF(1)$ assumption implies that $p_{\bar{x}}^{-1} \in \mathcal{P}_{\bar{x}}$, so that $\alpha_{\mathcal{M}}(m_{\bar{x}}) = \alpha_Y(m_{\bar{x}}, 1) = \alpha_{Y^\circ}(s_{\bar{x}} \cdot p_{\bar{x}}^{-1}) = 0$. More generally, the same argument works if $\mathcal{P}$ is valuative.

For more general puncturings, the second vanishing condition $\alpha_{\mathcal{M}}(m_{\bar{x}}) = 0$ in Definition 2.1 (2) is not automatic, but is needed to obtain good behavior under base-change (Proposition 2.8). Our log stacks $\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau)$ in Section 5.2.2 naturally carry such a more general puncturing. While these more general log structures have no further use in this paper, they may be of use elsewhere.

Note also that if $\mathcal{P}$ is a $DF(1)$ log structure and $\bar{y}$ is a geometric point of $\underline{Y}$, then

$$\overline{\mathcal{M}}_{Y,\bar{y}} \oplus \mathbb{N} \subseteq \overline{\mathcal{M}}_{Y^\circ,\bar{y}} \subset \overline{\mathcal{M}}_{Y,\bar{y}} \oplus \mathbb{Z}, \quad \overline{\mathcal{M}}_{Y^\circ,\bar{y}} \cap (\{0\} \times \mathbb{Z}_{<0}) = \emptyset. \tag{2.1}$$

We will see in Lemma 2.21 how such monoids can easily be encoded in the dual tropical picture.

Remark 2.3. Puncturings $\mathcal{M}^\circ$ of $\mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}$ are not unique. In a widely distributed early version of this manuscript as well as in [31], we found it instructive to work with

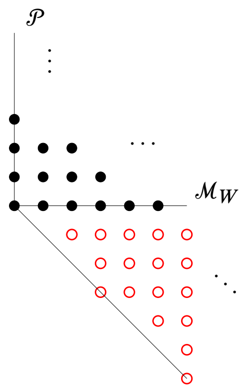

Figure 2.1. A puncturing Y° of a monoid $\mathcal{M} = \mathcal{M}_W$. Note that the part with negative projection in $\mathcal{P}^{\mathrm{gp}}$ (open circles) is not necessarily saturated.

a uniquely defined object $\mathcal{M}^{\mathcal{P}}$ we call here the *final puncturing*. It may be defined as the direct limit

$$\mathcal{M}^{\mathcal{P}} := \varinjlim_{\mathcal{M}^\circ \in \Lambda} \mathcal{M}^\circ,$$

over the collection Λ of all puncturings of $\mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}$. This exists in the category of quasi-coherent, not necessarily coherent, logarithmic structures. It has the advantage of being independent of any choice. Its disadvantage, apart from not being finitely generated, is in that its behavior under base change is subtle.

We emphasize that

(1) all puncturings used in this paper, with the exception of the remark above, are fine, and in particular they are finitely generated.

(2) On the other hand, the puncturings we use are rarely saturated, even though the logarithmic structure they puncture are themselves saturated. The reason is that base change of a saturated puncturing can lead to a non-saturated puncturing. Imposing a saturation condition would therefore lead to a subtle fiberwise saturation procedure. Instead, we find that the notion of pre-stability of Definition 2.6 below suffices to control these logarithmic structures and their moduli.

Remark 2.4. In the introduction, we motivated punctures as arising from restrictions of log structures on log smooth curves to irreducible components. Indeed, this is one way of producing punctures: see Proposition 5.2 for details. However, since we allow fine rather than fine saturated log structures for the puncturing, it is clear that not all the punctures we consider are of this form. See also Lemma 2.21 for a description of the submonoids of $\overline{\mathcal{M}}_{Y,\bar{y}} \oplus \mathbb{Z}$ that can arise.

It is worth making a historical remark here. When we began this project, we first considered what we called “pre-nodal” log structures in which we allowed precisely those log structures coming via restriction from a log smooth curve. However, we found the moduli space of pre-nodal log maps was very poorly behaved, almost never Deligne–Mumford. The notion of punctured points along with the notion of pre-stability of Definition 2.6 resolved these technical difficulties, and made gluing possible.

2.1.2 Pre-stable punctured log structures

In case a puncturing is equipped with a morphism to another fine log structure there is a canonical choice of puncturing. The following proposition follows immediately from the definitions.

Proposition 2.5. *Let X be a fine log scheme, and Y as in Definition 2.1, with a choice of puncturing Y° and a morphism $f : Y^\circ \to X$. Let $\widetilde{Y}^\circ$ denote the puncturing of Y given by the subsheaf of $\mathcal{M}_{Y^\circ}$ generated by $\mathcal{M}_Y$ and $f^\flat(f^*\mathcal{M}_X)$. Then*

(1) *We have $\mathcal{M}_{\widetilde{Y}^\circ}$ is a sub-logarithmic structure of $\mathcal{M}_{Y^\circ}$.*

(2) *There is a factorization*

$$\begin{tikzcd} Y^\circ \arrow[rr, "f"] \arrow[dr] & & X. \\ & \widetilde{Y}^\circ \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

(3) *Given $Y_1^\circ \to Y_2^\circ \to Y$ with both Y_1°, Y_2° puncturings of Y, and compatible morphisms $f_i : Y_i^\circ \to X$, then $\widetilde{Y}_1^\circ = \widetilde{Y}_2^\circ$.*

Definition 2.6. A morphism $f : Y^\circ \to X$ from a puncturing of a log scheme Y is said to be *pre-stable* if the induced morphism $Y^\circ \to \widetilde{Y}^\circ$ in the above proposition is the identity. In particular, one has $f = \tilde{f}$.

Proposition 2.5 yields the following criterion for pre-stability of a morphism from a punctured log scheme, see Figure 2.2.

Corollary 2.7. *A morphism $f : Y^\circ \to X$ is pre-stable if and only if the induced morphism of sheaves of monoids $f^*\overline{\mathcal{M}}_X \oplus \overline{\mathcal{M}}_Y \to \overline{\mathcal{M}}_{Y^\circ}$ is surjective.*

2.1.3 Pull-backs of puncturings

Proposition 2.8. *Let Z and Y be fs log schemes with log structures $\mathcal{M}_Z$ and $\mathcal{M}_Y$, and suppose given a morphism $g : Z \to Y$. Suppose also given an fs log structure*

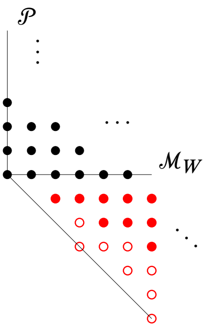

Figure 2.2. A morphism of the previous puncturing Y° which is not pre-stable, with $f^\flat\mathcal{M}_X$ generated by $(2,-1)$. The submonoid generated by $\mathcal{M}_Y$ and $f^\flat\mathcal{M}_X$, shown in solid dots, is a different puncturing $\widetilde{Y}^\circ$ which is pre-stable.

$\mathcal{P}_Y$ on $\underline{Y}$ and an induced log structure $\mathcal{P}_Z := g^*\mathcal{P}_Y$ on $\underline{Z}$. Set

$$Z' = \big(\underline{Z}, \mathcal{M}_Z \oplus_{\mathcal{O}_{\underline{Z}}^\times} \mathcal{P}_Z\big), \quad Y' = \big(\underline{Y}, \mathcal{M}_Y \oplus_{\mathcal{O}_{\underline{Y}}^\times} \mathcal{P}_Y\big).$$

Further, let Y° be a puncturing of Y' along $\mathcal{P}_Y$. Then there is a diagram

$$\begin{tikzcd} Z^\circ \arrow[r, "g^\circ"] \arrow[d] & Y^\circ \arrow[d] \\ Z' \arrow[r, "g'"] \arrow[d] & Y' \arrow[d] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

with all squares Cartesian in the category of underlying schemes, the lower square Cartesian in the category of fs log schemes, and the top square Cartesian in the category of fine log schemes. Furthermore, Z° is a puncturing of Z' along $\mathcal{P}_Z$, and g° is pre-stable.

Proof. We define Z° to be the fiber product $Z' \times^{\mathrm{f}}_{Y'} Y^\circ$ in the fine log category. The bottom square is Cartesian in all categories as $\mathcal{P}_Y$ is assumed saturated. Thus it is sufficient to show (1) the upper square is Cartesian in the ordinary category, that is, the underlying map of $Z^\circ \to Z'$ is the identity and (2) Z° is a puncturing of Z'.

Note that the fiber product $Z' \times_{Y'} Y^\circ$ in the category of log schemes is defined as $(\underline{Z}, \mathcal{M} := \mathcal{M}_{Z'} \oplus_{g^*\mathcal{M}_{Y'}} g^*\mathcal{M}_{Y^\circ})$. This pushout need not, in general, be integral, so we must integralize. Note there is a canonical isomorphism

$$\mathcal{M}^{\mathrm{gp}} = \mathcal{M}_{Z'}^{\mathrm{gp}} \oplus_{g^*\mathcal{M}_{Y'}^{\mathrm{gp}}} g^*\mathcal{M}_{Y^\circ}^{\mathrm{gp}} \cong \mathcal{M}_{Z'}^{\mathrm{gp}}$$

given by $(s_1, s_2) \mapsto s_1 \cdot (g')^\flat(s_2)$, where $(g')^\flat : g^*\mathcal{M}_{Y'}^{\mathrm{gp}} \to \mathcal{M}_{Z'}^{\mathrm{gp}}$ is induced by g'. The integralization $\mathcal{M}^{\mathrm{int}}$ of $\mathcal{M}$ is then the image of $\mathcal{M}$ in $\mathcal{M}^{\mathrm{gp}}$, which thus can be described as the subsheaf of $\mathcal{M}_{Z'}^{\mathrm{gp}}$ generated by $\mathcal{M}_{Z'}$ and $(g')^\flat(g^*\mathcal{M}_{Y^\circ})$. Note $\mathcal{M}_{Z'}$ and $(g')^\flat(g^*\mathcal{M}_{Y^\circ})$ both lie in $\mathcal{M}_Z \oplus_{\mathcal{O}_Z^\times} \mathcal{P}_Z^{\mathrm{gp}}$, and hence we can replace $\mathcal{M}^{\mathrm{gp}}$ with this subsheaf of $\mathcal{M}^{\mathrm{gp}}$ in describing $\mathcal{M}^{\mathrm{int}}$.

It is now sufficient to show that we can define a structure map $\alpha : \mathcal{M}^{\mathrm{int}} \to \mathcal{O}_Z$ compatible with the structure maps $\alpha_{Z'} : \mathcal{M}_{Z'} \to \mathcal{O}_Z$ and $\alpha_{Y^\circ} : g^*\mathcal{M}_{Y^\circ} \to \mathcal{O}_Z$. If $s \in \mathcal{M}^{\mathrm{int}}$ is of the form $s_1 \cdot (g')^\flat(s_2)$ for $s_1 \in \mathcal{M}_{Z'}$ and $s_2 \in g^*\mathcal{M}_{Y^\circ}$, then we define

$$\alpha(s) = \alpha_{Z'}(s_1) \cdot \alpha_{Y^\circ}(s_2).$$

We need to show this is well defined. If $s_2 \in g^*\mathcal{M}_{Y'}$, then $(g')^\flat(s_2) \in \mathcal{M}_{Z'}$, and thus as g' is a log morphism,

$$\alpha(s) = \alpha_{Z'}(s_1) \cdot \alpha_{Y^\circ}(s_2) = \alpha_{Z'}(s_1) \cdot \alpha_{Z'}((g')^\flat(s_2)) = \alpha_{Z'}(s).$$

In particular, $\alpha(s)$ only depends on s, and not on the particular representation of s as a product, provided that $s_2 \in g^*\mathcal{M}_{Y'}$.

On the other hand, if $s_2 \in (g^*\mathcal{M}_{Y^\circ}) \setminus (g^*\mathcal{M}_{Y'})$, then $\alpha_{Y^\circ}(s_2) = 0$ by definition of a puncturing. So in this case $\alpha(s) = 0$. Hence to check that α is well defined, it is enough to show that if $s = s_1 \cdot (g')^\flat(s_2) = s_1' \cdot (g')^\flat(s_2')$ with $s_2 \in g^*\mathcal{M}_{Y'}$ but $s_2' \notin g^*\mathcal{M}_{Y'}$, then $\alpha_{Z'}(s_1) \cdot \alpha_{Y^\circ}(s_2) = \alpha_{Z'}(s_1 \cdot (g')^\flat(s_2)) = 0$. Writing $s_i = (m_i, p_i)$, $s_i' = (m_i', p_i')$ using the descriptions $\mathcal{M}_{Z'} = \mathcal{M}_Z \oplus_{\mathcal{O}_Z^\times} \mathcal{P}_Z$, $g^*\mathcal{M}_{Y^\circ} \subset g^*\mathcal{M}_Y \oplus_{\mathcal{O}_Z^\times} \mathcal{P}_Z^{\mathrm{gp}}$, we note that we must have $m_1 g^\flat(m_2) = m_1' g^\flat(m_2')$. As $s_2' \notin g^*\mathcal{M}_{Y'}$, by Condition (2) of Definition 2.1 we necessarily have $\alpha_Y(m_2') = 0$. Hence $\alpha_Z(m_1' g^\flat(m_2')) = 0$, so $\alpha_Z(m_1 g^\flat(m_2)) = 0$. We deduce that $\alpha_{Z'}(s_1(g')^\flat(s_2)) = 0$, as desired. This shows α is well defined.

Finally, it is clear from the above description that Z° is a puncturing. By Corollary 2.7, the pre-stability of g° follows from the surjectivity of

$$g^{-1}(\bar{\mathcal{M}}_{Y^\circ}) \oplus \bar{\mathcal{M}}_Z \to g^{-1}(\bar{\mathcal{M}}_{Y^\circ}) \oplus^{\mathrm{f}}_{g^{-1}(\bar{\mathcal{M}}_Y)} \bar{\mathcal{M}}_Z = \bar{\mathcal{M}}_{Z^\circ},$$

where $\oplus^{\mathrm{f}}$ denotes the fibered coproduct in the category of fine monoids. ∎

Definition 2.9. In the situation of Proposition 2.8, we say that Z° is the *pullback of the puncturing* Y°.

Corollary 2.10. *Consider the situation of Proposition 2.8, and suppose in addition given a pre-stable morphism $f : Y^\circ \to X$. Then the composition $f \circ g^\circ : Z^\circ \to X$ is also pre-stable.*

Proof. This follows immediately from the definition of pre-stability and the construction of Z° in the proof of Proposition 2.8. ∎

2.1.4 Punctured curves

Throughout the paper, we will essentially only be interested in puncturing along logarithmic structures from designated marked points of logarithmic curves. Let $\pi : C \to W$ be a logarithmic curve in the sense of [41].

(1) The underlying morphism $\underline{\pi}$ is a family of ordinary pre-stable curves with pairwise disjoint sections $p_1, \ldots, p_n$ of $\underline{\pi}$ disjoint from the critical locus of $\underline{\pi}$.

(2) π is a proper logarithmically smooth and integral morphism of fine and saturated logarithmic schemes.

(3) If $\underline{U} \subset \underline{C}$ is the non-critical locus of $\underline{\pi}$ then $\bar{\mathcal{M}}_C|_{\underline{U}} \cong \underline{\pi}^* \bar{\mathcal{M}}_W \oplus \bigoplus_{i=1}^n p_{i*}\mathbb{N}_W$.

Note that by (3), all marked points receive a non-trivial logarithmic structure. We write $\alpha_C : \mathcal{M}_C \to \mathcal{O}_C$ for the structure map of the logarithmic structure on C. We call a geometric point of $\underline{C}$ *special* if it is either a marked or a nodal point.

Definition 2.11. A *punctured curve* over a fine and saturated logarithmic scheme W is given by the following data:

$$\left(C^\circ \xrightarrow{\mathrm{p}} C \xrightarrow{\pi} W, \mathbf{p} = (p_1, \ldots, p_n)\right) \tag{2.2}$$

where

(1) $C \to W$ is a logarithmic curve in the sense of [41] with its collection of pairwise disjoint sections $p_1, \ldots, p_n$ of the underlying curve as above.

(2) $C^\circ \to C$ is a puncturing of C along $\mathcal{P}$, where $\mathcal{P}$ is the divisorial logarithmic structure on $\underline{C}$ induced by the divisor $\bigcup_{i=1}^n p_i(\underline{W})$.

When there is no danger of confusion, we may call $C^\circ \to W$ a punctured curve. Sections in $\mathbf{p}$ are called *punctured sections*, or simply *punctures*. If $\underline{W} = \operatorname{Spec} \kappa$ with κ a field, we also speak of a *punctured point*. We also say C° is a puncturing of C along the punctured sections $\mathbf{p}$.

If locally around a punctured point p_i the puncturing is trivial, we say that the punctured point is a *marked point*. In this case, the theory will agree with the treatment of marked points in [2, 15, 30].

Examples 2.12. (1) Let $W = \operatorname{Spec} \Bbbk$ be the point with the trivial logarithmic structure, and C be a non-singular curve over W. Choose a closed point $p \in C$ and a puncturing $\mathcal{M}_{C^\circ}$ of C at p. Then $\mathcal{M}_{C^\circ} = \mathcal{P}$, as $\mathcal{M}_{C^\circ} \subset \mathcal{P}^{\mathrm{gp}}$ can have no sections s with $\alpha_{C^\circ}(s) = 0$. Thus, in this case the only puncturing $C^\circ \to C$ is the trivial one.

(2) Let $W = \operatorname{Spec}(\mathbb{N} \to \Bbbk)$ be the standard logarithmic point, and C be a non-singular curve over W, so that $\mathcal{M}_C = \mathcal{O}_C^\times \oplus \underline{\mathbb{N}}$, where $\underline{\mathbb{N}}$ denotes the constant sheaf on C with stalk $\mathbb{N}$. Again choose a closed point $p \in C$ with defining ideal (x). Let $\mathcal{M}_{C^\circ} \subset \pi^*\mathcal{M}_W \oplus_{\mathcal{O}_C^\times} \mathcal{P}^{\mathrm{gp}}$ be a puncturing. Let s be a local section of $\mathcal{M}_{C^\circ}$

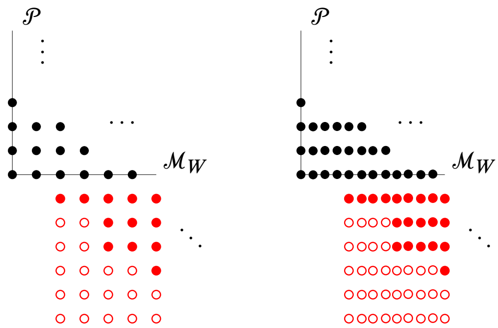

Figure 2.3. The solid puncturing on the left extends to $\Bbbk[\varepsilon]/(\varepsilon^2)$ but no further—the circled elements are the ones allowed for $k = 1$. Its pullback (see below) via $\mathcal{E}^2 = \varepsilon$ is pictured on the right—it is defined on $\Bbbk[\mathcal{E}]/(\mathcal{E}^4)$ but does not extend further.

near p. Write $s = ((\varphi, n), x^m)$ with $\varphi \in \mathcal{O}^\times_{C,p}$, $n \in \mathbb{N}$. If $m < 0$, then Condition (2) of Definition 2.1 implies that

$$\alpha_{\pi^*(\mathcal{M}_W)}(\varphi, n) = 0,$$

so we must have $n > 0$. Thus we see that

$$\overline{\mathcal{M}}_{C^\circ,p} \subset \{(n,m) \in \mathbb{N} \oplus \mathbb{Z} \mid m \geq 0 \text{ if } n = 0\}.$$

Conversely, any fine submonoid of the right-hand-side of the above inclusion which contains $\mathbb{N} \oplus \mathbb{N}$ can be realized as the stalk of the ghost sheaf at p for a puncturing. Note the monoid on the right-hand side is not finitely generated, and is the stalk of the ghost sheaf of the final puncturing, see Remark 2.3.

(3) Let $\underline{W} = \operatorname{Spec} \Bbbk[\varepsilon]/(\varepsilon^{k+1})$, and let W be given by the chart $\mathbb{N} \to \Bbbk[\varepsilon]/(\varepsilon^{k+1})$, $1 \mapsto \varepsilon$. Let C_0 be a non-singular curve over $\operatorname{Spec} \Bbbk$ with the trivial logarithmic structure, and let $C = W \times C_0$. Choose a section $p : W \to C$, with image locally defined by an equation $x = 0$. Condition (2) of Definition 2.1 now implies that a section s of a puncturing $\mathcal{M}_{C^\circ}$ near p takes the form $((\varphi, n), x^m)$ where $\varphi \in \mathcal{O}^\times_{C,p}$, and $0 \leq n \leq k$ implies $m \geq 0$. In particular,

$$\overline{\mathcal{M}}_{C^\circ,p} \subset \{(n,m) \in \mathbb{N} \oplus \mathbb{Z} \mid m \geq 0 \text{ if } n \leq k\},$$

and any fine submonoid of the right-hand side containing $\mathbb{N} \oplus \mathbb{N}$ can be realized as the stalk of the ghost sheaf at p of a puncturing. See Figure 2.3.

2.1.5 Pull-backs of punctured curves

Consider a punctured curve $(C^\circ \to C \to W, \mathbf{p})$ and a morphism of fine and saturated logarithmic schemes $h : T \to W$. Denote by $(C_T \to T, \mathbf{p}_T)$ the pullback of the log curve $C \to W$ via $T \to W$. By Proposition 2.8, we obtain a commutative diagram

$$\begin{tikzcd} C_T^\circ \arrow[r] \arrow[d, "\mathrm{p}_T"'] & C^\circ \arrow[d, "\mathrm{p}"] \\ C_T \arrow[r] \arrow[d, "\pi_T"'] & C \arrow[d, "\pi"] \\ T \arrow[r, "h"] & W \end{tikzcd} \tag{2.3}$$

where the bottom square is cartesian in the fine and saturated category, and the square on the top is cartesian in the fine category, and such that C_T° is a puncturing of the curve C_T along p_T. See again Figure 2.3.

Definition 2.13. We call $C_T^\circ \to T$ the *pullback* of the punctured curve $C^\circ \to W$ along $T \to W$.

2.1.6 Punctured maps

We now fix a morphism of fine and saturated logarithmic schemes $X \to B$.

Definition 2.14. A *punctured map to* $X \to B$ over a fine and saturated logarithmic scheme W over B consists of a punctured curve $(C^\circ \to C \to W, \mathbf{p})$ and a morphism f fitting into a commutative diagram

$$\begin{tikzcd} C^\circ \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d] \\ W \arrow[r] & B \end{tikzcd}$$

Such a punctured map is denoted by $(\pi : C^\circ \to W, \mathbf{p}, f)$ or $(C^\circ/W, \mathbf{p}, f)$.

The *pullback* of a punctured map $(C^\circ/W, \mathbf{p}, f)$ along a morphism of fine and saturated logarithmic schemes $T \to W$ is the punctured map $(C_T^\circ/T, \mathbf{p}_T, f_T)$ consisting of the pullback $C_T^\circ \to T$ of the punctured curve $C \to W$ and the pullback f_T of f.

Definition 2.15. A punctured map $(C^\circ \to W, \mathbf{p}, f)$ is called *pre-stable* if $f : C^\circ \to X$ is pre-stable in the sense of Definition 2.6.

A pre-stable punctured map is called *stable* if its underlying map, marked by the punctured sections, is stable in the usual sense.

Proposition 2.16. *Let* $(C^\circ/W, \mathbf{p}, f)$ *be a punctured map over* W.

(1) *The locus of points of* W *with pre-stable fibers forms an open sub-scheme of* $\underline{W}$.

(2) *If* $f : C^\circ \to X$ *is pre-stable, then the pullback* $f_T : C^\circ_T \to X$ *along any morphism of fine and saturated logarithmic schemes* $T \to W$ *is also pre-stable.*

Proof. The map $f : C^\circ \to X$ induces a morphism of fine logarithmic structures

$$f^\flat \oplus \mathrm{p}^\flat : f^*\mathcal{M}_X \oplus_{\mathcal{O}_C^\times} \mathcal{M}_C \to \mathcal{M}_{C^\circ}.$$

The pre-stability of f is equivalent to the condition that $f^\flat \oplus \mathrm{p}^\flat$ is surjective by Corollary 2.7. Statement (1) can be proved by applying Lemma 2.17 to the neighborhood of each puncture. Statement (2) follows immediately from Corollary 2.10. ∎

Lemma 2.17. *Let* $\underline{Y}$ *be a scheme, and* $\psi^\flat : \mathcal{M} \to \mathcal{N}$ *be a morphism of fine log structures on* $\underline{Y}$*. Then the locus* $\underline{Y}' \subset \underline{Y}$ *over which* $\psi^\flat$ *is surjective forms an open subscheme of* $\underline{Y}$*.*

Proof. We thank the anonymous referee for suggesting the following simplified proof. Since both $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{O}_{\underline{Y}}^\times$-torsors over $\overline{\mathcal{M}}$ and $\overline{\mathcal{N}}$ respectively, the surjectivity of $\psi^\flat$ is equivalent to the surjectivity of the induced morphism $\overline{\mathcal{M}} \to \overline{\mathcal{N}}$ of ghost sheaves. Since the statement is local on $\underline{Y}$, we may assume that $\overline{\mathcal{N}}$ is globally generated.

Suppose $y \in \underline{Y}$ is a geometric point over which $\overline{\mathcal{M}}_y \to \overline{\mathcal{N}}_y$ is surjective. Then each global section of $\overline{\mathcal{N}}$ lifts to a section of $\overline{\mathcal{M}}$ in an étale neighborhood of y. Since $\Gamma(\underline{Y}, \overline{\mathcal{N}})$ is finitely generated, there is a common étale neighborhood of y over which all the global sections of $\overline{\mathcal{N}}$ lift to $\overline{\mathcal{M}}$. This finishes the proof. ∎

The most interesting aspect of punctured curves is the appearance of negative contact orders, defined as follows.

Definition 2.18. The *contact order* of a punctured map $(C^\circ/W, \mathbf{p}, f)$ to $X \to B$ over a log point $W = \mathrm{Spec}(Q \to \kappa)$ at $p \in \mathbf{p}$ is the composition

$$u_p : \overline{\mathcal{M}}_{X,\underline{f}(p)} \xrightarrow{f^\flat} \overline{\mathcal{M}}_{C,p} \to Q \oplus \mathbb{Z} \xrightarrow{\mathrm{pr}_2} \mathbb{Z} \tag{2.4}$$

with the second map the canonical inclusion. We say that the contact order u_p is *negative* if $u_p(\overline{\mathcal{M}}_{X,\underline{f}(p)}) \not\subseteq \mathbb{N}$.

The difference with the case of logarithmic stable maps [30, Definition 1.8] is the appearance of $\mathbb{Z}$ instead of $\mathbb{N}$. The tropical interpretation of this condition will be discussed in Section 2.2 below. Note that if $(C^\circ/W, \mathbf{p}, f)$ is pre-stable, the contact order at $p \in \mathbf{p}$ is negative if and only if p is not a marked point.

Example 2.19. Here is a simple example featuring a negative contact order. Let $\underline{X}$ be a smooth surface, $\underline{D} \subseteq \underline{X}$ a non-singular rational curve with self-intersection -1

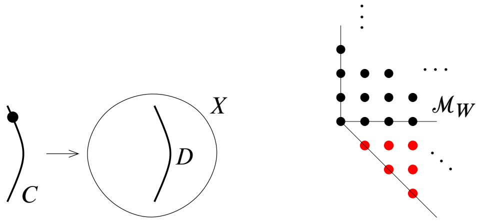

Figure 2.4. The (-1)-curve and its monoid.

inducing the divisorial log structure X on $\underline{X}$. Let $C \to W$ be the punctured curve of Example 2.12 (2), with $C \cong \mathbb{P}^1$. Let $\underline{f} : \underline{C} \to \underline{X}$ be an isomorphism of $\underline{C}$ with $\underline{D}$. This can be enhanced to a punctured map $C^\circ \to X$ as follows.

We first define $\bar{f}^\flat : \underline{f}^*\overline{\mathcal{M}}_X = \underline{\mathbb{N}} \to \overline{\mathcal{M}}_{C^\circ} \subseteq \overline{\mathcal{E}} = \underline{\mathbb{N}} \oplus \mathbb{Z}_p$ by $1 \mapsto (1, -1)$, where $\mathbb{Z}_p$ denotes the sky-scraper sheaf at p with stalk $\mathbb{Z}$. Note that the inverse image of $1 \in \Gamma(X, \overline{\mathcal{M}}_X)$ under the projection map $\mathcal{M}_X \to \overline{\mathcal{M}}_X$ is the $\mathcal{O}_X^\times$-torsor contained in $\mathcal{M}_X$ corresponding to the line bundle $\mathcal{O}_X(-D)$, and thus $1 \in \Gamma(C, \underline{f}^*\overline{\mathcal{M}}_X)$ similarly yields the $\mathcal{O}_C^\times$-torsor corresponding to $\mathcal{O}_C(1)$, using $-D^2 = 1$. On the other hand, the torsor contained in $\mathcal{M}_{C^\circ}$ corresponding to $(1, 0)$ is the torsor of $\mathcal{O}_C$, and the torsor corresponding to $(0, 1)$ is the torsor of the ideal $\mathcal{O}_C(-p)$. Hence $(1, -1) \in \Gamma(C, \overline{\mathcal{M}}_{C^\circ})$ corresponds to $\mathcal{O}_C(1)$. Choosing an isomorphism of torsors then lifts the map $\bar{f}^\flat$ to a map $f^\flat : \underline{f}^*\mathcal{M}_X \to \mathcal{M}_{C^\circ}$ inducing a morphism $f : C^\circ \to X$ (Figure 2.4).

Note this morphism does not lift to $C' \to W' = \operatorname{Spec}(\Bbbk[\varepsilon]/(\varepsilon^2))$ as in Example 2.12 (3), since we cannot even lift $\bar{f}^\flat$ at the level of ghost sheaves. Indeed, $(1, -1)$ is not a section of the ghost sheaf of $(C')^\circ$.

Remark 2.20 (Geometric implication of negative contact orders). Let $f : C^\circ/W \to X$ be a punctured map with $W = \operatorname{Spec}(Q \to \Bbbk)$. Suppose $p \in C$ is a punctured point which is not a marked point, and let C' be the irreducible component containing p, with generic point η. Then, intuitively, C' has negative order of tangency with certain strata in X, and this forces C' to be contained in those strata.

Explicitly, let $P_p = \overline{\mathcal{M}}_{X, \underline{f}(p)}$ and let $u_p : P_p \to \mathbb{Z}$ be as in Definition 2.2. Then if $\delta \in P_p$ with $u_p(\delta) < 0$, we must have $\mathrm{pr}_1 \circ \bar{f}_p(\delta) \neq 0$, as there is no element of $\overline{\mathcal{M}}_{C^\circ, p} \subset Q \oplus \mathbb{Z}$ of the form $(0, n)$ with $n < 0$. Thus if $\chi : P_p \to \overline{\mathcal{M}}_{X, \underline{f}(\eta)}$ denotes the generization map, we must have $u_p^{-1}(\mathbb{Z}_{<0}) \cap \chi^{-1}(0) = \emptyset$. This restricts the strata in which $f(C')$ can lie.

For example, if $X = (\underline{X}, D)$ for a simple normal crossings divisor D with irreducible components $D_1, \ldots, D_n$, then $P_p = \bigoplus_{i : f(p) \in D_i} \mathbb{N}$. The value u_p on the generator of P_p corresponding to D_i is the contact order with D_i. Then $f(C')$ must lie in the intersection of those D_i that have negative contact order at p.

A critical aspect of this phenomenon is discussed in Section 2.5, see especially Proposition 2.52 and Example 2.54.

2.2 The tropical interpretation

We now introduce the tropical picture, which gives the underlying organizing language for punctured Gromov–Witten theory. We assume familiarity with the discussion in ordinary logarithmic Gromov–Witten theory as presented in [3, Section 2]. We review in Section 2.2.1 the notations and basic concepts briefly while discussing the modifications needed for including non-trivial punctures.

2.2.1 Tropical punctured maps

In Appendix C we define tropicalization as a functor associating to a fine log algebraic stack a generalized cone complex $\Sigma(X)$. There is one stratum of $|\Sigma(X)|$ for each logarithmic stratum of X, the latter defined as a maximal connected locally closed subset $Z \subseteq |\underline{X}|$ with $\overline{\mathcal{M}}_X|_Z$ locally constant. For each logarithmic stratum Z we choose, once and for all, a geometric point $\bar{x}_Z$ with image in Z. Then $\Sigma(X)$ is defined as the diagram with only one cone

$$\sigma_Z = \mathrm{Hom}\big(\overline{\mathcal{M}}_{X,\bar{x}_Z}, \mathbb{R}_{\geq 0}\big) \tag{2.5}$$

for each logarithmic stratum Z, along with all its faces, and arrows induced by all sequences of generization morphisms and all face inclusions, including inverses of those that are isomorphisms. Note that due to monodromy, $\Sigma(X)$ may contain non-trivial arrows $\sigma \to \sigma$. The group

$$\mathrm{Aut}_{\Sigma(X)}(\sigma) = \{\sigma \to \sigma \text{ arrow in } \Sigma(X)\}$$

is a subgroup of the permutation group of the set of rays of σ, hence is always finite. Note that the map

$$\sigma_Z / \mathrm{Aut}_{\Sigma(X)}(\sigma_Z) \to |\Sigma(X)|$$

induced from $\sigma_Z \to |\Sigma(X)|$ may still not be injective due to arrows from strata of X whose closure intersect the closure of Z and that are not induced by monodromy on Z. Accordingly, the image of σ in $|\Sigma(X)|$ may be a finite quotient even on its interior.

By abuse of notation, $\Sigma(X)$ denotes both the distinguished presentation or the equivalence class as a generalized cone complex. When writing $\sigma \in \Sigma(X)$ we refer to the chosen presentation, so there is a unique logarithmic stratum $Z \subseteq X$ with $\sigma = \sigma_Z$. For any geometric point $\bar{x}$ with image in Z we have the cone

$$\sigma_{\bar{x}} = \mathrm{Hom}\big(\overline{\mathcal{M}}_{X,\bar{x}}, \mathbb{R}_{\geq 0}\big)$$

together with an isomorphism

$$\sigma_Z \to \sigma_{\bar{x}},$$

but this isomorphism is only unique up to pre-composition with arrows $\sigma_Z \to \sigma_Z$ in $\Sigma(X)$. In other words, the isomorphism $\sigma_Z \to \sigma_{\bar{x}}$ is unique up to the action of the monodromy group $\mathrm{Aut}_{\Sigma(X)}(\sigma_Z)$ of the logarithmic stratum Z.

For $\sigma \in \Sigma(X)$ we denote by

$$X_\sigma = \big\{x \in \underline{X} \mid \text{ there exists an arrow } \sigma \to \sigma_{\bar{x}} \text{ in } \Sigma(X)\big\} \subseteq \underline{X} \tag{2.6}$$

the closed set of points $x \in \underline{X}$ with σ connected to $\sigma_{\bar{x}} = \mathrm{Hom}(\overline{\mathcal{M}}_{X,\bar{x}}, \mathbb{R}_{\geq 0})$ by a sequence of generizations and inverses of invertible generizations of the stalks of $\overline{\mathcal{M}}_X$. We endow X_σ with the reduced induced scheme structure. In practice, say when X is log smooth over a log point, X_σ is the closure of the logarithmic stratum given by $\sigma \in \Sigma(X)$. For brevity, we refer to the X_σ as *strata* of X, but note that from the point of view of stratified spaces, and differing from the use in Appendix C, these are at best closures of strata. Note also that for $\sigma = \{0\}$ we obtain $X_{\{0\}} = \underline{X}$ assuming $\Sigma(X)$ connected, even if there is no geometric point $\bar{x}$ of X with $\overline{\mathcal{M}}_{X,\bar{x}} = 0$.

A stable logarithmic map $(C/W, \mathbf{p}, f)$ gives rise via functoriality of the tropicalization functor Σ to the diagram

$$\begin{tikzcd} \Sigma(C) \arrow[r, "\Sigma(f)"] \arrow[d, "\Sigma(\pi)"'] & \Sigma(X) \arrow[d] \\ \Sigma(W) \arrow[r] & \Sigma(B) \end{tikzcd} \tag{2.7}$$

We will almost exclusively consider such diagrams in which W is covered by a single chart and $\Sigma(W)$ has a single maximal cone $\omega = (\mathcal{M}^\vee_{W,\bar{w}})_{\mathbb{R}}$ for $\bar{w}$ some geometric point of $\underline{W}$. Then it is shown in [3, Proposition 2.25] that $\Sigma(\pi)$ along with the genera of the irreducible components of the geometric fiber $C_{\bar{w}}$ is a (family of) abstract tropical curves over ω, also written $(G, \mathbf{g}, \ell)$. Here G is the dual intersection graph of $C_{\bar{w}}$ with sets $V(G)$, $E(G)$, $L(G)$ of vertices, edges and legs, and the maps

$$\mathbf{g} : V(G) \to \mathbb{N}, \quad \ell : E(G) \to \mathrm{Hom}(\omega_{\mathbb{Z}}, \mathbb{N}) \setminus \{0\},$$

record the genera of the irreducible components of $C_{\bar{w}}$ and the lengths of edges as functions on ω, respectively, see [3, Definition 2.19]. If G arises from the tropicalization of a log curve over a geometric logarithmic point, we denote elements of $V(G)$, $E(G)$, $L(G)$ both by their graph-theoretic notations as vertices v, edges E, and legs L, or the corresponding algebraic geometric notations as generic points η, nodes q, and marked points p. By abuse of notation, we view homomorphisms $\omega_{\mathbb{Z}} \to \mathbb{N}$ also as homomorphisms $\omega \to \mathbb{R}_{\geq 0}$ respecting the integral structure. Conversely, from $(G, \mathbf{g}, \ell)$, the cone complex

$$\Gamma = \Gamma(G, \ell)$$

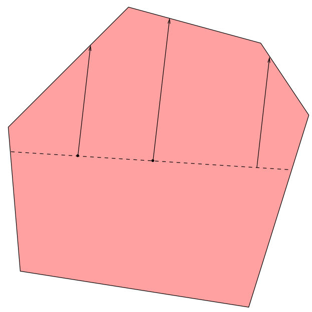

Figure 2.5. The length of a bounded leg varies piecewise linearly under linear variations of the adjacent vertex. The figure shows the intersection of the situation with an affine hyperplane.

recovering $\Sigma(C)$ has one copy of ω for each $v \in V(G)$, a cone

$$\omega_E = \big\{(s, \lambda) \in \omega \times \mathbb{R}_{\geq 0} \mid \lambda \leq \ell(E)(s)\big\} \tag{2.8}$$

for each edge $E \in E(G)$, and a copy of $\omega \times \mathbb{R}_{\geq 0}$ for each leg. Note that legs have infinite lengths for any parameter $s \in \omega_{\mathbb{R}}$ when viewing Γ as a family of metric graphs.

The only change in the punctured setup is that a leg may now have finite length. Indeed, if $L \in L(G)$ corresponds to a non-trivial puncture with puncturing submonoid $Q^\circ \subset Q \oplus \mathbb{Z}$, then $(Q^\circ)^\vee_{\mathbb{R}} = \omega_L$ with

$$\omega_L = \big\{(s, \lambda) \in \omega \times \mathbb{R}_{\geq 0} \mid \lambda \leq \ell(L)(s)\big\} \tag{2.9}$$

defined in analogy with (2.8) by a length function $\ell(L) : \omega \to \mathbb{R}_{\geq 0}$ with $\ell(L) \neq 0$. Note, however, that $\ell(L)$ is now only piecewise linear as illustrated in Figure 2.5. Here a continuous function $\ell : \omega \to \mathbb{R}_{\geq 0}$ on $\omega \in$ **Cones** is *piecewise linear* if there exists a fan subdivision of ω such that ℓ is the restriction of a linear form on each cone of the fan. For the following relation to monoids recall (2.1) from Remark 2.2.

Lemma 2.21. *Let Q be a sharp toric monoid and $\omega = Q^\vee_{\mathbb{R}}$. Assume further that $Q^\circ \subseteq Q \oplus \mathbb{Z}$ is a finitely generated submonoid with $Q \oplus \mathbb{N} \subsetneq Q^\circ$, $Q^\circ \cap (\{0\} \times \mathbb{Z}_{<0}) = \emptyset$. Then there exists a nonzero, concave, piecewise linear function*

$$\ell : \omega \to \mathbb{R}_{\geq 0}$$

with rational slopes such that

$$(Q^\circ)^\vee_{\mathbb{R}} = \{(s, \lambda) \in \omega \times \mathbb{R}_{\geq 0} \mid 0 \leq \lambda \leq \ell(s)\}. \tag{2.10}$$

Each such $\ell : \omega \to \mathbb{R}_{\geq 0}$ arises in this fashion, and two submonoids $Q_1^\circ, Q_2^\circ \subseteq Q \oplus \mathbb{Z}$ with $Q_i \oplus \mathbb{N} \subsetneq Q_i^\circ$, $Q_i^\circ \cap (\{0\} \times \mathbb{Z}_{<0}) = \emptyset$, $i = 1, 2$, lead to the same ℓ if and only if $(Q_1^\circ)^{\mathrm{sat}} = (Q_2^\circ)^{\mathrm{sat}}$.

Proof. Let $(s, \lambda) \in (Q^\circ)^\vee_{\mathbb{R}}$. Then since $Q \oplus \mathbb{N} \subseteq Q^\circ$, necessarily $s \in \omega = Q^\vee_{\mathbb{R}}$ and $\lambda \geq 0$. Conversely, $(s, 0) \in (Q^\circ)^\vee_{\mathbb{R}}$ for all $s \in \omega$, and in fact,

$$\omega \times \{0\} \subseteq (Q^\circ)^\vee_{\mathbb{R}}$$

is a facet. Since $Q^\circ \neq Q \oplus \mathbb{N}$ no ray of $(Q^\circ)^\vee_{\mathbb{R}}$ is vertical, that is, agrees with $\mathbb{R}_{\geq 0} \cdot (0, 1)$. Thus the union of the maximal cells of $\partial(Q^\circ)^\vee_{\mathbb{R}}$ neither contained in $\omega \times \{0\}$ nor in $\partial\omega \times \mathbb{R}$ form the graph of a piecewise linear function $\ell : \omega \to \mathbb{R}_{\geq 0}$ as in the statement of the lemma. Convexity of $(Q^\circ)^\vee_{\mathbb{R}}$ implies that ℓ is concave. Finally, $\ell \neq 0$ for otherwise $(0, -1) \in Q^\circ_{\mathbb{R}}$, contradicting $Q^\circ \cap (\{0\} \times \mathbb{Z}_{<0}) = \emptyset$.

Conversely, given a nonzero, concave, piecewise linear $\ell : \omega \to \mathbb{R}_{\geq 0}$ with rational slopes, the cone σ on the right-hand side of (2.10) contains $\omega \times \{0\}$ as a facet. Hence

$$\sigma^\vee \subseteq \omega^\vee \times \mathbb{R} = Q_{\mathbb{R}} \times \mathbb{R} \quad \text{and} \quad Q_{\mathbb{R}} \times \mathbb{R}_{\geq 0} \subseteq \sigma^\vee.$$

The case $Q_{\mathbb{R}} \times \mathbb{R}_{\geq 0} = \sigma^\vee$ does not occur since $\sigma \neq \omega \times \mathbb{R}_{\geq 0}$ by finiteness of the values of ℓ. Moreover, $\ell \neq 0$ implies σ is a full-dimensional cone, and hence $(0, -1) \notin \sigma^\vee$, or $\sigma^\vee \cap (\{0\} \times \mathbb{Z}_{<0}) = \emptyset$. This shows that knowing ℓ retrieves the convex hull of Q° in $Q_{\mathbb{R}} \times \mathbb{R}$, hence the set of integral points of its saturation $(Q^\circ)^{\mathrm{sat}}$. ■

Definition 2.22. (1) A (family of) *punctured tropical curves* over a cone $\omega \in$ **Cones** is a graph G together with two maps

$$\mathbf{g} : V(G) \to \mathbb{N}, \quad \ell : E(G) \cup L^\circ(G) \to \mathrm{Map}(\omega, \mathbb{R}_{\geq 0})$$

for some subset $L^\circ(G) \subseteq L(G)$, with $\ell(E) \in \mathrm{Hom}(\omega_{\mathbb{Z}}, \mathbb{N}) \setminus \{0\}$ for $E \in E(G)$ and $\ell(L) : \omega \to \mathbb{R}_{\geq 0}$ for $L \in L^\circ(G)$ nonzero, concave, piecewise linear, with rational slopes. We refer to elements of $L^\circ(G)$ as *finite or punctured legs*, all other legs as *infinite or marked*.

(2) A (family of) *punctured tropical maps* over $\omega \in$ **Cones** is a map of generalized cone complexes $h : \Gamma \to \Sigma(X)$ for $\Gamma = \Gamma(G, \ell)$ the cone complex defined by a punctured tropical curve $(G, \mathbf{g}, \ell)$ over ω.

For readability and as in [3] throughout, we assume for the rest of this subsection that $\Sigma(X)$ is simple [3, Definition 2.1]. This means that for each $\sigma \in \Sigma(X)$ the map $\sigma \to |\Sigma(X)|$ is injective. We will treat the general case in Section 2.6. As in [3, Proposition 2.26], it then follows readily from the definitions that the tropicalization of a punctured map to X over a logarithmic point $\mathrm{Spec}(Q \to \kappa)$ with κ algebraically closed is a punctured tropical map over $Q^\vee_{\mathbb{R}}$.

Given a punctured tropical map, one extracts associated discrete data as in [3, Remark 2.22]. These are an *image cone* map

$$\boldsymbol{\sigma} : V(G) \cup E(G) \cup L(G) \to \Sigma(X) \tag{2.11}$$

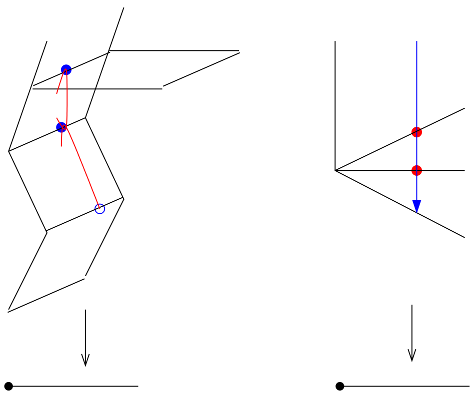

Figure 2.6. A curve in the fiber of a one-parameter family of surfaces (a threefold) and its tropicalization. There are two components, represented by vertices; one node represented by an edge; one regular marked point represented by an infinite leg and one puncture represented by a finite leg, which, by pre-stability, extends exactly as far as the cone allows.

associating to each object of G the (distinguished representative of the) minimal cone of $\Sigma(X)$ it maps to, and, referring again to the notation in Section 1.6, *contact orders*

$$u_q = u_E \in N_{\boldsymbol{\sigma}(E)}, \quad u_p = u_L \in N_{\boldsymbol{\sigma}(L)} \tag{2.12}$$

for edges $E = E_q \in E(G)$ and for legs $L = L_p \in L(G)$, respectively.

Contact orders are defined by the image of the tangent vector $(0,1)$ in the tangent space $N_\omega \times \mathbb{Z}$ of ω_E or ω_L under h. The contact order for an edge E depends, up to sign, on a choice of orientation on E, which we suppress in the notation. For legs, this definition is consistent with the definition of contact orders of punctured maps in Definition 2.18.

Note that the contact order $u_p \in N_{\boldsymbol{\sigma}(L_p)}$ of a marked point $p \in C_{\bar{w}}$ lies in $\boldsymbol{\sigma}(L_p)$. Conversely, a non-trivial puncture is forced by a leg $L = L_p$ if for any parameter value $s \in \omega$, the line segment $h(\{s\} \times [0, \ell(L)(s)])$ inside the image cone $\boldsymbol{\sigma}(L) \in \Sigma(X)$ does not extend to a half-line.

There is a simple tropical interpretation of pre-stability saying that images of legs extend as far as possible inside their image cones. See Figure 2.6 for an illustration. We call such tropical punctured maps *pre-stable*.

Proposition 2.23. *Let $(C^\circ/W, \mathbf{p}, f)$ be a pre-stable punctured map over a log point $W = \operatorname{Spec}(Q \to \kappa)$ and $h = \Sigma(f) : \Gamma(G, \ell) \to \Sigma(X)$ its tropicalization. For each finite leg $L \in L^\circ(G)$, we write $\omega_L \subseteq \omega \times \mathbb{R}_{\geq 0}$ as in (2.9). Then for all $s \in \omega$, we have*

$$h(s, \ell(L)(s)) = h(s, 0) + \ell(L)(s) \cdot u_L \in \partial \boldsymbol{\sigma}(L),$$

while $h(s, \ell(L)(s)) + \varepsilon u_L \notin \boldsymbol{\sigma}(L)$ for all $\varepsilon > 0$.

Proof. Let $\bar{p} \to C$ be the punctured point defined by L, and write $\omega = Q^\vee_{\mathbb{R}}$, $\sigma = P^\vee_{\mathbb{R}}$ for $P = \overline{\mathcal{M}}_{X,\underline{f}(\bar{p})}$. The map $h_L : \omega_L \to \sigma$ defined by h is dual to

$$\bar{f}^\flat_{\underline{f}(\bar{p})} : P \to \overline{\mathcal{M}}_{C,\bar{p}} = Q^\circ \subset Q \oplus \mathbb{Z}.$$

By pre-stability, Q° is generated by $Q \oplus \mathbb{N}$ and by the image of $\bar{f}^\flat_{\underline{f}(\bar{p})}$. Dually, we obtain

$$\omega_L = (Q^\circ)^\vee_{\mathbb{R}} = (\omega \times \mathbb{R}_{\geq 0}) \cap h_L^{-1}(\sigma).$$

Now ω_L is the convex hull of $\omega \times \{0\}$ and of $\{(s, \ell(L)(s)) \in \omega \times \mathbb{R}_{\geq 0}\}$, the graph of $\ell(L)$ as a map $\omega \to \mathbb{R}_{\geq 0}$. This shows that no point $(s, \ell(L)(s))$ maps to an interior point of σ, and the line segment in σ connecting $h(s, 0)$ with $h(s, \ell(L)(s))$ can not be extended, as claimed. ∎

Note that while $\omega_L^\vee \cap (N_\omega \times \mathbb{Z})^*$ only computes the saturation of Q°, the tropical picture also contains the map $P \to Q \oplus \mathbb{Z}$. In the pre-stable case, Q° is then the submonoid generated by the image of this map and by $Q \oplus \mathbb{N}$, so can be fully computed tropically.

2.2.2 Types of punctured maps

As in [3, Definition 2.23] for stable logarithmic maps, we now capture the combinatorics underlying punctured maps and their tropicalization by the notion of *type*.

Definition 2.24. (1) The *type of a family of tropical punctured maps* $h : \Gamma = \Gamma(G, \ell) \to \Sigma(X)$ over $\omega \in$ **Cones** is the tuple

$$\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$$

consisting of the associated genus-decorated connected graph $(G, \mathbf{g})$, the image cone map $\boldsymbol{\sigma}$ from (2.11) and the collection $\mathbf{u} = \{u_q, u_p\}_{p,q} = \{u_E, u_L\}_{E,L}$ of contact orders from (2.12). In particular, for $x \in E(G) \cup L(G)$ we require $u_x \in N_{\boldsymbol{\sigma}(x)}$. We also sometimes write $\mathbf{u}(x)$ instead of u_x when referring to a contact order given by a type rather than by a punctured map.

(2) The *type of a punctured map* $(C/W, \mathbf{p}, f)$ *to* X *at a geometric point* $\bar{w}$ *of* $\underline{W}$ is the type of the associated tropical map $\Gamma \to \Sigma(X)$ over $\omega = (\overline{\mathcal{M}}^\vee_{W,\bar{w}})_{\mathbb{R}}$.

Thus the type records the combinatorial data associated to $h : \Gamma \to \Sigma(X)$, but forgets the length function $\ell : E(G) \cup L^\circ(G) \to \mathrm{Map}(\omega, \mathbb{R}_{\geq 0})$.

For a punctured map over a logarithmic point, one sometimes also wants to keep the curve classes $\mathbf{A}(v) = \underline{f}_*([\underline{C}(v)])$ for $\underline{C}(v) \subset \underline{C}$ the irreducible component of $\underline{C}$ given by $v \in V(G)$. Here $\mathbf{A}(v)$ is a class of curves in singular homology of the corresponding stratum $X_{\boldsymbol{\sigma}(v)}$, or in some other appropriate monoid of curve classes, written $H_2^+(X_\sigma)$ for $\sigma \in \Sigma(X)$ in any case.[1] We refer to [33, Basic setup 1.6] for a listing of the properties of H_2^+ assumed throughout. Adding this information, one arrives at the notion of *decorated type*

$$\boldsymbol{\tau} = (\tau, \mathbf{A}) = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{A}). \tag{2.13}$$

Finally, just as in logarithmic Gromov–Witten theory, generization of punctured maps gives rise to contraction morphisms of graphs: Let $(C^\circ/W, \mathbf{p}, f)$ be a punctured map to X and let $\bar{w}' \to \bar{w}$ be a specialization arrow of geometric points of W. Denote by $h : \Gamma = \Gamma(G, \ell) \to \Sigma(X)$ and $h' : \Gamma' = \Gamma(G', \ell') \to \Sigma(X)$ the tropicalizations of the strict restrictions of $(C^\circ/W, \mathbf{p}, f)$ to $\bar{w}, \bar{w}'$. Then as in [3, eq. (2.15)], generization defines a contraction morphism of the associated decorated graphs

$$\phi : (G, \mathbf{g}) \to (G', \mathbf{g}'),$$

given by contracting those edges $E = E_q \in E(G)$ with corresponding node $\bar{q} \to \underline{C}_{\bar{w}}$ not contained in the closure of the nodal locus of $\underline{C}_{\bar{w}'}$. By abuse of notation we write ϕ also for the maps

$$V(G) \to V(G'), \quad L(G) \to L(G'), \quad E(G) \setminus E_\phi \xrightarrow{\text{bij}} E(G')$$

defining ϕ. Here $E_\phi \subseteq E(G)$ is the subset of contracted edges. Analogous to [3, Definition 2.24] there is a corresponding natural notion of contraction morphism of (decorated) types of tropical punctured maps

$$\begin{aligned} \tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}) &\to \tau' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}'), \\ \boldsymbol{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{A}) &\to \boldsymbol{\tau}' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}', \mathbf{A}'). \end{aligned} \tag{2.14}$$

Under such contraction morphisms, legs never get contracted. Moreover, identifying $L(G) = L(G')$, the contact order $\mathbf{u}(L) \in N_{\boldsymbol{\sigma}(L)}$ of a leg of G is the image of $\mathbf{u}'(L) \in N_{\boldsymbol{\sigma}'(L)}$ under the inclusion of lattices $N_{\boldsymbol{\sigma}'(L)} \to N_{\boldsymbol{\sigma}(L)}$ induced by the face map $\boldsymbol{\sigma}'(L) \to \boldsymbol{\sigma}(L)$. An analogous statement applies to contact orders of non-contracted edges.

Proposition 2.25. *Let $(C^\circ/W, \mathbf{p}, f)$ be a stable punctured map to X over some logarithmic scheme W and $(\tau_{\bar{w}}, \mathbf{A}_{\bar{w}})$ with $\tau_{\bar{w}} = (G_{\bar{w}}, \mathbf{g}_{\bar{w}}, \boldsymbol{\sigma}_{\bar{w}}, \mathbf{u}_{\bar{w}})$ its decorated type at the geometric point $\bar{w} \to W$ according to Definition 2.24 and (2.13).*

Then if $\bar{w}' \to \bar{w}$ is a specialization arrow of geometric points of W, the map

$$(\tau_{\bar{w}}, \mathbf{A}_{\bar{w}}) \to (\tau_{\bar{w}'}, \mathbf{A}_{\bar{w}'})$$

induced by generization is a contraction morphism.

[1] The notation allows *defining* $H_2^+(X_\sigma) := H_2^+(X)$ for all $\sigma \in \Sigma(X)$, by interpreting classes of curves in a stratum X_σ as classes of curves in X.

Proof. The proof is essentially identical to [3, Lemma 2.30], noting that the proof of [30, Lemma 1.11] also works for contact orders at punctures. ■

2.2.3 The balancing condition

The above discussion fits well with the tropical balancing condition at vertices of the dual graph of C°. In fact, the statement [30, Proposition 1.15] holds unchanged as there is no balancing condition at the endpoint of a leg $L \in L(G)$. As we will need the balancing condition to prove boundedness, we review this statement here. We note that the balancing conditions discussed here are heavily used in applications such as [33], [27, Section 4] or [32], as balancing severely limits the possible combinatorial types.

Suppose given a stable punctured map $(C^\circ/W, \mathbf{p}, f)$ with $W = \operatorname{Spec}(\mathbb{N} \to \kappa)$ the standard log point over an algebraically closed field, and denote by $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ its type. Let $g : \widetilde{D} \to C$ be the normalization of an irreducible component D with generic point η of C. One then obtains, with $\overline{\mathcal{M}} = \underline{f}^*\overline{\mathcal{M}}_X$, composed maps

$$\tau_\eta^X : \Gamma(\widetilde{D}, g^*\overline{\mathcal{M}}) \to \operatorname{Pic} \widetilde{D} \xrightarrow{\deg} \mathbb{Z}$$
$$\tau_\eta^C : \Gamma(\widetilde{D}, g^*\overline{\mathcal{M}}_{C^\circ}) \to \operatorname{Pic} \widetilde{D} \xrightarrow{\deg} \mathbb{Z}$$

with the first map on each line given by taking a section of the ghost sheaf to the corresponding $\mathcal{O}_{\widetilde{D}}^\times$-torsor, the inverse image of this section in $g^*\mathcal{M}$ or $g^*\mathcal{M}_{C^\circ}$. These are compatible: the pullback of $f^\flat$ to $\widetilde{D}$, $\varphi : g^*\mathcal{M} \to g^*\mathcal{M}_{C^\circ}$, induces $\bar{\varphi} : g^*\overline{\mathcal{M}} \to g^*\overline{\mathcal{M}}_{C^\circ}$ and a commutative diagram

$$\begin{tikzcd}
\Gamma(\widetilde{D}, g^*\overline{\mathcal{M}}) \arrow[r, "\bar{\varphi}"] \arrow[dr, "\tau_\eta^X"'] & \Gamma(\widetilde{D}, g^*\overline{\mathcal{M}}_{C^\circ}) \arrow[d, "\tau_\eta^C"] \\
& \mathbb{Z}
\end{tikzcd}$$

The map τ_η^X is given by $\underline{f}$ and $\mathcal{M}$, so depends on the logarithmic geometry of $f : C^\circ \to X$; however if $\underline{f}$ contracts D, then $\tau_\eta^X = 0$. On the other hand, τ_η^C is determined completely by the geometry of $D \subseteq C$ and $g^*\overline{\mathcal{M}}_{C^\circ}$ as follows. We use the notation in [30, Section 1.4]. For each point $q \in D$ over a node of $\underline{C}$ we have $\overline{\mathcal{M}}_{C^\circ, \bar{q}} = S_{e_q}$, the submonoid of $\mathbb{N}^2$ generated by $(0, e_q)$, $(e_q, 0)$ and $(1, 1)$. The generization map $\chi_q : \overline{\mathcal{M}}_{C^\circ, \bar{q}} \to \overline{\mathcal{M}}_{C^\circ, \bar{\eta}} = \mathbb{N}$ is given by projection to the second coordinate: $\chi_q(a, b) = b$. In what follows, we use q always to denote points over nodes and p to denote punctured points. We then have

$$\Gamma\big(\widetilde{D}, g^*\overline{\mathcal{M}}_{C^\circ}\big) \subseteq \big\{(n_q)_{q \in \widetilde{D}} \,|\, n_q \in S_{e_q} \text{and } \chi_q(n_q) = \chi_{q'}(n_{q'}) \text{ for } q, q' \in \widetilde{D}\big\} \oplus \bigoplus_{p \in \widetilde{D}} \mathbb{Z}.$$

This inclusion induces an equality at the level of groups. The equation $\chi_q(n_q) = \chi_{q'}(n_{q'})$ allows us to write $b = b_q = \chi_q(n_q)$ independent of q. We then obtain, with proof identical to that of [30, Lemma 1.14].

Lemma 2.26. $\tau_\eta^C(((a_q, b)_{q\in\tilde D}, (n_p)_{p\in\tilde D})) = -\sum_{p\in\tilde D} n_p + \sum_{q\in\tilde D} \frac{b-a_q}{e_q}$,

The equation $\tau_\eta^X = \tau_\eta^C \circ \varphi$ is a formula in $N_D := \Gamma(\tilde D, g^*\bar{\mathcal{M}}^{\mathrm{gp}})^*$, which is described in [30, eqs. (1.12), (1.13)] as follows. Let $\Sigma \subset \tilde D$ be the set of points x in $\tilde D$ mapping to a special point of C. Thus Σ can be identified with the subset of $E(G) \cup L(G)$ of edges or legs adjacent to the vertex v corresponding to η. For any point $x \in \tilde D$, we write $P_x := \bar{\mathcal{M}}_{X,g(x)}$. Then

$$N_D = \varinjlim_{x\in\tilde D} P_x^* = \Big(\bigoplus_{x\in\Sigma} P_x^*\Big)\Big/\sim$$

where for any $a \in P_\eta^*$ and any $x, x' \in \Sigma$,

$$(0, \ldots, 0, \iota_{x,\eta}(a), 0, \ldots, 0) \sim (0, \ldots, 0, \iota_{x',\eta}(a), 0, \ldots, 0).$$

Here $\iota_{x,\eta} : P_\eta^* \to P_x^*$ is the dual of generization, and the non-zero entries lie in the position indexed by x and x' respectively. Thus an element of N_D is represented by a choice of tangent vector $n_x \in N_{\boldsymbol{\sigma}(x)} = P_x^*$, one for each preimage $x \in \tilde D$ of a special point of C; and two such choices are identified if they can be related by repeatedly subtracting a tangent vector in $N_{\boldsymbol{\sigma}(v)} = P_\eta^*$ from one of the n_x and adding it to another.

We then have, exactly as in [30, Proposition 1.15], the balancing condition:

Proposition 2.27. *Suppose $(C^\circ/W, \mathbf{p}, f)$ is a stable punctured map to X/B with $W = \mathrm{Spec}(\mathbb{N} \to \kappa)$ a standard log point. Let $D \subseteq \underline{C}$ be an irreducible component with generic point η and $\Sigma \subset \tilde D$ the preimage of the set of special points. If $\tau_\eta^X \in \Gamma(\tilde D, g^*\bar{\mathcal{M}}^{\mathrm{gp}})^*$ is represented by $(\tau_x)_{x\in\Sigma}$, then*

$$(u_x)_{x\in\Sigma} + (\tau_x)_{x\in\Sigma} = 0$$

in $N_D = \Gamma(\tilde D, g^*\bar{\mathcal{M}}^{\mathrm{gp}})^*$.

Remark 2.28. With regard to the above interpretation of elements of N_D in terms of the type of $(C^\circ/W, \mathbf{p}, f)$, Proposition 2.27 says the following. The degree data of the $\mathcal{O}_C^\times$-torsors contained in $g^*\mathcal{M}$ defines a tuple of tangent vectors $\tau_x \in N_{\boldsymbol{\sigma}(x)}$, one for each edge or leg $x \in E(G) \cup L(G)$ adjacent to the vertex v corresponding to η, well-defined up to trading elements of $N_{\boldsymbol{\sigma}(v)}$ via the embedding $N_{\boldsymbol{\sigma}(v)} \hookrightarrow N_{\boldsymbol{\sigma}(x)}$ defined by the face morphism $\boldsymbol{\sigma}(v) \to \boldsymbol{\sigma}(x)$. Then (1) $\tau_x + u_x$ lies in the image of $P_\eta^* \to P_x^*$, and (2) the traditional tropical balancing condition holds in P_η^* for $\tau_x + u_x$, x running over the set of special points.

Traditional tropical geometry arises for the case that X is a toric variety with its toric log structure. Then $\mathcal{M}_X^{\mathrm{gp}}$ is the sheaf of rational functions that are invertible on the big torus. Monomial functions define trivial $\mathcal{O}_X^\times$-subtorsors of $\mathcal{M}_X^{\mathrm{gp}}$. Denoting by N the cocharacter lattice of the torus, we thus have a canonical monomorphism

$$N^* \to \Gamma\big(X, \mathcal{M}_X^{\mathrm{gp}}\big) \to \Gamma\big(X, \overline{\mathcal{M}}_X^{\mathrm{gp}}\big)$$

with composition with τ_η^X identically zero. Composing the equation displayed in Proposition 2.27 with the induced map $N_D \to N$ then yields the traditional balancing condition $\sum_x \bar{u}_x = 0$ for $\bar{u}_x$ the image of u_x under the embedding $N_{\sigma(x)} \to N$.

The following is an encapsulation of balancing which gives easy to use restrictions on curve classes realized by punctured maps with given contact orders. For the statement we denote by $\mathcal{L}_s^\times$ the torsor corresponding to $s \in \Gamma(X, \overline{\mathcal{M}}_X^{\mathrm{gp}})$, that is, the inverse image of s under the homomorphism $\mathcal{M}_X^{\mathrm{gp}} \to \overline{\mathcal{M}}_X^{\mathrm{gp}}$, and write $\mathcal{L}_s$ for the corresponding line bundle. Furthermore, the germ of s at $f(p_i)$ lies in $P_{p_i}^{\mathrm{gp}} = \overline{\mathcal{M}}_{X,f(p_i)}^{\mathrm{gp}}$ and hence defines a homomorphism $P_{p_i}^* \to \mathbb{Z}$, which we write as $\langle \cdot, s\rangle$.

Proposition 2.29. *Suppose given a punctured map $(C^\circ/W, \mathbf{p}, f)$ for W a log point. Let $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ be the type of this map, and let $\underline{D} \subseteq \underline{C}$ be an irreducible component of the domain, corresponding to $v \in V(G)$. Let $p_1, \ldots, p_n \in \mathbf{p}$ be the punctured points of C° contained in $\underline{D}$, and let $q_1, \ldots, q_m$ be the nodes of $\underline{C}$ contained in $\underline{D}$ but which are not nodes of $\underline{D}$. This gives rise to contact orders u_{p_i}, u_{q_j}, noting that for the contact orders of the nodes, we orient the corresponding edge away from v. Then we have*

$$\deg\big(\underline{f}^*\mathcal{L}_s\big)\big|_{\underline{D}} = -\sum_{i=1}^n \langle u_{p_i}, s\rangle - \sum_{i=1}^m \langle u_{q_i}, s\rangle.$$

Proof. First, by making a base-change, we can assume W is the standard log point. Note $\underline{f}^*\mathcal{L}_s$ must be isomorphic to the line bundle $\mathcal{L}_{\bar{f}^\flat(s)}$ associated to the torsor corresponding to $\bar{f}^\flat(s)$.

Now the total degree of $\mathcal{L}_{\bar{f}^\flat(s)}$ can be calculated using Lemma 2.26 and details of the proof of [30, Proposition 1.15]. Let $g : \underline{\widetilde{D}} \to \underline{C}$ be the normalization of $\underline{D}$, and let η be the generic point of $\underline{D}$. Then

$$\begin{aligned}\deg(f \circ g)^*\mathcal{L}_s &= \deg g^*\mathcal{L}_{\bar{f}^\flat(s)} = \tau_\eta^C(\varphi(s)) \\ &= \sum_{q \in \widetilde{D}} \frac{1}{e_q}(\langle V_\eta, s\rangle - \langle V_{\eta_q}, s\rangle) - \sum_{p_i \in \widetilde{D}} \langle u_{p_i}, s\rangle,\end{aligned}$$

in the notation of [30, Lemma 1.14, Proposition 1.15], and the last equality coming from the proof of [30, Proposition 1.15]. Here $V_\eta : P_\eta \to \mathbb{N}$ is the map $\bar{f}^\flat : \overline{\mathcal{M}}_{X,f(\eta)} \to \overline{\mathcal{M}}_{C,\eta}$, and similarly V_{η_q}, where η_q is the generic point of the other branch of C at the node q. By [30, eq. (1.9)], $\frac{1}{e_q}(V_\eta - V_{\eta_q}) = -u_q$, where u_q is the contact order of the node q with corresponding edge oriented away from v. Note that self-nodes of $\underline{D}$ appear twice in this sum, with opposite sign, and hence cancel. This then yields the desired formula. ∎

Corollary 2.30. *Suppose given a punctured curve $(C^\circ/W, \mathbf{p}, f)$ with W a log point, $\mathbf{p} = \{p_1, \ldots, p_n\}$. Then we have*

$$\deg \underline{f}^* \mathcal{L}_s = -\sum_{i=1}^{n} \langle u_{p_i}, s\rangle.$$

Proof. This is obtained from the previous proposition by summing over all irreducible components of $\underline{C}$. ∎

2.3 Basicness

A key concept in logarithmic moduli problems is the existence of *basic* or *minimal* logarithmic structures. The existence of such distinguished logarithmic structures on the base space of families is a necessary condition for a logarithmic moduli problem to be represented by a logarithmic algebraic stack. A good notion of basicness should be an open property, and hence is typically defined by a condition at geometric points.

The definition of basic stable logarithmic maps from [30, Section 1.5] is based on universality of the associated family of tropical maps. The original definition in [30, Definition 1.20] phrases this property in terms of the dual monoids and only indicates the tropical interpretation in [30, Remark 1.18]. A proof of the equivalence of the definitions in the present notation is given in [3, Proposition 2.28]. This equivalence of descriptions really only reflects the anti-equivalence between the categories of fs monoids and of rational polyhedral cones. In the following, we freely use this equivalence of categories when referring to material from [30].

The definition of basicness in the punctured case is formally the same as for stable logarithmic maps. Here we take the concrete, tropical view. For readability, we again assume that X is simple, deferring the general discussion to Section 2.6.

Definition 2.31. A pre-stable punctured map $(C/W, \mathbf{p}, f)$ is *basic at a geometric point $\bar{w}$ of $\underline{W}$* if the associated family of tropical maps

$$h : \Gamma = \Gamma(G, \ell) \to \Sigma(X)$$

over $(\overline{\mathcal{M}}_{W,\bar{w}})^\vee_{\mathbb{R}}$ is universal among tropical maps of the same type $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$. This means that each family of stable tropical maps of type $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ over some cone ω arises by pullback from $h : \Gamma \to \Sigma(X)$ via a unique map $\omega \to (\overline{\mathcal{M}}_{W,\bar{w}})^\vee_{\mathbb{R}}$ in **Cones**. Basicness without specifying $\bar{w}$ refers to basicness at all geometric points.

The monoids $\overline{\mathcal{M}}_{W,\bar{w}}$ obtained from basic punctured maps also formally have the same description as for stable logarithmic maps described in [30, Proposition 1.19] and [3, Proposition 2.28]. We provide a full proof of this description emphasizing the tropical perspective.

Proposition 2.32. *Let $(C^\circ/W, \mathbf{p}, f)$ be a basic, pre-stable punctured map over a logarithmic point $\operatorname{Spec}(Q \to \kappa)$ with κ an algebraically closed field, and let $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ be its type. For each generic point $\eta \in \underline{C}$ with $v = v_\eta \in V(G)$ the associated vertex write*

$$P_\eta = \overline{\mathcal{M}}_{X,\underline{f}(\eta)} = (\boldsymbol{\sigma}(v)_{\mathbb{Z}})^\vee.$$

Then the map

$$Q^\vee \to \Big\{((V_\eta)_\eta, (\ell_q)_q) \in \prod_\eta P_\eta^\vee \times \prod_q \mathbb{N} \mid V_\eta - V_{\eta'} = \ell_q \cdot \mathbf{u}(q)\Big\} \tag{2.15}$$

given by the duals of $(\overline{\pi_\eta^\flat})^{-1} \circ \bar{f}_\eta^\flat : P_\eta \to Q$ and of the classifying map $\prod_q \mathbb{N} \to Q$ of the log smooth curve C/W, is an isomorphism. Here q runs over the set of nodes of $\underline{C}$ and, in the equation, η, η' are the generic points of the adjacent branches, with the order chosen as in the definition of $\mathbf{u}$.

Proof. Denote by $\omega \in$ **Cones** the cone defined by the right-hand side of (2.15). We first construct a tropical punctured map

$$h_0 : \Gamma = \Gamma(G, \ell_0) \to \Sigma(X)$$

over ω as follows. Define

$$\ell_0(E) : \omega_{\mathbb{Z}} \to \mathbb{N}, \quad h_0(v) : \omega_{\mathbb{Z}} \to P_\eta^\vee \tag{2.16}$$

for $E = E_q \in E(G)$ and $v = v_\eta \in V(G)$ as the projections to the q-th factor in $\prod_q \mathbb{N}$ and to $P_\eta^\vee = \boldsymbol{\sigma}(v)_{\mathbb{Z}}$, respectively. For an edge $E = E_q$ with adjacent vertices v, v' and associated cone ω_E from (2.8), the map h_0 is defined by

$$\begin{aligned}
(h_0)_E : \omega_E &\to (P_q^\vee)_{\mathbb{R}} = \boldsymbol{\sigma}(E),\\
(s, \lambda) &\mapsto h_0(v)(s) + \lambda \cdot \mathbf{u}(E) = h_0(v')(s) + (\ell_0(E) - \lambda)(-\mathbf{u}(E)),
\end{aligned}$$

with the sign of $\mathbf{u}(E)$ chosen according to the orientation of E. In this definition, we view $h_0(v(s))$, $h_0(v'(s))$ as elements of $(P_q^\vee)_{\mathbb{R}}$ via the face inclusions $P_\eta^\vee, P_{\eta'}^\vee \to P_q^\vee$. The equality holds by the relation in the definition of ω by the right-hand side of equation (2.15). In particular, $(h_0)_E$ restricts to $h_0(v)$, $h_0(v')$ on its two faces defined by v, v'.

Finally, for a leg $L = L_p \in L(G)$ with adjacent vertex $v \in V(G)$, the length function $\ell_0(L)$ and the map $(h_0)_L$ defined on $(\omega_L)_{\mathbb{Z}}$ is uniquely determined by $h_0(v)$

and by the contact order $\mathbf{u}(L)$ via pre-stability (Proposition 2.23). This finishes our construction of a pre-stable tropical punctured map h_0 over ω.

Conversely, if $h : \Gamma = \Gamma(G, \ell) \to \Sigma(X)$ is a tropical punctured map of type $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ over some cone $\omega' \in$ **Cones**, the map

$$\omega' \to \omega, \quad s \mapsto (h(v_\eta(s)), \ell(E_q))_{\eta,q},$$

with $v_\eta : \omega' \to \Gamma$ the section of $\Gamma \to \omega'$ defined by $v_\eta \in V(G')$, is readily seen to be the unique morphism in **Cones** producing h by pullback from h_0. ∎

Definition 2.33. The fs monoid Q defined by (2.15) is called the *basic monoid associated to the type* $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$, while its dual $Q^\vee \in$ **Cones** (or $Q^\vee_{\mathbb{R}}$ with the integral structure understood) is called the *associated basic cone*.

Note that while the definition of the basic monoid makes sense for all types, the length function $\ell_0(E)$ constructed in (2.16) in the proof of Proposition 2.32 may be zero for some edge E. In this case, the universal tropical domain $\Gamma(G, \ell_0)$ in the proof of Proposition 2.32 is not the domain of a tropical punctured map according to Definition 2.22. The basic monoid is therefore only meaningful if there exists at least one tropical punctured map of the given type.[2] Observe also that just as marked points do not enter the definition of basicness, there is no role for punctures in the statement of Proposition 2.32.

Proposition 2.34. *Let $(C^\circ/W, \mathbf{p}, f)$ be a pre-stable punctured map. Then*

$$\Omega := \big\{\bar{w} \in |\underline{W}| \mid \{\bar{w}\} \times_{\underline{W}} (C^\circ/W, \mathbf{p}, f) \text{ is basic}\big\}$$

is an open subset of $|\underline{W}|$.

Proof. This is identical to [30, Proposition 1.22]. ∎

Proposition 2.35. *Any pre-stable punctured map to $X \to B$ arises as the pullback from a basic pre-stable punctured map to $X \to B$ with the same underlying ordinary pre-stable map. Both the basic pre-stable punctured map and the morphism are unique up to a unique isomorphism.*

Proof. The proof is almost identical to [30, Proposition 1.24]. Let $(\pi : C \to W, \mathbf{p}, f)$ be a pre-stable punctured map over B. For each geometric point $\bar{w} \to \underline{W}$ one obtains a tropical punctured map

$$h_{\bar{w}} : \Gamma_{\bar{w}} \to \Sigma(X)$$

over $\omega_{\bar{w}} = (\overline{\mathcal{M}}_{W,\bar{w}})^\vee_{\mathbb{R}}$, of some type $(G_{\bar{w}}, \mathbf{g}_{\bar{w}}, \boldsymbol{\sigma}_{\bar{w}}, \mathbf{u}_{\bar{w}})$. By Proposition 2.25, generization $\bar{w} \in \mathrm{cl}(\bar{w}')$ (i.e. existence of a specialization arrow $\bar{w}' \to \bar{w}$ as in Appendix C)

[2] The analogue of this statement in [30] is the condition $\underline{\mathrm{GS}}(\overline{\mathcal{M}}) \neq \emptyset$.

leads to a contraction morphism (2.14)

$$(G_{\bar{w}}, \mathbf{g}_{\bar{w}}, \boldsymbol{\sigma}_{\bar{w}}, \mathbf{u}_{\bar{w}}) \to (G_{\bar{w}'}, \mathbf{g}_{\bar{w}'}, \boldsymbol{\sigma}_{\bar{w}'}, \mathbf{u}_{\bar{w}'}).$$

This contraction morphism induces an embedding of $\Gamma_{\bar{w}'}$ as a subcomplex of $\Gamma_{\bar{w}}$ such that $h_{\bar{w}'}$ becomes the restriction of $h_{\bar{w}}$. These maps are compatible with the classifying maps to the dual of the respective basic monoids in Proposition 2.32, producing a cartesian diagram of pre-stable tropical punctured maps.

As in the proof of [30, Proposition 1.24], this situation produces monoid sheaves $\overline{\mathcal{M}}^{\mathrm{bas}}_{C^\circ}$, $\overline{\mathcal{M}}^{\mathrm{bas}}_{W}$ on $\underline{C}$ and $\underline{W}$, respectively, and a commutative diagram

$$\begin{tikzcd}
\underline{f}^*\overline{\mathcal{M}}_X \arrow[r] & \overline{\mathcal{M}}^{\mathrm{bas}}_{C^\circ} \arrow[r] & \overline{\mathcal{M}}_{C^\circ} \\
 & \underline{\pi}^*\overline{\mathcal{M}}^{\mathrm{bas}}_{W} \arrow[r] \arrow[u] & \underline{\pi}^*\overline{\mathcal{M}}_{W} \arrow[u]
\end{tikzcd} \tag{2.17}$$

In case B has a non-trivial log structure, all morphisms are compatible with morphisms from the pullback of $\overline{\mathcal{M}}_B$. Continuing as in [30, Proposition 1.24], we can now define the desired basic log structures by fiber product:

$$\mathcal{M}^{\mathrm{bas}}_{W} = \mathcal{M}_W \times_{\overline{\mathcal{M}}_W} \overline{\mathcal{M}}^{\mathrm{bas}}_{W}, \quad \mathcal{M}^{\mathrm{bas}}_{C^\circ} = \mathcal{M}_{C^\circ} \times_{\overline{\mathcal{M}}_{C^\circ}} \overline{\mathcal{M}}^{\mathrm{bas}}_{C^\circ}.$$

Each of these defines a log structure with the structure map being the composition of the projection to the first factor followed by the structure map for that log structure. The pair of induced morphisms

$$\pi_{\mathrm{bas}} : C^\circ_{\mathrm{bas}} = (\underline{C}, \mathcal{M}^{\mathrm{bas}}_{C^\circ}) \to W_{\mathrm{bas}} = (\underline{W}, \mathcal{M}^{\mathrm{bas}}_{W}), \quad f_{\mathrm{bas}} : C^\circ_{\mathrm{bas}} \to X$$

have tropicalizations at any geometric point $\bar{w}$ of $\underline{W}$ given by the universal pre-stable tropical punctured map to $\Sigma(X)$ over $\Sigma(B)$ of type $(G_{\bar{w}}, \mathbf{g}_{\bar{w}}, \boldsymbol{\sigma}_{\bar{w}}, \mathbf{u}_{\bar{w}})$. Thus $(C^\circ_{\mathrm{bas}}/W_{\mathrm{bas}}, \mathbf{p}, f)$ is a basic punctured map to X. By the construction by fiber products of monoid sheaves, it follows that f_{bas} commutes with the morphisms to B, and that $(C^\circ/W, \mathbf{p}, f)$ is the pullback of $(C^\circ_{\mathrm{bas}}/W_{\mathrm{bas}}, \mathbf{p}, f)$ by $W \to W_{\mathrm{bas}}$. The constructed basic punctured map is also pre-stable since $(C^\circ/W, \mathbf{p}, f)$ is and by the definition of $\mathcal{M}^{\mathrm{bas}}_{C^\circ}$ as a fiber product. Finally, the universal property of the basic monoid with regard to pre-stable tropical punctured maps in Proposition 2.32 implies uniqueness. ∎

Remark 2.36. Following [30], our construction of the basic pre-stable punctured map in the proof of Proposition 2.35 argues pointwise and uses compatibility with generizations to obtain the universal diagram of ghost sheaves. However, the existence of an étale sheaf with the stated stalks and generization maps is never checked, notably in the proof of [30, Lemma 1.23]. We use this occasion to close this gap.

The basic monoids and generization homomorphisms define a contravariant functor

$$\mathbf{Pt}(W) \to \mathbf{Mon}, \quad \bar{w} \mapsto Q_{\bar{w}} \tag{2.18}$$

from the category of geometric points $\mathbf{Pt}(W)$ with specialization arrows, recalled at the beginning of Appendix C, to the category of monoids. A specialization arrow $\bar{w} \to \bar{w}'$ maps to an epimorphism of monoids $Q_{\bar{w}'} \to Q_{\bar{w}}$ given by localization at a face and subsequently dividing out the subgroup of invertible elements. In any case, from a functor as in (2.18) one can define an étale sheaf $\overline{\mathcal{M}}^{\mathrm{bas}}$ by associating to an étale map $h : U \to X$ the monoid

$$\overline{\mathcal{M}}^{\mathrm{bas}}(U) = \mathrm{colim}_{\bar{w}\to h} Q_{\bar{w}},$$

together with the obvious restriction maps. Here the colimit is taken over all factorizations of $\bar{w} : \mathrm{Spec}\,\kappa(\bar{w}) \to X$ over h. The gap in [30] concerns the implicit claim that for a geometric point $\bar{w}$ of X the natural map

$$Q_{\bar{w}} \to \overline{\mathcal{M}}^{\mathrm{bas}}_{\bar{w}}$$

is an isomorphism.

This claim is étale local in $\underline{W}$. Hence we can assume that the given (non-basic) log structure $\mathcal{M}_W$ on W is a Zariski log structure with a global chart that is neat at some geometric point $\bar{w}$. We may also assume that the logarithmic stratum containing $\bar{w}$ lies in the closure of all other strata, and that the restriction map

$$\Gamma(W, \overline{\mathcal{M}}_W) \to \overline{\mathcal{M}}_{W,\bar{w}}$$

is an isomorphism. By [52, Proposition II.2.1.2] we obtain a continuous map

$$g : |\underline{W}| \to S = \mathrm{Spec}\,\overline{\mathcal{M}}_{W,\bar{w}}$$

from the topological space underlying $\underline{W}$ to the monoidal scheme of prime ideals of $\overline{\mathcal{M}}_{W,\bar{w}}$, a finite topological space, together with an isomorphism

$$g^{-1}\overline{\mathcal{M}}_S \to \overline{\mathcal{M}}_W.$$

Here $\overline{\mathcal{M}}_S$ is the structure sheaf of $\mathrm{Spec}\,\overline{\mathcal{M}}_{W,\bar{w}}$, a sheaf of sharp monoids.[3]

Note that a finite topological space is an Alexandrov space. Thus a subset is closed iff it is closed under specialization, and sheaves (of sets, say) are indeed given by contravariant functors from the category of points to **Sets**, see e.g. [46, Section 2].

[3] We have reinterpreted the statement in [52] as a statement for Kato fans to avoid dealing with invertible elements, which are irrelevant for our discussion.

The universal property of basic monoids provides a monoid homomorphism

$$Q_{\bar{w}} \to \overline{\mathcal{M}}_{W,\bar{w}},$$

hence a morphism of monoid spectra

$$k : S = \operatorname{Spec} \overline{\mathcal{M}}_{W,\bar{w}} \to S_{\mathrm{bas}} = \operatorname{Spec} Q_{\bar{w}}.$$

Compatibility of the basic monoids and their universal property with generization now shows first that $(k \circ g)^{-1}\overline{\mathcal{M}}_{S_{\mathrm{bas}}}$ is a sheaf of monoids with stalks equal to the basic monoids on W and having the expected generization homomorphisms, hence defines $\overline{\mathcal{M}}_W^{\mathrm{bas}}$, and second that the composition

$$\overline{\mathcal{M}}_W^{\mathrm{bas}} = (k \circ g)^{-1}\overline{\mathcal{M}}_{S_{\mathrm{bas}}} \to g^{-1}\overline{\mathcal{M}}_S \to \overline{\mathcal{M}}_W$$

stalkwise restricts to the classifying homomorphisms for $\overline{\mathcal{M}}_W$.

A similar argument on C provides the remaining parts of Diagram (2.17).

Proposition 2.37. *An automorphism $\varphi : C^\circ/W \to C^\circ/W$ of a basic pre-stable punctured map $(C^\circ/W, \mathbf{p}, f)$ with $\underline{\varphi} = \mathrm{id}_{\underline{C}^\circ}$ is trivial.*

Proof. This is identical to [30, Proposition 1.25]. ■

2.4 Global contact orders and global types

A fundamental ingredient in the definition of logarithmic Gromov–Witten invariants is the global specification of contact orders at the marked points. The local behaviour of contact orders in families of stable logarithmic maps is captured by the notion of morphism of types (2.14), implying that generization leads to the possible propagation of contact orders via face inclusions in $\Sigma(X)$. The global definition can be subtle in the presence of monodromy, as the following examples show.

Example 2.38. This example is modeled on the well-known toric construction of the Tate curve. Let Y be the three-dimensional toric variety (not of finite type) defined by the fan consisting of the collection of three-dimensional cones

$$\Sigma^{[3]} = \left\{\mathbb{R}_{\geq 0}(n,0,1) + \mathbb{R}_{\geq 0}(n+1,0,1) + \mathbb{R}_{\geq 0}(n,1,1) + \mathbb{R}_{\geq 0}(n+1,1,1) \mid n \in \mathbb{Z}\right\}$$

and their faces. Projection onto the third coordinate yields a toric morphism $Y \to \mathbb{A}^1$. After a base-change

$$\widehat{Y} = Y \times_{\mathbb{A}^1} \operatorname{Spec} \Bbbk[\![t]\!] \to \operatorname{Spec} \Bbbk[\![t]\!],$$

one may divide out $\widehat{Y}$ by the action of $\mathbb{Z}$ defined as follows. This action is generated by an automorphism of $\widehat{Y}$ induced by an automorphism of Y defined over $\mathbb{A}^1$. This

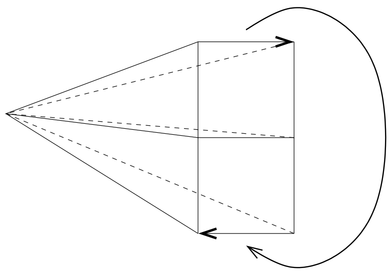

Figure 2.7. Tropicalization of a Zariski logarithmic scheme with contact order monodromy: $\ell = 2$.

automorphism is given torically via the linear transformation $\mathbb{Z}^3 \to \mathbb{Z}^3$ given by the matrix

$$\begin{pmatrix} 1 & 0 & \ell \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

where ℓ is a fixed positive integer. We then define $X = \widehat{Y}/\mathbb{Z}$, with log structure induced by the toric log structure on Y (Figure 2.7).

Then $X \to \operatorname{Spec} \Bbbk[\![t]\!]$ is a degeneration of the total space of a $\mathbb{G}_m$-torsor over an elliptic curve, the torsor corresponding to a 2-torsion element of the Picard group of the elliptic curve. As long as $\ell \geq 2$, X has a Zariski log structure. Further, $\Sigma(X)$ is a cone over a Möbius strip composed of ℓ squares. If one takes $u = (0, 1, 0) \in \sigma^{\mathrm{gp}}$ for any three-dimensional cone in $\Sigma(X)$, then propagating u via chains of face inclusions identifies u with $-u$ due to the twist in the Möbius strip.

Example 2.39. A variant of the previous example that we learnt from Jonathan Wise also produces monodromy of infinite order.

Let $\sigma \subset \mathbb{R}^4$ be the cone generated by the following column vectors:

$$\begin{aligned} v_1 &= (0,0,0,1)^t, & v_2 &= (0,1,0,1)^t, & v_3 &= (0,0,1,1)^t, & v_4 &= (0,1,1,1)^t, \\ v_5 &= (1,0,1,1)^t, & v_6 &= (1,1,1,1)^t, & v_7 &= (2,1,0,1)^t, & v_8 &= (2,2,0,1)^t. \end{aligned}$$

The linear transformation of $\mathbb{R}^4$ with matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

fulfills

$$Av_1 = v_7, \quad Av_2 = v_8, \quad Av_3 = v_5, \quad Av_4 = v_6.$$

Thus $A\tau_1 = \tau_2$ for the two facets

$$\tau_1 = \langle v_1, v_2, v_3, v_4\rangle, \quad \tau_2 = \langle v_5, v_6, v_7, v_8\rangle$$

of σ.

Now $\tau_1^{\mathrm{gp}} \cap \tau_2^{\mathrm{gp}}$ is the lattice spanned by

$$x = 2v_3 - v_1 = 2v_6 - v_8 = (0,0,2,1), \quad y = v_2 - v_1 = v_6 - v_5 = (0,1,0,0).$$

The restriction of A to this sublattice is a shear transformation, hence is of infinite order:

$$Ax = x - y, \quad Ay = y.$$

It is not hard to define a log structure on the nodal cubic curve $\underline{X}$ with $\overline{\mathcal{M}}_{X,q}^{\vee} \simeq \sigma \cap \mathbb{Z}^4$ at the node q, and the generization maps to the two branches of C at q dual to the inclusions $\tau_1, \tau_2 \hookrightarrow \sigma$. Then $X = (\underline{X}, \mathcal{M}_X)$ has infinite monodromy.

By pulling back $\mathcal{M}_X$ to a two-nodal curve of arithmetic genus 1, with the map to X contracting a $\mathbb{P}^1$, produces an example with Zariski log structure and infinite monodromy.

Note that the feature of infinite monodromy can not be seen from the underlying topological space of the tropicalization $\Sigma(X)$. In fact, as a topological space, σ is the cone over a polyhedron $\Xi \subset \mathbb{R}^3$ that is the convex hull of two disjoint facets with four vertices each, the intersections of τ_1, τ_2 with the affine hyperplane $x_4 = 1$ for $x_1, \ldots, x_4$ the coordinates on $\mathbb{R}^4$. Thus $|\Sigma(X)|$ is the cone over the cell complex obtained from Ξ by identifying these two facets. But replacing v_7, v_8 with $(2,0,0,1)^t$, $(2,1,0,1)^t$ and adapting A accordingly produces an example with homeomorphic $|\Sigma(X)|$ and without monodromy.

In the presence of monodromy as in Examples 2.38 and 2.39, the naïve definition of global contact orders by a reduced subscheme $\underline{Z} \subset \underline{X}$ and a section $s \in \Gamma(\underline{Z}, (\overline{\mathcal{M}}_X|_{\underline{Z}})^*)$ not extending to any larger subscheme from [30, Definition 3.1] does not work. We provide here an alternative treatment based on a notion of tangent vectors for the generalized cell complex $\Sigma(X)$ that suffices for the definition of finite type moduli spaces and of certain punctured Gromov–Witten invariants also in cases with monodromy. Some applications such as gluing (Theorem 5.8) in rare cases may require the more refined definition presented in Appendix A. For the sake of simplicity of presentation, we merely indicate what has to be modified to treat such rare cases.

2.4.1 Global contact orders

For $\sigma \in \Sigma(X)$ denote by $\Sigma_\sigma(X)$ the star of σ, considered as the category $\Sigma(X)$ under σ, i.e., the category with objects face embeddings $\sigma \to \sigma'$ in $\Sigma(X)$ and arrows given by morphisms $\sigma' \to \sigma''$ commuting with the given morphisms from σ. Thus the star $\Sigma_\sigma(X)$ is formed by all cones $\sigma_{\bar{x}} = \mathrm{Hom}(\overline{\mathcal{M}}_{X,\bar{x}}, \mathbb{R}_{\geq 0})$ with $\bar{x}$ running over the geometric points of X_σ. Associating to $(\sigma \to \sigma') \in \Sigma_\sigma(X)$ the free abelian group $N_{\sigma'}$, viewed as a set, gives a diagram in the category of sets indexed by $\Sigma_\sigma(X)$. This diagram can be viewed as the diagram of integral tangent vectors of $\Sigma_\sigma(X)$. Taking the colimit in the category of sets provides a set of homomorphisms $\overline{\mathcal{M}}_{X,\bar{x}} \to \mathbb{Z}$ for geometric points $\bar{x}$ of X_σ compatible with all generization homomorphisms. Elements of this colimit therefore provide a way to specify compatible sets of contact orders along the stratum X_σ independently of monodromy.

Definition 2.40. Let $\sigma \in \Sigma(X)$ and $\mathfrak{N}_\sigma : \Sigma_\sigma(X) \to \mathbf{Sets}$ be the diagram in the category of sets mapping $\sigma \to \sigma'$ to $N_{\sigma'}$. A *global contact order for* $\sigma \in \Sigma(X)$, or for the corresponding stratum $X_\sigma \subseteq \underline{X}$, is an element $\bar{u}$ of

$$\mathfrak{C}_\sigma(X) := \mathrm{colim}^{\mathbf{Sets}} \mathfrak{N}_\sigma = \mathrm{colim}^{\mathbf{Sets}}_{\sigma \to \sigma'} N_{\sigma'},$$

the *set of contact orders for* σ. For $\sigma' \in \Sigma_\sigma(X)$, or for a geometric point $\bar{x}$ of X_σ, we denote by

$$\iota_{\sigma\sigma'} : N_{\sigma'} \to \mathfrak{C}_\sigma(X), \quad \iota_{\sigma\bar{x}} : N_{\sigma_{\bar{x}}} \to \mathfrak{C}_\sigma(X)$$

the canonical maps.

A *global contact order* is a contact order for some $\sigma \in \Sigma(X)$. The set of global contact orders is denoted $\mathfrak{C}(X) := \coprod_{\sigma \in \Sigma(X)} \mathfrak{C}_\sigma(X)$.

We say a contact order $\bar{u}$ for $\sigma \in \Sigma(X)$ has *finite monodromy* if for all $(\sigma \to \sigma') \in \Sigma_\sigma(X)$ the set $\iota_{\sigma\sigma'}^{-1}(\bar{u}) \subseteq N_{\sigma'}$ is finite.

A global contact order $\bar{u} \in \mathfrak{C}_\sigma(X)$ is *monodromy-free* if for all $(\sigma \to \sigma') \in \Sigma_\sigma(X)$ there exists at most one $u \in N_{\sigma'}$ with $\bar{u} = \iota_{\sigma\sigma'}(u)$.

To be explicit, we spell out the definition of $\iota_{\sigma\bar{x}}$ for $\bar{x}$ a geometric point of X_σ. Let $Z \subseteq X$ be the smallest logarithmic stratum containing the image of $\bar{x}$. Then since $Z \cap X_\sigma \neq \emptyset$, the definition of $\Sigma(X)$ provides an isomorphism

$$\sigma_{\bar{x}} = \mathrm{Hom}\big(\overline{\mathcal{M}}_{X,\bar{x}}, \mathbb{R}_{\geq 0}\big) \xrightarrow{\simeq} \mathrm{Hom}\big(\overline{\mathcal{M}}_{X,\bar{x}_Z}, \mathbb{R}_{\geq 0}\big) = \sigma_Z$$

together with a face map $\sigma \to \sigma_Z$, unique up to arrows $\sigma \to \sigma$ and $\sigma_Z \to \sigma_Z$ in $\Sigma(X)$. Then $\iota_{\sigma\bar{x}}$ is defined by composing the induced isomorphism of lattices $N_{\sigma_{\bar{x}}} \simeq N_{\sigma_Z}$ with $\iota_{\sigma\sigma_Z}$. The definition of $\mathfrak{C}_\sigma(X)$ is designed to make all maps $\iota_{\sigma\bar{x}}$ independent of choices. In particular, a contact order as in (2.4) and (2.12) has an associated global contact order.

Note that if $\overline{\mathcal{M}}_X$ has monodromy along X_σ, there is a non-trivial group G of arrows $\sigma \to \sigma$ in $\Sigma(X)$. In this case, the map $\iota_{\sigma\sigma} : N_\sigma \to \mathfrak{C}_\sigma(X)$ factors over the quotient $N_\sigma \to N_\sigma/G$ of the induced linear action of G on N_σ. In particular, two tangent vectors $u, u' \in N_\sigma$ define the same global contact order $\bar{u} = \iota_{\sigma\sigma}(u) = \iota_{\sigma\sigma}(u')$ if they are related by monodromy along X_σ.

Given a punctured map $(C^\circ/W, \mathbf{p}, f)$ to X and $s : \underline{W} \to \underline{C}$ a punctured or nodal section, each geometric point $\bar{w}$ of $\underline{W}$ has an associated contact order $u_{s(\bar{w})}$ at $s(\bar{w})$, giving the contact orders u_p, u_q of (2.12) of the associated tropicalization:

$$u_{s(\bar{w})} : \overline{\mathcal{M}}_{X,\underline{f}(s(\bar{w}))} \to \mathbb{Z}.$$

Recall also that the contact order for a node, defined in [30, eq. (1.8)], depends on the choice of an ordering of the two branches of $\underline{C}_{\bar{w}}$ through the node q, just as $u_E = u_q$ depends on the choice of orientation of the edge E. Now for any $\sigma \in \Sigma(X)$ with $\mathrm{im}(\underline{f} \circ s) \subseteq X_\sigma$ and any $\bar{w} \to \underline{W}$, we obtain the induced global contact order

$$u_s^\sigma(\bar{w}) = \iota_{\sigma \underline{f}(s(\bar{w}))}(u_{s(\bar{w})}) \tag{2.19}$$

The following lemma shows that fixing global contact orders in families of punctured maps is both an open and closed condition. In particular, prescribing global contact orders for strata, formalized in the notion of marking below (Definition 3.4), works well in moduli problems.

Lemma 2.41. *Let $(C^\circ/W, \mathbf{p}, f)$ be a punctured map, $s : \underline{W} \to \underline{C}$ a punctured or nodal section, and $\sigma \in \Sigma(X)$ with $\mathrm{im}(\underline{f} \circ s) \subseteq X_\sigma$. Then the function $\bar{w} \mapsto u_s^\sigma(\bar{w})$ from (2.19), associating to a geometric point $\bar{w}$ of $\underline{W}$ the global contact order of $(C^\circ_{\bar{w}}/\bar{w}, \mathbf{p}_{\bar{w}}, f_{\bar{w}})$ for σ, is locally constant.*

Proof. The existence of neat charts for the punctured map $f : C^\circ \to X$ [52, Theorem III.1.2.7] shows that the composition

$$s^{-1} f^{-1} \overline{\mathcal{M}}_X \to s^{-1} \overline{\mathcal{M}}_{C^\circ} \to \underline{\mathbb{Z}},$$

is a morphism of constructible sheaves of sets. See also [52, Theorem II.2.5.4]. This composition defines the contact order as a function on $\underline{W}$. Hence the subset of $\underline{W}$ with f of a given contact order is a constructible set. It remains to show that contact orders are compatible with generization. Consider a specialization $\bar{w}'$ of $\bar{w}$, with $\underline{f} \circ s(\bar{w}') = \bar{x}'$ a specialization of $\underline{f} \circ s(\bar{w}) = \bar{x}$. By Proposition 2.25 the face embedding $N_{\sigma_{\bar{x}}} \to N_{\sigma_{\bar{x}'}}$ dual to generization, which is an arrow in $\mathfrak{N}_\sigma$, maps the contact order $u_{\bar{x}} \in N_{\sigma_{\bar{x}}}$ to $u_{\bar{x}'} \in N_{\sigma_{\bar{x}'}}$. Hence $\iota_{\sigma\bar{x}}(u_{\bar{x}}) = \iota_{\sigma\bar{x}'}(u_{\bar{x}'})$, as needed. ■

Definition 2.42. Let $(C^\circ/W, \mathbf{p}, f)$ be a punctured map, and $s : \underline{W} \to \underline{C}$ a punctured or nodal section with $\mathrm{im}(\underline{f} \circ s) \subseteq X_\sigma$ for some $\sigma \in \Sigma(X)$. Then $(C^\circ/W, \mathbf{p}, f)$ is said to have *global contact order* $\bar{u} \in \mathfrak{C}_\sigma(X)$ *for* σ *along* s if for each geometric point $\bar{w}$ of $\underline{W}$ the function in (2.19) fulfills $u_s^\sigma(\bar{w}) = \bar{u}$.

Remark 2.43. A previous version of this paper contained a notion of evaluation stratum for a global contact order. This was meant as the analogue of the pullback via $X \to \mathcal{A}_X$ of the image of $\mathcal{Z}_\sigma \to \mathcal{A}_X$ in the notion of contact orders based on the

Artin fan of X developed in Appendix A.2. We decided to remove this part for several reasons.

First, the given treatment was ad hoc since unlike in the notion based on Artin fans, there is no good functorial characterization of schematic evaluation strata based on families of punctured curves. This lack of a universal property is due to possible obstructions to deformations of punctured maps not coming from obstructions to deformations of the evaluation point.

Second, contact orders are naturally selected after fixing a reference stratum, see Section 3.2 below. In the most important case of realizable types of punctured maps (Definition 2.44 (2) below), the reference stratum already defines a reduced closed subscheme of the evaluation stratum for the given contact order. Thus defining a non-reduced evaluation stratum is pointless in this case. Indeed, so far there has not been any use of non-reduced evaluation strata in practice, and notably not in the applications mentioned in the introduction.

Third, should there ever be a need to define a non-reduced evaluation stratum, it can easily be defined via the theory of contact orders developed in Appendix A.

2.4.2 Global types

As emphasized throughout the paper, a central aspect of the theory of punctured maps involves the underlying combinatorics in terms of tropical geometry. On the level of moduli spaces, this aspect is captured by the notion of marking by tropical types.

For this purpose, we need a global version of the type of punctured maps (Definition 2.24). Crucially we replace contact orders by the global contact orders from Definition 2.40. For readability, we again restrict to the case of simple X first. The discussion of the additional data needed for the general case is contained in Section 2.6.

Definition 2.44. (1) A *global type (of a family of tropical punctured maps to $\Sigma(X)$)* is a tuple

$$\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$$

consisting of a genus-decorated connected graph $(G, \mathbf{g})$ and two maps

$$\boldsymbol{\sigma} : V(G) \cup E(G) \cup L(G) \to \Sigma(X), \quad \bar{\mathbf{u}} : E(G) \cup L(G) \to \mathfrak{C}(X)$$

with $\bar{\mathbf{u}}(x) \in \mathfrak{C}_{\boldsymbol{\sigma}(x)}$ for each $x \in E(G) \cup L(G)$. A (type of) punctured maps has an *associated global type* by replacing the contact orders by the associated global contact orders. *Morphisms of global types* are defined analogously to morphisms of types of tropical punctured maps in (2.14).

If the composition of $\bar{\mathbf{u}}$ with the natural map $\mathfrak{C}(X) \to \mathfrak{C}(B)$ equals 0, we say τ is a global type for X/B or relative B.

(2) A global type τ is *realizable*[4] if there exists a tropical map to $\Sigma(X)$ with associated global type τ.

(3) A *decorated global type* $\boldsymbol{\tau} = (\tau, \mathbf{A})$ of tropical punctured maps is obtained by adding a curve class $\mathbf{A}$ as in (2.13).

(4) A *class of tropical punctured maps* for a connected X is a decorated global type with a graph G with only one vertex v, no edges, and all strata $\boldsymbol{\sigma}(x) = \{0\}$. We write a class of tropical punctured maps as $\beta = (g, \bar{\mathbf{u}}, A)$ with $g \in \mathbb{N}$, $A \in H_2^+(X)$ and $\bar{\mathbf{u}} : L(G) \to \mathfrak{C}_{\{0\}}(X)$. The *class of a decorated global type* is the class of tropical punctured maps obtained by contracting all edges and keeping the set of legs, but with associated strata $0 \in \Sigma(X)$ and each global contact order the image under the canonical map

$$\mathfrak{C}_{\boldsymbol{\sigma}(L)}(X) \to \mathfrak{C}_{\{0\}}(X).$$

For a class β of a global type we write $\underline{\beta} = (g, k, A)$ with $k = |L(G)|$ for the class of the underlying ordinary stable map.

If X is disconnected, one takes one class of tropical punctured map for each connected component of X.

We will often drop the adjective *"tropical"* and refer to a global type, decorated global type, or class of punctured maps.

The following lemma will only be used in the proof of Proposition 3.24, which in turn is only used in the dimension formulas of Proposition 3.30.

Lemma 2.45. *Let $(G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ be a realizable global type, and assume all logarithmic strata $Z_\sigma \subseteq X$ for $\sigma \in \operatorname{im}(\boldsymbol{\sigma})$ are monodromy-free. Then there is a unique type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ of punctured maps with associated global type $(G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$.*

Proof. Indeed, realizability implies in particular that for each $x \in E(G) \cup L(G)$, the contact order $u_x \in \mathfrak{C}_{\boldsymbol{\sigma}(x)}(X)$ lies in the image of the natural map $N_{\boldsymbol{\sigma}(x)} \to \mathfrak{C}_{\boldsymbol{\sigma}(x)}(X)$. However, it follows immediately from the definition of $\mathfrak{C}_\sigma(X)$ that the map $N_\sigma \to \mathfrak{C}_\sigma(X)$ is injective for each $\sigma \in \Sigma(X)$. ∎

A sufficient condition for the absence of monodromy in Lemma 2.45 is of course that X is simple.

Remark 2.46 (Relation with types). There are two differences of the notion of global type to the notion of type in Definition 2.24. First, contact orders are replaced by global contact orders. Second, the requirement $\bar{\mathbf{u}}(x) \in \mathfrak{C}_{\boldsymbol{\sigma}(x)}(X)$ for $x \in E(G) \cup L(G)$ does not imply $u_x \in N_{\boldsymbol{\sigma}(x)}$. The lack of the latter condition for edges makes it impossible to define a basic monoid just depending on a global type.

[4]The term signifies that the combinatorial data underlies a tropical object. It should not be confused with realizability in tropical algebraic geometry, which signifies that a tropical object is the tropicalization of an algebraic object.

However, some useful discrete data remain. For simplicity we assume X is simple again, deferring the discussion of the general case to Section 2.6.4. Consider a tropical punctured map $\Gamma \to \Sigma(X)$, with associated type $\tau' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}')$, basic monoid $Q_{\tau'}$ as in (2.15), and dual monoid $Q_{\tau'}^\vee$ underlying the corresponding moduli of tropical maps. We have an associated *global* type $\overline{\tau}' = (G', \mathbf{g}', \boldsymbol{\sigma}', \bar{\mathbf{u}}')$ as in Definition 2.44 (1) obtained by replacing the contact orders $\mathbf{u}'(x)$ with their images in $\mathfrak{C}_{\boldsymbol{\sigma}(x)}(X)$.

Now fix a contraction morphism $\phi : \overline{\tau}' \to \tau$ to a global type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$, with set of contracted edges E_ϕ. We claim that there is a well-defined face $Q_{\tau\tau'}^\vee$ of $Q_{\tau'}^\vee$, see (2.20), with dual localization (2.21), not requiring a morphism of types lifting $\overline{\tau}' \to \tau$. Fix a point of $Q_{\tau'}^\vee$ given as a tuple $(V_v, \ell_E)_{v \in V(G), E \in E(G)}$. Then $(V_v, \ell_E)_{v,E} \in Q_{\tau\tau'}^\vee$ if and only if

(1) the position V_v of any vertex v maps to the cell $\boldsymbol{\sigma}(\phi(v))$ associated to $\phi(v) \in V(G)$ by τ, and
(2) if $E \in V(G')$ is an edge contracted by ϕ then $\ell_E = 0$.

Here we replaced generic points η and nodal points q in (2.15) by vertices $v \in V(G')$ and edges $E \in E(G')$. It is critical that $\boldsymbol{\sigma}(\phi(v))$ is a well-defined face of $\boldsymbol{\sigma}(v)$. This is where we use the simplicity assumption. Define $Q_{\tau\tau'}$ as the dual of this face, given precisely as:

$$Q_{\tau\tau'}^\vee = \big\{(V_v, \ell_E) \in Q_{\tau'}^\vee \mid \forall v \in V(G') : V_v \in \boldsymbol{\sigma}(\phi(v)) \quad \forall E \in E_\phi : \ell_E = 0\big\}. \tag{2.20}$$

We then obtain a localization morphism

$$\chi_{\tau\tau'} : Q_{\tau'} \to Q_{\tau\tau'}, \tag{2.21}$$

just as for basic monoids associated to types of tropical punctured maps [3, Definition 2.31 (3)]. The difference is that now both $Q_{\tau\tau'}$ and $\chi_{\tau\tau'}$ depend not only on the morphism $\phi : \overline{\tau}' \to \tau$ of global types, but also on the lift of $\overline{\tau}'$ to a type τ' of tropical punctured maps.

2.5 Puncturing log-ideals

The punctured points which are not marked points impose extra important constraints on the possible deformations of a punctured curve, hence of punctured stable maps, captured by an ideal in the base monoid. This is a key new feature of the theory which we now describe.

2.5.1 Review of idealized log schemes

We review here the notion of idealized log schemes from [52], as this notion is considerably less common in the literature.

Given a sheaf of monoids $\mathcal{M}$ on a scheme X, we use the term *log-ideal* for a sheaf of monoid ideals $\mathcal{K} \subseteq \mathcal{M}$. The sheaf of monoid ideals $\mathcal{K}$ is said to be *coherent* (see [52, Proposition II.2.6.1]) if locally on X, $\mathcal{K}$ is generated by a finite set of sections.

An *idealized log scheme* is data $(X, \mathcal{M}_X, \alpha_X, \mathcal{K}_X)$ where $(X, \mathcal{M}_X, \alpha_X)$ is an ordinary log scheme, with $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ the structure map, and $\mathcal{K}_X \subseteq \mathcal{M}_X$ a log-ideal such that $\mathcal{K}_X \subseteq \alpha_X^{-1}(0)$. A *morphism of idealized log schemes* $f : (X, \mathcal{K}_X) \to (Y, \mathcal{K}_Y)$ is a morphism $f : X \to Y$ of log schemes such that

$$f^\flat(f^{-1}(\mathcal{K}_Y)) \subseteq \mathcal{K}_X.$$

See [52, Definition III.1.3.1].

If $f : X \to Y$ is a morphism of log schemes and $\mathcal{K}_Y \subseteq \mathcal{M}_Y$ is a log-ideal, we adopt the notation of [52] by writing $f^\bullet(\mathcal{K}_Y) \subseteq \mathcal{M}_X$ as the log-ideal generated by $f^\flat(f^{-1}(\mathcal{K}_Y))$. We say a morphism $f : X \to Y$ of idealized log schemes is *idealized strict* [52, Definition III.1.3.2] if $\mathcal{K}_X = f^\bullet\mathcal{K}_Y$.

If W is a fine log scheme and $\mathcal{K} \subseteq \mathcal{M}_W$ is a log-ideal, then $\mathcal{K}$ is invariant under the multiplicative action of $\mathcal{O}_W^\times$, and the quotient $\overline{\mathcal{K}} = \mathcal{K}/\mathcal{O}_W^\times$ is a log-ideal in $\overline{\mathcal{M}}_W$. As the stalks of $\overline{\mathcal{M}}_W$ are finitely generated monoids, the stalks of $\overline{\mathcal{K}}$ are then finitely generated ideals.

Lemma 2.47. *Let $(W, \mathcal{M}_W)$ be a fine log scheme and $\mathcal{K} \subseteq \mathcal{M}_W$ a log-ideal. Then the following are equivalent:*

(1) *$\mathcal{K}$ is a coherent sheaf of ideals;*

(2) *for any geometric points $\bar{x}$, $\bar{y}$ of W with $\bar{y} \to \bar{x}$ a specialization arrow, the stalk $\mathcal{K}_{\bar{y}}$ is generated by the image of the generization map $\mathcal{K}_{\bar{x}} \to \mathcal{K}_{\bar{y}}$.*

Proof. (1)⇒(2): Suppose $\mathcal{K}$ is a coherent sheaf of ideals. Then given geometric points as in the statement of the lemma, there is an open neighborhood U of $\bar{x}$ and a finite set of sections $S \subseteq \Gamma(U, \mathcal{M}_W)$ generating $\mathcal{K}|_U$. In particular, $\bar{y}$ lifts to a geometric point of U and hence $\mathcal{K}_{\bar{x}}$ and $\mathcal{K}_{\bar{y}}$ are both generated by S. In particular, the generization map $\mathcal{K}_{\bar{x}} \to \mathcal{K}_{\bar{y}}$ is surjective.

(2)⇒(1): Suppose the generatedness statement always holds. Since $\mathcal{M}_W$ is fine, for any geometric point $\bar{x}$ of W, one may find an étale neighborhood U with a chart $\phi : Q \to \mathcal{M}_W|_U$ inducing an isomorphism $Q \to \overline{\mathcal{M}}_{W,\bar{x}}$. Let $K \subseteq Q$ be the inverse image of $\overline{\mathcal{K}}_{\bar{x}}$ under this isomorphism, and let $S \subseteq K$ be a finite generating set. Then $\phi(S)$ provides a subset of $\Gamma(U, \mathcal{M}_W)$, necessarily generating an ideal subsheaf $\mathcal{K}'$ of $\mathcal{K}$. However, because of the assumed surjectivity, it follows immediately that $\mathcal{K}' = \mathcal{K}$. ∎

Many notions in log geometry have idealized versions. In particular, there are notions of idealized log étale and idealized log smooth morphisms, defined using idealized versions of formal lifting. We send the reader to [52, Section IV.3.1] for details.

Morally, an idealized log smooth morphism is one modeled on a morphism between torus invariant subschemes of toric varieties; alternatively it is a morphism $X \to Y$ such that there is a closed substack $\mathcal{Z}_{X/Y}$ of a relative Artin fan $\mathcal{A}_{X/Y}$ [5, Corollary 3.3.5] defined by a monomial ideal such that the induced morphism $X \to \mathcal{A}_{X/Y}$ factors through a smooth morphism $X \to \mathcal{Z}_{X/Y}$. See Proposition B.2 for precise statements as needed in this paper.

Proposition 2.48. *If $X \to B$ is log smooth, and B is log smooth over $\Bbbk$ or is a log point, then every stratum X_σ of X is idealized log smooth over B, where $\sigma \in \Sigma(X)$. Here we endow X_σ with its reduced induced subscheme structure, and with the log structure induced by the closed embedding $X_\sigma \hookrightarrow X$.*

Proof. Since the statement is étale local in B, we may assume there exists a global chart $B \to A_Q = \operatorname{Spec} \Bbbk[Q]$. Note also that by Proposition C.11, X_σ is irreducible, hence is set-theoretically the closure of a geometric generic point $\overline{\eta}$ of X_σ.

Define the log ideal $\mathcal{K} \subseteq \mathcal{M}_{X_\sigma}$ on X_σ by

$$\mathcal{K}(U) := \left\{ s \in \mathcal{M}_{X_\sigma}(U) \mid \alpha_{X_\sigma}(s) = 0 \right\}.$$

To check that $(X_\sigma, \mathcal{K}) \to (B, \emptyset)$ is idealized log smooth near a point $x \in X_\sigma$, we consider a chart for $X \to B$ as in Proposition B.4, an étale neighborhood $h : U \to X$ of x fitting into a commutative diagram

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[dr] & B \times_{\mathcal{A}_Q} \mathcal{A}_P \arrow[r] \arrow[d] & \mathcal{A}_P \arrow[d] \\ & B \arrow[r] & \mathcal{A}_Q, \end{tikzcd}$$

with all horizontal arrows strict, $g : U \to B \times_{\mathcal{A}_Q} \mathcal{A}_P$ smooth, $P^\times = \{0\}$, and a lift $\tilde{x}$ of x to U mapping to the closed (deepest) stratum of $\mathcal{A}_P$. Then we obtain an isomorphism $\psi : P \to \overline{\mathcal{M}}_{X,\bar{x}} = (\sigma_{\bar{x}}^\vee)_{\mathbb{Z}}$. Each specialization arrow $\overline{\eta} \to \bar{x}$ defines a face inclusion $\sigma \to \sigma_{\bar{x}}$, hence a closed reduced substack $\mathcal{Z} \subset \mathcal{A}_P$ with

$$h(g^{-1}(B \times_{\mathcal{A}_Q} \mathcal{Z})) \subseteq \mathcal{S}_\sigma,$$

where $\mathcal{Z}_\sigma$ is the logarithmic stratum of X with closure X_σ. Thus if $F_i \subseteq P$ denotes the dual faces of P defined by such specializations, then by the definitions of $\mathcal{K}$ and X_σ,

$$\psi\Big(P \setminus \bigcup_i F_i\Big) = \overline{\mathcal{K}}_{\bar{x}} \subseteq \overline{\mathcal{M}}_{X,\bar{x}}. \tag{2.22}$$

Note this gives an alternative, stalkwise definition of the log ideal $\mathcal{K}$, using the reasoning in Remark 2.36.

To show the claim on idealized smoothness, it thus remains to show that the preimage in U of the closed reduced substacks of $\mathcal{A}_P$ are reduced for then the subscheme of U defined by $P \setminus \bigcup_i F_i$ agrees with $h^{-1}(X_\sigma)$.

Now a closed reduced substack $\mathcal{Z} \subseteq \mathcal{A}_P$ maps onto a closed reduced substack $\mathcal{T}$ of $\mathcal{A}_Q$, which by our assumptions on B pulls back to a reduced subscheme $S \subseteq B$. Therefore $B \times_{\mathcal{A}_Q} \mathcal{Z} = S \times_{\mathcal{T}} \mathcal{Z}$ is reduced since $S \to \mathcal{T}$ is smooth, and so is its preimage in U. ∎

2.5.2 Log-ideals of punctured curves

Let $(\pi : C^\circ \to W, \mathbf{p})$ be a punctured curve. For each of the punctures $p : \underline{W} \to \underline{C}$ consider the composition

$$v_p : p^*\mathcal{M}_{C^\circ} \to \overline{\mathcal{M}}_W \oplus \underline{\mathbb{Z}} \to \underline{\mathbb{Z}} \tag{2.23}$$

of fine monoid sheaves, with the first map induced by the canonical monoid inclusion $p^*\overline{\mathcal{M}}_{C^\circ} \to \overline{\mathcal{M}}_W \oplus \underline{\mathbb{Z}}$ and the second map the projection. Denote by $\mathcal{I}_p \subseteq p^*\mathcal{M}_{C^\circ}$ the sheaf of ideals generated by $(v_p)^{-1}(\underline{\mathbb{Z}}_{<0})$.

Definition 2.49. The *puncturing log-ideal* $\mathcal{K}_W \subseteq \mathcal{M}_W$ of the punctured curve $(\pi : C^\circ \to W, \mathbf{p})$ is the ideal sheaf

$$\bigcup_p (\pi^\flat)^{-1}(\mathcal{I}_p) \subseteq \mathcal{M}_W,$$

with p running over all punctures.

In the context of the definition we abuse notation when writing $\pi^\flat$ for the composition

$$\mathcal{M}_W \xrightarrow{\pi^\flat} \pi_*\mathcal{M}_{C^\circ} \to \pi_* p_* p^*\mathcal{M}_{C^\circ} = p^*\mathcal{M}_{C^\circ},$$

where as usual $p^*\mathcal{M}_{C^\circ}$ denotes the pullback log structure, while the right arrow is induced by the adjunction unit morphism $1 \to p_* p^{-1}$ of the associated abelian sheaves.

We sometimes also refer to the quotient $\overline{\mathcal{K}}_W$ of $\mathcal{K}_W$ by $\mathcal{O}_W^\times$ as the puncturing log-ideal, but will then write $\overline{\mathcal{K}}_W \subseteq \overline{\mathcal{M}}_W$ for clarity.

An illustration for the definition is contained in Figure 2.8.

This picture indicates an equivalent way to identify $\overline{\mathcal{K}}_W$. For the stalkwise characterization we may do a strict base change to a geometric point of $\underline{W}$ and hence assume W is a log point. For a marking p on a component of C° with generic point η, consider the generization map $\phi_{p,\eta} : \overline{\mathcal{M}}_{C^\circ,p} \to \overline{\mathcal{M}}_{C^\circ,\eta} \simeq \overline{\mathcal{M}}_W$. Identify $\overline{\mathcal{M}}_W$ as a submonoid of $\overline{\mathcal{M}}_{C^\circ,p}$ via $\pi^\flat$, making $\phi_{p,\eta}$ an idempotent homomorphism on $\overline{\mathcal{M}}_{C^\circ,p}$ with image $\overline{\mathcal{M}}_W$. An element $m \in \overline{\mathcal{M}}_W$ is in $\overline{\mathcal{K}}_W$ if and only if there is a marking p

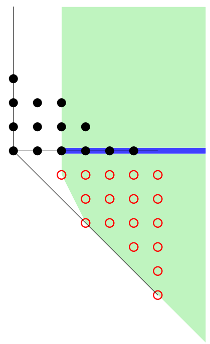

Figure 2.8. An idealized punctured point (ideal lightly shaded) and the resulting log ideal (the horizontal shaded ray). If there are several punctures, one takes the ideal generated by these horizontal regions.

and an element $n \in (u_p)^{-1}(\mathbb{Z}_{<0})$ such that $\phi_{p,\eta}(n) = m$, where $u_p : \overline{\mathcal{M}}_{C^\circ,p} \to \mathbb{Z}$ is the contact order associated to the identity morphism. Indeed, if there is $n \in (u_p)^{-1}(\underline{\mathbb{Z}}_{<0})$ and $n' \in \overline{\mathcal{M}}_{C^\circ,p}$ with $\pi^\flat(m) = n + n'$ then, writing $n'' = n + \phi_{p\eta}(n')$ we have $m = \phi_{p\eta}(n'')$; conversely, if $m = \phi_{p\eta}(n'')$ with $u_p(n'') = -b < 0$ then, using the notation of (2.23), we have $\pi^\flat(m) = n'' + b \cdot (0, 1)$.

Lemma 2.50. *The puncturing log-ideal $\mathcal{K}_W$ of a punctured curve $(\pi : C^\circ \to W, \mathbf{p})$ is coherent.*

Proof. We verify the characterization of Lemma 2.47. Let $\bar{x}, \bar{y} \to W$ with $\bar{x} \in \mathrm{cl}(\bar{y})$. Fix a generization map $\chi_{\bar{x}\bar{y}} : \overline{\mathcal{K}}_{\bar{x}} \to \overline{\mathcal{K}}_{\bar{y}}$ and let $m_{\bar{y}} \in \overline{\mathcal{K}}_{\bar{y}}$. We wish to construct $m_{\bar{x}} \in \overline{\mathcal{K}}_{\bar{x}}$ with $\chi_{\bar{x}\bar{y}}(m_{\bar{x}}) = m_{\bar{y}}$.

We refer to the following commutative diagram of generizations and contact orders:

$$\begin{tikzcd}
\overline{\mathcal{M}}_{W,\bar{x}} = \overline{\mathcal{M}}_{C^\circ,\eta_{\bar{x}}} \arrow[rr, "\chi_{\bar{x}\bar{y}} = \chi_{\eta_{\bar{x}}\eta_{\bar{y}}}"] & & \overline{\mathcal{M}}_{C^\circ,\eta_{\bar{y}}} = \overline{\mathcal{M}}_{W,\bar{y}} \\
\overline{\mathcal{M}}_{C^\circ,p_{\bar{x}}} \arrow[u, "\phi_{p_{\bar{x}}\eta_{\bar{x}}}"] \arrow[rr, "\chi_{p_{\bar{x}}p_{\bar{y}}}"'] \arrow[d, "u_{p_{\bar{x}}}"'] & & \overline{\mathcal{M}}_{C^\circ,p_{\bar{y}}} \arrow[u, "\phi_{p_{\bar{y}}\eta_{\bar{y}}}"'] \arrow[d, "u_{p_{\bar{y}}}"] \\
\mathbb{Z} \arrow[rr, equal] & & \mathbb{Z}
\end{tikzcd}$$

Note that $m_{\bar{y}} \in \overline{\mathcal{K}}_{\bar{y}}$ means that there is a puncture $p_{\bar{y}}$ lying on a component with generic point $\eta_{\bar{y}}$ of $C_{\bar{y}}$ and an element $m_{p_{\bar{y}}} \in (u_{p_{\bar{y}}})^{-1}(\underline{\mathbb{Z}}_{<0})$ whose generization is $\phi_{p_{\bar{y}}\eta_{\bar{y}}}(m_{p_{\bar{y}}}) = m_{\bar{y}}$.

Since $\mathcal{M}_{C^\circ}$ is coherent, there is an element $m_{p_{\bar{x}}} \in \overline{\mathcal{M}}_{C^\circ,\bar{x}}$ such that

$$\chi_{p_{\bar{x}}p_{\bar{y}}}(m_{p_{\bar{x}}}) = m_{p_{\bar{y}}}.$$

Note that $u_{p_{\bar y}} \circ \chi_{p_{\bar x} p_{\bar y}} = u_{p_{\bar x}}$, see Lemma 2.41. This implies $m_{p_{\bar x}} \in (u_{p_{\bar x}})^{-1}(\underline{\mathbb{Z}}_{<0})$. Write $m_{\bar x} := \phi_{p_{\bar x}\eta_{\bar x}}(m_{p_{\bar x}})$. By definition $m_{\bar x} \in \bar{\mathcal{K}}_{\bar x}$.

We obtain that $m_{\bar y} = \phi_{p_{\bar y}\eta_{\bar y}} \circ \chi_{p_{\bar x} p_{\bar y}}(m_{p_{\bar x}}) = \chi_{\eta_{\bar x}\eta_{\bar y}} \phi_{p_{\bar x}\eta_{\bar x}}(m_{p_{\bar x}}) = \chi_{\eta_{\bar x}\eta_{\bar y}}(m_{\bar x}) = \chi_{\bar x \bar y}(m_{\bar x})$, as needed. ■

Puncturing log-ideals behave well under pull-backs.

Proposition 2.51. *Let $(\pi : C^\circ \to W, \mathbf{p})$ be a punctured curve, $(\pi_T : C^\circ_T \to T, \mathbf{p}_T)$ its pullback via $h : T \to W$ and $\mathcal{K}_W$, $\mathcal{K}_T$ the respective puncturing log-ideals. Then $\mathcal{K}_T = h^\bullet \mathcal{K}_W$.*

Proof. Denote by $g : C^\circ_T \to C^\circ$ the pullback of h to the curves. By coherence of $\mathcal{K}_W$ and $\mathcal{K}_T$ it suffices to check that for each geometric point $\bar t \to T$, the image of $\bar{\mathcal{K}}_{W,\underline{h}(\bar t)}$ under $\bar h^\flat_{\bar t}$ generates $\bar{\mathcal{K}}_{T,\bar t}$. Denote by $\bar w = h(\bar t)$. For a puncture p of C° consider the commutative diagram

$$\begin{tikzcd}
\bar{\mathcal{M}}_{W,\bar w} \arrow[r, "\bar\pi^\flat"] \arrow[d, "\bar h^\flat"] & \bar{\mathcal{M}}_{C^\circ, p(\bar w)} \arrow[r] \arrow[d, "\bar g^\flat"'] & \bar{\mathcal{M}}_{W,\bar w} \oplus \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d, "="] \\
\bar{\mathcal{M}}_{T,\bar t} \arrow[r, "\bar\pi_T^\flat"] & \bar{\mathcal{M}}_{C^\circ_T, p_T(\bar t)} \arrow[r] & \bar{\mathcal{M}}_{T,\bar t} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

The two left squares are cocartesian in the category of fine monoids by the definition of pullback of punctured curves. This shows first that $\bar g^\flat(\bar{\mathcal{I}}_{p,\bar w})$ generates $\bar{\mathcal{I}}_{p_T,\bar t}$, and in turn that $\bar h^\flat((\bar\pi^\flat)^{-1}(\bar{\mathcal{I}}_{p,\bar w}))$ generates $(\bar\pi_T^\flat)^{-1}(\bar{\mathcal{I}}_{p_T,\bar t})$. Taking the union over all punctures finishes the proof. ■

Here comes the crucial vanishing property putting restrictions on deformations of punctured curves.

Proposition 2.52. *Let $(C^\circ/W, \mathbf{p})$ be a punctured curve and $\mathcal{K}_W \subseteq \mathcal{M}_W$ its puncturing log-ideal. Then it holds*

$$\alpha_W(\mathcal{K}_W) = 0.$$

Proof. Let $\mathcal{I}_p = v_p^{-1}(\mathbb{Z}_{<0}) \subseteq p^*\mathcal{M}_{C^\circ}$ be the ideal sheaf defined after (2.23). Definition 2.1 (2) implies $(p^*\alpha_{C^\circ})(\mathcal{I}_p) = 0$. Pulling back via $\pi^\flat : \mathcal{M}_W \to p^*\mathcal{M}_{C^\circ}$ thus yields

$$\alpha_W((\pi^\flat)^{-1}(\mathcal{I}_p)) = (p^*\alpha_{C^\circ})(\mathcal{I}_p) = 0.$$

The claimed vanishing follows by taking the union over the punctures p. ■

Proposition 2.52 demonstrates the announced statement that the base of a family of punctured curves is naturally an idealized log scheme (or stack).

Corollary 2.53. *For a punctured curve $(C^\circ/W, \mathbf{p})$ with $\mathcal{K}_W$ its puncturing log-ideal, the triple $(W, \mathcal{M}_W, \mathcal{K}_W)$ is a coherent idealized log scheme.*

Example 2.54. Let $(C^\circ/W, \mathbf{p})$ be a punctured curve over the logarithmic point $W = \operatorname{Spec}(Q \to \Bbbk)$, with $Q = \mathbb{N}^2$, $\underline{C}$ a smooth and connected curve and with only one punctured point p with

$$\bar{\mathcal{M}}_{C^\circ,p} = (Q \oplus \mathbb{N}) + \mathbb{N} \cdot (a, 0, -1) + \mathbb{N} \cdot (0, b, -1) \subset Q \oplus \mathbb{Z},$$

for some $a, b \in \mathbb{N} \setminus \{0\}$. Then the puncturing log-ideal $\bar{\mathcal{K}}_W$ is generated by $(a, 0)$, $(0, b)$. This implies that if we view W as the strict closed subspace of $\mathbb{A}^2 = \operatorname{Spec} \Bbbk[t_1, t_2]$ with its toric log structure, then the maximal subscheme of $\mathbb{A}^2$ to which $(C/W, \mathbf{p})$ extends is given by the ideal $(t_1^a, t_2^b) \subset \Bbbk[t_1, t_2]$.

2.5.3 Log-ideals of punctured maps

We define puncturing log-ideals only for pre-stable punctured maps.[5]

Definition 2.55. The *puncturing log ideal* $\mathcal{K}_W$ of a pre-stable punctured map $(C^\circ/W, \mathbf{p}, f)$ is the puncturing log-ideal of the punctured domain curve $(C^\circ/W, \mathbf{p})$, as defined in Definition 2.49.

It is clear from the definition and Proposition 2.51 that puncturing log ideals of punctured maps are stable under base change, and they also enjoy the vanishing property $\alpha_W(\mathcal{K}_W) = 0$ from Proposition 2.52.

We finish this subsection by giving a tropical interpretation in the spirit of Proposition 2.23 of the radical of the puncturing log-ideal $\mathcal{K}_W$ of a pre-stable punctured map, see Proposition 2.57. This interpretation is based on the following technical result concerning monoid ideals.

Lemma 2.56. *Suppose given a sharp toric monoid Q, and a collection of sharp toric monoids $P_{p_1}, \ldots, P_{p_r}$ along with monoid homomorphisms $\varphi_{p_i} : P_{p_i} \to Q \oplus \mathbb{Z}$ with $u_{p_i} := \mathrm{pr}_2 \circ \varphi_{p_i}$. Let $\mathrm{ev}_i := (\mathrm{pr}_1 \circ \varphi_{p_i})^t : Q^\vee_{\mathbb{R}} \to (P_{p_i})^\vee_{\mathbb{R}}$. Let the ideal $I \subset Q$ be the monoid ideal*

$$I = \bigcup_{i=1}^{r} \big\langle \mathrm{pr}_1 \circ \varphi_{p_i}(m) \mid m \in P_{p_i} \text{ and } u_{p_i}(m) < 0 \big\rangle.$$

For σ a face of the cone $Q^\vee_{\mathbb{R}}$, let $A_\sigma = \operatorname{Spec} \Bbbk[\sigma^\perp \cap Q]$ be the closed toric stratum of $\operatorname{Spec} \Bbbk[Q]$ corresponding to σ. Then there is a decomposition

$$\operatorname{Spec} \Bbbk[Q]/\sqrt{I} = \bigcup_{\sigma} A_\sigma$$

[5] If $(C^\circ/W, \mathbf{p}, f)$ has associated pre-stable map $(\tilde{C}^\circ/W, \mathbf{p}, \tilde{f})$ (Proposition 2.5), the ideal $\mathcal{K}_W$ of Definition 2.49 associated to C°/W may strictly include the corresponding ideal associated to $\tilde{C}^\circ/W$.

where the union is over all faces σ of $Q^\vee_{\mathbb{R}}$ such that if $x \in \mathrm{Int}(\sigma)$, then $\mathrm{ev}_i(x) + \varepsilon u_{p_i} \in (P_{p_i})^\vee_{\mathbb{R}}$ for $\varepsilon > 0$ sufficiently small and $1 \le i \le r$.[6]

Proof. Let $I_{p_i} \subset Q$ be the monoid ideal

$$I_{p_i} = \big\langle \mathrm{pr}_1 \circ \varphi_{p_i}(m) \mid m \in P_{p_i} \text{ satisfies } u_{p_i}(m) < 0 \big\rangle.$$

Of course $V(I) = \bigcap_i V(I_{p_i})$. We first show that if σ satisfies the given condition, then $A_\sigma \subseteq V(I_{p_i})$ for each i. The monomial ideal defining A_σ is $Q \setminus (\sigma^\perp \cap Q)$, so it is enough to show that $\sigma^\perp \cap I_{p_i} = \emptyset$. Choose an $x \in \mathrm{Int}(\sigma)$. Let $q \in I_{p_i}$ be a generator of I_{p_i}, that is, there exists an $m \in P_{p_i}$ such that $q = \mathrm{pr}_1(\varphi_{p_i}(m))$ and $u_{p_i}(m) < 0$. Since $m \in P_{p_i}$ and $\mathrm{ev}_i(x) + \varepsilon u_{p_i} \in (P_{p_i})^\vee_{\mathbb{R}}$ for some $\varepsilon > 0$, we have

$$0 \le \langle \mathrm{ev}_i(x) + \varepsilon u_{p_i}, m\rangle.$$

Thus $\langle u_{p_i}, m\rangle < 0$ implies $\langle \mathrm{ev}_i(x), m\rangle > 0$, or $\langle x, \mathrm{pr}_1(\varphi_{p_i}(m))\rangle = \langle x, q\rangle > 0$, as desired.

Conversely, suppose that $A_\sigma \subseteq V(I)$ for some face σ of $Q^\vee_{\mathbb{R}}$, but there exists an i and some $x \in \mathrm{Int}(\sigma)$ such that $\mathrm{ev}_i(x) + \varepsilon u_{p_i} \notin (P_{p_i})^\vee_{\mathbb{R}}$ for any $\varepsilon > 0$. Then there exists an $m \in P_{p_i}$ such that $\langle \mathrm{ev}_i(x) + \varepsilon u_{p_i}, m\rangle < 0$ for all $\varepsilon > 0$. Since $\langle \mathrm{ev}_i(x), m\rangle \ge 0$, we must have $\langle \mathrm{ev}_i(x), m\rangle = 0$ and $u_{p_i}(m) < 0$. Thus $q = \mathrm{pr}_1(\varphi_{p_i}(m))$ lies in I_{p_i}. We have

$$\langle x, q\rangle = \langle \mathrm{ev}_i(x), m\rangle = 0,$$

so $q \in \sigma^\perp$. In particular, z^q does not vanish on A_σ, contradicting $A_\sigma \subseteq V(I)$. ∎

Proposition 2.57. *Let $(C^\circ/W, \mathbf{p}, f)$ be a punctured map to X over the logarithmic point $W = \mathrm{Spec}(Q \to \kappa)$,*

$$h : \Gamma = \Gamma(G, \ell) \to \Sigma(X)$$

the associated tropical curve over $\omega = Q^\vee_{\mathbb{R}}$, and $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ its type. Denote by $\sqrt{\bar{\mathcal{K}}_W} \subset Q$ the radical of the puncturing log-ideal of $(C^\circ/W, \mathbf{p}, f)$.

Then a face $Q' \subseteq Q$ lies in $Q \setminus \sqrt{\bar{\mathcal{K}}_W}$ if and only if for any punctured leg $L \in L(G)$ it holds[7]

$$\ell(L)((Q')^\perp \cap \omega_{\mathbb{Z}}) \neq 0.$$

In other words, Q' determines a face $(Q')^\perp \cap \omega_{\mathbb{Z}}$ of $\omega_{\mathbb{Z}}$, and each point of this face corresponds to a tropical map. Thus we require that the length function $\ell(L)$ of each punctured leg be non-vanishing on this face of $\omega_{\mathbb{Z}}$.

[6] See Example 2.58 below.
[7] Again, see Example 2.58 below.

Proof. By pre-stability, $\bar{\mathcal{K}}_W$ is generated by those $q \in Q$ such that there exists a puncture $p_i \to \underline{C}$ of C° and $m \in \bar{\mathcal{M}}_{X,\underline{f}(p_i)}$ with $\bar{f}^\flat(m) = (q, a)$ and $a = u_{p_i}(m) < 0$. Thus $\bar{\mathcal{K}}_W = I$ in Lemma 2.56 applied with $P_{p_i} = \bar{\mathcal{M}}_{X,\underline{f}(p_i)}$. Using the characterization of punctured legs in the pre-stable case in Proposition 2.23, the statement to be proved is then a reformulation of the conclusion of Lemma 2.56 in terms of tropical maps. ∎

Phrased more geometrically, the conclusion of Proposition 2.57 says that exactly those faces of the basic cone of a tropical punctured map (Definition 2.33) can possibly arise from a generization of punctured maps if the puncturing legs remain of positive length.

We end this section with an example highlighting the fact that the natural base spaces in punctured Gromov–Witten theory are possibly reducible spaces due to the puncturing ideals. See Theorem 3.25 and Remark 3.27 for the general picture underlying this phenomenon.

Example 2.58. *Algebraic setup.* Take $B = \operatorname{Spec} \Bbbk$, and consider X a smooth surface with log structure coming from a smooth rational curve $D \subseteq X$ with $D^2 = 2$. Consider a type of punctured maps of genus 0, underlying curve class $[D]$, and four punctures, $p_1, \ldots, p_4$, with contact orders $-1, -1, 2$ and 2 respectively. Consider a punctured curve $f : C^\circ \to X$ where $C = C_1 \cup C_2 \cup C_3$ has three irreducible components and two nodes $q_1 = C_1 \cap C_2$, $q_2 = C_1 \cap C_3$. We assume $p_1, p_3 \in C_2$, $p_2, p_4 \in C_3$. Finally, $\underline{f}$ identifies C_1 with D and contracts C_2 and C_3. Orienting the node q_i from C_1 to C_i, it is not difficult to check such a curve exists with $u_{q_1} = u_{q_2} = 1$ (Figure 2.9).

The tropical curve. The corresponding tropical curve Γ has three vertices, v_1, v_2, v_3, edges E_{q_1}, E_{q_2}, and legs $E_{p_1}, \ldots, E_{p_4}$. The moduli space of tropical curves of this type is $\mathbb{R}^3_{\geq 0}$, with coordinates ρ, ℓ_1, ℓ_2, where ρ gives the distance of the image of v_1 from the origin of $\Sigma(X) = \mathbb{R}_{\geq 0}$, and ℓ_1, ℓ_2 give the lengths of the edges E_{q_1}, E_{q_2}. In particular, the basic monoid for this punctured log curve is $Q = \mathbb{N}^3$, generated by ρ, ℓ_1, ℓ_2.

The punctured ideal. In this case we may easily calculate the puncturing ideal (Definition 2.55). We have contributions from each of the two punctures. Using the definition, we note that at the puncture p_i, $i = 1$ or 2, the map $\varphi_{\bar{\eta}} \circ \chi_{\eta,p_i} : P_{p_i} = \mathbb{N} \to Q$ is dual to $\mathrm{ev}_i : Q^\vee_{\mathbb{R}} \to (P_{p_i})^\vee = \mathbb{R}_{\geq 0}$ evaluating the tropical curve parametrized by a point at $Q^\vee_{\mathbb{R}}$ at v_2 or v_3, see Lemma 2.56. Thus for $m \in Q^\vee_{\mathbb{R}}$, $\mathrm{ev}_i(m) = \rho(m) + \ell_i(m)$. Dually $\varphi_{\bar{\eta}} \circ \chi_{\eta,p_i} : P \to Q$ is given by $1 \mapsto \rho + \ell_i$. As $u_{p_i}(1) = -1$, $i = 1, 2$, we see the puncturing ideal K is generated by $\rho + \ell_1, \rho + \ell_2$. Writing $\Bbbk[Q] = \Bbbk[x, y, z]$, with the three variables corresponding to ρ, ℓ_1, ℓ_2 respectively, we see $\operatorname{Spec} \Bbbk[Q]/K = \operatorname{Spec} \Bbbk[x, y, z]/(xy, xz)$, which has two irreducible components of differing dimension.

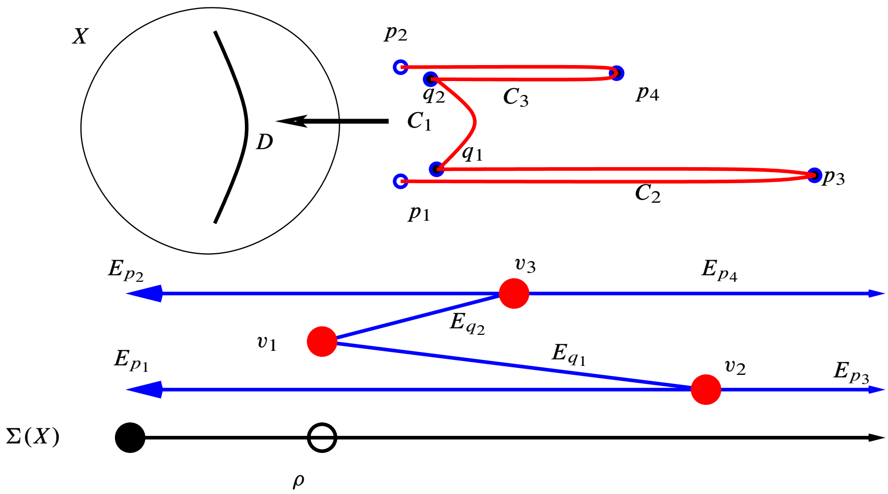

Figure 2.9. The algebraic map and its tropical counterpart. Here $\rho = 1$, $\ell_1 = 2$, and $\ell_2 = 1$.

The participating and excluded cones. The decomposition $\bigcup_\sigma A_\sigma$ of Lemma 2.56 translates to the statement that the cones *excluded* in this decomposition are the origin, the ℓ_1-axis, and the ℓ_2-axis. Indeed, these are the cones where at least one puncture is positioned with its tail at the origin, hence forced to have length 0, which is excluded by Proposition 2.57.

The components of the algebraic moduli space. Note that deformation theory provides two deformation classes of the punctured map. The first smooths one or both of the nodes, resulting in a punctured map with at least one pair p_1, p_3 or p_2, p_4 now being distinct points on the component of the domain mapping surjectively to D. Since this component contains a negative contact order point, its image cannot be deformed away from D by Remark 2.20.

The second deformation class keeps the domain of f fixed, but deforms the image of C_1 away from D, so that it meets D transversally in two points. The remaining components C_2 and C_3 are then contracted to the points of intersection of $f(C_1)$ with D. It is then no longer possible to smooth the nodes.

The data captured by the ideal. This local reducibility of moduli space happens despite the obstruction group $H^1(C, f^*\Theta_X)$ for deformations with fixed domain (see Chapter 4) being zero. The point of the puncturing ideal is that it captures these intrinsic singularities of the moduli space. These obstructions really come from obstructions to deforming the punctured domain curve.

The general picture explaining this phenomenon is developed in Section 3.5. In particular, Example 3.32 revisits the present example from the general perspective.

2.6 Targets with monodromy

We now drop the assumption that X is simple and discuss what is needed to treat the general case.

2.6.1 Tropicalization of punctured maps with non-simple targets

Let $(C^\circ/W, \mathbf{p}, f)$ be a punctured map over a logarithmic point $W = \operatorname{Spec}(Q \to \kappa)$ with κ algebraically closed. Then the inclusion of a nodal point q or punctured point p into C° is a geometric point of $\underline{C}$ that we denote by $\bar{q}$ and $\bar{p}$, respectively. For a node q of $\underline{C}$, the generic points $\eta, \eta' \in \operatorname{Spec} \mathcal{O}_{\underline{C},\bar{q}}$ of the two branches of $\underline{C}$ at $\bar{q}$ provide two specialization arrows of geometric points (see Appendix C)

$$\overline{\eta} \to \bar{q}, \quad \overline{\eta}' \to \bar{q},$$

unique up to order and precomposition with an isomorphism in the category of geometric points in $\operatorname{Spec} \mathcal{O}_{\underline{C},\bar{q}}$. The node q is a self-intersection point of $\underline{C}$ iff $\overline{\eta}, \overline{\eta}'$ have the same image in $\underline{C}$, that is, iff they are isomorphic as geometric points of $\underline{C}$. In any case, denoting by G the dual intersection graph of C°, each specialization arrow $\overline{\eta} \to \bar{x}$ with $x \in E(G) \cup L(G)$ gives rise to a face inclusion

$$Q^\vee = \overline{\mathcal{M}}^\vee_{C,\overline{\eta}} \to \overline{\mathcal{M}}^\vee_{C,\bar{x}}. \tag{2.24}$$

The equality on the left-hand side is the canonical isomorphism obtained since C° is a log smooth curve over $\operatorname{Spec}(Q \to \kappa)$.

Applying f yields a specialization arrow $f(\overline{\eta}) \to f(\bar{x})$ and a corresponding face embedding

$$\overline{\mathcal{M}}^\vee_{X,f(\overline{\eta})} \to \overline{\mathcal{M}}^\vee_{X,f(\bar{x})} \tag{2.25}$$

Our tropicalization procedure for $f : C^\circ \to W$ requires us to choose, for each $x \in V(G) \cup E(G) \cup L(G)$ with associated geometric point $\bar{x}$ of $\underline{C}$, an isomorphism

$$\operatorname{Hom}\big(\overline{\mathcal{M}}_{X,\underline{f}(\bar{x})}, \mathbb{R}_{\geq 0}\big) \to \boldsymbol{\sigma}(x) \tag{2.26}$$

in $\Sigma(X)$. Composing these isomorphisms or their inverses with the arrow in (2.25) defines an arrow

$$\iota_{x\eta} : \boldsymbol{\sigma}(\eta) \to \boldsymbol{\sigma}(x)$$

in $\Sigma(X)$. If $\Sigma(X)$ is simple there is only one arrow $\boldsymbol{\sigma}(\eta) \to \boldsymbol{\sigma}(x)$ in $\Sigma(X)$. In the general case, the $\iota_{x\eta}$ are part of the data defining the tropicalization, up to the simultaneous action of

$$\mathbb{G} = \prod_{x \in V(G)\cup E(G)\cup L(G)} \operatorname{Aut}_{\Sigma(X)}(\boldsymbol{\sigma}(x)) \tag{2.27}$$

on the choices of isomorphisms (2.26). Note that $\mathbb{G}$ may not act transitively on the set of arrows $\boldsymbol{\sigma}(\eta) \to \boldsymbol{\sigma}(x)$, and then the specialization morphism $\overline{\eta} \to \bar{x}$ in $\underline{C}$ at a node or marked point distinguishes a $\mathbb{G}$-orbit of such arrows.

We emphasize that if $x = q$ is a node there are two such arrows, regardless if q is self-intersecting or not, one for each branch of $\underline{C}$ at q. Thus the proper labelling would not be by pairs (η, q) but by half-edges of the dual intersection graph G of C°. By abuse of notation we nevertheless denote these two half-edges by (q, η) and (q, η').

Given a node q with adjacent geometric generic point $\overline{\eta}$, we can compose $f^\flat_{\overline{\eta}} : \overline{\mathcal{M}}_{X,\underline{f}(\overline{\eta})} \to \overline{\mathcal{M}}_{C,\overline{\eta}}$ with the identification $\overline{\mathcal{M}}_{C,\overline{\eta}} = Q$ and the isomorphisms (2.26), and dualize to obtain the map of cones

$$V_\eta : Q^\vee \to \boldsymbol{\sigma}(\eta).$$

The defining equation [3, eq. (2.22)] of the contact order $u_q \in \boldsymbol{\sigma}(q)$ at q now takes the form

$$\iota_{q\eta'} \circ V_{\eta'} - \iota_{q\eta} \circ V_\eta = \ell(E_q) \cdot u_q, \tag{2.28}$$

an equality in $\mathrm{Hom}(Q^\vee, \boldsymbol{\sigma}(q))$. Here η' is the other geometric generic point of $\mathrm{Spec}\,\mathcal{O}_{C,\bar{q}}$ as above.

The pair $(V_\eta, V_{\eta'})$, or equivalently $(V_\eta, \ell(E_q), u_q)$, determines the tropicalization of $(C^\circ/W, \mathbf{p}, f)$ at q. At a marked point p, the tropicalization is similarly defined by V_η and the contact order u_p.

Taken together, we obtain the following description of the tropicalization of $(C^\circ/W, \mathbf{p}, f)$.

Proposition 2.59. *The tropicalization of a punctured map $(C^\circ/W, \mathbf{p}, f)$ to X with $W = \mathrm{Spec}(Q \to \kappa)$ an algebraically closed logarithmic point is given by the abstract tropical curve $(G, \mathbf{g}, \ell)$, i.e. the tropicalization of C°/W, and the tuple*

$$(V_\eta, u_x, \iota_{x\eta})_{\eta,x},$$

as discussed. Here $\eta \in V(G)$, $x \in E(G) \cup L(G)$, with η adjacent to x for $\iota_{x\eta}$, and the data is subject to (2.28). A self-intersecting node q produces two arrows $\iota_{x\eta}$, as commented on above. The tuple $(V_\eta, u_x, \iota_{x\eta})_{\eta,x}$ is unique up to the obvious action of $\mathbb{G}$ from (2.27) on the set of tuples.

Conversely, a tropical punctured map over $\omega \in \mathbf{Cones}$ consists of two maps $\Gamma \to \omega$ and $\Gamma \to \Sigma(X)$ of generalized cone complexes. Lifting both maps locally near the strata of $|\Gamma|$ labeled by vertices, edges and legs to maps of cone complexes provides a tuple $(V_\eta, u_x, \iota_{x\eta})_{\eta,x}$ that is again unique up to the action of $\mathbb{G}$. Thus we have a one-to-one correspondence between tropical punctured maps and $\mathbb{G}$-orbits of tuples $(V_\eta, u_x, \iota_{\eta x})_{\eta,x}$. Note in particular that each individual contact order $u_x \in \boldsymbol{\sigma}(x)$,

$x \in E(G) \cup L(G)$, is only defined up to the action of $\mathrm{Aut}_{\Sigma(X)}(\boldsymbol{\sigma}(x))$, but more information is retained when considering contact orders simultaneously and together with the set of face inclusions $\iota_{x\eta}$. Here is a simple example illustrating the effect of monodromy on the procedure.

Example 2.60. This is a modification of the Whitney umbrella example in [4, Section 5.4.1]. Let C be the nodal cubic with its log smooth structure over the standard log point $\mathrm{Spec}(\mathbb{N} \to \Bbbk)$. Define X as the quotient of $(\mathbb{A}^1 \setminus \{0\}) \times C$ by the $\mathbb{Z}/2$-action that swaps the two branches of C at the node and acts by multiplication by -1 on $\mathbb{A}^1 \setminus \{0\}$. We can view X as a non-trivial, log smooth fibration over $(\mathbb{A}^1 \setminus \{0\}) \times \mathrm{Spec}(\mathbb{N} \to \Bbbk)$ with all fibers X_s isomorphic to the nodal cubic C. Thus X is irreducible with two logarithmic strata with closures $\underline{X}$ and $\underline{X}_{\mathrm{sing}}$, respectively. Denoting by $\overline{\eta}_0$, $\overline{\eta}_1$ geometric generic points for these strata, we have $\overline{\mathcal{M}}_{X,\overline{\eta}_0} = \mathbb{N}$, $\overline{\mathcal{M}}_{X,\overline{\eta}_1} = \mathbb{N}^2$. The tropicalization $\Sigma(X)$ has a presentation with two non-zero cones

$$\sigma_0 = \mathbb{R}_{\geq 0}, \quad \sigma_1 = \mathbb{R}^2_{\geq 0},$$

and non-trivial arrows the two face inclusions $\sigma_0 \to \sigma_1$ and the automorphism $\sigma_1 \to \sigma_1$ swapping the two coordinates.

The inclusion $C \to X$ of a closed fiber defines a stable log map with unique generic point η, one node q, and no marked points. We have $\boldsymbol{\sigma}(\eta) = \sigma_0$, $\boldsymbol{\sigma}(q) = \sigma_1$, and a unique arrow (2.26) in $\Sigma(X)$ for $x = \eta$, hence a unique map of cones $V_\eta : Q^\vee = \mathbb{R}_{\geq 0} \to \sigma_0$. There are, however, two choices of isomorphisms

$$\mathrm{Hom}\big(\overline{\mathcal{M}}_{X,\underline{f}(\bar{q})}, \mathbb{R}_{\geq 0}\big) \to \boldsymbol{\sigma}(q) = \sigma_1.$$

Each such choice gives two arrows $\iota_{q\eta}, \iota_{q\eta'} : \sigma_0 \to \sigma_1$ and a contact order u_q. If one choice gives

$$(V_\eta, u_q, \iota_{q\eta}, \iota_{q\eta'})$$

for the tuple in Proposition 2.59, the other choice swaps $\iota_{q\eta}$, $\iota_{q\eta'}$ and replaces u_q by $-u_q$. This is indeed the action of $\mathbb{G} = \mathbb{Z}/2$ on the set of tuples as stated in the same proposition.

The relation to the Whitney umbrella $Y = V(x^2 z - y^2) \subseteq \mathbb{A}^3$ is as follows. Endow Y with the restriction of the divisorial log structure on $\mathbb{A}^3$ defined by Y. We view $Y \setminus V(z)$ as a fibration over $\mathbb{A}^1 \setminus \{0\}$ by one-nodal rational curves via projection to the z-coordinate. Then there is an étale map $Y \setminus V(z) \to X$ of degree two of fiber spaces over $\mathbb{A}^1 \setminus \{0\}$ that separates the branches of the fibers of $X \to \mathbb{A}^1 \setminus \{0\}$.

2.6.2 Types of punctured maps with non-simple targets

One way to define the type of a punctured map in general is as an equivalence class of tropicalizations which identifies two tropical punctured maps whenever they fit into one family. The action of the automorphism group $\mathbb{G}$ on a face map $\iota_{x\eta}$ in Proposition 2.59 is induced by propagation along appropriate families. Thus in the general case, the type of a punctured map at a geometric point, or of a tropical punctured map, in addition to $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ needs to specify these face maps $\iota_{x\eta}$, at least up to the overall action by $\mathbb{G}$. This leads to the following modification of Definition 2.24.

Definition 2.61. (1) A *framed type (of a family of tropical punctured maps)* is a tuple $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ with $\mathbf{u}(x) \in N_{\boldsymbol{\sigma}(x)}$ for all $x \in E(G) \cup L(G)$ as in Definition 2.24, together with arrows[8] in $\Sigma(X)$,

$$\iota_{xv} : \boldsymbol{\sigma}(v) \to \boldsymbol{\sigma}(x),$$

for all $x \in E(G) \cup L(G)$ and $v \in V(G)$ an adjacent vertex.

(2) The *type (of a family of tropical punctured maps)* is an equivalence class of framed types under the obvious action of $\mathbb{G}$ on the set of framed types, as obtained from Proposition 2.59. The notation for a framed type is $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\iota})$ with $\boldsymbol{\iota} = (\iota_{xv})_{x,v}$.

The *type of a punctured map* $(C^\circ/W, \mathbf{p}, f)$ *to* X *at a geometric point* $\bar{w}$ *of* W is the type of the associated tropical map $\Gamma \to \Sigma(X)$ over $\omega = (\overline{\mathcal{M}}^\vee_{W,\bar{w}})_{\mathbb{R}}$.

Note that $\mathbb{G}$ acts trivially on the domain data $(G, \mathbf{g})$, the strata map $\boldsymbol{\sigma}$ and on global contact orders. So for framed types the action is on the tuple $(\mathbf{u}(x), \iota_{xv})$ with x running through $E(G) \cup V(G)$ and v through vertices adjacent to x. In particular, since the group $\mathbb{G}$ acts also trivially on the space $\mathfrak{C}_\sigma$ of global contact orders for $\sigma \in \Sigma(X)$, the definition of global type in Definition 2.44 remains unchanged.

We skip the obvious decorated versions of the notions of types in the general case. These just add the data of curve classes to vertices.

2.6.3 Contraction morphisms of types for non-simple targets

The definition of contraction morphism of types

$$\phi : \tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}) \to \tau' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}')$$

from [3, Definition 2.24] imposes the condition that $\boldsymbol{\sigma}'(\phi(x))$ is a face of $\boldsymbol{\sigma}(x)$ for all $x \in V(G) \cup (E(G) \setminus E_\phi) \cup L(G)$. In the general case, this condition has to be replaced by the choice of an arrow

$$\boldsymbol{\sigma}'(\phi(x)) \to \boldsymbol{\sigma}(x)$$

in $\Sigma(X)$ as part of the data defining ϕ. We obtain the following definition.

[8] In the case of a self-intersecting node $x = q$ there are two such arrows, which as before we do not distinguish by the notation.

Definition 2.62. (1) Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\iota})$, $\tau' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}', \boldsymbol{\iota}')$ be two framed types. A *contraction morphism of framed types* $\tau \to \tau'$ is a contraction morphism $\phi : (G, \mathbf{g}) \to (G', \mathbf{g}')$ of genus-decorated graphs together with arrows

$$\iota_x : \boldsymbol{\sigma}'(\phi(x)) \to \boldsymbol{\sigma}(x)$$

in $\Sigma(X)$ for all $x \in V(G) \cup (E(G) \setminus E_\phi) \cup L(G)$. We require that the ι_x are compatible with $\boldsymbol{\iota}$, $\boldsymbol{\iota}'$, that is, the diagrams

$$\begin{tikzcd} \boldsymbol{\sigma}'(\phi(v)) \arrow[r, "\iota_v"] \arrow[d, "\iota'_{\phi(x)\phi(v)}"] & \boldsymbol{\sigma}(v) \arrow[d, "\iota_{xv}"] \\ \boldsymbol{\sigma}'(\phi(x)) \arrow[r, "\iota_x"] & \boldsymbol{\sigma}(x) \end{tikzcd} \tag{2.29}$$

commute, for all $x \in (E(G) \setminus E_\phi) \cup L(G)$ and all $v \in V(G)$ an adjacent vertex.[9]

(2) An equivalence class for the obvious action of the group $\mathbb{G}$ from (2.27) acting on the set of contraction morphisms with domain framed types with given $(G, \mathbf{g}, \boldsymbol{\sigma})$ defines the notion of *contraction morphism of types*.

There is again no change in the definition of contraction morphism of global types compared to the case with simple X.

As in the discussion of types in the preceding Section 2.6.2, we have again skipped spelling out the trivial generalization to the decorated versions.

Contraction morphisms arise from specializations in families of punctured maps, as proved in the case of simple X in Proposition 2.25. Here is the version for the general case.

Proposition 2.63. *Let $(C^\circ/W, \mathbf{p}, f)$ be a stable punctured map to X over some logarithmic scheme W, and let $\bar{w}' \to \bar{w}$ be a specialization arrow of geometric points of W. Let $(\tau, \mathbf{A})$ with $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\iota})$ be the decorated framed type of $(C/W, \mathbf{p}, f)$ at the geometric point $\bar{w}$ of $\underline{W}$ according to Definition 2.61 (2) by a choice of arrows (2.26). Let similarly $(\tau', \mathbf{A}')$ with $\tau' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}')$ be the decorated framed type of $(C^\circ/W, \mathbf{p}, f)$ at $\bar{w}'$, for the induced choice of arrows (2.26).*

Then the map

$$(\tau, \mathbf{A}) \to (\tau', \mathbf{A}')$$

induced by generization is a contraction morphism.

Proof. The proof is again identical to the proof of [3, Lemma 2.30] save keeping track of the choices of arrows in $\Sigma(X)$. ∎

[9] Note that $\iota'_{\phi(x)\phi(v)}$ is uniquely determined by the diagram from $\iota_v, \iota_{xv}, \iota_x$.

2.6.4 The basic monoid and tropical moduli in general

The definition of basicness (Definition 2.31) makes sense in complete generality by replacing "type" by "a framed type representing the type of $(C^\circ/W, \mathbf{p}, f)$ at the geometric point $\bar{w}$". Indeed, given a framed type, the space of tropical curves of the given framed type is a subspace of the set of tuples (V_η, ℓ_q) with entries taking values in strongly convex rational polyhedral cones and subject to some integral equalities, hence is parametrized by a strongly convex rational polyhedral cone itself. This cone has been made explicit in Proposition 2.32 in the case of simple X. Here is the restatement of this proposition with reference to a framed type.

Proposition 2.64. *Let $(\pi : C^\circ/W, \mathbf{p}, f)$ be a basic, pre-stable punctured map over a logarithmic point* $\operatorname{Spec}(Q \to \kappa)$ *with κ an algebraically closed field. Denote by G the dual intersection graph of C°. For each $x \in V(G) \cup E(G) \cup L(G)$ with associated geometric point $\bar{x}$ of $\underline{C}^\circ$ and smallest stratum $\boldsymbol{\sigma}(x) \in \Sigma(X)$ containing $f(\bar{x})$ choose an isomorphism*

$$\mu_x : \overline{\mathcal{M}}_{X,f(\bar{x})} \to (\boldsymbol{\sigma}(x)_{\mathbb{Z}})^\vee,$$

dual to an arrow in $\Sigma(X)$ as in (2.26). Denote by $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\iota})$ the framed type of $(\pi : C^\circ/W, \mathbf{p}, f)$ defined by this choice according to the discussion leading to Proposition 2.59. Then the map

$$Q^\vee \to \Big\{((V_\eta)_\eta, (\ell_q)_q) \in \prod_\eta \boldsymbol{\sigma}(\eta)_{\mathbb{Z}} \times \prod_q \mathbb{N} \mid \iota_{q\eta} \circ V_\eta - \iota_{q\eta'} \circ V_{\eta'} = \ell_q \cdot \mathbf{u}(q)\Big\} \tag{2.30}$$

with V_η-entry the dual of $(\overline{\pi_\eta^\flat})^{-1} \circ \bar{f}_{\bar{\eta}}^\flat \circ \mu_\eta^{-1} : \boldsymbol{\sigma}(\eta)_{\mathbb{Z}}^\vee \to Q$ and ℓ_q-entries given by the dual of the classifiying map $\prod_q \mathbb{N} \to Q$ of the log smooth curve C/W, is an isomorphism. Here η and q run over the set of generic points and nodes of $\underline{C}$, respectively. The equation in the bracket holds in $N_{\boldsymbol{\sigma}(q)}$ for all nodal points q with adjacent generic points η, η' ordered according to the orientation of E_q (with the usual ambiguity of notation concerning self-intersecting nodes).

Proof. The proof is identical to the proof of Proposition 2.32 once the refined tropicalization procedure of Section 2.6.1 is taken into account. ∎

With this description of the basic monoid in the general case the proof of Proposition 2.34, which proves that basicness is an open condition, generalizes without problems.

The final point we want to discuss concerns the monoid quotient

$$\chi_{\tau\tau'} : Q_{\tau'} \to Q_{\tau\tau'}, \tag{2.31}$$

of basic monoids from (2.21) obtained from a framed type τ' and contraction morphism $\overline{\tau}' \to \tau$ of the associated global type. The basic monoid $Q_{\tau'}$ depends only on the framed type, as spelled out in (2.30). But note that the group $\mathbb{G}$ from (2.27) generally acts non-trivially on the right-hand side of (2.30), so the basic monoid is *not* intrinsic to the type.

Similarly, the description of $Q_{\tau\tau'}$ in (2.20) requires the knowledge of the image of the arrows $\iota_v : \boldsymbol{\sigma}(\phi(v)) \to \boldsymbol{\sigma}'(v)$, hence works only for a contraction morphism of framed types as follows. Let $(C^\circ/W, \mathbf{p}, f)$ be a basic punctured map and $\bar{w}$ a geometric point of $\underline{W}$. Then a choice of isomorphisms in (2.26), or equivalently of $\boldsymbol{\mu} = (\mu_x)$ in Proposition 2.64, provides a framed type $\tau' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}', \boldsymbol{\iota}')$ and an isomorphism of $\bar{\mathcal{M}}^\vee_{W,\bar{w}}$ with the submonoid $Q^\vee_{\tau'} \subseteq \prod_\eta \boldsymbol{\sigma}'(\eta)^\vee_{\mathbb{Z}} \times \prod_q \mathbb{N}$ on the right-hand side of (2.30). Let $\phi : \overline{\tau}' \to \tau$ be a contraction morphism of the global type $\overline{\tau}'$ associated to τ' to some other global type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$. Then each choice $\boldsymbol{\iota}_\bullet$ of arrows

$$\iota_v : \boldsymbol{\sigma}(\phi(v)) \to \boldsymbol{\sigma}'(v), \qquad v \in V(G)$$

in $\Sigma(X)$ provides a face $Q^\vee_{\tau\tau'}(\boldsymbol{\iota}_\bullet) \subseteq Q_{\tau'}$ as in (2.20), hence a dual localization morphism

$$\chi_{\tau\tau'}(\boldsymbol{\mu}, \boldsymbol{\iota}_\bullet) : \bar{\mathcal{M}}_{W,\bar{w}} \xrightarrow{\simeq} Q_{\tau'} \to Q_{\tau\tau'}(\boldsymbol{\iota}_\bullet)$$

as in (2.31). Thus this quotient of $\bar{\mathcal{M}}_{W,\bar{w}}$ depends on both the choices of $\boldsymbol{\mu}$ and $\boldsymbol{\iota}_\bullet$. Note that $Q_{\tau\tau'} \neq 0$ only if there exists a degeneration of tropical punctured maps of framed type τ compatible with the restriction on the images of vertices given by $\boldsymbol{\iota}_\bullet$.

The schematic restriction to punctured maps of global type τ is then locally reflected in the monoid ideal

$$I_{\tau\tau'} = \bigcap_{\boldsymbol{\iota}} (\chi_{\tau\tau'}(\boldsymbol{\mu}, \boldsymbol{\iota}_\bullet))^{-1}(Q_{\tau\tau'}(\boldsymbol{\iota}) \setminus \{0\}) \subseteq \bar{\mathcal{M}}_{W,\bar{w}}. \tag{2.32}$$

Note that unlike in the simple case, $\operatorname{Spec} \Bbbk[Q_{\tau'}]/I_{\tau\tau'}$ may now be a reducible scheme. See Definition 3.4 (3) for the use of this ideal in a moduli context.

Chapter 3

The stack of punctured maps

Throughout this chapter we fix as the target a morphism $X \to B$ locally of finite type between separated, locally noetherian fs logarithmic schemes over $\Bbbk$. We assume further that X is connected and that $X \to B$ fits into a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{A}_X \arrow[d] \\ B \arrow[r] & \mathcal{A}_B \end{tikzcd}$$

with strict horizontal arrows, $\mathcal{A}_B$ the Artin fan of B, and $\mathcal{A}_X$ an Artin fan representable over Log or over Log^1. If X has a Zariski log structure and $X \to B$ is log smooth then [3, Proposition 2.8] shows that we can take the Artin fan of X for $\mathcal{A}_X$, which is representable over Log by definition. In general, [5, Corollary 3.3.5] provides the desired diagram with $\mathcal{A}_X$ representable over Log^1.[1] We define

$$\mathcal{X} = B \times_{\mathcal{A}_B} \mathcal{A}_X,$$

which by abuse of notation we refer to as *the relative Artin fan* of $X \to B$.

3.1 Stacks of punctured curves

The purpose of this section is the introduction of stacks of punctured curves as domains for punctured maps.

3.1.1 Stacks of marked pre-stable curves

For a genus-decorated graph $(G, \mathbf{g})$ recall from [3, Section 2.4] the logarithmic stacks $\mathbf{M}(G, \mathbf{g})$ of $(G, \mathbf{g})$-marked pre-stable curves over the ground field $\Bbbk$ with its basic log structure as a nodal curve, and $\mathfrak{M}_B(G, \mathbf{g}) = \mathrm{Log}_{\mathbf{M}(G,\mathbf{g})\times B}$ of $(G, \mathbf{g})$-marked log smooth curves over B with arbitrary fs log structures on the base. For a leg $L \in L(G)$ denote by p_L the associated marked section.

3.1.2 The nodal log-ideal on $\mathbf{M}(G, \mathbf{g})$

Since the basic monoid of an r-nodal curve is $\mathbb{N}^r$, each $(G, \mathbf{g})$-marked nodal curve $C \to W$ comes with a homomorphism $\mathbb{N}^r \to \overline{\mathcal{M}}_W$ with $r = |E(G)|$. The image of

[1] The representability assumption is used in the proof of Lemma 3.11.

$\mathbb{N}^r \setminus \{0\}$ generates a coherent sheaf of ideals $\overline{\mathcal{I}} \subset \overline{\mathcal{M}}_W$ with preimage $\mathcal{I} \subset \mathcal{M}_W$ mapping to 0 under the structure homomorphism $\mathcal{M}_W \to \mathcal{O}_W$. Thus $\mathcal{I}$ endows $\mathbf{M}(G, \mathbf{g})$ with the structure of an idealized log stack.

Definition 3.1. We refer to $\mathcal{I}$ and to any pullback of $\mathcal{I}$ to a stack over $\mathbf{M}(G, \mathbf{g})$ such as $\mathfrak{M}(G, \mathbf{g})$ (and $\breve{\mathfrak{M}}(G, \mathbf{g})$ below) as the *nodal log-ideal*.

The local structure of moduli spaces of nodal curves implies that $\mathbf{M}(G, \mathbf{g})$ with the nodal log-ideal is idealized logarithmically smooth over the trivial log point $\operatorname{Spec} \Bbbk$. If $(C/W, \mathbf{p})$ is a $(G, \mathbf{g})$-marked curve, the log ideal is generated at a geometric point $\bar{w}$ of $\underline{W}$ by those standard basis vectors of $\overline{\mathcal{M}}_{W,\bar{w}} \simeq \mathbb{N}^r$ mapping to the smoothing parameters of the nodes labeled by $E(G)$.

3.1.3 Enter stacks of punctured curves

We now define a stack $\breve{\mathfrak{M}}_B(G, \mathbf{g})$ of punctured curves by admitting arbitrary puncturings at these marked sections.

Definition 3.2. Let $(G, \mathbf{g})$ be a genus-decorated graph. A $(G, \mathbf{g})$-*marking* of a punctured curve $(C^\circ/W, \mathbf{p})$ is a $(G, \mathbf{g})$-marking of the underlying marked curve $(\underline{C}/\underline{W}, \mathbf{p})$. The stack $\breve{\mathfrak{M}}_B(G, \mathbf{g})$ is the fibered category over $(\mathbf{Sch}/\underline{B})$ with objects $(G, \mathbf{g})$-marked punctured curves $(C^\circ/W, \mathbf{p})$ over B. Morphisms are given by *strict* fiber diagrams of punctured curves respecting the markings by $(G, \mathbf{g})$.

Note that the morphisms in $\breve{\mathfrak{M}}_B(G, \mathbf{g})$ are pull-backs of punctured curves as defined in Definition 2.13.

The maps associating to a $(G, \mathbf{g})$-marked punctured curve the underlying $(G, \mathbf{g})$-marked nodal curve with its *basic* log structure defines a morphism of logarithmic stacks

$$\breve{\mathfrak{M}}_B(G, \mathbf{g}) \to \mathbf{M}(G, \mathbf{g}). \tag{3.1}$$

3.1.4 The stacks of punctured curves are algebraic

Proposition 3.3. (1) *The stack $\breve{\mathfrak{M}}_B(G, \mathbf{g})$ is a logarithmic algebraic stack.*

(2) *Endowing $\breve{\mathfrak{M}}_B(G, \mathbf{g})$ with the idealized log structure defined by the union of its puncturing log-ideal (Definition 2.49) and its nodal log-ideal (Definition 3.1) and $\mathfrak{M}_B(G, \mathbf{g})$ with its nodal log-ideal, the strict morphism*

$$\breve{\mathfrak{M}}_B(G, \mathbf{g}) \to \mathfrak{M}_B(G, \mathbf{g})$$

forgetting the puncturing, is locally of finite type, quasi-separated, representable, unramified, and idealized logarithmically étale.

Proof. We argue by showing that the morphism $\breve{\mathfrak{M}}_B(G,\mathbf{g}) \to \mathfrak{M}_B(G,\mathbf{g})$ is representable by algebraic spaces, satisfying the adjectives spelled out in (2).[2] This is sufficient as $\mathfrak{M}_B(G,\mathbf{g})$ is a logarithmic algebraic stack.

The stack $\mathfrak{M}_B(G,\mathbf{g})$ is locally noetherian, so it has a covering $\sqcup W_\alpha \to \mathfrak{M}_B(G,\mathbf{g})$ in the strict smooth topology, where W_α are noetherian logarithmic schemes. Letting W be one of these, define

$$\breve{W} = W \times_{\mathfrak{M}_B(G,\mathbf{g})} \breve{\mathfrak{M}}_B(G,\mathbf{g}),$$

viewed as a category fibered in groupoids over $\underline{W}$, or, equivalently, over the category of strict morphisms $T \to W$. It suffices to prove that $\breve{W}$ is an algebraic space satisfying the conditions of (2).

We show this directly by exhibiting $\breve{W}$ as a *sheaf of sets*, with representable diagonal, having an étale covering by a scheme, and satisfying the above conditions.

The morphism $W \to \mathfrak{M}_B(G,\mathbf{g})$ corresponds to a $(G,\mathbf{g})$-marked logarithmic curve $\pi : C \to W$. Spelled out, the formation of $\breve{W}$ means that for any strict morphism $T \to W$, the objects in $\breve{W}(T)$ are punctured curves $(C_T^\circ \to C_T \to T, \mathbf{p}_T)$ with punctures at the markings of C_T. Here $C_T = C \times_W T \to T$ is the pullback of the logarithmic curve $C \to W$. Pull-backs in $\breve{W}$ are defined as pull-backs of punctured curves along strict morphisms over W. The markings by $(G,\mathbf{g})$ are inherited from C/W and do not play any further role.

First, we note that $\breve{W}$ is a sheaf of sets over W. We have to show that any automorphism of the log curve parametrized by W induces at most one automorphism of any corresponding punctured curve above it. Indeed, an isomorphism of punctured curves over the identity of a given logarithmic curve is a pullback diagram as in Diagram (2.3), with $h : T = W \to W$ and $C_T = C \to C$ the identity. Such an isomorphism is an equality of the submonoids of $\mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}^{\mathrm{gp}}$ in the notation of Definition 2.1. In particular, such an isomorphism is unique when it exists.

Second, Isom functors are representable, in fact by open subschemes of the base T. Indeed, the locus on C_T where two logarithmic structures inside $\mathcal{M}_{C_T}^{\mathrm{gp}}$ coincide is open in C_T (as can be deduced from Lemma 2.17), and its complement is a closed subscheme of the markings of C_T, whose image in T is closed. The complement is the desired open subscheme of T. In particular, $\breve{W} \to \breve{W} \times_W \breve{W}$ is an open embedding; once we prove $\breve{W}$ is locally of finite type over W, we will know the diagonal $\breve{W} \to \breve{W} \times_W \breve{W}$ is quasi-compact. This will prove the quasi-separatedness in (2).

Third, it now remains to construct an étale atlas by a scheme, and verify the various adjectives in (2).

We note that the statements of the proposition are both local on W. Further shrinking W, we may assume that the Artin fan $\mathcal{A}_W$ equals $\mathcal{A}_Q$ for an fs and sharp monoid Q.

[2] A simple reduction to known stacks would be welcome.

To prove both statements of the proposition, it suffices to proceed as follows: For any object $C_T^\circ \to C_T \to T$ in $\breve{W}(T)$,

(1) we will construct a locally of finite type, unramified, idealized logarithmically étale, and strict morphism $V \to W$, for V some log scheme,

(2) show that $T \to W$ factors through V,

(3) construct a punctured curve $C_V^\circ \to C_V \to V$, and

(4) show that $C_T^\circ \to C_T \to T$ is the pullback of $C_V^\circ \to C_V \to V$.

(5) Finally, we will show that the tautological morphism $V \to \breve{W}$ defined by the family $C_V^\circ \to C_V \to V$ is étale.

In particular, we obtain an étale cover $\sqcup V \to \breve{W}$ of the sheaf $\breve{W}$ by ordinary schemes, or equivalently, by strict étale morphisms of log schemes.

Since the statements above are étale local on T, we may assume the Artin fan $\mathcal{A}_T$ equals $\mathcal{A}_{Q'}$ for some fs sharp monoid Q'. Since the puncturing ideal $\mathcal{K}_T$ of $(C_T^\circ \to T, \mathbf{p}_T)$ is coherent, further shrinking T we may assume that there is a monoid ideal $K \subset Q'$ such that the corresponding log ideal $\mathcal{K}$ on $\mathcal{A}_{Q'}$ pulls-back to $\mathcal{K}_T$.

The strict morphism $T \to W$ induces a strict open embedding $\mathcal{A}_{Q'} \to \mathcal{A}_Q$. Replacing W by its strict open subscheme $W \times_{\mathcal{A}_Q} \mathcal{A}_{Q'}$, we may assume that $Q = Q'$.

Step 1. Construction of $V \to W$. Fix any point $t \in T$ over the unique closed point of $\mathcal{A}_Q$. Consider the monoid ideal $K = \overline{\mathcal{K}}_T|_t \subset Q$. Let $\mathcal{V} \to \mathcal{A}_Q$ be the strict closed embedding defined by the ideal K, and $\mathcal{K}_\mathcal{V}$ be the corresponding log ideal over $\mathcal{V}$. Then $\mathcal{V} \to \mathcal{A}_Q$ is finite type, strict, and idealized logarithmically étale. Thus the projection $V := \mathcal{V} \times_{\mathcal{A}_Q} W \to W$ with the log ideal $\mathcal{K}_V := \mathcal{K}_\mathcal{V}|_V$ is a finite type, strict closed embedding and idealized logarithmically étale.

Step 2. $T \to W$ factors through V. Recall that $\mathcal{K}_T$ is the pullback of $\mathcal{K}$. By Proposition 2.52 applied to C_T°/T the image $\alpha_T(\mathcal{K}_T) = (\alpha_{\mathcal{A}_Q}(\mathcal{K}))_T$ is the zero ideal. Hence the morphism $T \to \mathcal{A}_Q$ factors through $\mathcal{V}$. Consequently, $T \to W$ factors through V, as claimed.

For the point t as in Step 1, we denote its image in V by w.

Step 3. Construction of the punctured curves $C_V^\circ \to C_V \to V$. To construct the sheaf of monoids $\overline{\mathcal{M}}_{C_V^\circ}$, first notice that the inclusion $\overline{\mathcal{M}}_{C_V} \subseteq \overline{\mathcal{M}}_{C_V^\circ}$ is an isomorphism away from the points of $\mathbf{p}$. For each puncture $p_w \in \mathbf{p}_w$ over w, we define $\overline{\mathcal{M}}_{C_V^\circ, p_w} := \overline{\mathcal{M}}_{C_T^\circ, p_t}$ using the fiber over t. Let p_T, p_V be the punctured sections corresponding to p_w of $\underline{C}_T/\underline{T}$, $\underline{C}_V/\underline{V}$ respectively. Note that we have

$$\overline{\mathcal{M}}_{C_T^\circ, p_t} \xrightarrow{\cong} \Gamma\big(\underline{T}, p_T^* \overline{\mathcal{M}}_{C_T^\circ}\big), \quad \overline{\mathcal{M}}_{C_T, p_t} = \overline{\mathcal{M}}_{C_V, p_w} = Q \oplus \mathbb{N} \xrightarrow{\cong} \Gamma\big(\underline{V}, p_V^* \overline{\mathcal{M}}_{C_V}\big).$$

Define $\overline{\mathcal{M}}_{C_V^\circ} \subset \overline{\mathcal{M}}_{C_V}^{\mathrm{gp}}$ as the subsheaf of fine monoids generated by the image of $\overline{\mathcal{M}}_{C_V^\circ, p_w} \subset \overline{\mathcal{M}}_{C_V, p_w}^{\mathrm{gp}}$ under this isomorphism.

Consider $\mathcal{M}_{C_V^\circ} := \mathcal{M}_{C_V}^{\mathrm{gp}} \times_{\overline{\mathcal{M}}_{C_V}^{\mathrm{gp}}} \overline{\mathcal{M}}_{C_V^\circ}$. Observe that $\mathcal{M}_{C_V} \subseteq \mathcal{M}_{C_V^\circ}$. We define the structure morphism $\alpha_{C_V^\circ} : \mathcal{M}_{C_V^\circ} \to \mathcal{O}_{C_V}$ as follows. First, we require $\alpha_{C_V^\circ}|_{\mathcal{M}_{C_V}} = \alpha_{C_V}$. Second, for a local section δ of $\mathcal{M}_{C_V^\circ}$ not contained in $\mathcal{M}_{C_V}$, we define $\alpha_{C_V^\circ}(\delta) = 0$.

This defines a monoid homomorphism. Indeed, using the decomposition $\mathcal{M}_{C_V^\circ} \subseteq \mathcal{M} \oplus_{\mathcal{O}_{C_v}^\times} \mathcal{P}^{\mathrm{gp}}$ as in Definition 2.1, write $\delta = (\delta', \delta'')$ with δ' the pullback of a section of $\mathcal{M}_V$. It is sufficient to check that when $\delta \notin \mathcal{M}_{C_V}$ we have $\alpha_V(\delta') = 0$.

In the notation of Section 2.5.2 the assumption $\delta \notin \mathcal{M}_{C_V}$ implies $\delta \in \mathcal{I}_p$. Hence according to Definition 2.49 we have $\delta' \in \mathcal{K}_V$. As V is defined by $\alpha_V(\mathcal{K}_V) = 0$, we have $\alpha_V(\delta') = 0$ as needed.

This defines a logarithmic structure $\mathcal{M}_{C_V^\circ}$ over $\underline{C}_V$. The inclusion of logarithmic structures $\mathcal{M}_{C_V} \subseteq \mathcal{M}_{C_V^\circ}$ is a puncturing, hence defines a punctured curve $C_V^\circ \to C_V \to V$.

Step 4. $C_T^\circ \to C_T \to T$ is the pullback of $C_V^\circ \to C_V \to V$ via $i : T \to V$. Denote by $\underline{j} : \underline{C}_T \to \underline{C}_V$ the pullback of $\underline{i}$. Since $C_T \to T$ is given by base change from $C_V \to V$, it suffices to show that $\underline{j}^* \mathcal{M}_{C_V^\circ} = \mathcal{M}_{C_T^\circ}$ as sub-sheaves of monoids in $\mathcal{M}_{C_T}^{\mathrm{gp}}$. Away from the punctures, the equality clearly holds. Along each puncture $p \in \mathbf{p}_T$, we have the equality $j^* \overline{\mathcal{M}}_{C_V^\circ, p_w} = \overline{\mathcal{M}}_{C_T^\circ, p_t}$ at p_t by the construction in Step 3, which extends along the marking p by generization. This proves the desired equality.

Step 5. Étale covering. Consider a strict, square-zero extension $T \to T'$ over W and a family of punctured curves $C_{T'}^\circ \to C_{T'} \to T'$ such that $C_{T'} = C \times_W T'$, and $C_T^\circ \to C_T \to T$ is the pullback of $C_{T'}^\circ \to C_{T'} \to T'$. Since the strict morphism $T' \to \mathcal{A}_Q$ again factors through $\mathcal{A}_{Q'}$, we may continue to assume $Q = Q'$. Applying Step 2 again, we see that $T' \to W$ factors through V uniquely.

Denote by $t' \in T'$ the image of t via $T \to T'$. The family $C_V^\circ \to C_V \to V$ is constructed using the same geometric fiber over t. Applying Step 4 again, we see that $C_{T'}^\circ \to C_{T'} \to T'$ can be obtained via pulling back $C_V^\circ \to C_V \to V$.

This shows that $V \to \breve{W}$ is formally étale, and we claim it is actually étale, in other words, for any scheme T'' and morphism $T'' \to \breve{W}$, we need to show that $T'' \times_{\breve{W}} V \to T''$ is locally of finite presentation. The question being local, we may assume $T'' \to \breve{W}$ factors through some $V'' \to \breve{W}$ in our covering, and may as well replace T'' by V''. In this case $V \times_{\breve{W}} V'' \to V \times_W V''$, the pullback of the diagonal $\breve{W} \to \breve{W} \times_W \breve{W}$ along $V \times_W V'' \to \breve{W} \times_W \breve{W}$, is an open embedding. As V, V'' and W are noetherian, the map $V \times_{\breve{W}} V'' \to V''$ is of finite presentation.

Moreover, since $\sqcup V \to W$ is locally of finite presentation and $\sqcup V \to \breve{W}$ is étale and surjective, we have that $\breve{W} \to W$ is locally of finite presentation, see [67, Section 06Q1]. As indicated earlier, this implies that the diagonal is quasi-separated, completing the proof. ∎

3.2 Stacks of punctured maps marked by tropical types

3.2.1 Weak markings and markings

In analogy with [3, Definition 2.31] we define the following notion.

Definition 3.4. Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ be a global type of punctured maps (Definition 2.44). A *weak marking by* τ of a basic punctured map $(C^\circ/W, \mathbf{p}, f)$ to X is a $(G, \mathbf{g})$-marking of the domain curve $(C^\circ/W, \mathbf{p})$ (Definition 3.2) with the following properties:

(1) The restriction of $\underline{f}$ to the closed subscheme $Z \subseteq \underline{C}$ (a subcurve or punctured or nodal section of $\underline{C}$) defined by $x \in V(G) \cup E(G) \cup L(G)$ factors through the closed stratum $X_{\boldsymbol{\sigma}(x)} \subseteq \underline{X}$ (Section 2.2.1).

(2) For each geometric point $\bar{w}$ of $\underline{W}$ with $\tau_{\bar{w}} = (G_{\bar{w}}, \mathbf{g}_{\bar{w}}, \boldsymbol{\sigma}_{\bar{w}}, \mathbf{u}_{\bar{w}})$ the associated type of $(C^\circ/W, \mathbf{p}, f)$ at $\bar{w}$ (Definition 2.24), the contraction morphism $(G_{\bar{w}}, \mathbf{g}_{\bar{w}}) \to (G, \mathbf{g})$ of decorated graphs given by the marking defines a contraction morphism of the associated global types

$$(G_{\bar{w}}, \mathbf{g}_{\bar{w}}, \boldsymbol{\sigma}_{\bar{w}}, \bar{\mathbf{u}}_{\bar{w}}) \to \tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}}). \tag{3.2}$$

A weak marking of $(C^\circ/W, \mathbf{p}, f)$ by τ is a *marking* if in addition the following condition holds.

(3) For all geometric points $\bar{w}$ of $\underline{W}$, the ideal in $\mathcal{M}_{W,\bar{w}}$ defined by the monoid ideal $I_{\tau\tau_{\bar{w}}} \subseteq \overline{\mathcal{M}}_{W,\bar{w}}$ in (2.32) maps to 0 under the structure morphism $\mathcal{M}_{W,\bar{w}} \to \mathcal{O}_{W,\bar{w}}$.

A *marking of* $(C^\circ/W, \mathbf{p}, f)$ *by a decorated global type* $\boldsymbol{\tau} = (\tau, \mathbf{A})$ is defined analogously, with the associated types replaced by associated decorated types introduced in (2.13).

In the definition, basicness is not necessary for (1) and (2), but is needed when referring to (2.32) in (3).

Note that a marking of a punctured map by a global type τ does not mean that τ is realizable. It just means that there is a contraction morphism $\overline{\tau}' \to \tau$ from the global type $\overline{\tau}'$ associated to a realizable type τ', the type of the given punctured map.

Remark 3.5. The difference between weak markings and markings is fairly subtle and is related to saturation in the definition of the basic monoid. Recall first the construction of the basic monoid from [30, Construction 1.16]. Let $f : C/W \to X$ be a punctured map defined over a log point. The basic monoid Q associated to this log map was constructed as the saturation of a quotient of the monoid $\prod_{\eta \in \underline{C}} P_\eta \times \prod_{q \in \underline{C}} \mathbb{N}$. Here η runs over generic points of $\underline{C}$ and q runs over the nodes of $\underline{C}$. Denote by Q^{fine} this quotient before saturating, so that Q is the saturation of Q^{fine}, as in [30, eq. (1.14)].

Now suppose that $f : C/W \to X$ is a weakly τ-marked log map with W an arbitrary fs log scheme, but suppose in addition that for every geometric point $\bar{w}$ of W, $C_{\bar{w}} \to X$ is of type τ. Thus $\overline{\mathcal{M}}_W$ is locally constant with stalk Q. The proof of Lemma 3.21 below implies in particular that if s is any section of $\mathcal{M}_W$ whose image $\bar{s}$ in $\overline{\mathcal{M}}_W$ has stalk lying in $Q^{\text{fine}} \setminus \{0\}$ at each geometric point, then $\alpha_W(s) = 0$. However, the condition for being marked requires this vanishing even when $\bar{s}$ lies in $Q \setminus \{0\}$.

For an explicit example where Q^{fine} is not saturated, see [30, Example 1.17 (3)]. There, Q^{fine} is the submonoid of $\mathbb{Z}^2$ generated by $(1, -6)$, $(0, 2)$ and $(0, 3)$. In such a situation, it is not difficult to construct an example of a weakly τ-marked but not τ-marked curve, as follows.

Start with a basic τ-marked log map $f : C/W \to X$ with W a log point, and assume that $Q \neq Q^{\text{fine}}$. Let $W^{\text{fine}} = \operatorname{Spec}(Q^{\text{fine}} \to \Bbbk)$. Since all nodal generators $\rho_E \in Q$ already lie in Q^{fine} by construction, we may find a sub-log structure $\mathcal{M}_{C^{\text{fine}}} \subseteq \mathcal{M}_C$ so that $C^{\text{fine}} \to W^{\text{fine}}$ is a log smooth curve (in the category of fine log schemes) and f induces a morphism $C^{\text{fine}} \to X$. Saturating W^{fine} may yield a non-reduced scheme W^{sat} with reduction W. The composition

$$C^{\text{sat}} := C^{\text{fine}} \times_{W^{\text{fine}}} W^{\text{sat}} \to C^{\text{fine}} \to X$$

yields a stable log map in the category of fs log schemes which is weakly marked, but not marked, by τ.

In the cited example [30, Example 1.17 (3)], Q is the submonoid of $\mathbb{Z}^2$ generated by $(1, -6)$ and $(0, 1)$, and one checks that

$$\underline{W}^{\text{sat}} \cong \operatorname{Spec} \Bbbk[Q]/\langle z^{(1,-6)}, z^{(0,2)} \rangle,$$

which is a scheme of length two.

Under the presence of monodromy, the following more refined version of marked punctured maps using framed types rather than global types is sometimes more appropriate, notably in gluing. Note however that framed types work with contact orders living on a single stratum X_σ. Hence this refined notion is inappropriate when studying punctured maps with a contact order propagating into several X_σ not contained in a single stratum.

Definition 3.6. Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\iota})$ be a framed type of a family of tropical punctured maps (Definition 2.24). A *weak marking by* τ of a basic punctured map $(C^\circ/W, \mathbf{p}, f)$ to X is a weak marking by the global type $(G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ associated to τ, along with, for each $x \in E(G) \cup L(G)$ with associated nodal or punctured locus $Z_x \subseteq \underline{C}$, a homomorphism of sheaves of monoids

$$\mu(x) : (\underline{f}|_{Z_x})^{-1} \overline{\mathcal{M}}_X \to \underline{\boldsymbol{\sigma}(x)^\vee_{\mathbb{Z}}},$$

whose stalkwise duals at all geometric points $\bar{w}$ of W are arrows in $\Sigma(X)$, and which lift the contraction morphism of global types (3.2) to a contraction morphism of framed types (Definition 2.62). Here $\underline{\boldsymbol{\sigma}(x)^\vee_{\mathbb{Z}}}$ is the constant sheaf with stalks the dual of the set of integral points of $\boldsymbol{\sigma}(x)$.

A *marking by a framed type* is then defined by replacing $I_{\tau\tau'}$ in Definition 3.2 (3) by $\chi^{-1}_{\tau\tau'}(Q_{\tau\tau'} \setminus \{0\})$, noting that $\mu(x)$ makes it possible to define $Q_{\tau\tau'}$ and $\chi_{\tau\tau'}$ unambiguously and consistently.

Remark 3.7. We expect that all results that we formulate for (weak) markings by global types hold for (weak) markings by framed types. Since the framed notions have only been included in a late revision of the paper, we nevertheless decided to leave the full development of this modified theory to other occasions. We emphasize that in most applications one is either interested in simple X from the outset or one can reduce to this situation, and in this case the framed perspective does not provide any additional information.

3.2.2 Enter stacks of punctured maps

We continue to assume that $X \to B$ is a morphism of fs log algebraic schemes fulfilling the assumptions stated at the beginning of Chapter 3.

Definition 3.8. Let $\boldsymbol{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}}, \mathbf{A}) = (\tau, \mathbf{A})$ be a decorated global type (Definition 2.44). Then

$$\mathcal{M}(X/B, \boldsymbol{\tau}) \quad \text{and} \quad \mathcal{M}(X/B, \tau)$$

are defined as the stacks over $(\mathbf{Sch}/\underline{B})$ with objects basic stable punctured maps to X over B (Definition 2.15) marked by $\boldsymbol{\tau}$ and by τ, respectively (Definition 3.4).

Weakening stability to pre-stability, the analogous stacks to the relative Artin fan $\mathcal{X}$ of X over B, as defined at the beginning of Chapter 3, are denoted[3]

$$\mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau}) \quad \text{and} \quad \mathfrak{M}(\mathcal{X}/B, \tau).$$

The corresponding stacks with markings replaced by weak markings are denoted by the same symbols adorned with primes:

$$\mathcal{M}'(X/B, \boldsymbol{\tau}), \quad \mathcal{M}'(X/B, \tau), \quad \mathfrak{M}'(\mathcal{X}/B, \boldsymbol{\tau}), \quad \mathfrak{M}'(\mathcal{X}/B, \tau).$$

An important special case is that $\boldsymbol{\tau}$ is the class $\beta = (g, \bar{\mathbf{u}}, A)$ of a punctured map (Definition 2.44). Then G is the graph with only one vertex v of some genus g, stratum $\boldsymbol{\sigma}(v) = 0 \in \Sigma(X)$, and curve class A, no edges, and any number of legs.

[3] Stability being a concept for graphs decorated by genera and curve classes, there does exist a stable version of $\mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau})$. We omit this variant.

Recalling from Section 2.2.1 that the stratum of X associated to the origin $0 \in \Sigma(X)$ equals $\underline{X}$, the resulting stacks

$$\mathcal{M}'(X/B, \beta) = \mathcal{M}(X/B, \beta), \quad \mathfrak{M}'(\mathcal{X}/B, \beta) = \mathfrak{M}(\mathcal{X}/B, \beta) \tag{3.3}$$

restrict only the total genus and total curve class, as well as the number of punctures and their global contact orders.

Remark 3.9. We will see in Proposition 3.30 that for a realizable global type τ the moduli spaces $\mathfrak{M}(\mathcal{X}/B, \tau)$ of τ-marked punctured maps to $\mathcal{X}/B$ are reduced and pure-dimensional, at least for simple X. For a general global type the reduction of $\mathfrak{M}(\mathcal{X}/B, \tau)$ is stratified by the images of the morphisms $\mathfrak{M}(\mathcal{X}/B, \tau') \to \mathfrak{M}(\mathcal{X}/B, \tau)$ for realizable types τ' dominating τ, see Remark 3.31 below. Thus from the stratified point of view, markings as in Definition 3.4 (3) are the correct notion. This feature explains their appearance in [3, Definition 2.31].

However, the notion of weak marking, as in Definition 3.4 (1)–(2), appears naturally in gluing situations. Notably the commutative square in Theorem 5.8 is only cartesian with weak markings. For applications in Gromov–Witten theory, one works with cycles in the moduli spaces of punctured maps appearing in this diagram and the difference between markings and weak markings disappears, possibly up to computable multiplicities. See for example [71] where this approach is taken.

3.2.3 The stacks are algebraic

Theorem 3.10. *Let $X \to B$ be a morphism of fs logarithmic schemes fulfilling the assumptions stated at the beginning of Chapter 3, and let $\boldsymbol{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}}, \mathbf{A}) = (\tau, \mathbf{A})$ be a decorated global type of punctured maps to X. Then the stacks*

$$\mathcal{M}(X/B, \boldsymbol{\tau}), \quad \mathcal{M}(X/B, \tau), \quad \mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau}), \quad \mathfrak{M}(\mathcal{X}/B, \tau)$$

are logarithmic algebraic stacks locally of finite type over B. Moreover, $\mathcal{M}(X/B, \boldsymbol{\tau})$ and $\mathcal{M}(X/B, \tau)$ are Deligne–Mumford, and the forgetful morphisms to the stack $\mathcal{M}(\underline{X}/\underline{B})$ of ordinary stable maps are representable.

Analogous results hold for the weakly marked versions $\mathcal{M}'(X/B, \boldsymbol{\tau})$, $\mathcal{M}'(\mathcal{X}/B, \tau)$, $\mathfrak{M}'(\mathcal{X}/B, \boldsymbol{\tau})$, $\mathfrak{M}'(\mathcal{X}/B, \tau)$.

Proof. We first restrict to $\mathcal{M}'(X/B, \boldsymbol{\tau})$ and then comment on the minor changes for the other cases.

Step 1: An algebraic stack of prestable maps. Denote by

$$\breve{\mathfrak{C}} = \breve{\mathfrak{C}}(G, \mathbf{g}) \to \breve{\mathfrak{M}} = \breve{\mathfrak{M}}(G, \mathbf{g})$$

the universal curve over the logarithmic algebraic stack $\breve{\mathfrak{M}}(G, \mathbf{g})$ of $(G, \mathbf{g})$-marked punctured curves from Definition 3.2 and Proposition 3.3. This morphism is proper, flat, integral, of finite type and has geometrically reduced fibers. Hence [70, Corollary 1.1.1] applies to show that

$$\mathrm{Hom}_{\breve{\mathfrak{M}}}\big(\breve{\mathfrak{C}}, \breve{\mathfrak{M}} \times^{\mathrm{f}}_{B} X\big)$$

is representable by a logarithmic algebraic stack, locally of finite type.[4]

The rest of the proof is analogous to [3, Proposition 2.34].

Step 2: Carving out weakly marked basic stable maps. Condition (1) in Definition 3.4 of marking by $\boldsymbol{\tau}$ defines a closed substack of $\mathrm{Hom}_{\breve{\mathfrak{M}}}(\breve{\mathfrak{C}}, \breve{\mathfrak{M}} \times^{\mathrm{f}}_{B} X)$, while all the remaining conditions in Definition 3.4 (2) are open, see Proposition 2.63. Note here we are using that curve classes are locally constant in flat families. The condition on a map being basic is open by Proposition 2.34; stability is open since it is open on the underlying stable maps. Thus the morphism

$$\mathcal{M}'(X/B, \boldsymbol{\tau}) \to \mathrm{Hom}_{\breve{\mathfrak{M}}}\big(\breve{\mathfrak{C}}, \breve{\mathfrak{M}} \times^{\mathrm{f}}_{B} X\big)$$

forgetting all parts of the marking except the $(G, \mathbf{g})$-marking of the domain curve identifies the stack $\mathcal{M}'(X/B, \boldsymbol{\tau})$ with an open substack of a strict closed substack of $\mathrm{Hom}_{\breve{\mathfrak{M}}}(\breve{\mathfrak{C}}, \breve{\mathfrak{M}} \times^{\mathrm{f}}_{B} X)$.

Step 3: Verifying properties. By Proposition 2.37, logarithmic automorphisms of basic stable maps acting trivially on underlying maps are trivial. Hence $\mathcal{M}'(X/B, \boldsymbol{\tau}) \to \mathcal{M}(\underline{X}/\underline{B})$ is representable. Since $\mathcal{M}(\underline{X}/\underline{B})$ is a Deligne–Mumford stack, so is $\mathcal{M}'(X/B, \boldsymbol{\tau})$. Ignoring curve classes yields the statement for $\mathcal{M}'(X/B, \tau)$.

Step 4: Weakly marked maps to $\mathcal{X}$. The morphism $\mathcal{X} \to B$ from the relative Artin fan is well behaved.

Lemma 3.11. *The morphism $\mathcal{X} \to B$ is quasi-separated, locally of finite type, and has affine stabilizers.*

Proof. It suffices to verify these properties for the morphism $\mathcal{A}_X \to \mathcal{A}_B$. This is shown in [6, Lemma 2.5.5] in case $X \to B$ is logarithmically smooth, and we indicate here why the argument applies here. Since the properties claimed are local in B (or $\mathcal{A}_B$), we may assume $\mathcal{A}_B$ is an Artin cone $\mathcal{A}_\tau$. Since $\mathcal{A}_X$ has a cover by étale maps from Artin cones $\mathcal{A}_{\sigma\to\tau}$, we have that $\mathcal{A}_X$ is locally of finite type.

Quasiseparation follows in the same way as in [6, Lemma 2.3.8 (ii)], applied to $\mathcal{A}_X \to \mathcal{A}_X \times_{\mathcal{A}_B} \mathcal{A}_X$ instead of $\mathcal{A}_X \to \mathcal{A}_X \times \mathcal{A}_X$ and using representability over Log^1 instead of Log: one needs to show, for two charts $\mathcal{A}_{\sigma_1\to\tau}$ and $\mathcal{A}_{\sigma_2\to\tau}$ of $\mathcal{A}_X$, that $\mathcal{A}_{\sigma_1\to\tau} \times_{\mathcal{A}_X} \mathcal{A}_{\sigma_2\to\tau}$ is quasicompact. By [6, Lemma 2.3.8 (i)] and representability

[4] This last property is not explicitly stated in [70], but follows by inspection of the proof.

it suffices to show that the stack $\mathcal{A}_{\sigma_1\to\tau} \times_{\mathrm{Log}^1} \mathcal{A}_{\sigma_2\to\tau}$ has finitely many points. The argument of [6, Lemma 2.3.8 (ii)] then applies as stated.

The claim about stabilizers follows as in [6, Lemma 2.5.5]. ■

It follows that [70, Corollary 1.1.2] still applies. The rest of the proof for the stacks $\mathfrak{M}'(\mathcal{X}/B, \boldsymbol{\tau})$ and $\mathfrak{M}'(\mathcal{X}/B, \tau)$ is the same, except we can not conclude the Deligne–Mumford property due to the absence of stability.

Step 5: Marked maps. Stacks of marked maps are closed substacks of stacks of weakly marked maps, locally defined by the log-ideal $I_{\tau\tau_{\bar{w}}}$ in Definition 3.4 (3).[5] Hence the result also holds for these cases. ■

3.3 Boundedness

For ordinary stable logarithmic maps, boundedness of $\mathcal{M}(X/B, \beta)$ is established in [2, 30] for projective $X \to B$ under the technical assumption that $\overline{\mathcal{M}}_X$ is globally generated. Reference [5] removed the technical assumption by showing that there is a logarithmic blowing up $Y \to X$ with $\overline{\mathcal{M}}_Y$ globally generated and then using birational invariance of the moduli spaces $\mathfrak{M}(\mathcal{X}/B, \beta)$ under this process. Since this birational invariance seems to be rather more subtle in the punctured case, we content ourselves with a statement assuming global generatedness, which suffices for most practical applications. Throughout this and the next subsections we assume that the log structure on X is Zariski as in [30], which we follow. We believe this assumption could be removed by minor adaptations of the proof.

Theorem 3.12. *Suppose the underlying family $X \to B$ is projective, and the sheaf $\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by its global sections.*[6] *Then the projection $\mathcal{M}(X/B, \beta) \to B$ is of finite type.*

Proof. We split the proof into several steps. The theorem follows from Propositions 3.16 and 3.17 below. ■

Global generatedness of $\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ can be easily read off from the cone complex $\Sigma(X)$ as follows.

Proposition 3.13. *The sheaf $\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by global sections if and only if there exists a continuous map*

$$|\Sigma(X)| \to \mathbb{R}^r$$

with restriction to each $\sigma \in \Sigma(X)$ an injective homomorphism of additive monoids.

[5] For a much more detailed discussion of this point, in terms of the idealized structure defined by markings, see Section 3.5 below, and notably Theorem 3.25.

[6] Samuel Johnston in [39] has meanwhile removed the global generatedness assumptions along the same line as [5].

Proof. A map $|\Sigma(X)| \to \mathbb{R}^r$ which is injective when restricted to any $\sigma \in \Sigma(X)$ is dual to a system of surjective homomorphisms

$$\varphi_\sigma : \mathbb{R}^r \to \operatorname{Hom}(\sigma, \mathbb{R}),$$

compatible with the dual of the face maps defining $\Sigma(X)$. But such a compatible system $(\varphi_\sigma)_{\sigma\in\Sigma(X)}$ of surjections is equivalent to a linear map

$$\mathbb{R}^r \to \Gamma\big(\underline{X}, \overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}\big) = \Gamma\big(\underline{X}, \overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}\big) \otimes_{\mathbb{Q}} \mathbb{R}$$

with composition to $\overline{\mathcal{M}}_{X,x}^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$ surjective for each $x \in X$. The claim follows. ∎

Remark 3.14. We remark that if $\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by global sections, then all global contact orders of X are monodromy free, which we see as follows. The map of Proposition 3.13 gives a well-defined map $\mathfrak{C}_\sigma(X) \to \mathbb{Z}^r \subseteq \mathbb{R}^r$. Indeed, if $\sigma' \in \Sigma_\sigma(X)$ and $u \in N_{\sigma'}$, we may view u as an integral tangent vector (i.e., an element of $N_{\sigma'}$) to $\sigma' \in \Sigma(X)$ and take its image under the map $|\Sigma(X)| \to \mathbb{R}^r$. Since u is compatible with inclusion of faces, this provides a point v of $\mathbb{Z}^r \subseteq \mathbb{R}^r$ only depending on $\iota_{\sigma\sigma'}(u)$ (see Definition 2.40 for notation). Since $|\Sigma(X)| \to \mathbb{R}^r$ is injective on cones, v arises, for each σ', as the image of at most one $u \in N_{\sigma'}$. Hence all global contact orders are monodromy free.

3.3.1 Boundedness of $\mathcal{M}(X/B, \beta)$

Definition 3.15. A class β of a punctured map (Definition 2.44) is called *combinatorially finite* if the set of types (Definition 2.24) of stable punctured maps with associated class β is finite.

Proposition 3.16. *Suppose β is combinatorially finite. Then the forgetful map*

$$\mathcal{M}(X/B, \beta) \to \mathcal{M}(\underline{X}/\underline{B}, \underline{\beta}) \tag{3.4}$$

is of finite type.

Proof. The strategy of the proof is similar to those in [30, Section 3.2] and [15, Section 5.4] by showing that each stratum with constant combinatorial structure is bounded. The proof is largely the same, with extra care needed only in the proof of [30, Proposition 3.17].

By Theorem 3.10, $\mathcal{M}(X/B, \beta) \to B$ is locally of finite type, and hence so is the morphism $\mathcal{M}(X/B, \beta) \to \mathcal{M}(\underline{X}/\underline{B}, \underline{\beta})$. Thus it is sufficient to prove the latter morphism is quasi-compact. We thus need to show that $\underline{W} \times_{\mathcal{M}(\underline{X}/\underline{B}, \underline{\beta})} \mathcal{M}(X/B, \beta)$ is quasi-compact for any quasi-compact scheme $\underline{W}$ and morphism $\underline{W} \to \mathcal{M}(\underline{X}/\underline{B}, \underline{\beta})$. Using [30, Lemma 3.14], it is enough to find a weak cover in the sense of [30, Definition 3.13] of $\underline{W} \to \mathcal{M}(\underline{X}/\underline{B}, \underline{\beta})$ by finitely many quasi-compact subsets. We may weakly cover $\underline{W}$ by a finite number of locally closed strata on which the corresponding ordinary stable map is combinatorially constant (in the sense of [30, Definition 3.15]), and replace $\underline{W}$ with one of these locally closed strata. Thus we may assume given $\mathfrak{f} = (\underline{C}/\underline{W}, \mathbf{p}, \underline{f})$ a combinatorially constant ordinary stable map over an integral, quasi-compact scheme $\underline{W}$. Then $\underline{W} \times_{\mathcal{M}(\underline{X}/\underline{B},\underline{\beta})} \mathcal{M}(X/B,\beta)$ classifies punctured enhancements of the ordinary stable maps parametrized by $\underline{W}$, and we need to show this fiber product is quasi-compact.

As the combinatorial type of a log curve with constant dual intersection graph is locally constant, we have a decomposition

$$\underline{W} \times_{\mathcal{M}(\underline{X}/\underline{B},\underline{\beta})} \mathcal{M}(X/B,\beta) = \coprod_{\mathbf{u}} \mathcal{M}(X,\mathfrak{f},\mathbf{u})$$

into disjoint open substacks according to the type $\mathbf{u}$. As β is assumed combinatorially finite, this is a finite union. Hence it is sufficient to show that each $\mathcal{M}(X,\mathfrak{f},\mathbf{u})$ is quasi-compact. As in the proof of [30, Proposition 3.17], it is sufficient to construct a quasi-compact stack Z with a morphism $Z \to \mathcal{M}(X,\mathfrak{f},\mathbf{u})$ which is surjective on geometric points.

To do so, set $Q_1 := \mathbb{N}^k$, where k is the number of nodes of any fiber of $\underline{C} \to \underline{W}$. By Proposition 2.32 and the fact we have fixed the type $\mathbf{u}$, the basic monoid Q is constant on $\mathcal{M}(X,\mathfrak{f},\mathbf{u})$, and there is a canonical morphism $Q_1 \to Q$. The latter induces a morphism of Artin cones $\mathcal{A}_{Q^\vee} \to \mathcal{A}_{Q_1^\vee}$. We equip $\underline{W}$ with the canonical log structure coming from the family of pre-stable curves $\underline{C} \to \underline{W}$, and consider $Z_1 = \mathcal{A}_{Q^\vee} \times_{\mathcal{A}_{Q_1^\vee}} W$. Pulling back the universal family from W, we obtain a family of log curves $C_1 \to Z_1$ and an ordinary stable map $\underline{f} : \underline{C}_1 \to \underline{X}/\underline{B}$. Observe that there is a global chart $Q \to \overline{\mathcal{M}}_{Z_1}$. To check Z_1 is quasi-compact we can, and do, replace Z_1 with its underlying reduced substack.

The type $\mathbf{u}$ prescribes, for each marked section $p \in \mathbf{p}$, an ideal sheaf

$$\overline{\mathcal{I}}_p \subseteq \overline{\mathcal{M}}_W \oplus \mathbb{Z} \subseteq p^*\overline{\mathcal{M}}_C^{\mathrm{gp}}$$

generated by $u_p^{-1}(\mathbb{Z}_{<0})$, which, we note, is constant along Z_1. These ideals produce an ideal $\overline{\mathcal{K}} \subseteq Q$ as in Definition 2.49 by taking into account all punctures in $\mathbf{p}$. Denote by $\mathcal{K} = \overline{\mathcal{K}} \times_{\overline{\mathcal{M}}_{Z_1}} \mathcal{M}_{Z_1}$ the resulting log ideal, where the arrow on the left is given by the composition $\overline{\mathcal{K}} \to Q \to \overline{\mathcal{M}}_{Z_1}$ with the last arrow the global chart.

To obtain a family of punctured stable maps of type $\mathbf{u}$ over Z_1 then requires that $\alpha_{Z_1}(\mathcal{K}) = 0$ by Proposition 2.52. Thus in particular if $0 \in \overline{\mathcal{K}}$, then there are no punctured maps of type $\mathbf{u}$ and we can ignore such a $\mathbf{u}$; otherwise, as Z_1 is reduced and $\overline{\mathcal{M}}_{Z_1}$ is locally constant with stalk Q, necessarily $\alpha_{Z_1}(\mathcal{K}) = 0$. Indeed, any local section m of $\mathcal{K}$ maps to a nowhere zero section of $\overline{\mathcal{M}}_{Z_1}$, and hence $\alpha_{Z_1}(m)$ is nowhere invertible, thus zero, since Z_1 is reduced.

We now construct a punctured family of curves $C_1^\circ \to Z_1$. First, the ghost sheaf $\overline{\mathcal{M}}_{C_1^\circ}$ is identical to $\overline{\mathcal{M}}_{C_1}$ away from the punctures. Along each puncture $p \in \mathbf{p}$, we take $\overline{\mathcal{M}}_{C_1^\circ,p} \subset \overline{\mathcal{M}}^{\mathrm{gp}}_{C_1,p}$ to be the smallest fine submonoid generated by $\overline{\mathcal{M}}_{C_1,p}$ and the image of $f^{-1}\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}^{\mathrm{gp}}_{C_1,p}$ determined by the type $\mathbf{u}$. As all the ghost sheaves and morphisms between them are constant along Z_1, this yields a well-defined sheaf of monoids $\overline{\mathcal{M}}_{C_1^\circ}$, hence $\mathcal{M}_{C_1^\circ} := \overline{\mathcal{M}}_{C_1^\circ} \times_{\overline{\mathcal{M}}^{\mathrm{gp}}_{C_1}} \mathcal{M}^{\mathrm{gp}}_{C_1}$ over $\underline{C}_1$.

We define the structure homomorphism $\alpha_{C_1^\circ} : \mathcal{M}_{C_1^\circ} \to \mathcal{O}_{C_1}$ by $\alpha_{C_1^\circ}|_{\mathcal{M}_{C_1}} = \alpha_{C_1}$ and $\alpha_{C_1^\circ}|_{\mathcal{M}_{C_1^\circ} \setminus \mathcal{M}_{C_1}} = 0$. The same argument as in the proof of Proposition 3.3, Step 3, shows that this defines a logarithmic structure $\mathcal{M}_{C_1^\circ}$, hence the desired punctured curve $C_1^\circ \to Z_1$.

The remainder of the proof is now identical to that of [30, Proposition 3.17]. ∎

3.3.2 Finiteness of the combinatorial data

In order to complete the proof that $\mathcal{M}(X/B,\beta)$ is finite type, it remains to bound the combinatorial data.

Proposition 3.17. *Suppose $\overline{\mathcal{M}}^{\mathrm{gp}}_X \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by its global sections. Then any class of punctured map β is combinatorially finite.*

Proof. Arguing stratawise as in [30, Section 3.2], it is sufficient to show that for any combinatorially constant family of ordinary stable maps $(\underline{C}/\underline{W}, \mathbf{p}, \underline{f})$ in the sense of [30, Definition 3.15], there are only finitely many combinatorial types of liftings of such a family to a punctured log curve of type β. Since types are constant along a combinatorially constant family, we may further assume that $\underline{W}$ is the spectrum of a field. Finiteness of the number of types of a logarithmic stable map with a given underlying stable map over a field with fixed contact orders u_p is proved in [30, Theorem 3.9].

One small difference in our setup concerns the definition of contact orders. In [30] these were given by a sheaf homomorphism $\overline{\mathcal{M}}_Z \to \mathbb{N}$, hence were fixed at $\underline{f}(p)$ by the underlying ordinary stable map and the contact orders u_p. In contrast, a global contact order may give an infinite set of maps $\overline{\mathcal{M}}_{X,\underline{f}(p)} \to \mathbb{Z}$. The argument is saved under the assumption that $\overline{\mathcal{M}}^{\mathrm{gp}}_X \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by its global sections: The injectivity statement in Proposition 3.13 implies that there is at most one local representative of u_p. ∎

3.4 Valuative criterion

We now show stable reduction for basic stable punctured maps, which allows us to conclude properness of the moduli spaces of such maps. Recall that for a given class

$\beta = (g, \bar{\mathbf{u}}, A)$ of stable punctured maps to $X \to B$, we have the class $\underline{\beta} = (g, k, A)$ for ordinary stable maps to $\underline{X} \to \underline{B}$ by removing contact orders. We will show that

Theorem 3.18. *Assume that the log structure on X is defined in the Zariski topology. Then the tautological morphism removing all logarithmic structures*

$$\mathcal{M}(X/B, \beta) \to \mathcal{M}(\underline{X}/\underline{B}, \underline{\beta})$$

satisfies the valuative criterion for properness.

Proof. In what follows, we assume given R a discrete valuation ring over $\underline{B}$ with maximal ideal $\mathfrak{m}$, residue field $\kappa = R/\mathfrak{m}$, and fraction field K. Suppose we have a commutative square of solid arrows of the underlying stacks:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{M}(X/B, \beta) \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \mathcal{M}(\underline{X}/\underline{B}, \underline{\beta}).
\end{tikzcd}$$

We want to show that there is a dashed arrow making the above diagram commutative, which is unique up to a unique isomorphism.

The top arrow of the above diagram yields a stable punctured map

$$(\pi_K : C_K^\circ \to \operatorname{Spec}(Q_K \to K), \mathbf{p}_K, f_K)$$

over the logarithmic point $\operatorname{Spec}(Q_K \to K)$. The bottom arrow of the above diagram yields an ordinary stable map $(\underline{C}/\operatorname{Spec} R, \mathbf{p}, \underline{f})$ with its generic fiber given by the underlying stable map of f_K. To construct the dashed arrow, it suffices to extend the stable punctured map f_K across the closed point $0 \in \operatorname{Spec} R$ with the given underlying stable map $\underline{f}$. The task is to then extend the logarithmic structures and morphisms thereof. The proof is almost identical to that of [30, Theorem 4.1]. Since that proof is quite long, we only note the salient differences.

Reference [30, Section 4.1] accomplishes this extension at the level of ghost sheaves; in particular, [30, Proposition 4.3], which states that the type of the central fiber is uniquely determined by the stable log map on the generic fiber, carries through with u_p for a puncture p determined as for marked points. Indeed, if p is a punctured point on $\underline{C}_0$ in the closure of the punctured point p_K on $\underline{C}_K$, then u_p must be the composition

$$P_p \to P_{p_K} \xrightarrow{u_{p_K}} \mathbb{Z}, \tag{3.5}$$

where the first map is the generization map $(\underline{f}^* \overline{\mathcal{M}}_X)_p \to (\underline{f}^* \overline{\mathcal{M}}_X)_{p_K}$. In particular, the contact orders u_p and u_{p_K} both have global contact order as specified in β.

By Proposition 2.32, the type of the central fiber then determines the extension $\overline{\mathcal{M}}_{C^\circ}$ of $\overline{\mathcal{M}}_{C_K^\circ}$ and a map $\bar{f}^\flat : \underline{f}^*\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_{C^\circ}$ extending the corresponding map on the generic fiber. Here $\overline{\mathcal{M}}_{C^\circ}$ is defined at punctures by pre-stability via Corollary 2.7.

Next, [30, Section 4.2][7] shows that the logarithmic structure on the base Spec R is uniquely defined. In this argument, marked points play no role, and the argument remains unchanged in the punctured case. In particular, this produces a unique choice of logarithmic structure $\mathcal{M}_R$ on Spec R, which in addition comes with a morphism of logarithmic structures $\mathcal{M}_R^0 \to \mathcal{M}_R$ where $\mathcal{M}_R^0$ is the basic logarithmic structure (pulled back from the moduli space of pre-stable curves **M** with its basic logarithmic structure, see [30, Appendix A]) associated to the family $\underline{C} \to \operatorname{Spec} R$. In particular, one obtains a logarithmic structure $(\underline{C}, \mathcal{M}'_C) = (\operatorname{Spec} R, \mathcal{M}_R) \times_{(\operatorname{Spec} R, \mathcal{M}_R^0)} (\underline{C}, \mathcal{M}_C^0)$, where $\mathcal{M}_C^0$ is the logarithmic structure pulled back from the basic logarithmic structure of the universal curve over $\mathfrak{M}(\underline{X}/\underline{B}, \beta)$. The logarithmic structure $\mathcal{M}'_C$ then has logarithmic marked points along the punctures p, but there is a sub-logarithmic structure $\mathcal{M}_C \subset \mathcal{M}'_C$ which only differs in that we remove the marked points, that is, we make $(\underline{C}, \mathcal{M}_C) \to (\operatorname{Spec} R, \mathcal{M}_R)$ strict away from the nodes.

By Corollary 2.7, there is a natural inclusion $\overline{\mathcal{M}}_{C^\circ} \subset (\overline{\mathcal{M}}'_C)^{\mathrm{gp}}$. We form $\mathcal{M}_{C^\circ} := \overline{\mathcal{M}}_{C^\circ} \times_{(\overline{\mathcal{M}}'_C)^{\mathrm{gp}}} (\mathcal{M}'_C)^{\mathrm{gp}}$ and define a structure homomorphism $\alpha_{C^\circ} : \mathcal{M}_{C^\circ} \to \mathcal{O}_C$ by $\alpha_{C^\circ}|_{\mathcal{M}_{C'}} = \alpha_{C'}$ and $\alpha_{C^\circ}(\mathcal{M}_{C^\circ} \setminus \mathcal{M}'_C) = 0$, as in Proposition 3.3, Step 3. To show that this is a homomorphism, it is enough to show that if $s \in \mathcal{M}_{C^\circ,p} \setminus \mathcal{M}'_{C,p}$, writing $s = (s_1, s_2)$ as a stalk of $\mathcal{M}_C \oplus_{\mathcal{O}_C^\times} \mathcal{P}^{\mathrm{gp}}$, then $\alpha_C(s_1) = 0$. But necessarily $(\bar{s}_1, \bar{s}_2) = \bar{f}^\flat(m) + (\bar{s}'_1, \bar{s}'_2)$ for some $m \in P_p$ with $u_p(m) < 0$ and $(\bar{s}'_1, \bar{s}'_2) \in \overline{\mathcal{M}}_{C,p} \oplus \mathbb{N}$. Write for points $x, x' \in \underline{C}$ with x in the closure of x' the generization map $\chi_{x',x} : P_x \to P_{x'}$. Then $u_{p_K}(\chi_{p_K,p}(m)) = u_p(m)$ by (3.5). Thus $u_{p_K}(\chi_{p_K,p}(m)) < 0$ and necessarily $\alpha_{C_K}(s_1|_{C_K}) = 0$. But since C is reduced and C_K is dense in C, this implies $\alpha_C(s_1) = 0$, as desired. Thus we have a punctured log scheme C°.

We can now extend $f_K^\flat : f_K^*\mathcal{M}_X \to \mathcal{M}_{C_K^\circ}$ to $f^\flat : f^*\mathcal{M}_X \to \mathcal{M}_{C^\circ}$ as in [30, Section 4.3]. ∎

Corollary 3.19. *Let $\boldsymbol{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}}, \mathbf{A})$ be a decorated global type of punctured maps (Definition 2.44) and assume $X \to B$ is projective, the log structure on X is Zariski, and $\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is globally generated.*[8] *Then $\mathfrak{M}(X/B, \boldsymbol{\tau}) \to B$ is proper. In particular, $\mathfrak{M}(X/B, \beta)$ is proper for any $\beta = (g, \bar{\mathbf{u}}, A)$.*

[7] We take the opportunity to correct an error, pointed out by the referee of the current paper, in the first paragraph of [30, Section 4.2]. Two descriptions of a set $U(\eta)$ are given. The first description, as the set of generizations of points in A, the set of non-special points in $\underline{C}_0$, is not correct (it is not necessarily an open set). Thus the reader should rely only on the second description of the set $U(\eta)$.

[8] Again, the latter assumption has been removed by [39].

Proof. Theorem 3.12 shows that $\mathcal{M}(X/B, \beta) \to B$ is of finite type. Properness for $\boldsymbol{\tau} = \beta$ now follows from the valuative criterion verified in Theorem 3.18.

For general $\boldsymbol{\tau}$, the proof of [3, Proposition 2.34] generalizes to the present punctured setup to exhibit $\mathcal{M}(X/B, \boldsymbol{\tau})$ as a closed substack of the base change of the stack $\mathcal{M}(X/B, \beta)$ by the finite map $\mathbf{M}(G, \mathbf{g}) \to \mathbf{M}$. ∎

3.5 Idealized smoothness of $\mathfrak{M}(\mathcal{X}/B, \tau) \to B$

For simplicity of presentation, we restrict to X simple throughout this section. Thus for any $\sigma, \tau \in \Sigma(X)$ there is at most one arrow $\sigma \to \tau$ in $\Sigma(X)$.

3.5.1 Marking log-ideals

Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ be a global type of punctured maps. Recall from the discussion after Definition 3.1 that the moduli stack $\mathbf{M}(G, \mathbf{g})$ of $(G, \mathbf{g})$-marked pre-stable curves with its nodal log ideal sheaf is idealized logarithmically smooth over the trivial log point $\operatorname{Spec} \Bbbk$. A similar result holds for our moduli spaces $\mathfrak{M}(\mathcal{X}/B, \tau)$. To introduce the idealized structure let $(\pi : C^\circ \to W, \mathbf{p}, f)$ be a τ-marked basic punctured map and let $\bar{w}$ of $\underline{W}$ be a geometric point. Let $\tau_{\bar{w}} = (G_{\bar{w}}, \mathbf{g}_{\bar{w}}, \boldsymbol{\sigma}_{\bar{w}}, \mathbf{u}_{\bar{w}})$ be the type of the punctured map over $\bar{w}$, equipped with its marking contraction morphism $\phi : \tau_{\bar{w}} \to \tau$ (Definitions 2.24 and 3.4 (2)), with set of contracted edges E_ϕ. For the sake of Definition 3.20 below, we introduce the following notation. For $x \in V(G_{\bar{w}}) \cup L(G_{\bar{w}}) \cup (E(G_{\bar{w}}) \setminus E_\phi)$ the face inclusion $\boldsymbol{\sigma}(\phi(x)) \to \boldsymbol{\sigma}_{\bar{w}}(x)$ is dual to a localization map

$$\chi_x : P_x \to P_{\phi(x)}$$

of stalks of $\overline{\mathcal{M}}_X$. We also have homomorphisms

$$\varphi_x : P_x \to \overline{\mathcal{M}}_{C^\circ, x}, \quad u_x : P_x \to \mathbb{Z}$$

defined by $\bar{f}_{\bar{w}}^\flat$ and by the contact order $\mathbf{u}_{\bar{w}}$. For uniformity of notation we define $u_x = 0$ for $x \in V(G_{\bar{w}})$. Moreover, by Definition 2.18 of contact order, $\varphi_x(u_x^{-1}(0)) \subseteq \overline{\mathcal{M}}_{C^\circ, x}$ is contained in the image of $\bar{\pi}_x^\flat : \overline{\mathcal{M}}_{W, \bar{w}} \to \overline{\mathcal{M}}_{C^\circ, x}$. For the following definition recall also the homomorphism $\chi_{\tau\tau_{\bar{w}}} : Q_{\tau_{\bar{w}}} \to Q_{\tau\tau_{\bar{w}}}$ from (2.21).

Definition 3.20. The *τ-marking ideal* $\bar{\mathcal{I}}_W^\tau$ of the τ-marked basic punctured map $(\pi : C \to W, \mathbf{p}, f)$ is the sheaf of ideals in $\overline{\mathcal{M}}_W$ with stalk at the geometric point $\bar{w}$ of $\underline{W}$ generated by the following subsets:

(i) (Target stratum generators) the preimage under $\bar{\pi}_x^\flat$ of $\varphi_x(P_x \setminus \chi_x^{-1}(0))$ for $x \in V(G_{\bar{w}}) \cup L(G_{\bar{w}}) \cup (E(G_{\bar{w}}) \setminus E_\phi)$;

(ii) (Nodal generators) the nodal generators $\rho_E \in \overline{\mathcal{M}}_{W, \bar{w}} = Q_{\tau_{\bar{w}}}$ for $E \in E(G_{\bar{w}}) \setminus E_\phi$;

(iii) (Basic monoid generators) $\chi_{\tau\tau_{\bar{w}}}^{-1}(Q_{\tau\tau_{\bar{w}}} \setminus \{0\})$.

The collection of stalks $\overline{\mathcal{I}}^\tau_{W,\bar{w}} \subset \overline{\mathcal{M}}_{W,\bar{w}}$ in Definition 3.20 form a coherent ideal $\overline{\mathcal{I}}^\tau_W \subset \overline{\mathcal{M}}_W$. Indeed, we obtain a sheaf by the method of Remark 2.36, and, as W is fine and saturated, we may apply Lemma 2.47, noting that all generating sets are compatible with generization. As usual, we also refer to the preimage $\mathcal{I}_W \subset \mathcal{M}_W$ of $\overline{\mathcal{I}}_W$ under $\mathcal{M}_W \to \overline{\mathcal{M}}_W$ as the *τ-marking ideal*. Without the generators specified in (iii) we speak of the *weak τ-marking ideal*.

3.5.2 The base of a punctured map is idealized by the marking log-ideal

The τ-marking ideal defines an idealized log structure on base spaces of τ-marked punctured maps as follows.

Lemma 3.21. *Let $(C^\circ/W, \mathbf{p}, f)$ be a τ-marked basic punctured map. Then the τ-marking ideal $\mathcal{I}_W \subset \mathcal{M}_W$ maps to* 0 *under the structure homomorphism $\mathcal{M}_W \to \mathcal{O}_W$.*

Proof. It is enough to show that any lift $s \in \mathcal{M}_{W,\bar{w}}$ of an element of one of the generating sets satisfies $\alpha_W(s) = 0$. This holds for elements described in (iii) of Definition 3.20 by Definition 3.4 (3).

Similarly, Definition 3.4 (1) guarantees the required vanishing for elements described in (i) of Definition 3.20. Indeed, consider first the case of $x = v \in V(G_{\bar{w}})$, where we defined $u_v = 0$. Then

$$u_v^{-1}(0) \setminus \chi_v^{-1}(0) = P_v \setminus \chi_v^{-1}(0),$$

and $\alpha_X(P_v \setminus \chi_v^{-1}(0)) \subset \mathcal{O}_X$ locally generates the ideal $\mathcal{I}_{X_{\boldsymbol{\sigma}(\phi(v))}} \subset \mathcal{O}_X$ of the stratum $X_{\boldsymbol{\sigma}(\phi(v))}$ in X. Thus, the condition that the restriction of $\underline{f}$ to the closed subscheme of $\underline{C}$ corresponding to $\phi(v)$ factors through $\underline{X}_{\boldsymbol{\sigma}(\phi(v))}$ implies the desired vanishing in this case. A similar argument works for legs and edges.

Finally, the lift to $\mathcal{M}_{W,\bar{w}}$ of a nodal generator $\rho_E \in \overline{\mathcal{M}}_{W,\bar{w}}$ lies in the nodal log-ideal (Definition 3.1) of the $(G, \mathbf{g})$-marked curve C/W, which maps to zero in $\mathcal{O}_W$ by Proposition 3.3 (2). ∎

Remark 3.22. Omitting the last set (iii) of generators in Definition 3.20 leads to the idealized structure for moduli spaces of *weakly* marked punctured maps (Definition 3.4).

As shown in Proposition 2.52, the base W is also idealized by the puncturing log ideal $\mathcal{K}$. It is therefore natural to combine the two.

Definition 3.23. We call the union $\mathcal{I}^\tau \cup \mathcal{K}$ of the τ-marking and the puncturing log ideals the *canonical idealized structure* on our τ-marked moduli spaces such as $\mathfrak{M}(\mathcal{X}/B, \tau)$.

3.5.3 The realizable case

While the definition of the τ-marking ideal may seem complicated, in fact in the case we most frequently need it, namely the realizable case, the canonical idealized structure has a simpler description: By Lemma 2.45 there is a unique lift to a type, and the associated basic monoid already knows about marked strata, non-deforming nodes and punctures.

Proposition 3.24. *If τ is a realizable global type, then $\overline{\mathcal{I}}^{\tau}_{W,\bar{w}} + \overline{\mathcal{K}}_{W,\bar{w}}$ with $\overline{\mathcal{K}}_W$ the puncturing log ideal (Definition 2.55) is given by the set* (iii) *in Definition 3.20.*

Proof. Denote by $\chi : Q_{\tau_{\bar{w}}} \to Q_{\tau\tau_{\bar{w}}}$ the localization homomorphism from (2.21) defined by the τ-marking of $(C^\circ/W, \mathbf{p}, f)$. By Lemma 2.45 there is a unique type of punctured map with associated global type τ. Hence in particular $Q_{\tau\tau_{\bar{w}}}$ agrees with the basic monoid for a tropical punctured map of this type and does not depend on $\bar{w}$. We write this basic monoid as Q_τ. Denote by $R \subset Q_{\tau_{\bar{w}}}$ the ideal $\chi^{-1}(Q_\tau \setminus \{0\})$.

We need to show that R contains the elements listed in (i) and (ii) of Definition 3.20 as well as generators of the puncturing log ideal stated in Definition 2.49. Adopting the notation given in Definition 3.20, for $v \in V(G_{\bar{w}})$ we have a commutative diagram

$$\begin{tikzcd} P_v \arrow[r, "\varphi_v"] \arrow[d, "\chi_v"'] & Q_{\tau_{\bar{w}}} \arrow[d, "\chi"] \\ P_{\phi(v)} \arrow[r, "\varphi_{\phi(v)}"'] & Q_\tau \end{tikzcd}$$

The fact that τ is realizable implies that $\varphi_{\phi(v)}$ is a local homomorphism, i.e. $\varphi^{-1}_{\phi(v)}(0) = \{0\}$. Indeed, dually, the map $Q_\tau^\vee \to P^\vee_{\phi(v)}$ is given by evaluation of the tropical map at the vertex v, and realizability implies the image of this map intersects the interior of $P^\vee_{\phi(v)}$. This is equivalent to the local homomorphism statement. But this implies that $\varphi_v(P_v \setminus \chi_v^{-1}(0)) \subseteq \chi^{-1}(Q_\tau \setminus \{0\}) = R$.

In the case of a leg L, we similarly have a diagram

$$\begin{tikzcd} P_L \arrow[r, "\varphi_L"] \arrow[d, "\chi_L"'] & Q_{\tau_{\bar{w}}} \oplus \mathbb{Z} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\chi\oplus\mathrm{id}"] & Q_{\tau_{\bar{w}}} \arrow[d, "\chi"] \\ P_{\phi(L)} \arrow[r, "\varphi_{\phi(L)}"'] & Q_\tau \oplus \mathbb{Z} \arrow[r, "\mathrm{pr}_1"'] & Q_\tau \end{tikzcd}$$

Again, $\varphi_{\phi(L)}$ is necessarily local by realizability. Note that, with $\iota : Q_{\tau_{\bar{w}}} \to Q_{\tau_{\bar{w}}} \oplus \mathbb{Z}$ given by $m \mapsto (m, 0)$,

$$\iota^{-1}\big(\varphi_L\big(P_L \setminus \chi_L^{-1}(0)\big)\big) = \iota^{-1}\big(\varphi_L\big(u_L^{-1}(0) \setminus \chi_L^{-1}(0)\big)\big) = \mathrm{pr}_1 \circ \varphi_L\big(u_L^{-1}(0) \setminus \chi_L^{-1}(0)\big).$$

Thus $\mathrm{pr}_1 \circ \varphi_L(u_L^{-1}(0) \setminus \chi_L^{-1}(0)) \subseteq \chi^{-1}(Q_\tau \setminus \{0\}) = R$, as desired. In fact we obtain more from this. If instead $p \in P_L$ with $u_L(p) < 0$, then $\mathrm{pr}_1(\varphi_L(p))$ is a generator of $\bar{\mathcal{K}}_{W,\bar{w}}$, and $\chi(\mathrm{pr}_1(\varphi_L(p)))$ is a generator of the puncturing ideal for the type τ. But as the type is realizable, this ideal does not contain 0. Thus $\mathrm{pr}_1(\varphi_L(p)) \in R$, so $\bar{\mathcal{K}}_{W,\bar{w}} \subseteq R$.

For an edge $E \in V(G)$, the argument that $\phi_E(u_E^{-1}(0) \setminus \chi_E^{-1}(0)) \subseteq R$ is similar and we leave the details to the reader. Finally, for the corresponding nodal generator $\rho_E \in Q_{\tau_{\bar{w}}}$ from Definition 3.20 (ii), observe that $\chi(\rho_E)$ is the edge length function of the edge E. Again, since τ is realizable, $\chi(\rho_E) \neq 0$ and $\rho_E \in R$. ■

3.5.4 The stacks are idealized log smooth

Theorem 3.25. *Assume that X is simple. Then the forgetful morphisms*

$$\mathfrak{M}(\mathcal{X}/B, \tau) \to \mathbf{M}(G, \mathbf{g}) \times B$$

remembering only the domain curve as a family of marked curves over B, is idealized logarithmically étale for the canonical idealized structures. An analogous result holds for τ replaced by a decorated global type $\boldsymbol{\tau} = (\tau, \mathbf{A})$ of a punctured map, and for weak markings.

Proof. Step 1. Lifting to the stack of punctured curves. We first note that the morphism in question is in fact idealized. Indeed, the generators of the nodal log-ideal (Definition 3.1) on $\mathbf{M}(G, \mathbf{g}) \times B$ are pulled back to the nodal generator ρ_E of Definition 3.20 (ii) for $E \in E(G)$. The morphism then factors over the idealized logarithmically étale morphism

$$\breve{\mathfrak{M}}_B(G, \mathbf{g}) \to \mathfrak{M}_B(G, \mathbf{g}) = \mathrm{Log}_{\mathbf{M}(G,\mathbf{g})\times B}$$

from Proposition 3.3 (2). Moreover, by [53, Theorem 4.6 (iii)], the morphism

$$\mathrm{Log}_{\mathbf{M}(G,\mathbf{g})\times B} \to \mathbf{M}(G, \mathbf{g}) \times B$$

is also logarithmically étale. It thus suffices to prove the statement with $\mathbf{M}(G, \mathbf{g}) \times B$ replaced by the stack $\breve{\mathfrak{M}}_B(G, \mathbf{g})$ of $(G, \mathbf{g})$-marked punctured curves. Note that the morphism $\mathfrak{M}(\mathcal{X}/B, \tau) \to \breve{\mathfrak{M}}_B(G, \mathbf{g})$ is strict, but not in general idealized strict: the nodal log-ideal of $\breve{\mathfrak{M}}_B(G, \mathbf{g})$ from Definition 3.1 involves only the nodes of the domain curves, whereas the τ-marking ideal of $\mathfrak{M}(\mathcal{X}/B, \tau)$ from Definition 3.20, in particular part (i), also records target data.

Step 2. Lifting to the prestable map. According to the definition of idealized log étale, it is sufficient to consider a diagram of solid arrows in the category of idealized log spaces

$$\begin{tikzcd} T_0 \arrow[r, "g_0"] \arrow[d] & \mathfrak{M}(\mathcal{X}/B,\tau) \arrow[d] \\ T \arrow[r, "g"'] \arrow[ur, dashed] & \breve{\mathfrak{M}}_B(G,\mathbf{g}) \end{tikzcd} \tag{3.6}$$

where $T_0 \hookrightarrow T$ is an idealized strict closed embedding defined by a square-zero ideal. Denote by $\mathcal{K}_{T_0}$ and $\mathcal{K}_T$ the log-ideals of T_0 and T respectively. We wish to show that there is a unique dashed arrow making the above diagram commutative.

Denote by $f_{T_0} : C^\circ_{T_0} \to \mathcal{X}$ the punctured map over T_0 corresponding to the morphism g_0, and by $C^\circ_{T_0} \hookrightarrow C^\circ_T$ the extension given by g. Write also $\pi_{T_0} : C^\circ_{T_0} \to T_0$, $\pi_T : C^\circ_T \to T$. Thus the lifting problem (3.6) reduces to the following:

$$\begin{tikzcd} C^\circ_{T_0} \arrow[r, "f_{T_0}"] \arrow[d] & \mathcal{X} \arrow[d] \\ C^\circ_T \arrow[r] \arrow[ur, dashed, "f_T"] & B \end{tikzcd}$$

Since $\mathcal{X} \to B$ is log étale, by the infinitesimal lifting property of log étale morphisms in the category of idealized log schemes [52, p. 399], such f_T exists and is unique.

It remains to check that f_T is also a τ-marked curve. Item (2) of Definition 3.4 is automatic as T_0 and T have the same geometric points. As a preparation for establishing (1) and (3), we first check the vanishing of the τ-marking ideal.

Step 3. The marking ideal vanishes. Fix a geometric point $\bar{t}$ of T_0. Let $I_0^\tau \subseteq \mathcal{M}_{T_0,\bar{t}}$ be the stalk of the log-ideal $g_0^\bullet \mathcal{I}^\tau_{\mathfrak{M}(\mathcal{X}/B,\tau)}$ at $\bar{t}$, and write $\bar{I}_0^\tau \subseteq \bar{\mathcal{M}}_{T_0,\bar{t}}$ for its image. As $\bar{\mathcal{M}}_{T_0,\bar{t}} = \bar{\mathcal{M}}_{T,\bar{t}}$, we also obtain an ideal $I^\tau \subseteq \mathcal{M}_{T,\bar{t}}$ as the inverse image of $\bar{I}_0^\tau$ under the map $\mathcal{M}_{T,\bar{t}} \to \bar{\mathcal{M}}_{T,\bar{t}}$. As g_0 is idealized, necessarily $I_0^\tau \subseteq \mathcal{K}_{T_0,\bar{t}}$. Since $T_0 \to T$ is idealized strict, we thus have $I^\tau \subseteq \mathcal{K}_{T,\bar{t}}$ and hence $\alpha_T(I^\tau) = 0$. This finishes Step 3.

Now let $x \in V(G) \cup E(G) \cup L(G)$, and let $Z \subseteq C_T$ be the corresponding closed subscheme. To verify condition (1) of Definition 3.4, we need to show that $f_T|_Z$ factors through $\mathcal{X}_{\boldsymbol{\sigma}(x)}$. Let $\bar{w} = g_0(\bar{t})$, with corresponding type of tropical curve $\tau_{\bar{w}}$, equipped with a contraction morphism $\phi : \tau_{\bar{w}} \to \tau$. We now check the needed factorization for each kind of x in the following steps.

Step 4. The marking lifts at a vertex. First consider the case that x is a vertex. In this case Z is a sub-curve of $\underline{C}_T$, flat over $\underline{T}$. Let $U \subseteq Z$ be the open subset of non-special points; it is then sufficient to show that $\underline{f}_T|_U$ factors through the closed substack $\mathcal{X}_{\boldsymbol{\sigma}(x)}$. So let $\bar{u}$ be a geometric point of U lying over $\bar{t}$, contained in an irreducible component of $Z_{\bar{t}}$ indexed by a vertex $v \in V(G_{\tau_{\bar{w}}})$. Note then that $\phi(v) = x$. It is enough to show that $f_T^\sharp : \mathcal{O}_{\mathcal{X},\underline{f}_T(\bar{u})} \to \mathcal{O}_{C_T,\bar{u}}$ takes the stalk $\mathcal{J}_{\underline{f}_T(\bar{u})}$ of the ideal $\mathcal{J}$

of $\mathcal{X}_{\boldsymbol{\sigma}(x)}$ in $\mathcal{X}$ to 0. Using the notation of Definition 3.20, we have $\overline{\mathcal{M}}_{\mathcal{X},\underline{f}_T(\bar{u})} = P_v$ and a generization map $\chi_v : P_v \to P_{\phi(v)}$. If $p \in P_v$, write $s_p \in \mathcal{M}_{\mathcal{X},\underline{f}_T(\bar{u})}$ for a lift of p. We next observe that since B is a log point or is log smooth over $\operatorname{Spec}\Bbbk$ and X is simple, the ideal $\mathcal{J}_{\underline{f}_T(\bar{u})}$ is generated by the set $\{\alpha_{\mathcal{X}}(s_p) \mid p \in P_v \setminus \chi_v^{-1}(0)\}$. Indeed, this is the idealized smoothness statement of the strata in Proposition 2.48, applied on a smooth chart of $\mathcal{X}$, together with the stalkwise characterization (2.22) of the log ideal $\mathcal{K}$ in the proof of that proposition. Note that due to simplicity, the only face map is $\chi_v^t : \boldsymbol{\sigma}(x) \to P_{\mathbb{R}}^\vee$ in the present case, and hence $\overline{\mathcal{K}}_{\underline{f}_T(\bar{u})} = P_v \setminus \chi_v^{-1}(0)$.

Now by Definition 3.20 (i) and strictness of π_T at $\bar{u}$, for each $p \in P_v$ there exists $s'_p \in I^\tau \subseteq \mathcal{M}_{T,\bar{t}}$ with $f_T^\flat(s_p) = h \cdot \pi_T^\flat(s'_p)$ for some $h \in \mathcal{O}^\times_{C_T,\bar{u}}$. Thus

$$f_T^\sharp(\alpha_{\mathcal{X}}(s_p)) = \alpha_{C_T}(f_T^\flat(s_p)) = \alpha_{C_T}(h \cdot \pi_T^\flat(s'_p)) = h \cdot \pi_T^\sharp(\alpha_T(s'_p)) = 0.$$

This shows that $\underline{f}_T|_U$ factors through $\mathcal{X}_{\boldsymbol{\sigma}(x)}$.

Step 5. The marking lifts at a leg. Second consider the case that $x = L \in L(G)$. In this case Z is the image of a section of π_T, with $\underline{Z} \cong \underline{T}$. Let $\bar{u}$ be the unique geometric point of $\underline{Z}$ over $\bar{t}$. We now have a generization map $\chi_L : P_L = \overline{\mathcal{M}}_{\mathcal{X},\underline{f}_T(\bar{u})} \to P_{\phi(L)}$. Following the same notation as in the previous paragraph, it is then sufficient to show that for each $p \in P_L \setminus \chi_L^{-1}(0)$, we have $0 = \alpha_{C_T}(f_T^\flat(s_p))|_Z \in \mathcal{O}_{Z,\bar{u}}$. As in the previous paragraph, this is forced by the generators of the puncturing ideal in Definition 3.20 (i) in case $u_L(p) = 0$. If $u_L(p) > 0$, then $\alpha_{C_T}(f_T^\flat(s_p))$ contains a positive power of the defining equation of Z as a subscheme of C_T, and hence vanishes along Z. If $u_L(p) < 0$, then we achieve vanishing by Definition 2.1 (2). Thus we obtain the desired vanishing.

Step 6. The marking lifts at an edge. The third case is $x = E \in E(G)$. The argument is similar to the second case, and we leave the details to the reader. This verifies that f_T satisfies condition (1) of τ-marked curve.

Step 7. Base marking, decoration and weak marking. Finally, condition (3) holds. Indeed, the generators in Definition 3.20 (iii) guarantee the desired vanishing.

This completes the proof for markings by τ. The proof for τ replaced by $\boldsymbol{\tau}$ is identical. The weakly marked case is obtained by the same proof omitting (iii) in Definition 3.20. ∎

Remark 3.26. The proof in the weakly marked case uses simplicity only when arguing that the ideal defining $\mathcal{X}_{\boldsymbol{\sigma}(x)}$ locally is generated by expressions $\alpha_{\mathcal{X}}(s_p)$, $p \in P_v \setminus \chi_v^{-1}(0)$ for the unique generization map $\chi_v : P_v \to P_{\phi(v)}$, $P_v = \overline{\mathcal{M}}_{\underline{f}_T(\bar{u})}$. In general there is still always a log ideal $K \subseteq \overline{\mathcal{M}}_{\underline{f}_T(\bar{u})}$ with this property, as we saw in the proof of Proposition 2.48. This larger log ideal can be accounted for by modifying Definition 3.20 (i) accordingly. In the marked case, we also need to refine $Q_{\tau\tau_{\bar{w}}}$ in Definition 3.20 (iii) to the version stated in (2.32) in Section 2.6.4.

Thus we expect the statement of Theorem 3.25 to hold true in the non-simple case with these adjustments. Details are left to the interested reader.

Remark 3.27 (Local structure of stacks of prestable maps). Theorem 3.25 gives the following local description of $\mathfrak{M}(\mathcal{X}/B,\tau)$. Let $(C^\circ/W,\mathbf{p},f)$ be a basic stable punctured map over a log point $W = \operatorname{Spec}(Q \to \kappa)$ over B marked by τ. Denote by s the number of edges of the graph G given by $\tau = (G,\mathbf{g},\boldsymbol{\sigma},\bar{\mathbf{u}})$ and assume that $\underline{C}$ has $s+r$ nodes. Thus r nodes of $\underline{C}$ can be smoothed while keeping a marking by $(G,\mathbf{g})$.

The underlying object $(\underline{C}/\underline{W},\mathbf{p})$, viewed as a pre-stable curve with its basic log structure, is a point $\operatorname{Spec}\kappa \to \mathbf{M}(G,\mathbf{g}) \times B$.

By the deformation theory of nodal curves, there exists a strict smooth neighborhood of this point étale locally isomorphic to

$$\mathbb{A}^r \times U \times B. \tag{3.7}$$

Here $\mathbb{A}^r$ is endowed with the idealized log structure obtained by restricting the toric log structure of $\mathbb{A}^{s+r}$ to an intersection of s coordinate hyperplanes, and corresponds to deforming the r smoothable nodes; U is smooth with trivial log structure corresponding to equisingular deformations of $\underline{C}$; and the étale local isomorphism is a product of an étale local isomorphism of $\mathbb{A}^r \times U$ with an open substack of $\mathbf{M}(G,\mathbf{g})$ and id_B.

Note that the image of $(C^\circ/W,\mathbf{p})$ in $\mathbf{M}(G,\mathbf{g})$ is defined by the underlying marked nodal curve $(\underline{C}/\underline{W},\mathbf{p})$ endowed with its basic log structure of marked nodal curves.

Consider the point $W \to \mathfrak{M}(\mathcal{X}/B,\tau)$ corresponding to the object $(C^\circ/W,\mathbf{p},f)$. Pulling back the neighborhood (3.7) along $\mathfrak{M}(\mathcal{X}/B,\tau) \to \mathbf{M}(G,\mathbf{g}) \times B$ gives a smooth neighborhood V of $W \to \mathfrak{M}(X/B,\tau)$ equipped with a morphism $\phi : V \to \mathbb{A}^r \times U \times B$. We may now apply Proposition B.4 to describe this neighborhood explicitly étale locally, as follows. We use the notation $\mathcal{A}_P$ and $\mathcal{A}_{P,I}$ defined in (B.1), for P a monoid and $I \subseteq P$ a monoid ideal.

The log-ideal $\mathcal{I}^\tau_{\mathfrak{M}(\mathcal{X}/B,\tau)} \cup \mathcal{K}_{\mathfrak{M}(\mathcal{X}/B,\tau)}$ induces a monoid ideal $I \subseteq Q$, as constructed in Definition 3.20, with associated idealized Artin fan $\mathcal{A}_{Q,I}$. Let Q_B be the stalk of $\overline{\mathcal{M}}_B$ at the image point of the composition $W \to \mathfrak{M}(\mathcal{X}/B,\tau) \to B$. We may first replace B with an étale neighborhood of this image point and so assume given a map $Q_B \to \overline{\mathcal{M}}_B$, or equivalently a strict morphism $B \to \mathcal{A}_{Q_B}$. Then by Proposition B.4, possibly after passing to an étale neighborhood of V, there is a diagram

$$\begin{tikzcd}
V \arrow[dr, "\theta"] \arrow[drr, bend left, "\psi"] \arrow[ddr, bend right, "\phi"'] & & \\
& V' \arrow[r] \arrow[d] & \mathcal{A}_{Q,I} \arrow[d, "\iota"] \\
& \mathbb{A}^r \times U \times B \arrow[r] & \mathcal{A}_{\mathbb{N}^{s+r},J} \times \mathcal{A}_{Q_B}
\end{tikzcd} \tag{3.8}$$

with the square Cartesian in the log, fine and fs categories, ψ and both horizontal arrows strict and idealized strict, and θ étale and strict. Further, ι is induced by the map on stalks of ghost sheaves $\mathbb{N}^{s+r} \oplus Q_B \to Q$ given by the morphism $\mathfrak{M}(\mathcal{X}/B,\tau) \to \mathbf{M}(G,\mathbf{g}) \times B$. Finally, $J \subseteq \mathbb{N}^{s+r}$ is the ideal generated by the first s generators of $\mathbb{N}^{s+r}$, so that the morphism $\mathbb{A}^r \to \mathcal{A}_{\mathbb{N}^{s+r},J}$ is strict and idealized strict.

In conclusion, we see that V is étale locally isomorphic to

$$V' \cong U \times \big((\mathbb{A}^r \times B) \times_{\mathcal{A}_{\mathbb{N}^{s+r},J} \times \mathcal{A}_{Q_B}} \mathcal{A}_{Q,I}\big). \tag{3.9}$$

Thus the local models of $\mathfrak{M}(\mathcal{X}/B,\tau)$ and their idealized structures are explicitly described from the types of tropical punctured maps admitting a contraction morphism to τ.

3.5.5 Dimension formulas

Example 2.58 exhibits a case where $\mathfrak{M}(\mathcal{X}/B,\tau)$ is not pure-dimensional. Before revisiting this example, we give a useful condition which implies $\mathfrak{M}(\mathcal{X}/B,\tau)$ is pure-dimensional, of the expected dimension. The statement involves a refinement of the notion of realizability of global types from Definition 2.44 (2) relative to B.

Definition 3.28. Let τ be a global type of punctured map to X. We say that τ *is realizable over* B if there exists a geometric point $\bar{w}$ of $\mathfrak{M}(\mathcal{X}/B,\tau)$ such that the corresponding punctured map has global type τ.

Proposition 3.29. *Suppose the Artin fan $\mathcal{A}_X$ of X is Zariski (Definition A.7). Then a global type $\tau = (G,\mathbf{g},\boldsymbol{\sigma},\bar{\mathbf{u}})$ is realizable over B if and only if the following conditions hold:*

(1) *τ is realizable, hence there is a universal family $h : \Gamma = \Gamma(G,\ell) \to \Sigma(X)$ of type τ, parametrized by $\omega_\tau := Q^\vee_{\tau,\mathbb{R}}$, where Q_τ is the basic monoid for tropical maps of type τ.*

(2) *The universal family of tropical maps of type τ is defined over $\Sigma(B)$, i.e., there is a map $\omega_\tau \to \Sigma(B)$ making the diagram*

$$\begin{tikzcd} \Gamma \arrow[r, "h"] \arrow[d] & \Sigma(X) \arrow[d] \\ \omega_\tau \arrow[r] & \Sigma(B) \end{tikzcd}$$

commute.

(3) *Let $\sigma \in \Sigma(B)$ be the minimal cone containing the image of ω_τ. Then there exists a point $b \in B_\sigma$ such that $\sigma = \mathrm{Hom}(\overline{\mathcal{M}}_{B,b},\mathbb{R}_{\geq 0})$.*

Proof. That conditions (1)–(3) are necessary is clear. Conversely, suppose (1)–(3) hold. Let $\underline{C}/\operatorname{Spec}\Bbbk$ be a pre-stable curve with dual intersection graph G. Pull-back the basic log structure on $\underline{C}/\operatorname{Spec}\Bbbk$ by the canonical morphism $\mathbb{N}^{|E(G)|}\to Q_\tau$ from the nodal parameters to the basic monoid for τ to define a log smooth curve C/W over the log point $W=\operatorname{Spec}(Q_\tau\to\Bbbk)$. We may then construct a morphism $W\to B$ with image a point $b\in B_\sigma$ given by item (3) in the statement of the proposition. Note we may take b to be a closed point, so that $b=\operatorname{Spec}\Bbbk$. At the logarithmic level, this morphism can be taken so its induced tropicalization is the given map $\omega_\tau\to\sigma$.

Next apply the correspondence [3, Proposition 2.10] (it is here we need the hypothesis that $\mathcal{A}_X$ is Zariski) between morphisms from a logarithmic space to an Artin fan and their tropicalizations to first construct a saturated puncturing $\widetilde{C}^\circ\to C$ and then a logarithmic map $\widetilde{C}^\circ\to\mathcal{A}_X$ with tropicalization of type τ. Prestabilizing then leads to a basic pre-stable punctured map $(C^\circ/W,\mathbf{p},f)$ to $\mathcal{A}_X$ of type τ. Note that C° is not necessarily saturated. On the other hand, we have a composed morphism $C^\circ\to W\to B$, with $W\to B$ constructed in the previous paragraph. The compositions $C^\circ\to\mathcal{A}_X\to\mathcal{A}_B$ and $C^\circ\to B\to\mathcal{A}_B$ agree by item (2) of the proposition, and hence we obtain a punctured map $C^\circ\to\mathcal{X}=\mathcal{A}_X\times_{\mathcal{A}_B}B$ defined over B with the necessary properties. ∎

Proposition 3.30. *Let $\tau=(G,\mathbf{g},\boldsymbol{\sigma},\bar{\mathbf{u}})$ be a global type (Definition 2.44) and assume X is simple and B is either log smooth over* $\operatorname{Spec}\Bbbk$ *or $B=\operatorname{Spec}\Bbbk^\dagger$, the standard log point. Assume further τ is realizable over B. Then $\mathfrak{M}(\mathcal{X}/B,\tau)$ is non-empty, reduced and pure-dimensional. If B is log smooth over* $\operatorname{Spec}\Bbbk$*, then*

$$\dim\mathfrak{M}(\mathcal{X}/B,\tau)=3|\mathbf{g}|-3+|L(G)|-\operatorname{rk}Q_\tau^{\mathrm{gp}}+\dim B,$$

while if $B=\operatorname{Spec}\Bbbk^\dagger$, then

$$\dim\mathfrak{M}(\mathcal{X}/B,\tau)=3|\mathbf{g}|-3+|L(G)|-\operatorname{rk}Q_\tau^{\mathrm{gp}}+1.$$

Proof. By Proposition 3.24, as τ is a realizable type, the τ-marked ideal at a point $\bar{w}'$ of $\mathfrak{M}(\mathcal{X}/B,\tau)$ takes the form $\chi^{-1}_{\tau\tau_{\bar{w}'}}(Q_\tau\setminus\{0\})$. Thus, in the description of a smooth neighborhood V of $\bar{w}'$ as given in (3.8), $\mathcal{A}_{Q,I}$ is reduced, and if B is log smooth over $\operatorname{Spec}\Bbbk$, the bottom horizontal arrow is smooth, and hence V' is also reduced as the square is Cartesian. This shows that $\mathfrak{M}(\mathcal{X}/B,\tau)$ is reduced in this case.

If on the other hand $B=\operatorname{Spec}\Bbbk^\dagger$, we may take $Q_B=\mathbb{N}$ in (3.8). Since $\mathfrak{M}(\mathcal{X}/B,\tau)$ is defined over B, the induced morphism of stalks of ghost sheaves $\mathbb{N}\to Q_\tau$ is local and hence $\mathbb{N}\setminus\{0\}$ maps into $Q_\tau\setminus\{0\}$, and thus more generally $\mathbb{N}\to Q_{\tau_{\bar{w}'}}$ maps $\mathbb{N}\setminus\{0\}$ into $\chi^{-1}_{\tau\tau_{\bar{w}'}}(Q_\tau\setminus\{0\})$ by compatibility of these maps with generization. Hence we may replace $\mathcal{A}_{Q_B}$ with the closed substack $\mathcal{A}_{\mathbb{N},\mathbb{N}\setminus\{0\}}$ in (3.8) without affecting this diagram in any other way. In particular, the bottom horizontal arrow is now still smooth. So V' is again reduced.

Let $\bar{w}$ be a point as in Definition 3.28. We may now calculate dimensions by looking at the description of (3.8) for a neighborhood of $\bar{w}$ in $\mathfrak{M}(\mathcal{X}/B,\tau)$. Since the corresponding curve $C^\circ/\bar{w}$ now has no smoothable nodes, we may take $r=0$ and $s=|E(G)|$ in (3.8). Further, since $I=Q_\tau\setminus\{0\}$, necessarily $\dim \mathcal{A}_{Q_\tau,I}=-\operatorname{rank} Q_\tau^{\mathrm{gp}}$. Thus we may calculate, with the cases being for B log smooth and $B=\operatorname{Spec}\Bbbk^\dagger$ respectively,

$$\begin{aligned}\dim\mathfrak{M}(\mathcal{X}/B)-\dim\mathbf{M}(G,\mathbf{g})\times B &= \dim V'-\dim U\times B\\ &=\begin{cases}\dim\mathcal{A}_{Q_\tau,I}-\dim\mathcal{A}_{\mathbb{N}^s,J}\times\mathcal{A}_{Q_B}\\ \dim\mathcal{A}_{Q_\tau,I}-\dim\mathcal{A}_{\mathbb{N}^s,J}\times\mathcal{A}_{\mathbb{N},\mathbb{N}\setminus\{0\}}\end{cases}\\ &=\begin{cases}-\operatorname{rank} Q_\tau^{\mathrm{gp}}-(-s)\\ -\operatorname{rank} Q_\tau^{\mathrm{gp}}-(-s-1)\end{cases}\end{aligned}$$

As $\dim\mathbf{M}(G,\mathbf{g})=3|\mathbf{g}|-3+|L(G)|-|E(G)|$, and $s=|E(G)|$, we then obtain the desired dimension formulas in the two cases. ∎

Remark 3.31 (*Stratified structure of* $\mathfrak{M}(\mathcal{X}/B,\tau)$). If $\tau'\to\tau$ is a morphism of global types (Definition 2.44), a marking by τ' induces a marking by τ by composition of the marking morphism with $\tau'\to\tau$. The same arguments as for ordinary logarithmic maps [3, Proposition 2.34] shows that the corresponding morphism of stacks

$$j_{\tau\tau'}:\mathfrak{M}(\mathcal{X}/B,\tau')\to\mathfrak{M}(\mathcal{X}/B,\tau)$$

is finite and unramified. If τ' is realizable over B, then Proposition 3.30, under the assumptions on B stated there, further shows that $\operatorname{im}(j_{\tau\tau'})$ defines a pure-dimensional substack of $\mathfrak{M}(\mathcal{X}/B,\tau)$. Conversely, if there is no τ'' which is realizable over B mapping to τ' then $\mathfrak{M}(\mathcal{X},\tau')=\emptyset$. Thus the images of $j_{\tau\tau'}$ for morphisms of global types $\tau'\to\tau$ with τ' realizable over B define a stratification of $\mathfrak{M}(\mathcal{X}/B,\tau)$ into pure-dimensional strata.

In particular, the closure of a maximal stratum is the image of $\mathfrak{M}(\mathcal{X}/B,\tau')$ for τ' a *minimal* global type realizable over B dominating τ. Minimality here means that the morphism $\tau'\to\tau$ does not factor over any other global type realizable over B.

Note, however, that $\mathfrak{M}(\mathcal{X}/B,\tau')$ is not in general irreducible even for realizable τ', due to saturation phenomena already present in ordinary stable logarithmic maps. In the logarithmic enhancement question for transverse stable logarithmic maps of [3, Theorem 4.13], this reducibility is reflected in various choices of roots of unity.

Example 3.32 (Example 2.58 revisited, see Figure 3.1). Let τ be the global type with G having just one vertex of genus 0, no edges, and four legs, all image cones equal to $0\in\Sigma(X)=\{0,\mathbb{R}_{\geq 0}\}$ and global contact orders $-1,-1,2,2$. This global type is not realizable because there can be no positive length legs for the two punctures, but there are several minimal realizable global types marked by τ. Here are two of

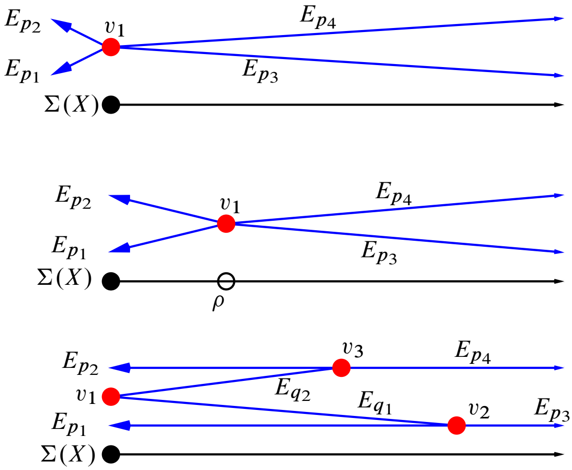

Figure 3.1. The top combinatorial map is not tropically realizable since E_{p_1}, E_{p_2} have nowhere to stretch. The first realizable type has no nodes, with $\ell_1 = \ell_2 = 0$, but with v_1 positioned at $\rho > 0$. The second has v_1 positioned at $\rho = 0$ but then $\ell_1, \ell_2 > 0$.

them. The first, τ_1, has the same $(G, \mathbf{g})$ as τ, but all image cones are $\mathbb{R}_{\geq 0}$. In the notation of Example 2.58, the tropical punctured map realizing this type has $\rho > 0$ and $\ell_1 = \ell_2 = 0$. The other minimal realizable type, τ_2, has G with three vertices v_1, v_2, v_3 with $\boldsymbol{\sigma}(v_1) = \{0\}$, $\boldsymbol{\sigma}(v_2) = \boldsymbol{\sigma}(v_3) = \mathbb{R}_{\geq 0}$ and two edges, connecting v_1 to v_2 and v_3, respectively, and one positive and one negative leg attached to each of v_2 and v_3. This global type is realizable by tropical punctured maps with $\rho = 0$ and $\ell_1, \ell_2 > 0$. Note that by Proposition 3.30, $\dim \mathfrak{M}(\mathcal{X}/B, \tau_1) = 0$ but $\dim \mathfrak{M}(\mathcal{X}/B, \tau_2) = -1$, showing non-pure-dimensionality of $\mathfrak{M}(\mathcal{X}/B, \tau)$.

3.5.6 Comparing marked and weakly marked stacks

We end this section by showing that the marked and weakly marked moduli spaces have the same reduction.

Proposition 3.33. *Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ be a global type of punctured maps and assume X is simple. Then the canonical morphism*

$$\mathfrak{M}(\mathcal{X}/B, \tau) \to \mathfrak{M}'(\mathcal{X}/B, \tau)$$

is a closed embedding defined by a nilpotent ideal. Analogous statements hold for moduli spaces of punctured maps to X/B and for decorated global types.

Proof. By the idealized description in Theorem 3.25 of the moduli spaces in question, the statement amounts to showing that the τ-marked ideal from Definition 3.20 is contained in the radical of the weakly τ-marked ideal defined in Remark 3.22.

Let $(C/W, \mathbf{p}, f)$ be a punctured map weakly marked by τ and $\bar{w}$ of $\underline{W}$ a geometric point. We adopt the notation from Definition 3.20 and in particular write $\phi : \tau_{\bar{w}} \to \tau$ for the contraction morphism given by the marking and

$$\chi_{\tau\tau_{\bar{w}}} : Q_{\tau_{\bar{w}}} \to Q_{\tau\tau_{\bar{w}}}$$

for the localization morphism of basic monoids. We have to show that for each $q \in Q_{\tau_{\bar{w}}}$ with $\chi_{\tau\tau_{\bar{w}}}(q) \neq 0$ a multiple kq lies in the monoid ideal generated by the elements listed in Definition 3.20 (i) and (ii). The description of the dual basic monoids in Proposition 2.32 provides the following commutative diagram with horizontal arrows surjective up to saturation, with as usual P_v denoting the monoid dual to the cone $\boldsymbol{\sigma}(v)$:

$$\begin{tikzcd}
\prod_{v\in V(G_{\bar{w}})} P_v \times \prod_{E\in E(G_{\bar{w}})} \mathbb{N} \arrow[r] \arrow[d] & Q_{\tau_{\bar{w}}} \arrow[d, "\chi_{\tau\tau_{\bar{w}}}"] \\
\prod_{v\in V(G)} P_v \times \prod_{E\in E(G)} \mathbb{N} \arrow[r] & Q_{\tau\tau_{\bar{w}}}
\end{tikzcd}$$

The left vertical homomorphism is as follows:

$$\Big((p_v)_{v\in V(G_{\bar{w}})}, (\ell_E)_{E\in E(G_{\bar{w}})}\Big) \mapsto \Big(\Big(\sum_{\phi(v')=v} \chi_{v'}(p_{v'})\Big)_v, (\ell_{\phi^{-1}(E)})_E\Big).$$

As the top arrow is surjective up to saturation, there exists $k \geq 0$ such that $kq \in Q_{\tau_{\bar{w}}}$ lifts to an element (p_v, ℓ_E) in the left upper corner. Since $\chi_{\tau\tau_{\bar{w}}}(q) \neq 0$, the image of this lift in the lower left corner is non-zero. We conclude that there exists (1) $v \in V(G_{\bar{w}})$ with $\chi_v(p_v) \neq 0$ or (2) $E \in E(G_{\bar{w}}) \setminus E_\phi$ with $\ell_E \neq 0$. In the first case kq lies in the ideal generated by $\varphi_v(P_v \setminus \chi_v^{-1}(0))$, part of Definition 3.20 (i), while in the second case kq lies in the ideal generated by the nodal generator q_E from Definition 3.20 (ii). ∎

Chapter 4

The perfect obstruction theory

Throughout this chapter, we fix a log smooth morphism $X \to B$ of fs logarithmic schemes fulfilling the assumptions stated at the beginning of Chapter 3 and $n \in \mathbb{N}$. Crucial for the following discussion is the factorization of $X \to B$ over the relative Artin fan $\mathcal{X} \to B$.

Denote by $\mathcal{M}_n(X/B)$ (resp. $\mathfrak{M}_n(\mathcal{X}/B)$) the stack of marked or weakly marked punctured maps to $X \to B$ (resp. $\mathcal{X} \to B$), with n the number of punctured or nodal sections, fixing and suppressing all other decorations in the notation. In Sections 4.1 and 4.2, we construct two perfect relative obstruction theories, in the sense of [13, Definition 4.4], one for $\mathcal{M}_n(X/B) \to \mathfrak{M}_n(\mathcal{X}/B)$ and one for a related morphism $\mathcal{M}_n(X/B) \to \mathfrak{M}_n^{\mathrm{ev}}(\mathcal{X}/B)$; the latter space incorporates data of maps to X at a set of special points on the domain curve, see (4.13). Working over $\mathfrak{M}_n^{\mathrm{ev}}(\mathcal{X}/B)$ is crucial for understanding gluing at a virtual level in Section 5.3.

We will avail ourselves of the dualizing complex of various Gorenstein morphisms π. To avoid adjusting for shifts of dimension in the formulas, we denote by ω_π the relative dualizing complex, usually denoted $\omega_\pi^\bullet$, of a relatively Gorenstein morphism π, that is, the complex with the invertible relative dualizing sheaf defined in [36, Example III.9.7] (see also [20, p. 157]) shifted to the left by the relative dimension.

4.1 Obstruction theories for logarithmic maps from pairs

All cases of interest fit into the following general setup. For this subsection we do not enforce the assumptions on B from the Conventions, Section 1.6.

4.1.1 Source family

Let S be a log stack over B and assume we are given a proper and representable morphism of fine log stacks

$$Y \to S,$$

with underlying map of ordinary stacks $\underline{Y} \to \underline{S}$ flat and relatively Gorenstein. The fibers of this morphism serve as domains for a space of logarithmic maps.

In the application, Y is either the universal curve over $S = \mathfrak{M}_n(\mathcal{X}/B)$ or over $S = \mathfrak{M}_n^{\mathrm{ev}}(\mathcal{X}/B)$, or a union of sections of the universal curve with induced log structure.

4.1.2 Target family

As a target, we take a composition of morphisms of fine log stacks

$$V \to W \to B,$$

with $V \to W$ log smooth. In applications this will be the sequence[1] $X \to \mathcal{X} \to B$. We assume further given a B-morphism $Y \to W$ defining a commutative square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & W \arrow[d] \\ S \arrow[r] & B; \end{tikzcd}$$

In our applications this is the universal family of maps to the Artin fan, either prestable maps of curves or the corresponding maps of the union of sections, as the case may be.

4.1.3 Moduli of lifted maps

Let M be an open algebraic substack of the following algebraic stack over S. An object over an affine S-scheme T, considered as a log scheme by pulling back the log structure from S, consists of a commutative diagram

$$\begin{tikzcd} Y_T \arrow[rr] \arrow[d] \arrow[dr] & & V \arrow[d] \\ T \arrow[dr] & Y \arrow[r] \arrow[d] & W \arrow[d] \\ & S \arrow[r] & B \end{tikzcd} \tag{4.1}$$

where the square formed by Y_T, T, S and Y is cartesian. Thus we are interested in lifting the map $Y \to W$ to V fiberwise relative to S. We endow M with the log structure making the morphism $M \to S$ strict. The pullback of Y to M defines the universal domain $\pi : Y_M \to M$. We have the following 2-commutative diagram of stacks

$$\begin{tikzcd} Y_M \arrow[rr, "f"] \arrow[d, "\pi"'] \arrow[dr] & & V \arrow[d] \\ M \arrow[dr] & Y \arrow[r] \arrow[d] & W \arrow[d] \\ & S \arrow[r] & B \end{tikzcd} \tag{4.2}$$

[1]In this case, $V \to W$ is strict and we could indeed work with ordinary cotangent complexes throughout, but for possible other applications we do not make this assumption.

In the main application, with $Y \to S$ the family of prestable curves, M is an open substack of the stack of punctured maps of interest; thus our deformation theory fixes both the domain of the punctured map to X and the map to the relative Artin fan $\mathcal{X}$. In the secondary application, with $Y \to S$ the family of sections with logarithmic structures, the stack M parametrizes liftings of the sections from $\mathcal{X}$ to X.

4.1.4 An obstruction theory

Functoriality of log cotangent complexes [54, Property 1.1 (iv)] yields the morphism

$$f^*\Omega_{V/W} = Lf^*\mathbb{L}_{V/W} \to \mathbb{L}_{Y_M/Y} = \pi^*\mathbb{L}_{M/S}. \tag{4.3}$$

The equality on the left holds by [54, Property 1.1 (iii)] since $V \to W$ is log smooth, while the equality on the right follows since

$$\mathbb{L}_{M/S} = \mathbb{L}_{\underline{M}/\underline{S}} \quad \text{and} \quad \mathbb{L}_{Y_M/Y} = \mathbb{L}_{\underline{Y}_M/\underline{Y}}$$

by strictness of $M \to S$ [54, Property 1.1 (ii)] and then using compatibility of the ordinary cotangent complexes with flat pullback by $\underline{\pi}$.

Since $\underline{Y} \to \underline{S}$ is relatively Gorenstein by assumption, so is $\underline{Y}_M \to \underline{M}$ and we have a natural isomorphism of exact functors $\pi^! = \pi^* \otimes \omega_\pi$. Thus (4.3) is equivalent to a morphism $f^*\Omega_{V/W} \otimes \omega_\pi \to \pi^!\mathbb{L}_{M/S}$, which by adjunction is equivalent to a morphism

$$\Phi : \mathbb{E} \to \mathbb{L}_{M/S} \tag{4.4}$$

with

$$\mathbb{E} = R\pi_*(f^*\Omega_{V/W} \otimes \omega_\pi).$$

4.1.5 Functoriality

We will show in Proposition 4.2 that Φ is a perfect obstruction theory for M over S. A most transparent proof that Φ is a perfect obstruction theory for M over S relies on the fact that the construction of Φ is functorial. For lack of reference we provide a proof for this well-known property in the following lemma. If $T \to M$ is any map, denote by

$$\Phi_T : \mathbb{E}_T \to \mathbb{L}_{T/S}$$

the morphism in (4.4) constructed from (4.1) instead of (4.2).

Lemma 4.1. *The construction of Φ in (4.4) is functorial in the following sense: Let $\underline{T} \to \underline{M}$ be a morphism of stacks. Denoting $T \to M$ the associated strict morphism* *of log stacks, we obtain the commutative diagram*

$$\begin{tikzcd}
Y_T \arrow[r, "\tilde h"'] \arrow[d, "\pi_T"'] \arrow[rrr, bend left, "f_T"] & Y_M \arrow[rr, "f"] \arrow[d, "\pi"'] \arrow[dr] & & V \arrow[d] \\
T \arrow[r, "h"] & M \arrow[dr] & Y \arrow[r] \arrow[d] & W \arrow[d] \\
& & S \arrow[r] & B
\end{tikzcd}$$

with the two squares of domains (i.e., the left-most square and the parallelogram) cartesian. Then we have a commutative square

$$\begin{tikzcd}
Lh^*\mathbb{E} \arrow[r, "Lh^*\Phi"] \arrow[d, "\beta"'] & Lh^*\mathbb{L}_{M/S} \arrow[d] \\
\mathbb{E}_T \arrow[r, "\Phi_T"] & \mathbb{L}_{T/S},
\end{tikzcd}$$

with left-hand vertical arrow a natural isomorphism and the right-hand vertical arrow defined by functoriality of cotangent complexes.

Proof. Naturality of the base change map [67, Remark 07A7] applied to $f^*\Omega_{V/W} \otimes \omega_\pi \to \mathbb{L}_{Y_M/Y} \otimes \omega_\pi$ together with $f \circ \tilde h = f_T$ and $\tilde h^*\omega_\pi = \omega_{\pi_T}$ [20, Theorem 3.6.1], leads to the commutative square

$$\begin{tikzcd}
Lh^*\mathbb{E} = Lh^*R\pi_*(f^*\Omega_{V/W} \otimes \omega_\pi) \arrow[r] \arrow[d, "\beta"'] & Lh^*R\pi_*(\mathbb{L}_{Y_M/Y} \otimes \omega_\pi) \arrow[d, "b"] \\
\mathbb{E}_T = R\pi_{T*}(f_T^*\Omega_{V/W} \otimes \omega_{\pi_T}) \arrow[r] & R\pi_{T*}\big(L\tilde h^*\mathbb{L}_{Y_M/Y} \otimes \omega_{\pi_T}\big).
\end{tikzcd} \tag{4.5}$$

Now $\mathbb{L}_{Y_M/Y} \simeq \pi^*\mathbb{L}_{M/S}$, as remarked after (4.3), and hence the adjunction counit $R\pi_*\pi^! \to \mathrm{id}$ applied in the construction of Φ in (4.4) is given by the projection formula followed by the trace morphism,

$$R\pi_*(\pi^*\mathbb{L}_{M/S} \otimes \omega_\pi) \xrightarrow{\simeq} \mathbb{L}_{M/S} \otimes R\pi_*(\omega_\pi) \xrightarrow{\mathrm{Tr}_{\omega_\pi}} \mathbb{L}_{M/S}.$$

Thus the upper horizontal map of (4.5) composed with Lh^* of this adjunction counit isomorphism yields $Lh^*\Phi$.

Similarly, extending the lower horizontal arrow by the map induced by functoriality of cotangent complexes,

$$L\tilde h^*\mathbb{L}_{Y_M/Y} \to \mathbb{L}_{Y_T/Y} = \pi_T^*\mathbb{L}_{T/S},$$

composed with the adjunction counit morphism

$$R\pi_{T*}(\pi_T^*\mathbb{L}_{T/S} \otimes \omega_{\pi_T}) \to \mathbb{L}_{T/S}$$

for π_T retrieves the definition of Φ_T.

Moreover, by compatibility of both the projection formula [67, Lemma 0B6B] and the trace morphism [67, Lemma 0E6C] with base change, the following diagram continuing (4.5) on the right is commutative:

$$\begin{tikzcd}
Lh^*R\pi_*(\pi^*\mathbb{L}_{M/S} \otimes \omega_\pi) \arrow[r, "\simeq"] \arrow[d, "b"'] & Lh^*\mathbb{L}_{M/S} \otimes Lh^*R\pi_*\omega_\pi \arrow[d] \arrow[dr, "\mathrm{tr}"] & \\
R\pi_{T*}(\pi_T^*Lh^*\mathbb{L}_{M/S} \otimes \omega_{\pi_T}) \arrow[r, "\simeq"] \arrow[d] & Lh^*\mathbb{L}_{M/S} \otimes R\pi_{T*}\omega_{\pi_T} \arrow[r] \arrow[d] & Lh^*\mathbb{L}_{M/S} \arrow[d] \\
R\pi_{T*}(\pi_T^*\mathbb{L}_{T/S} \otimes \omega_{\pi_T}) \arrow[r, "\simeq"] & \mathbb{L}_{T/S} \otimes R\pi_{T*}\omega_{\pi_T} \arrow[r] & \mathbb{L}_{T/S}.
\end{tikzcd}$$

The three left horizontal isomorphisms are defined by projection formulas, the diagonal and the two horizontal morphisms on the right induced by trace homomorphisms, the two upper vertical arrows defined by base change, and the three lower vertical arrows defined by functoriality of cotangent complexes. For the identification of the upper left vertical arrow with the right vertical arrow labeled b in (4.5) note that

$$L\tilde{h}^*\mathbb{L}_{Y_M/Y} \simeq L\tilde{h}^*\pi^*\mathbb{L}_{M/S} \simeq \pi_T^*Lh^*\mathbb{L}_{M/S}.$$

This establishes the claimed commutative diagram.

It remains to show that β is a natural isomorphism. This follows from the general base change statement [67, Lemma 0A1K] applied to $\pi : Y_M \to M$, with $f^*\Omega_{V/W}$ for the object in $D_{\mathrm{QCoh}}(\mathcal{O}_{Y_M})$ and with ω_π as complex of π-flat quasi-coherent sheaves. ∎

Proposition 4.2 (Φ is a perfect obstruction theory). *The morphism $\Phi : \mathbb{E} \to \mathbb{L}_{M/S}$ constructed in* (4.4) *is an obstruction theory for $M \to S$ in the sense of* [13, Definition 4.4].

Proof. We check the obstruction-theoretic criterion [13, Theorem 4.5.3], applied in the setting relative to S, similarly to ordinary logarithmic maps carried out in [30, Proposition 5.1].

Assume given a morphism $h : T \to M$, a square-zero extension $T \to \overline{T}$ with ideal sheaf $\mathcal{J}$ and a morphism $\overline{T} \to S$, with log structures turning all three morphisms strict.

This situation leads to the following commutative diagram:

$$
\begin{tikzcd}
 & Y_T \arrow[rr, "\tilde{h}"] \arrow[dl] \arrow[dd, "\pi_T"] \arrow[rrrr, bend left, "f_T"] & & Y_M \arrow[rr, "f"] \arrow[dl] \arrow[dd, "\pi"] & & V \arrow[dl] \\
Y_{\overline{T}} \arrow[rr] \arrow[dd] & & Y \arrow[rr] \arrow[dd] & & W \arrow[dd] & \\
 & T \arrow[rr, "h"] \arrow[dl] & & M \arrow[dl] & & \\
\overline{T} \arrow[rr] & & S \arrow[rr] & & B. &
\end{tikzcd}
$$

All sides of the cube on the left are cartesian, but not in general the bottom and top faces.

The obstruction class $\omega(h) \in \mathrm{Ext}^1(Lh^*\mathbb{L}_{\underline{M}/\underline{S}}, \mathcal{J})$ for extending h to an S-morphism $\overline{T} \to M$ is the composition

$$Lh^*\mathbb{L}_{\underline{M}/\underline{S}} \to \mathbb{L}_{\underline{T}/\underline{\overline{T}}} \to \tau_{\geq -1}\mathbb{L}_{\underline{T}/\underline{\overline{T}}} = \mathcal{J}[1],$$

the first arrow defined by functoriality of cotangent complexes, see [38, Proposition 2.2.4] with $X_0 = T$, $X = \overline{T}$, $Y_0 = Y = M$ and $Z_0 = Z = S$. Because $T \to \overline{T}$ and $M \to S$ are strict we can replace the ordinary cotangent complex with the log cotangent complex in this construction [54, Property 1.1 (ii)].

Now $\Phi^*\omega(h)$ is the composition of this morphism with $Lh^*\Phi : Lh^*\mathbb{E} \to Lh^*\mathbb{L}_{M/S}$. By functoriality of our obstruction theory (Lemma 4.1), this composition also has the factorization

$$\mathbb{E}_T = R\pi_{T*}(f_T^*\Omega_{V/W} \otimes \omega_{\pi_T}) \xrightarrow{\Phi_T} \mathbb{L}_{T/S} \to \tau_{\geq -1}\mathbb{L}_{T/\overline{T}} = \mathcal{J}[1],$$

which by adjunction is equivalent to the composition

$$f_T^*\Omega_{V/W} \otimes \omega_{\pi_T} \to \mathbb{L}_{Y_T/Y} \otimes \omega_{\pi_T} \to \tau_{\geq -1}\pi_T^!\mathbb{L}_{T/\overline{T}} = \pi_T^*\mathcal{J}[1] \otimes \omega_{\pi_T}.$$

Up to tensoring with ω_{π_T} this is the obstruction class for extending $f_T : Y_T \to V$ to $Y_{\overline{T}}$, as a morphism over W. By our assumption on the objects of M, this extension exists if and only if $T \to M$ extends to $\overline{T}$. This shows the part of the criterion concerning the obstruction.

A similar argument shows that once $\omega(h) = 0$, the space of extensions form a torsor under $\mathrm{Ext}^0(Lh^*\mathbb{L}_{\underline{M}/\underline{S}}, \mathcal{J})$, showing the second part of the criterion. ■

4.1.6 The dualizing complex of the embedding of markings

After this recapitulation of obstruction theories for logarithmic maps with proper and relatively Gorenstein domains, we are now in position to bring in point conditions.

Abstractly we consider a composition of proper, representable morphisms of fine log stacks

$$Z \xrightarrow{\iota} Y \to S, \tag{4.6}$$

with maps of algebraic stacks underlying $Z \to S$ and $Y \to S$ flat and relatively Gorenstein as before. Note that while ι may not be flat and hence cannot be considered relatively Gorenstein following the usual convention, one can still define a relative dualizing sheaf

$$\omega_\iota = \omega_{Z/S} \otimes \iota^* \omega_{Y/S}^\vee \tag{4.7}$$

fulfilling relative duality, hence defining a right-adjoint functor $\iota^!$ to $R\iota_*$. This works as in the case of smooth morphisms discussed e.g. in [37, Section 3.4].

4.1.7 Obstruction for markings

We now have another algebraic stack N, an open substack of the stack over S with objects given by diagrams as in (4.1) with Y replaced by Z. We assume the open substack N is chosen large enough so that composition with $\iota : Z \to Y$ defines a morphism of stacks

$$\varepsilon : M \to N. \tag{4.8}$$

As in (4.4) we now obtain two obstruction theories, one for $M \to S$, the other for $N \to S$,

$$\Phi : \mathbb{E} \to \mathbb{L}_{M/S}, \quad \Psi : \mathbb{F} \to \mathbb{L}_{N/S}. \tag{4.9}$$

In our application, $Y \to S$ is some universal curve and $Z \to Y$ a strict closed embedding with morphism to S scheme-theoretically étale. In this case, Ψ is simply the obstruction theory for a number of points in V/W, i.e., a trivial obstruction theory in the sense that there are no obstructions. In particular, étale locally $\mathbb{F}$ can be taken as the direct sum of the pullback of $\Omega_{V/W}$ by scheme-theoretic maps from $\underline{N}$ to $\underline{V}$.

Proposition 4.3 (Compatibility of obstruction theories). *The two obstruction theories Φ and Ψ in (4.9) fit into a commutative square*

$$\begin{tikzcd} L\varepsilon^*\mathbb{F} \arrow[r, "L\varepsilon^*\Psi"] \arrow[d] & L\varepsilon^*\mathbb{L}_{N/S} \arrow[d] \\ \mathbb{E} \arrow[r, "\Phi"] & \mathbb{L}_{M/S}, \end{tikzcd}$$

with the right-hand vertical morphism given by functoriality of the cotangent complex.

Proof. Consider the following commutative diagram with the left four squares cartesian.

$$\begin{tikzcd}
Z \arrow[d, "\iota"'] & Z_N \arrow[l] \arrow[d] \arrow[dd, bend right, "p"'] \arrow[rrr, bend left, "g"] & Z_M \arrow[l, "\widetilde{\varepsilon}"'] \arrow[d, "\iota_M"] \arrow[dd, bend right, "p_M"'] \arrow[rr, "h"] \arrow[drr] & & V \arrow[d] \\
Y \arrow[d] & Y_N \arrow[l] \arrow[d] & Y_M \arrow[l] \arrow[d, "\pi"] \arrow[rr] \arrow[urr, "f"'] & & W \arrow[d] \\
S & N \arrow[l] & M \arrow[l, "\varepsilon"] \arrow[rr] & & B
\end{tikzcd}$$

The left column is the given morphism (4.6) of domains, the lower horizontal row contains the restriction morphism ε from (4.8) and the morphisms to B and S, while $f : Y_M \to V$ and $g : Z_N \to V$ are the respective universal morphisms defined on the universal domains $Y_M \to M$ and $Z_N \to N$.

The obstruction theory Φ in (4.9) was defined by applying $R\pi_*(\cdot \otimes \omega_\pi)$ to

$$f^*\Omega_{V/W} \to \mathbb{L}_{Y_M/Y} = \pi^*\mathbb{L}_{M/S}$$

followed by the adjunction counit $R\pi_*\pi^! \to \mathrm{id}$, using $\pi^! = \pi^* \otimes \omega_\pi$. For Ψ one analogously takes $Rp_*(\cdot \otimes \omega_p)$ of $g^*\Omega_{V/W} \to \mathbb{L}_{Z_N/Z} = p^*\mathbb{L}_{N/S}$ followed by $Rp_*p^! \to \mathrm{id}$. By functoriality of obstruction theories (Lemma 4.1), the pullback $L\varepsilon^*\Psi$ is similarly obtained by $Rp_{M*}(\cdot \otimes \omega_{p_M})$ of

$$h^*\Omega_{V/W} \to L\widetilde{\varepsilon}^*\mathbb{L}_{Z_N/Z} = L\widetilde{\varepsilon}^* p^*\mathbb{L}_{N/S} = p_M^* L\varepsilon^*\mathbb{L}_{N/S}, \tag{4.10}$$

followed by $Rp_{M*}p_M^! \to \mathrm{id}$.

From $h = f \circ \iota_M = g \circ \widetilde{\varepsilon}$ we can extend (4.10) to the commutative diagram

$$\begin{tikzcd}
\widetilde{\varepsilon}^* g^*\Omega_{V/W} \arrow[r] \arrow[d, equal] & L\widetilde{\varepsilon}^*\mathbb{L}_{Z_N/W} \arrow[r] \arrow[d] & L\widetilde{\varepsilon}^*\mathbb{L}_{Z_N/Z} = p_M^* L\varepsilon^*\mathbb{L}_{N/S} \arrow[d] \\
h^*\Omega_{V/W} \arrow[r] \arrow[d, equal] & \mathbb{L}_{Z_M/W} \arrow[r] \arrow[d] & \mathbb{L}_{Z_M/Z} = p_M^*\mathbb{L}_{M/S} \arrow[d, "\simeq"] \\
\iota_M^* f^*\Omega_{V/W} \arrow[r] & L\iota_M^*\mathbb{L}_{Y_M/W} \arrow[r] & L\iota_M^*\mathbb{L}_{Y_M/Y} = p_M^*\mathbb{L}_{M/S}
\end{tikzcd}$$

The last row in this diagram is $L\iota_M^*$ of the morphism $f^*\Omega_{V/W} \to \pi^*\mathbb{L}_{M/S}$ that gives rise to the obstruction theory Φ for M. The essential part of this diagram is the square

$$\begin{tikzcd}
h^*\Omega_{V/W} \arrow[r] \arrow[d, equal] & p_M^* L\varepsilon^*\mathbb{L}_{N/S} \arrow[d] \\
\iota_M^* f^*\Omega_{V/W} \arrow[r] & p_M^*\mathbb{L}_{M/S}
\end{tikzcd} \tag{4.11}$$

Next observe that $\omega_{p_M} = \iota_M^* \omega_\pi \otimes \omega_{\iota_M}$, $h = f \circ \iota_M$, and $\iota_M^! = \iota_M^* \otimes \omega_{\iota_M}$ show that

$$Rp_{M*}(h^*\Omega_{V/W} \otimes \omega_{p_M}) = R\pi_* R\iota_{M*}\iota_M^!(f^*\Omega_{V/W} \otimes \omega_\pi).$$

Thus $Rp_{M*}(\cdot \otimes \omega_{p_M})$ applied to (4.11) yields the upper left square of the following commutative diagram:

$$\begin{tikzcd}
L\varepsilon^*\mathbb{F} = Rp_{M*}(h^*\Omega_{V/W} \otimes \omega_{p_M}) \arrow[r] \arrow[d, equal] & Rp_{M*}p_M^! L\varepsilon^*\mathbb{L}_{N/S} \arrow[r] \arrow[d, "a"] & L\varepsilon^*\mathbb{L}_{N/S} \arrow[d] \\
R\pi_* R\iota_{M*}\iota_M^!(f^*\Omega_{V/W} \otimes \omega_\pi) \arrow[r, "b"'] \arrow[d] & Rp_{M*}p_M^!\mathbb{L}_{M/S} \arrow[r] \arrow[d] & \mathbb{L}_{M/S} \arrow[d] \\
\mathbb{E} = R\pi_*(f^*\Omega_{V/W} \otimes \omega_\pi) \arrow[r] & R\pi_*\pi^!\mathbb{L}_{M/S} \arrow[r] & \mathbb{L}_{M/S}.
\end{tikzcd} \tag{4.12}$$

The upper right square is from functoriality of adjunction $Rp_{M*}p_M^! \to \mathrm{id}$ applied to the arrow marked a, the lower left one similarly from $R\iota_{M*}\iota_M^! \to \mathrm{id}$ applied to the arrow marked b. The lower right square is from the natural isomorphism of the adjunction counit $Rp_{M*}p_M^! \to \mathrm{id}$ with the composition

$$R\pi_* R\iota_{M*}\iota_M^! \pi^! \to R\pi_*\pi^! \to \mathrm{id},$$

see [36, Proposition VII.3.4 (b)], [20, Lemma 3.4.3].

The outer square of (4.12) provides the claimed commutative diagram. ∎

4.2 Obstruction theories for punctured maps with point conditions

We are now in position to define obstruction theories for moduli spaces of punctured maps with prescribed point conditions. Recall the log smooth morphism $X \to B$ and its factorization over the relative Artin fan $\mathcal{X} \to B$ from the beginning of this chapter. We want to work relative to a stack S of stable punctured maps to $\mathcal{X}/B$. Adopting the notation used elsewhere in the paper, we now write $\mathfrak{M}$ instead of S for the algebraic stack of domains together with the tuple of points at which to impose point conditions. For example, $\mathfrak{M}$ could be $\mathfrak{M}(\mathcal{X}/B, \tau)$ from Definition 3.8. Then $Y \to S = \mathfrak{M}$ is the universal curve, $Z \to Y$ the strict closed embedding of a union of sections, one for each point condition to be imposed, assumed ordered, and we have a universal diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathfrak{M} \arrow[r] & B.
\end{tikzcd}$$

As our target we now take the composition

$$X \to \mathcal{X} \to B.$$

Note that $\mathcal{X} \to B$ is log étale and $X \to \mathcal{X}$ is strict and log smooth. Hence $\underline{X} \to \underline{\mathcal{X}}$ is smooth as a morphism of stacks and we have a sequence of canonical isomorphisms

$$\mathbb{L}_{X/B} = \Omega_{X/B} = \Omega_{X/\mathcal{X}} = \Omega_{\underline{X}/\underline{\mathcal{X}}} = \mathbb{L}_{\underline{X}/\underline{\mathcal{X}}}.$$

For easier reference later on we also write $\mathcal{M}$ instead of M for the algebraic stack of punctured maps to X to be considered.

For the moduli space N of point conditions we take the space of factorizations of the composition $Z \to Y \to \mathcal{X}$ via $X \to \mathcal{X}$. Note that since $X \to \mathcal{X}$ is strict, it is enough to provide the lift for $\underline{X} \to \underline{\mathcal{X}}$, that is, ignoring the log structure. Thinking of these factorizations as providing evaluation maps $\mathfrak{M} \to \underline{X}$ at the marked points given by the sections Z of $Y \to S$, we denote the stack of such factorizations by $\mathfrak{M}^{\mathrm{ev}}$. This stack is algebraic by the fiber product description

$$\mathfrak{M}^{\mathrm{ev}} = \mathfrak{M} \times_{\underline{\mathcal{X}} \times_{\underline{B}} \cdots \times_{\underline{B}} \underline{\mathcal{X}}} (\underline{X} \times_{\underline{B}} \cdots \times_{\underline{B}} \underline{X}). \tag{4.13}$$

Here the map $\mathfrak{M} \to \underline{\mathcal{X}} \times_{\underline{B}} \cdots \times_{\underline{B}} \underline{\mathcal{X}}$ is defined by composing the sections $\mathfrak{M} \to \underline{\mathfrak{M}} \to \underline{Z}$ with the composition $\underline{Z} \to \underline{Y} \to \underline{\mathcal{X}}$ in the given order of the sections.

With this notation, the composition $M \to N \to S$ considered in the proof of Proposition 4.3 reads

$$\mathcal{M} \xrightarrow{\varepsilon} \mathfrak{M}^{\mathrm{ev}} \to \mathfrak{M}. \tag{4.14}$$

In Section 4.1 we recalled the construction of obstruction theories for $\mathcal{M}/\mathfrak{M}$ and for $\mathfrak{M}^{\mathrm{ev}}/\mathfrak{M}$, which in the situation at hand are perfect of amplitude contained in $[-1, 0]$, and showed their compatibility (Proposition 4.3). As in [50, Construction 3.13], this situation provides perfect obstruction theories for $\mathcal{M}/\mathfrak{M}^{\mathrm{ev}}$ by completing the compatibility diagram in Proposition 4.3 to a morphism of distinguished triangles:

$$\begin{tikzcd}
L\varepsilon^*\mathbb{F} \arrow[r] \arrow[d] & \mathbb{E} \arrow[r] \arrow[d] & \mathbb{G} \arrow[r] \arrow[d, dashed] & L\varepsilon^*\mathbb{F}[1] \arrow[d] \\
L\varepsilon^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}/\mathfrak{M}} \arrow[r] & \mathbb{L}_{\mathcal{M}/\mathfrak{M}} \arrow[r] & \mathbb{L}_{\mathcal{M}/\mathfrak{M}^{\mathrm{ev}}} \arrow[r] & L\varepsilon^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}/\mathfrak{M}}[1]
\end{tikzcd} \tag{4.15}$$

Remark 4.4. Note that while the isomorphism class of $\mathbb{G}$ is unique, the dashed arrow is not, so this recipe potentially provides several different obstruction theories for $\mathcal{M}/\mathfrak{M}^{\mathrm{ev}}$. On the other hand, any two dashed arrows differ by an element of the image of

$$\mathrm{Hom}(\mathbb{G}, \mathbb{L}_{\mathcal{M}/\mathfrak{M}}) \to \mathrm{Hom}(\mathbb{G}, \mathbb{L}_{\mathcal{M}/\mathfrak{M}^{\mathrm{ev}}}).$$

Thus the space of obstruction theories $\mathbb{G} \to \mathbb{L}_{\mathcal{M}/\mathfrak{M}^{\mathrm{ev}}}$ constructed as dashed arrow in (4.15) is parametrized by an affine space. This shows that the virtual classes constructed from any two such obstruction theories agree.[2]

For the sake of being explicit and for later use we now work out $\mathbb{G}$. For simplicity of notation write $\pi : C \to \mathcal{M}$ for the pullback $Y_{\mathcal{M}}$ of the universal curve $Y \to \mathfrak{M}$ to $\mathcal{M}$, and, in disagreement with our usual conventions, write $\iota : Z \to C$ for the strict closed substack of special points rather than $Z_{\mathcal{M}}$. We assume that $Z = Z' \amalg Z''$ with Z' disjoint from the critical locus of $\underline{C} \to \underline{\mathcal{M}}$ and Z'' the images of a set of nodal sections, as reviewed in Definition 5.1 below. Denote by $\kappa : \widetilde{C} \to C$ the partial normalization of $\widetilde{C}$ along the nodal sections exhibiting $\underline{C}$ as the fibered sum

$$\underline{C} = \underline{Z}'' \amalg_{\underline{\widetilde{Z}}''} \underline{\widetilde{C}}$$

with $\widetilde{Z}'' = \kappa^{-1}(Z'') \to Z''$ the two-fold unbranched cover induced by κ. Write $\widetilde{\pi} = \pi \circ \kappa : \widetilde{C} \to \mathcal{M}$, $\widetilde{f} = f \circ \kappa : \widetilde{C} \to X$ and $\widetilde{Z} = \kappa^{-1}(Z)$, with the log structures making $\widetilde{C} \to C$ and $\widetilde{Z} \to \widetilde{C}$ strict.[3]

For simplicity of the following statement *we now assume the two-fold covering* $\widetilde{Z}'' \to Z''$ *is trivial*, that is, that there is an isomorphism

$$\widetilde{Z}'' \simeq Z'' \amalg Z''$$

over Z''. This is sufficient for all applications we can currently think of. The general case can be treated by going over to an orientation covering or by twisting with an orientation sheaf.

Proposition 4.5. *For the tangent-obstruction bundle in* (4.15) *it holds*

$$\begin{aligned}\mathbb{G} &\simeq R\pi_*(f^*\Omega_{X/B} \otimes \kappa_*(\omega_{\widetilde{\pi}}(\widetilde{Z}))) \\ &\simeq R\widetilde{\pi}_*(\widetilde{f}^*\Omega_{X/B} \otimes \omega_{\widetilde{\pi}}(\widetilde{Z})) \\ &\simeq (R\widetilde{\pi}_*\widetilde{f}^*\Theta_{X/B}(-\widetilde{Z}))^\vee.\end{aligned}$$

Moreover, $\mathbb{G}$ *is perfect of amplitude* $[-1, 0]$.

Proof. The second isomorphism follows by the projection formula, the third isomorphism by relative duality.

For the first isomorphism we first claim there exists the following exact sequence of complexes, all concentrated in degree -1:

$$0 \to \omega_\pi \to \kappa_*(\omega_{\widetilde{\pi}}(\widetilde{Z})) \to \iota_*\mathcal{O}_Z[1] \to 0. \tag{4.16}$$

[2] We learnt this argument from Tom Graber.

[3] The log structures on $\widetilde{C}$ and $\widetilde{Z}$ are irrelevant for the following discussion and are merely chosen for the sake of uncluttering the notation.

On the complement of the nodal locus Z'', this sequence is defined by

$$0 \to \omega_\pi \to \omega_\pi(Z') \to \omega_\pi \otimes_{\mathcal{O}_C} \iota_*\mathcal{O}_{Z'}(Z') \to 0$$

by means of the canonical isomorphism

$$\omega_\pi \otimes_{\mathcal{O}_C} \iota_*\mathcal{O}_{Z'}(Z') = \iota_*\big(\iota^*\omega_\pi \otimes_{\mathcal{O}_{Z'}} \omega_\iota\big) \simeq \iota_*\mathcal{O}_{Z'}[1]$$

coming from the definition of $\omega_\iota = \iota^*\omega_\pi^\vee \simeq \mathcal{O}_{Z'}(Z')$ in (4.7). Explicitly, the homomorphism $\omega_\pi(Z') \to \iota^*\mathcal{O}_{Z'}[1]$ takes the residue along Z'.

Near the nodal locus, (4.16) is defined by

$$0 \to \omega_\pi \xrightarrow{\kappa^*} \kappa_*(\omega_{\tilde\pi}(\tilde Z)) \to \iota_*\mathcal{O}_{Z''}[1] \to 0.$$

To obtain this sequence, recall that étale locally $\omega_\pi = \Omega_{C/\mathcal{M}}[1]$ with $\Omega_{C/\mathcal{M}}$ the sheaf of relative logarithmic differentials for $C/\mathcal{M}$, while $\omega_{\tilde\pi} = \Omega_{\underline{\tilde C}/\underline{\mathcal{M}}}[1]$ with $\Omega_{\underline{\tilde C}/\underline{\mathcal{M}}}$ the sheaf of relative ordinary differentials for $\underline{\tilde C}/\underline{\mathcal{M}}$. In fiberwise coordinates z, w for the two branches of C along Z'' on an étale neighborhood, $\Omega_{C/\mathcal{M}}$ is locally generated by $z^{-1}dz = -w^{-1}dw$, hence pulls back to ordinary differentials with simple poles along $\kappa^{-1}(Z'') \subseteq \tilde Z$. The map to $\mathcal{O}_Z$ takes the difference of the residues of such rational differential forms on $\tilde C$ along the two preimages of the nodal locus. Note that this map depends on an order of the two branches along each connected component of Z'', hence relies on the assumption $\tilde Z'' = Z'' \amalg Z''$. This establishes sequence (4.16).

Next note that $\omega_{p_\mathcal{M}} \simeq \mathcal{O}_Z$ since $p_\mathcal{M} : Z \to \mathcal{M}$ is étale. Using the projection formula we can thus rewrite

$$L\varepsilon^*\mathbb{F} = Rp_{\mathcal{M}*}(h^*\Omega_{X/B} \otimes \omega_{p_\mathcal{M}}) = R\pi_*\iota_*\iota^* f^*\Omega_{X/B} = R\pi_*(f^*\Omega_{X/B} \otimes \iota_*\mathcal{O}_Z).$$

Finally, apply $R\pi_*$ to (4.16) tensored with $f^*\Omega_{X/B}$ to produce the upper triangle of (4.15) with the claimed middle term $\mathbb{G} = R\pi_*(f^*\Omega_{X/B} \otimes \kappa_*(\omega_{\tilde\pi}(\tilde Z)))$:

$$\begin{array}{ccccc}
\mathbb{E} & & \mathbb{G} & & L\varepsilon^*\mathbb{F}[1] \\
\| & & \| & & \| \\
R\pi_*(f^*\Omega_{X/B} \otimes \omega_\pi) & \longrightarrow & R\pi_*(f^*\Omega_{X/B} \otimes \kappa_*(\omega_{\tilde\pi}(\tilde Z))) & \longrightarrow & p_{\mathcal{M}*}(h^*\Omega_{X/B})[1]
\end{array} \tag{4.17}$$

Taking cohomologies, this diagram also shows the statement about the amplitude of $\mathbb{G}$. ∎

4.3 Punctured Gromov–Witten invariants

Using properness of $\mathcal{M}(X/B, \beta)$ over B (Corollary 3.19) and the obstruction theory, we can now define punctured Gromov–Witten invariants. To be explicit, we assume the ground field $\Bbbk$ to be a subfield of $\mathbb{C}$ and take $H_2(X)$ to be singular homology of the base change to $\mathbb{C}$. Since $\mathfrak{M}(\mathcal{X}/B,\beta)$ is typically non-equidimensional due to the puncturing ideal, the general definition demands a stratum-by-stratum treatment. Sometimes one can show independence of certain choices, e.g. in the setting of [33], but presently our understanding of the intersection theory of $\mathfrak{M}(\mathcal{X}/B)$ and in logarithmic geometry is too limited to make general statements. Some steps in this direction have been taken in [10, 71].

Let $X \to B$ be projective and log smooth, with Zariski logarithmic structure on X. Let $\boldsymbol{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}}, \mathbf{A})$ be a decorated global type (Definition 2.44). Denote by g the total genus and $k = |L(G)|$. We assume $\bar{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ to be generated by global sections to apply Corollary 3.19, or otherwise $\mathcal{M}(X/B,\boldsymbol{\tau}) \to B$ to be proper. Denote by $Z_L = X_{\boldsymbol{\sigma}(L)} \subseteq X$ the evaluation stratum for $L \in L(G)$.

Considering for simplicity evaluations at all punctures rather than at a subset of punctures, we then have an evaluation map

$$\underline{\mathrm{ev}} : \underline{\mathcal{M}}(X/B,\boldsymbol{\tau}) \to \prod_{L \in L(G)} \underline{Z}_L,$$

and, by Section 4.2 and notably (4.15), a perfect relative obstruction theory $\mathbb{G}$ for

$$\varepsilon : \mathcal{M}(X/B,\boldsymbol{\tau}) \to \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau}).$$

The relative virtual dimension is given by the Riemann–Roch formula applied to the virtual bundle in Proposition 4.5 as

$$d(g,k,A,n) = c_1(\Theta_{X/B}) \cdot A + n \cdot (1 - g - k). \tag{4.18}$$

Here $A = |\mathbf{A}|$ and $g = |\mathbf{g}|$ are the total curve class and total genus of $\boldsymbol{\tau}$, $k = |L(G)|$ the number of point conditions imposed and $n = \dim X - \dim B$ the relative dimension of X over B. Denote by $\varepsilon^!_{\mathbb{G}}$ the associated virtual pullback from [50], an operational Chow class for ε.

Definition 4.6. The *punctured Gromov–Witten correspondence* defined by the global decorated type $\boldsymbol{\tau}$ is the homomorphism

$$(\underline{\mathrm{ev}} \times p)_* \varepsilon^!_{\mathbb{G}} : A_*(\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})) \to A_{*+d(g,k,A,n)}\Big(\prod_L \underline{Z}_L \times \mathcal{M}_{g,k}\Big)$$

of rational Chow groups.

Here $\prod_L$ denotes the cartesian product of spaces over B. As usual, pairing with cohomology classes in $\prod_L \underline{Z}_L \times \mathcal{M}_{g,k}$ and taking degrees then produces Gromov–Witten invariants. Note also that Proposition 3.30 defines pure-dimensional cycles in $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})$ as the images of the fundamental classes of $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau}')$ for $\boldsymbol{\tau}' \to \boldsymbol{\tau}$ a contraction morphism from a realizable global type.

Chapter 5

Splitting and gluing

As discussed in the introduction, one crucial motivation for the introduction of the notion of punctured maps is the desire to treat logarithmic Gromov–Witten invariants by splitting the domain curves along nodal sections, in situations where such sections occur uniformly in the moduli space.

After briefly formalizing this splitting operation, we present the second series of main results of this paper, the reverse procedure of gluing a pair of punctured sections, followed by its treatment in punctured Gromov–Witten theory. We end this chapter with an application to the degeneration situation of [3].

Throughout this chapter, $X \to B$ denotes a morphism of fs logarithmic schemes fulfilling the assumptions stated at the beginning of Chapter 3.

5.1 Splitting punctured maps

We first discuss the operation of splitting of punctured curves along nodal sections.

Definition 5.1. A *nodal section* of a family of nodal curves $\underline{\pi} : \underline{C} \to \underline{W}$ is a section $s : \underline{W} \to \underline{C}$ of $\underline{\pi}$ that étale locally in $\underline{W}$ factors over the closed embedding defined by the ideal (x, y) in the domain of an étale map

$$\mathbf{Spec}\, \mathcal{O}_W[x, y]/(xy) \to \underline{C}.$$

The *partial normalization of* $\underline{C}/\underline{W}$ *along* s is the map

$$\underline{\kappa} : \underline{\widetilde{C}} \to \underline{C} \tag{5.1}$$

that étale locally is given by base change from the normalization of the plane nodal curve $\operatorname{Spec} \Bbbk[x, y]/(xy)$. We say s is *of splitting type* if the two-fold unbranched cover $\underline{\kappa}^{-1}(\operatorname{im}(s)) \to \operatorname{im}(s)$ is trivial.

A nodal section of a punctured curve $(C^\circ/W, \mathbf{p})$ or punctured map $(C^\circ/W, \mathbf{p}, f)$ is a nodal section of the underlying curve $\underline{C}/\underline{W}$.

Note that a nodal section s of a nodal curve $\underline{C}/\underline{W}$ with partial normalization $\underline{\kappa} : \underline{\widetilde{C}} \to \underline{C}$ and nodal locus $\underline{Z} = \operatorname{im}(s)$ exhibits $\underline{C}$ as the fibered sum

$$\underline{Z} \amalg_{\underline{\kappa}^{-1}(\underline{Z})} \underline{\widetilde{C}} \xrightarrow{\simeq} \underline{C}. \tag{5.2}$$

A punctured curve can be split along a nodal section of splitting type.

Proposition 5.2. *Let $\underline{\kappa} : \underline{\widetilde{C}} \to \underline{C}$ be the partial normalization of a punctured curve $(\pi : C^\circ \to W, \mathbf{p})$ defined by the splitting at a nodal section s of splitting type. Let $p_1, p_2 : \underline{W} \to \underline{\widetilde{C}}$ be two sections of $\underline{\kappa}^{-1}(\mathrm{im}(s)) \to \mathrm{im}(s)$ with disjoint images.*

Then

$$(\widetilde{C}^\circ, \tilde{\mathbf{p}}) = \big(\tilde{\pi} : (\underline{\widetilde{C}}, \underline{\kappa}^*\mathcal{M}_{C^\circ}) \xrightarrow{\kappa} C^\circ \to W, \{\hat{\mathbf{p}}, p_1, p_2\}\big)$$

with $\hat{\mathbf{p}} : \underline{W} \to \underline{\widetilde{C}}$ the unique set of sections with $\mathbf{p} = \underline{\kappa} \circ \hat{\mathbf{p}}$, is a (possibly disconnected) punctured curve.

Proof. Since $\kappa : \widetilde{C}^\circ \to C^\circ$ is an isomorphism away from $\mathrm{im}(p_1) \cup \mathrm{im}(p_2)$, it suffices to consider a neighborhood of a geometric point $\bar{p} \to \underline{\widetilde{C}}$ of one of $\mathrm{im}(p_i)$, say $i = 1$. Denote by $\bar{q} = \underline{\kappa} \circ \bar{p}$ the corresponding geometric point of $\underline{C}$, thus a geometric point of the image of the nodal section. By the structure of log smooth curves, $\mathcal{M}_{C^\circ,\bar{q}}$ is generated by $(\pi^*\mathcal{M}_W)_{\bar{q}}$, s_x and s_y, where $s_x, s_y \in \mathcal{M}_{C^\circ,q}$ are induced by the coordinates x, y in Definition 5.1. These are subject to the relation $s_x s_y = s_\rho$ for some $s_\rho \in (\pi^*\mathcal{M}_W)_{\bar{q}}$. Hence $(\pi^*\mathcal{M}_W)_{\bar{q}}$ and s_y locally generate $\mathcal{M}^{\mathrm{gp}}_{C^\circ}$ as a group, with $s_x = s_\rho s_y^{-1}$. Pulling back to $\underline{\widetilde{C}}$, along the branch $x = 0$, hence with $y = 0$ giving $\mathrm{im}(p_1)$, we see that $(\kappa^*\mathcal{M}_{C^\circ})^{\mathrm{gp}}$ is locally generated by $(\tilde{\pi}^*\mathcal{M}_W)_{\bar{p}}$ and $\kappa^\flat s_y$. Further, $\kappa^\flat s_y$ is also a section of $\mathcal{P}$, the divisorial log structure given by p_1, and the image of $\kappa^\flat s_y$ in $\overline{\mathcal{P}}$ generates $\overline{\mathcal{P}}$ as a monoid. Thus locally near $\bar{p}$,

$$\tilde{\pi}^*\mathcal{M}_W \oplus_{\mathcal{O}^\times_{\widetilde{C}}} \mathcal{P} \subseteq \kappa^*\mathcal{M}_{C^\circ} \subset \tilde{\pi}^*\mathcal{M}_W \oplus_{\mathcal{O}^\times_{\widetilde{C}}} \mathcal{P}^{\mathrm{gp}}.$$

Further, any local section of $\kappa^*\mathcal{M}_{C^\circ}$ not contained in $\tilde{\pi}^*\mathcal{M}_W \oplus_{\mathcal{O}^\times_{\widetilde{C}}} \mathcal{P}$ can be written in the form $s_x^a s_y^b s_W$ with $a > 0$, $b \geq 0$ and s_W a local section of $\tilde{\pi}^*\mathcal{M}_W$. Since $\alpha(s_x) = 0$ when $x = 0$, we see that α applied to any such element is zero. Thus $(\widetilde{C}^\circ/W, \tilde{\mathbf{p}})$ is a punctured curve near $\bar{p}$. ∎

For the application to moduli spaces of punctured maps we formalize the splitting procedure as an operation on graphs, hence on (global) types of punctured maps.

Definition 5.3. Let G be a connected graph and $\mathbf{E} \subseteq E(G)$ a subset of edges. Replacing each $E \in \mathbf{E}$ by a pair of legs L_E, L'_E leads to a graph $\widehat{G}$ with

$$V(\widehat{G}) = V(G), \quad E(\widehat{G}) = E(G) \setminus \mathbf{E}, \quad L(\widehat{G}) = L(G) \cup \{L_E, L'_E\}_{E\in\mathbf{E}}.$$

We call the collection of connected subgraphs $G_1, \dots, G_r$ of $\widehat{G}$ *the graphs obtained from G by splitting along* $\mathbf{E}$.

There is an obvious induced notion of splitting of a genus-decorated graph $(G, \mathbf{g})$, of a (global) type τ, or of a (global) decorated type $\boldsymbol{\tau}$ of a punctured map along a subset of edges of the corresponding graphs.

Proposition 5.4. *Let $X \to B$ be a morphism of fs logarithmic schemes over $\Bbbk$ fulfilling the assumptions stated at the beginning of Chapter 3. Let $\tau_1, \dots, \tau_r$ be obtained*

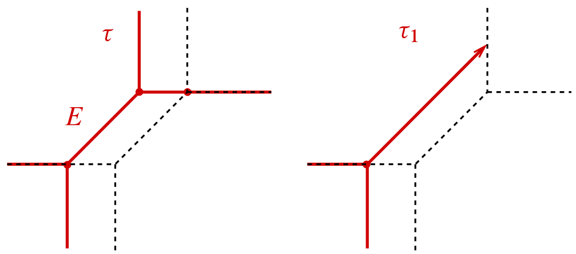

Figure 5.1. Tropical splitting.

from splitting a global type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ of a punctured map to X/B along a subset of edges $\mathbf{E} \subseteq E(G)$. Then the splitting morphism from Proposition 5.2 followed by pre-stabilization (Proposition 2.5) defines morphisms of stacks

$$\mathfrak{M}(\mathcal{X}/B, \tau) \to \prod_i \mathfrak{M}(\mathcal{X}/B, \tau_i), \quad \mathcal{M}(X/B, \tau) \to \prod_i \mathcal{M}(X/B, \tau_i),$$

with the products understood as fiber products over B.

Analogous results hold for decorated types and for moduli spaces of weakly marked punctured maps.

Proof. The statement is immediate from Propositions 5.2 and 2.5. ∎

Example 5.5. As an illustration of the splitting procedure consider the degeneration of $\mathbb{P}^1 \times \mathbb{P}^1$ to two copies of $\mathbb{P}^2$ constructed as follows. Take the polyhedral decomposition $\mathcal{P}$ of $\mathbb{R}^2$ with two vertices at $(0,0)$, $(1,1)$ and four maximal cells given by the dashed part of Figure 5.1. Embed $\mathbb{R}^2$ as affine hyperplane $\mathbb{R}^2 \times \{1\}$ in $\mathbb{R}^3$ and take the closures of the cones over cells of $\mathcal{P}$ to define a fan Σ in $\mathbb{R}^3$ with support $|\Sigma| = \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. The corresponding toric threefold X comes with a flat morphism

$$\pi : X \to \mathbb{A}^1$$

induced by the projection $|\Sigma| \to \mathbb{R}_{\geq 0}$ to the last coordinate. It is not hard to show that $\pi^{-1}(\mathbb{A}^1 \setminus \{0\}) = (\mathbb{P}^1 \times \mathbb{P}^1) \times (\mathbb{A}^1 \setminus \{0\})$, a trivial family, and $\pi^{-1}(0) = \mathbb{P}^2 \amalg_{\mathbb{P}^1} \mathbb{P}^2$, a gluing of two copies of $\mathbb{P}^2$ along a pair of toric divisors.

Figure 5.1 on the left shows the tropicalization of a family of curves of bidegree $(1,1)$ giving a type τ. The figure shows the intersection with the affine hyperplane $\mathbb{R}^2 \times \{1\}$. Splitting along the edge E yields the two types τ_1, τ_2 whose general members are depicted on the right. Note also that the leg in τ_2 obtained from splitting τ at E extends to the boundary of the cell, while this is not true for τ_1. This illustrates the necessity of pre-stabilization in the splitting procedure.

The opposite process of shrinking legs to an edge of a tropical domain curve appears in gluing, see Remark 5.12.

5.2 Gluing punctured maps to $\mathcal{X}/B$

5.2.1 Notation for splitting edges

In this section we work in categories of spaces over $\underline{B}$ or B. In particular, products are to be understood as fiber products over $\underline{B}$ or B, as appropriate.

Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ be a global type of punctured tropical maps and $\tau_i = (G_i, \mathbf{g}_i, \boldsymbol{\sigma}_i, \bar{\mathbf{u}}_i)$, $i = 1, \ldots, r$, the global types obtained by splitting τ at a subset $\mathbf{E} \subseteq E(G)$ of edges (Definition 5.3). We choose an orientation on each edge $E \in \mathbf{E}$ and refer to the two legs obtained by splitting the edge E with vertices v, v' by the corresponding half-edges (E, v), (E, v'), with E oriented from v to v'.[1] Denote by $\mathbf{L} \subseteq \bigcup_i L(G_i)$ the subset of all legs obtained from splitting edges, and by $i(v) \in \{1, \ldots, r\}$ for $v \in V(G)$ the index i with $v \in V(G_i)$.

5.2.2 The stack $\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau)$ and its evaluation morphism

Evaluation at the nodal sections for $\mathbf{E}$ defines the morphism

$$\underline{\mathrm{ev}}_{\mathbf{E}} : \underline{\mathfrak{M}'}(\mathcal{X}/B, \tau) \to \prod_{E \in \mathbf{E}} \underline{\mathcal{X}}.$$

For each $E \in \mathbf{E}$ denote by $\mathfrak{M}'_E(\mathcal{X}/B, \tau)$ the image of the corresponding nodal section $s_E : \underline{\mathfrak{M}'}(\mathcal{X}/B, \tau) \to \underline{\mathfrak{C}'}^{\circ}(\mathcal{X}/B, \tau)$ with the restriction of the log structure on the universal domain $\mathfrak{C}'^{\circ}(\mathcal{X}/B, \tau)$. Denote further by $\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau)$ the fs fiber product

$$\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau) = \mathfrak{M}'_{E_1}(\mathcal{X}/B, \tau) \times^{\mathrm{fs}}_{\mathfrak{M}'(\mathcal{X}/B,\tau)} \cdots \times^{\mathrm{fs}}_{\mathfrak{M}'(\mathcal{X}/B,\tau)} \mathfrak{M}'_{E_r}(\mathcal{X}/B, \tau), \quad (5.3)$$

where $E_1, \ldots, E_r \in E(G)$ are the edges in $\mathbf{E}$.[2] With this enlarged log structure, the pullback $\widetilde{\mathfrak{C}}'^{\circ}(\mathcal{X}/B, \tau) \to \widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau)$ of the universal domain has sections $\tilde{s}_E$, $E \in \mathbf{E}$, in the category of log stacks. Moreover, $\underline{\mathrm{ev}}_{\mathbf{E}}$ lifts to a logarithmic evaluation morphism

$$\mathrm{ev}_{\mathbf{E}} : \widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau) \to \prod_{E \in \mathbf{E}} \mathcal{X}, \quad (5.4)$$

with E-component equal to

$$\tilde{f} \circ \tilde{s}_E \quad \text{for } \tilde{f} : \widetilde{\mathfrak{C}}'^{\circ}(\mathcal{X}/B, \tau) \to \mathcal{X}$$

the universal punctured morphism.

[1] We use this notation as it is easy to parse, but note that (E, v) is ambiguous if E is a loop. It will always be clear from the context how to fix this ambiguity with a heavier notation.

[2] Note that we have suppressed the dependence of the stack on $\mathbf{E}$ from the notation.

5.2.3 The stacks $\widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau_i)$, evaluation and splitting morphisms

Similarly, for each of the global types $\tau_i = (G_i, \mathbf{g}_i, \boldsymbol{\sigma}_i, \bar{\mathbf{u}}_i)$ obtained by splitting and $L \in L(G_i)$, denote by $\mathfrak{M}'_L(\mathcal{X}/B,\tau_i)$ the image of the punctured section $s_L : \underline{\mathfrak{M}}'(\mathcal{X}/B,\tau_i) \to \underline{\mathfrak{C}'}^{\circ}(\mathcal{X}/B,\tau_i)$ defined by L, again endowed with the pullback of the log structure on $\mathfrak{C}'^{\circ}(\mathcal{X}/B,\tau_i)$. With $L_1,\ldots,L_s$ the legs of G_i obtained from splitting, define the stack

$$\widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau_i) = \left(\mathfrak{M}'_{L_1}(\mathcal{X}/B,\tau_i) \times^{\mathrm{f}}_{\mathfrak{M}'(\mathcal{X}/B,\tau_i)} \cdots \times^{\mathrm{f}}_{\mathfrak{M}'(\mathcal{X}/B,\tau_i)} \mathfrak{M}'_{L_s}(\mathcal{X}/B,\tau_i)\right)^{\mathrm{sat}},$$

where sat denotes saturation, bearing in mind that the log structures on the stacks $\mathfrak{M}'_{L_j}(\mathcal{X}/B,\tau_i)$ are not saturated.

This stack differs from $\mathfrak{M}'(\mathcal{X}/B,\tau_i)$ by adding the pullback of the log structure of each puncture obtained from splitting, so that the pullback $\widetilde{\mathfrak{C}'}^{\circ}(\mathcal{X}/B,\tau_i) \to \widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau_i)$ of the universal curve now has punctured sections in the category of log stacks. We define the evaluation morphism

$$\mathrm{ev}_{\mathbf{L}} : \prod_{i=1}^{r} \widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau_i) \to \prod_{E\in\mathbf{E}} \mathcal{X}\times\mathcal{X}, \tag{5.5}$$

by taking as E-component the evaluation at the corresponding two sections $s_{E,v}$, $s_{E,v'}$, observing the chosen orientation of E.

Lemma 5.6. *The splitting morphism $\mathfrak{M}(\mathcal{X}/B,\tau) \to \prod_i \mathfrak{M}(\mathcal{X}/B,\tau_i)$ in Proposition 5.4 lifts to a morphism*

$$\widetilde{\mathfrak{M}}(\mathcal{X}/B,\tau) \to \prod_{i=1}^{r} \widetilde{\mathfrak{M}}(\mathcal{X}/B,\tau_i). \tag{5.6}$$

Analogous statements hold for weak markings and for the moduli spaces of stable maps to X rather than $\mathcal{X}$.

Proof. We only treat the case of marked moduli spaces of punctured maps to $\mathcal{X}$, the other cases being completely analogous.

It suffices to produce a morphism

$$\mathfrak{M}_E(\mathcal{X}/B,\tau) \to \mathfrak{M}_L(\mathcal{X}/B,\tau_i)$$

lifting $\mathfrak{M}(\mathcal{X}/B,\tau) \to \mathfrak{M}(\mathcal{X}/B,\tau_i)$ whenever $L = (E,v) \in L(G_i)$ is one of the two legs obtained from splitting E. Indeed, this then provides a morphism of fibered products, which lifts to the saturation by functoriality of saturation.

To construct this lifting let $\mathfrak{C}^\circ \to \mathfrak{M} := \mathfrak{M}(\mathcal{X}/B,\tau)$ be the universal curve, and $\widetilde{\mathfrak{C}}^\circ \to \mathfrak{C}^\circ$ the splitting of all nodes labeled by an element of $\mathbf{E}$, strict as a morphism of log stacks. The graph G_i given by τ_i selects a connected component $\widetilde{\mathfrak{C}}^\circ_i \subset \widetilde{\mathfrak{C}}^\circ$, and the nodal section s_E lifts to a punctured section $\tilde{s}_i : \underline{\mathfrak{M}} \to \underline{\widetilde{\mathfrak{C}}_i^\circ}$. Let similarly $\mathfrak{C}_L^\circ \to \mathfrak{M}_i := \mathfrak{M}(\mathcal{X}/B, \tau_i)$ and $s_L : \underline{\mathfrak{M}}_i \to \underline{\mathfrak{C}}_L$ the corresponding universal curve and punctured section over $\mathfrak{M}_i$. Then $\mathfrak{M}_E(\mathcal{X}/B, \tau) = \underline{\mathfrak{M}} \times_{\underline{\widetilde{\mathfrak{C}}_i^\circ}} \widetilde{\mathfrak{C}}_i^\circ$ since $\widetilde{\mathfrak{C}}_i^\circ \to \mathfrak{C}^\circ$ is strict, and similarly $\mathfrak{M}_L(\mathcal{X}/B, \tau_i) = \underline{\mathfrak{M}}_i \times_{\underline{\mathfrak{C}_L^\circ}} \mathfrak{C}_L^\circ$. Now there is a canonical morphism $\widetilde{\mathfrak{C}}_i^\circ \to \mathfrak{C}_L^\circ$ lifting $\mathfrak{M}(\mathcal{X}/B, \tau) \to \mathfrak{M}(\mathcal{X}/B, \tau_i)$—the prestabilization morphism as a punctured map. Pulling back we obtain the desired morphism $\mathfrak{M}_E(\mathcal{X}/B, \tau) \to \mathfrak{M}_L(\mathcal{X}/B, \tau_i)$. ■

We next show that enlarging the log structures for the punctures may change the structure of the underlying stacks, but only by nilpotents in the structure sheaf.

For $\mathbf{S} \subseteq E(G) \cup L(G)$ we unify the notation, denoting by $\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau) \to \mathfrak{M}'(\mathcal{X}/B, \tau)$ the corresponding fiber product over both nodal and punctured sections. In this generality we have:

Proposition 5.7. *Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ be a global type of punctured maps, $\mathbf{S} \subseteq E(G) \cup L(G)$ and $\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau)$ the corresponding stack of weakly τ-marked punctured maps to $\mathcal{X}/B$ with sections. Then the canonical map*

$$\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau) \to \mathfrak{M}'(\mathcal{X}/B, \tau)$$

induces an isomorphism on the reductions of their underlying stacks. If moreover $\mathbf{S} \subseteq E(G)$, the canonical map is an isomorphism on underlying stacks.

Analogous results hold for the marked and decorated versions.

Proof. Going inductively, it suffices to treat the case that $\mathbf{S}$ has only one element. The case $\mathbf{S} = \{E\}$ is an edge leads to the problem of going over from a monoid Q to the saturation of a monoid of the form $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ with $1 \in \mathbb{N}$ mapping to $(1,1) \in \mathbb{N}^2$. Since the morphism $\mathbb{N} \to \mathbb{N}^2$ is saturated and integral, $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ is saturated and integral as well by [52, Propositions I.4.8.5, I.4.6.3]. In particular, the fs fiber product in (5.3) agrees with the ordinary fiber product, and only changes the log structure.

For $\mathbf{S} = \{L\}$ a leg, we need to take the saturation of the strict subspace given by a punctured section. Let $(C^\circ/W, \mathbf{p}, f)$ be a punctured map to $\mathcal{X}/B$ with $W = \operatorname{Spec}(Q \to A)$. Let $\underline{W} \to \operatorname{Spec} \Bbbk[Q^\circ]$ be a chart for the log structure induced by the punctured section corresponding to the leg L, with $Q^\circ \subset Q \oplus \mathbb{Z}$. Then necessarily the induced map $Q^\circ \to A$ takes $Q^\circ \setminus (Q \oplus 0)$ to zero.

The saturation of $\operatorname{Spec}(Q^\circ \to A)$ equals $W' = \operatorname{Spec}(Q' \to A')$ with Q' the saturation of Q° and $A' = A \otimes_{k[Q^\circ]} k[Q']$. Necessarily, if $m \in Q' \setminus Q^\circ$ then $m \in Q \oplus \mathbb{Z}_{<0}$, and so its image $z^m \in A'$ is nilpotent (following the notation of Section 1.6). It is then immediate that $A \to A'_{\mathrm{red}}$ is surjective. This map factors through $A_{\mathrm{red}} \to A'_{\mathrm{red}}$, so the latter is surjective. Thus W'_{red} is a closed subscheme of W_{red}. On the other hand, by [52, Proposition III.2.1.5] saturation is always a surjective morphism, and hence $W'_{\mathrm{red}} \to W_{\mathrm{red}}$ is an isomorphism. ■

By Proposition 5.7 the Chow theories of the moduli stacks of punctured maps do not change by enlarging the log structures. We can thus freely use the enlarged log structures in discussing gluing.

We are now in position to state the central technical gluing result. It explains how a τ-marked punctured map is equivalent to giving a collection of τ_i-marked punctured maps obeying a logarithmic matching condition.

Theorem 5.8. *Let $X \to B$ be a morphism of fs logarithmic schemes over $\Bbbk$ fulfilling the assumptions stated at the beginning of Chapter 3, and assume X is simple. Let $\tau_1, \ldots, \tau_r$ be the global types of punctured maps (Definition 2.44) obtained by splitting a global type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ along a subset of edges $\mathbf{E}$. Then the commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau) & \xrightarrow{\delta_{\mathfrak{M}}} & \prod_{i=1}^{r} \widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau_i) \\ {\scriptstyle \mathrm{ev}_{\mathbf{E}}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_{\mathbf{L}}} \\ \prod_{E\in\mathbf{E}} \mathcal{X} & \xrightarrow{\Delta} & \prod_{E\in\mathbf{E}} \mathcal{X}\times\mathcal{X} \end{array}$$

with Δ the product of diagonal embeddings and the other arrows defined in (5.4), (5.5), and (5.6), is cartesian in the category of fs log stacks. We remind the reader that all products in this square are taken over B.

An analogous statement holds for τ replaced by a decorated global type $\boldsymbol{\tau} = (\tau, \mathbf{A})$.

Remark 5.9. We note that it is important that we use the weakly marked moduli spaces here. Indeed, there exist simple examples of (strongly) marked punctured maps which may be glued to obtain a punctured map which is only weakly marked. This arises as saturation issues in the above fiber product description may introduce nilpotents. For an explicit example, see [26, Example 4.5]. We also note that this is essentially the same saturation issue as in Remark 3.5, and the examples are closely related.

The proof of the theorem, given further below, is based on the following gluing result for punctures with a section.

Lemma 5.10. *Let W be an fs log scheme and U_i° a puncturing along $\{0\}\times W$ of strict open neighborhoods $U_1, U_2 \subseteq \mathbb{A}^1 \times W$ of $\{0\}\times W$, $i = 1, 2$. Here $\mathbb{A}^1$ is endowed with its toric log structure. Furthermore, let $s_i : W \to U_i^\circ$ be sections with schematic image $\{0\}\times \underline{W}$ of the composition $U_i^\circ \to U_i \to W$ of the puncturing map and the projection.*

Then there exists an enlarged puncturing $\widehat{U}_i^\circ \to U_i^\circ \to U_i$ through which the sections s_i factor, and a unique log smooth curve $\pi : U \to W$ with maps

$$\iota_1 : \widehat{U}_1^\circ \to U, \quad \iota_2 : \widehat{U}_2^\circ \to U$$

over W inducing an isomorphism of underlying schemes $\underline{U}_1 \amalg_{\{0\}\times \underline{W}} \underline{U}_2 \simeq \underline{U}$, strict away from $\{0\} \times \underline{W}$, and such that $\iota_1 \circ \hat{s}_1 = \iota_2 \circ \hat{s}_2$, with $\hat{s}_i$ the lifts of s_i.

Remark 5.11. The lifting of s_i to $\widehat{U}_i^\circ$ is unique. The enlarged puncturing $\widehat{U}_i^\circ$ is not unique, but may be chosen uniquely if we require that $\widehat{U}_i^\circ \to U_i^\circ \times U$ is prestable. We obtain a pushout diagram up to unique punctured enlargement:

$$\begin{tikzcd}
 & U_1^\circ & \widehat{U}_1^\circ \arrow[l, dotted] \arrow[dr, dotted, "\iota_1"] & \\
W \arrow[ur, "s_1"] \arrow[urr, dotted, "\hat{s}_1"'] \arrow[dr, "s_2"'] \arrow[drr, dotted, "\hat{s}_2"] & & & U \\
 & U_2^\circ & \widehat{U}_2^\circ \arrow[l, dotted] \arrow[ur, dotted, "\iota_2"'] &
\end{tikzcd}$$

Proof of Lemma 5.10. The statement is about the unique definition of the log structure on $\underline{U}$ near the nodal locus $\{0\} \times \underline{W} \subset \underline{U}$. Since this is a local question we can restrict attention to a neighborhood of a geometric point $\bar{q} = (0, \bar{w})$ of $\{0\} \times \underline{W}$. By the definition of puncturing, the linear coordinate of $\mathbb{A}^1$ defines elements $\sigma_x \in \mathcal{M}_{U_1^\circ,\bar{q}}$, $\sigma_y \in \mathcal{M}_{U_2^\circ,\bar{q}}$.

Now assume that $U = (\underline{U}, \mathcal{M}_U) \to W$ is a log smooth curve with the required properties for some $\widehat{U}_i^\circ$. Since $\widehat{U}_i^\circ$, U_i° are both puncturings of U_i we may identify $\overline{\mathcal{M}}^{\mathrm{gp}}_{\widehat{U}_i^\circ} = \overline{\mathcal{M}}^{\mathrm{gp}}_{U_i^\circ} = \overline{\mathcal{M}}^{\mathrm{gp}}_{U_i}$. Then

$$\bar{\iota}_i^\flat : \overline{\mathcal{M}}^{\mathrm{gp}}_{U,\bar{q}} \to \overline{\mathcal{M}}^{\mathrm{gp}}_{\widehat{U}_i^\circ,\bar{q}} = \overline{\mathcal{M}}^{\mathrm{gp}}_{U_i^\circ,\bar{q}} = \overline{\mathcal{M}}^{\mathrm{gp}}_{W,\bar{w}} \oplus \mathbb{Z}$$

is an isomorphism with $\overline{\mathcal{M}}_{W,\bar{w}} \oplus \mathbb{N} \subseteq \bar{\iota}_i^\flat(\overline{\mathcal{M}}_{U,\bar{q}})$. Thus there exist $\widetilde{\sigma}_x, \widetilde{\sigma}_y \in \mathcal{M}_{U,\bar{q}}$ with

$$\sigma_x = \iota_1^\flat(\widetilde{\sigma}_x), \quad \sigma_y = \iota_2^\flat(\widetilde{\sigma}_y).$$

An important property of log smooth structures at nodes is that logarithmic lifts of given local coordinates at the two branches of the node become unique if one requires their product to lie in $\pi^\flat(\mathcal{M}_{W,\bar{w}})$ [51, Section 3.8]. With this condition imposed on $\widetilde{\sigma}_x, \widetilde{\sigma}_y$, we now obtain a unique element $\sigma_q \in \mathcal{M}_{W,\bar{w}}$ with

$$\widetilde{\sigma}_x \cdot \widetilde{\sigma}_y = \pi^\flat(\sigma_q). \tag{5.7}$$

Under the assumption of the existence of factorizations $\hat{s}_1$, $\hat{s}_2$ of the sections s_1, s_2, we can compute σ_q from σ_x and σ_y as follows: With $\iota_1 \circ \hat{s}_1 = \iota_2 \circ \hat{s}_2$ we obtain

$$\sigma_q = (\iota_1 \circ \hat{s}_1)^\flat(\pi^\flat(\sigma_q)) = (\iota_1 \circ \hat{s}_1)^\flat(\widetilde{\sigma}_x) \cdot (\iota_2 \circ \hat{s}_2)^\flat(\widetilde{\sigma}_y) = s_1^\flat(\sigma_x) \cdot s_2^\flat(\sigma_y).$$

Note also that $\mathcal{M}_{U,\bar{q}}$ is generated by $(\pi^*\mathcal{M}_W)_{\bar{q}}$ and $\widetilde{\sigma}_x, \widetilde{\sigma}_y$, with single relation (5.7).

Conversely, we can define the structure of a log smooth curve at $\bar{q} \to \underline{U}$ with the requested properties simply by defining

$$\sigma_q = s_1^\flat(\sigma_x) \cdot s_2^\flat(\sigma_y), \tag{5.8}$$

and

$$\mathcal{M}_{U,\bar{q}} := (\pi^*\mathcal{M}_W)_{\bar{q}} \oplus_{\mathbb{N}} \mathbb{N}^2,$$

with the generator $1 \in \mathbb{N}$ in the fibered sum mapping to $\pi^\flat(\sigma_q) \in (\pi^*\mathcal{M}_W)_q$ and to $(1,1) \in \mathbb{N}^2$, respectively. The structure morphism

$$\mathcal{M}_{U,\bar{q}} \to \mathcal{O}_{U,\bar{q}}$$

is defined by the structure morphism of W on the first summand, and by mapping $(a,b) \in \mathbb{N}^2$ to $x^a y^b$ when writing $\underline{U} \subseteq W \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}[x,y]/(xy)$. Since the projection $U_i^\circ \setminus (\{0\} \times \underline{W}) \to W$ is strict, this log structure near $\bar{q}$ patches uniquely to the given log structure on $U_i^\circ \setminus (\{0\} \times \underline{W})$ to define the desired log smooth curve $U \to W$.

The morphisms $\iota_i : \widehat{U}_i^\circ \to U$ are then given by

$$\begin{aligned} (\iota_1^\flat)_{\bar{q}} &: \mathcal{M}_{U,\bar{q}} \to \mathcal{M}^{\mathrm{gp}}_{U_1^\circ,\bar{q}}, \quad (1,0) \mapsto \sigma_x, \qquad\qquad (0,1) \mapsto \sigma_x^{-1}\pi^\flat(\sigma_q), \\ (\iota_2^\flat)_{\bar{q}} &: \mathcal{M}_{U,\bar{q}} \to \mathcal{M}^{\mathrm{gp}}_{U_2^\circ,\bar{q}}, \quad (1,0) \mapsto \sigma_y^{-1}\pi^\flat(\sigma_q), \quad (0,1) \mapsto \sigma_y. \end{aligned} \tag{5.9}$$

These definitions are forced upon us by the structure homomorphisms on U_i° and by the defining relation (5.8) for $\mathcal{M}_{U,\bar{q}}$. If $\sigma_x^{-1}\pi^\flat(\sigma_q) \notin \mathcal{M}_{U_1^\circ,q}$, we may have to enlarge the puncturing of U_1° for this map to define $\widehat{U}_1^\circ \to U$, and similarly for $\widehat{U}_2^\circ$; if we choose the enlargement to be generated by $\sigma_x^{-1}\pi^\flat(\sigma_q)$ it is uniquely defined. Note that by (5.8), the image of σ_q under the structure morphism is $xy = 0$, and hence this enlargement of puncturing is possible. Note also that s_1 factors uniquely over this extension of puncturing since by (5.8),

$$(s_1^\flat)^{\mathrm{gp}}\big(\sigma_x^{-1}\pi^\flat(\sigma_q)\big) = s_2^\flat(\sigma_y),$$

and similarly for s_2. Finally, to check the equality $\iota_1 \circ s_1 = \iota_2 \circ s_2$ we compute

$$(s_1^\flat \circ \iota_1^\flat)(1,0) = s_1^\flat(\sigma_x) = s_2^\flat(\sigma_y)^{-1}\sigma_q = s_2^\flat(\sigma_y^{-1}\pi^\flat(\sigma_q)) = (s_2^\flat \circ \iota_2^\flat)(1,0),$$

and similarly for $(1,0)$ replaced by $(0,1)$. This shows the claimed properties for $U \to W$ and ι_1, ι_2. Uniqueness follows from the discussion at the beginning of the proof. ∎

Remark 5.12. It is worthwhile to understand the gluing construction of a pair of punctured points to a node on the level of ghost sheaves and in terms of the dual tropical picture. The relevant monoids are

$$Q = \overline{\mathcal{M}}_{W,\bar{w}}, \quad Q_i = \overline{\mathcal{M}}_{U_i^\circ,\bar{q}} \subset Q \oplus \mathbb{Z},$$

and their duals

$$\omega = \mathrm{Hom}(Q, \mathbb{R}_{\geq 0}), \quad \tau_i = \mathrm{Hom}(Q_i, \mathbb{R}_{\geq 0}) \subset \omega \times \mathbb{R}_{\geq 0}.$$

We choose the embedding $Q_i \subset Q \oplus \mathbb{Z}$ such that $\bar{\pi}^\flat$ identifies Q with $Q \oplus \{0\}$, while the puncturing log structure is generated by $(0, 1) \in Q \oplus \mathbb{Z}$. The sections s_i define left-inverses

$$\bar{s}_i^\flat : Q_i \to Q$$

to $\bar{\pi}^\flat$. Now the point of the gluing construction is that there are exactly two automorphisms ϕ of $Q^{\mathrm{gp}} \oplus \mathbb{Z}$ making the following diagram of monoids commutative:

$$\begin{tikzcd}
Q \arrow[rr, "\mathrm{id}"] \arrow[d, "\bar{\pi}^\flat"'] & & Q \arrow[d, "\bar{\pi}^\flat"] \\
Q_1 \arrow[d] & & Q_2 \arrow[d] \\
Q^{\mathrm{gp}} \oplus \mathbb{Z} \arrow[rr, "\phi"] \arrow[dr, "\bar{s}_1^\flat"'] & & Q^{\mathrm{gp}} \oplus \mathbb{Z} \arrow[dl, "\bar{s}_2^\flat"] \\
& Q^{\mathrm{gp}} &
\end{tikzcd}$$

Indeed, by commutativity of the square, $\phi(m, 0) = (m, 0)$ for all $m \in Q^{\mathrm{gp}}$. Define $\rho_i = \bar{s}_i^\flat(0, 1)$, $i = 1, 2$, and ρ_q by $\phi(0, 1) = (\rho_q, d)$. Then $d = \pm 1$ since $\phi(0, 1)$ together with $Q^{\mathrm{gp}} \oplus \{0\}$ generates $Q^{\mathrm{gp}} \oplus \mathbb{Z}$. This sign determines the two possibilities. Commutativity of the triangle now shows

$$\rho_1 = \bar{s}_1^\flat(0, 1) = \bar{s}_2^\flat(\rho_q, \pm 1) = \rho_q \pm \rho_2.$$

The situation obtained by splitting a node into two punctures produces the negative sign. With this choice we obtain an isomorphism of the submonoid $Q_{12} \subset Q^{\mathrm{gp}} \oplus \mathbb{Z}$ generated by $Q \oplus \mathbb{N}$ and $\phi^{-1}(Q \oplus \mathbb{N})$ with $Q \oplus_{\mathbb{N}} \mathbb{N}^2$, with $1 \in \mathbb{N}$ mapping to ρ_q and $(1, 1)$, respectively. The defining equation $\rho_q = \rho_1 + \rho_2$ retrieves (5.8) in the proof of Lemma 5.10 on the level of ghost sheaves. The change of puncturing of U_1° becomes necessary if $Q_{12} \not\subset Q_1$, and similarly if $\phi(Q_{12}) \not\subset Q_2$ for U_2°. Figure 5.2 provides an illustration.

For the tropical interpretation, illustrated in Figure 5.3, we have two factorizations

$$\omega \xrightarrow{\Sigma(s_i)} \tau_i \xrightarrow{\Sigma(\pi)} \omega,$$

of id_ω. Here the second map is the projection to the first component when writing $\tau_i \subseteq \omega \times \mathbb{R}_{\geq 0}$. Thus $\Sigma(s_i)(h) = (h, \ell_i(h))$ for some piecewise linear map

$$\ell_i : \omega \to \mathbb{R}_{\geq 0}.$$

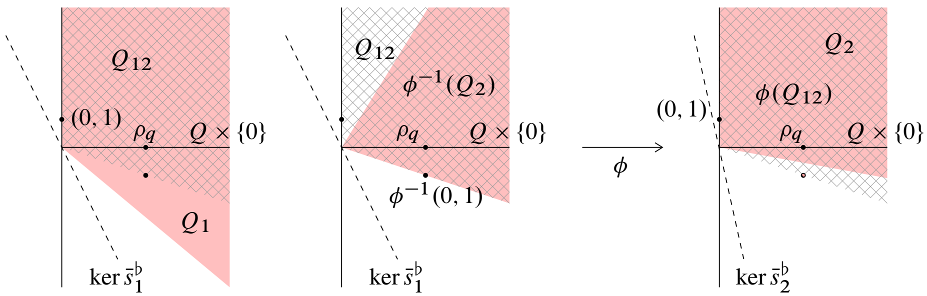

Figure 5.2. The monoids $Q_1, Q_2, Q_{12} \subset Q \oplus \mathbb{Z}$ and their comparison under $\phi : Q \oplus \mathbb{Z} \to Q \oplus \mathbb{Z}$. The hatched area depicts Q_{12}, the solid shading Q_1, Q_2 or $\phi^{-1}(Q_2)$. Note that $\phi(Q_{12}) = Q_{12}$ because both cones are spanned by $Q \times \{0\}$, $(0,1)$ and $\rho_q - (0,1)$. In the sketched situation, the puncturing for U_2° has to be enlarged, the one for U_1° does not.

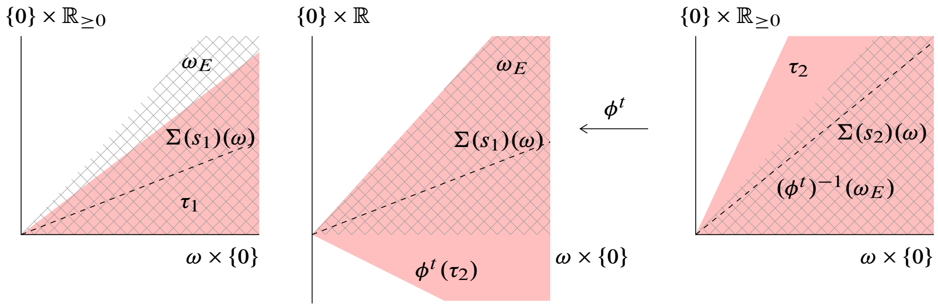

Figure 5.3. The dual tropical picture of Figure 5.2. The hatched area covers ω_E, the solid shading τ_1, τ_2 and $\phi^t(\tau_2)$. The dashed line indicates the image of ω under $\Sigma(s_1)$ or $\Sigma(s_2)$.

Thinking of h as parametrizing a punctured tropical curve, $\ell_i(h)$ specifies a point on the puncturing interval or ray emanating from the unique vertex v_i. The tropical glued curve then produces the metric graph with two vertices v_1, v_2 by joining the two intervals at the specified points, hence producing an edge E of length $\ell_1(h) + \ell_2(h)$. The tropical glued curve over ω thus has edge function $\ell : \omega \to \mathbb{R}_{\geq 0}$ simply defined by

$$\ell = \ell_1 + \ell_2. \tag{5.10}$$

The process of producing the glued cone $\omega_E \subset \omega \times \mathbb{R}_{\geq 0}$ over ω is dual to the statement $Q_{12} = (Q \oplus \mathbb{N}) + \phi^{-1}(Q \oplus \mathbb{N})$:

$$\omega_E = \mathrm{Hom}(Q_{12}, \mathbb{R}_{\geq 0}) = (\omega \times \mathbb{R}_{\geq 0}) \cap \phi^t(\omega \times \mathbb{R}_{\geq 0}).$$

The change of puncturing is necessary if $\ell(h)$ is smaller than either of the length functions obtained by tropicalizing the puncturing, or if either one of $Q_{12} \cap Q_1$, $\phi(Q_{12}) \cap Q_2$ is not saturated.

We now turn to the proof of the gluing theorem for punctured maps to $\mathcal{X}/B$.

Proof of Theorem 5.8. Write $\tau_i = (G_i, \mathbf{g}_i, \sigma_i, \bar{\mathbf{u}}_i)$. We check the universal property of cartesian diagrams.

Step 1: An object of the fibered product. Consider an fs log scheme W with two morphisms

$$W \to \prod_{E \in \mathbf{E}} \mathcal{X}, \quad W \to \prod_{i=1}^{r} \widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau_i) \tag{5.11}$$

together with an isomorphism of the compositions to $\prod_E \mathcal{X} \times \mathcal{X}$. Spelled out this means that (1) for each $i = 1, \ldots, r$ we have given a weakly τ_i-marked, pre-stable punctured map

$$(\pi_i : C_i^\circ \to W, \mathbf{p}_i, f_i : C_i^\circ \to \mathcal{X})$$

over W and for each leg $(E, v) \in L(G_i)$ a section $s_{E,v} : W \to C_i^\circ$ with image the puncture labeled by the leg in G_i generated by E; and (2) the sections fullfill the logarithmic matching property

$$f_{i(v)} \circ s_{E,v} = f_{i(v')} \circ s_{E,v'}, \tag{5.12}$$

for each edge $E \in \mathbf{E}$ with adjacent vertices v, v'. Write $p_{E,v}$ for the strict closed subspace of C_i° defined by $(E, v) \in L(G_i)$.

Step 2. The glued curve. Denote by $\underline{C}$ the family of nodal curves over $\underline{W}$ obtained by gluing $\coprod_i \underline{C}_i$ schematically along pairs of punctures. Let $E \in \mathbf{E}$ be an edge with vertices v, v', and q_E the nodal section of $\underline{C} \to \underline{W}$ given by the image of the pair of punctures $p_{E,v}$, $p_{E,v'}$. Applying Lemma 5.10 étale locally near the image of q_E provides a local extension of the log structure defined by the C_i° away from q_E to a log smooth curve over W. Thus there is a punctured curve

$$(\pi : C^\circ \to W, \mathbf{p})$$

with underlying scheme $\underline{C}$ that replaces each pair of punctures $p_{E,v}$, $p_{E,v'}$ in $\coprod_i C_i^\circ$, for an edge $E \in \mathbf{E}$ with vertices v, v', by a node q_E. The lemma also provides a morphism of punctured curves $\widehat{C}_i^\circ \to C_i^\circ$ with unique liftings $\hat{s}_{E,v}$ of each section $s_{E,v}$ to $\widehat{C}_{i(v)}^\circ$, and morphisms

$$\iota_i : \widehat{C}_i^\circ \to C^\circ$$

with $\iota_{i(v)} \circ \hat{s}_{E,v} = \iota_{i(v')} \circ \hat{s}_{E,v'}$, and $\widehat{C}_i^\circ$ equal to C_i° possibly up to enlargement of the puncturing. For each edge $E \in \mathbf{E}$ we can thus define the nodal section

$$s_E := \iota_{i(v)} \circ \hat{s}_{E,v} = \iota_{i(v')} \circ \hat{s}_{E,v'} : W \to C^\circ.$$

Step 3. Gluing the tropical map. Denote by $(\widehat{C}_i^\circ, \hat{\mathbf{p}}_i, \hat{f}_i)$ with $\hat{f}_i = f_i \circ (\widehat{C}_i^\circ \to C_i^\circ)$ the punctured stable map with the enlarged punctured structure. It follows from the tropical description of the gluing construction in Remark 5.12 that the tropicalizations

$$\Sigma(\hat{f}_i) : \Sigma(\widehat{C}_i^\circ) \to \Sigma(\mathcal{X})$$

of $\hat{f}_i$ glue to a map of generalized cone complexes

$$\Sigma(C^\circ) \to \Sigma(\mathcal{X})$$

which commutes with the map to $\Sigma(B)$. In fact, restricting to a geometric point $\bar{w}$ of $\underline{W}$ and adopting the notation from Remark 5.12, at an edge $E \in \mathbf{E}$ with vertices v_1, v_2, the cone $\omega_E \subseteq \omega \times \mathbb{R}_{\geq 0}$ of $\Sigma(C_{\bar{w}}^\circ)$ is defined by the length function $\ell_E = \ell_1 + \ell_2$. Denote further $\sigma = (\overline{\mathcal{M}}_{\mathcal{X},\bar{y}}^\vee)_\mathbb{R}$ for

$$\bar{y} = f_{i(v_1)}(s_{E,v_1}(\bar{w})) = f_{i(v_2)}(s_{E,v_2}(\bar{w})).$$

Assuming E oriented from v_1 to v_2, the contact orders obtained from splitting τ at E are related by

$$u_{E,v_1} = u_E = -u_{E,v_2} \in \sigma_\mathbb{Z}^{\mathrm{gp}}.$$

Now the map $\omega_E \to \sigma \in \Sigma(\mathcal{X})$ can be defined by

$$\omega_E \ni (h, \lambda) \mapsto V_1(h) + \lambda \cdot u_{E,v_1} = V_2(h) + (\ell_E(h) - \lambda) \cdot u_{E,v_2}, \tag{5.13}$$

where $V_\mu : \omega \to \sigma$ is the map for the vertex v_μ given by $\Sigma(f_{i(v_\mu)})$, $\mu = 1, 2$. The image of this map lies in σ since the line segment $\{h\} \times [0, \ell_1(h)]$ is contained in $\tau_1 \subseteq \omega \times \mathbb{R}_{\geq 0}$ and $V_1(h) + \lambda \cdot u_{E,v_1} = \Sigma(f_{i(v_1)})(h, \lambda)$, and similarly for the line segment $\{h\} \times [\ell_1(h), \ell(h)]$ and $\Sigma(f_{i(v_2)})$. The equality in (5.13) holds because

$$\begin{aligned} V_1(h) + \ell_1(h) \cdot u_{E,v_1} &= \Sigma(f_{i(v_1)} \circ s_{E,v_1})(h) \\ &= \Sigma(f_{i(v_2)} \circ s_{E,v_2})(h) = V_2(h) + \ell_2(h) \cdot u_{E,v_2}. \end{aligned}$$

Note this last argument uses the assumption that $\bar{u}_E$ is monodromy-free to assure that $u_{E,v_1} = -u_{E,v_2}$. This finishes the construction of the map $\Sigma(C^\circ) \to \Sigma(\mathcal{X})$.

Step 4. Gluing the punctured map. In view of [3, Proposition 2.10][3], we thus obtain a morphism $C^\circ \to \mathcal{A}_X$ over $\mathcal{A}_B$. By the same token, the composition $C^\circ \to \mathcal{A}_X \to \mathcal{A}_B$ agrees with $C^\circ \to B \to \mathcal{A}_B$. We thus obtain an induced morphism

$$f : C^\circ \to \mathcal{X} = B \times_{\mathcal{A}_B} \mathcal{A}_X,$$

[3] While [3, Proposition 2.10] assumes a more restricted context, the proof only uses that the Artin fan of the codomain is Zariski (Definition A.7). This is true here by simplicity of X and our standing assumptions on B.

commuting with the maps to B. By functoriality of this construction and the tropical description of the gluing process, it holds $f \circ \iota_i = \hat{f}_i$ for all i.

The data $(C^\circ \to W, f, \mathbf{p})$ and the collection of nodal sections s_E now define the desired morphism

$$W \to \widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau).$$

Indeed, splitting the domain C° at the nodes for edges $E \in \mathbf{E}$ and pre-stabilizing obviously retrieves the collection of pre-stable maps $(C_i^\circ \to W, \mathbf{p}_i, f_i)$ with compatible evaluation maps to $\mathcal{X}$ and sections $s_{E,v}$ that we started with. This finishes the existence part in checking cartesianity.

Uniqueness follows from the uniqueness statement in Lemma 5.10. ■

5.2.4 Relative and absolute maps

We end this section by remarking that in many situations, working with all fiber products over B may be burdensome, as each product in the diagram of Theorem 5.8 is over B. In the standard degeneration situation considered in Section 5.4 below, we might be working over a standard log point b_0, and saturation issues even over b_0 can complicate the fiber product. Thus the following is generally useful.

Proposition 5.13. *Let B be an affine log scheme equipped with a global chart $P \to \mathcal{M}_B$ inducing an isomorphism $P \cong \Gamma(B, \overline{\mathcal{M}}_B)$. Let τ be a global type of punctured tropical map for X/B (Definition 2.44 (1)), with underlying graph connected.*[4] *Then there are isomorphisms $\mathfrak{M}(\mathcal{X}/B, \tau) \cong \mathfrak{M}(\mathcal{X}/\operatorname{Spec} \Bbbk, \tau)$ and $\mathcal{M}(X/B, \tau) \cong \mathcal{M}(X/\operatorname{Spec} \Bbbk, \tau)$.*

Proof. We show the first isomorphism, the second being similar. There is a canonical forgetful morphism $\mathfrak{M}(\mathcal{X}/B, \tau) \to \mathfrak{M}(\mathcal{X}/\operatorname{Spec} \Bbbk, \tau)$, and we need to show it is an isomorphism. For this purpose, it is enough to demonstrate that given a punctured map $f : C^\circ/W \to \mathcal{X}$, there is a unique morphism $h : W \to B$ which fits into a commutative diagram

$$\begin{tikzcd} C^\circ \arrow[r, "f"] \arrow[d, "\pi"'] & \mathcal{X} \arrow[d, "g"] \\ W \arrow[r, "h"'] & B \end{tikzcd}$$

First, to define the underlying $\underline{h} : \underline{W} \to \underline{B}$ it is sufficient to define $\underline{h}^\# : \Gamma(B, \mathcal{O}_B) \to \Gamma(W, \mathcal{O}_W) \cong \Gamma(W, \pi_* \mathcal{O}_C)$, the latter isomorphism from the fact that π is flat, proper with connected and reduced fibers and [67, Lemma 0E0S]. We take this map to coincide with $(g \circ f)^\# : \Gamma(B, \mathcal{O}_B) \to \Gamma(C, \mathcal{O}_C)$.

[4] Connectedness is generally assumed in this paper, although usually not necessary, but here the result is not true without it.

We next enhance $\underline{h}$ to a log morphism, first by describing the map at the level of ghost sheaves, or equivalently, at the tropical level. Fix $\bar{w}$ a geometric point of $\underline{W}$, and let $\tau' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}')$ be the type of $C_{\bar{w}} \to \mathcal{X}$, so that in particular there is a contraction morphism $\tau' \to \tau$. Since τ' and τ have the same set of legs with the same contact orders, the fact that τ is defined over B implies that the composed map $\Sigma(g \circ f) : \Sigma(C_{\bar{w}}) \to \Sigma(B) = P^\vee_{\mathbb{R}}$ contracts all legs. However, $g \circ f$ is a punctured map with underlying schematic map constant, and thus by Proposition 2.27, the restriction of $\Sigma(g \circ f)$ to any fiber of $\Sigma(\pi)$ is a balanced tropical map. Since all legs are contracted, the image of this tropical map is compact. Hence, there must be a hyperplane H in the vector space $P^*_{\mathbb{R}}$ containing the image of a vertex of this map and with the entire image contained in a half-space bounded by H. By balancing, this is impossible unless the tropical map is constant. Hence the desired diagram exists at the tropical level. This shows that the map $P = \Gamma(B, \overline{\mathcal{M}}_B) \to \Gamma(C^\circ_{\bar{w}}, \overline{\mathcal{M}}_{C^\circ_{\bar{w}}})$ factors uniquely over $\overline{\mathcal{M}}_{W,\bar{w}}$. In particular, we obtain a map $\bar{h}^\flat : \Gamma(B, \overline{\mathcal{M}}_B) \to \Gamma(W, \overline{\mathcal{M}}_W)$.

Finally, there is a unique lifting of $\bar{h}^\flat$ to $h^\flat : \Gamma(B, \mathcal{M}_B) \to \Gamma(W, \mathcal{M}_W)$. Indeed, let $s \in \Gamma(B, \mathcal{M}_B)$ be a section which maps to $\bar{s} \in \Gamma(B, \overline{\mathcal{M}}_B)$. Then because the desired diagram exists at the level of ghost sheaves, $(\bar{f}^\flat \circ \bar{g}^\flat)(\bar{s}) = \bar{\pi}^\flat(\bar{t})$ for some $\bar{t} \in \Gamma(W, \overline{\mathcal{M}}_W)$. Thus étale locally on W, we may choose a lift $t \in \mathcal{M}_W$ of $\bar{t}$, and write $(f^\flat \circ g^\flat)(s) = \psi \cdot \pi^\flat(t)$ for some $\psi \in \Gamma(\mathcal{O}^\times_C)$. However, again by properness of π and connectivity and reducedness of the fibers of π, $\psi = \pi^\#(\psi')$ for some invertible function ψ' on W, and we may define $h^\flat(s) = \psi' \cdot t$. Because this choice of $h^\flat(s)$ is determined uniquely by $(g \circ f)^\flat$, this local description patches to give a section $h^\flat(s) \in \Gamma(W, \mathcal{M}_W)$, making the diagram commute.

We have thus defined a functor $\mathfrak{M}(\mathcal{X}/\operatorname{Spec}\Bbbk, \tau) \to \mathfrak{M}(\mathcal{X}/B, \tau)$ at the level of objects. By the uniqueness of the construction of the morphism $W \to B$ given $f : C^\circ/W \to \mathcal{X}$ above, a morphism in the category $\mathfrak{M}(\mathcal{X}/\operatorname{Spec}\Bbbk, \tau)$ defines a morphism in the category $\mathfrak{M}(\mathcal{X}/B, \tau)$, hence completely defining the functor. This defines the desired morphism $\mathfrak{M}(\mathcal{X}/\operatorname{Spec}\Bbbk, \tau) \to \mathfrak{M}(\mathcal{X}/B, \tau)$ which is inverse to the forgetful morphism $\mathfrak{M}(\mathcal{X}/B, \tau) \to \mathfrak{M}(\mathcal{X}/\operatorname{Spec}\Bbbk, \tau)$. ∎

We note that the two moduli problems, with isomorphic moduli spaces $\mathcal{M}(X/B, \tau)$ and $\mathcal{M}(X/\operatorname{Spec}\Bbbk, \tau)$, have different obstruction theories.

5.3 Evaluation stacks and gluing at the virtual level

While Theorem 5.8 transparently describes the process of gluing a collection of punctured maps at pairs of punctures with matching contact orders, it lacks two crucial properties needed for applications in punctured Gromov–Witten theory. First, since the diagonal map $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is not proper except in trivial cases and neither is the splitting map $\delta_{\mathfrak{M}}$, it is impossible to push forward cycles via $\delta_{\mathfrak{M}}$ for the purpose of splitting computations according to the splitting of τ along the chosen set of edges $\mathbf{E} \subseteq E(G)$. And second, the obvious commutative square lifting the splitting map $\delta_{\mathfrak{M}}$

to a map $\mathcal{M}'(X/B,\tau) \to \prod_i \mathcal{M}'(X/B,\tau_i)$ is far from being cartesian even on the underlying stacks of (pre-) stable maps since it imposes matching at the nodes only on $\mathcal{X}$ rather than on X. (We remind the reader that the products such as $\prod_i$ are all over the base log scheme B in this discussion.) Hence this approach has no hope to be compatible with the virtual formalism.

Both problems are solved by enriching the stacks $\mathfrak{M}(\mathcal{X}/B,\tau)$ and $\mathfrak{M}(\mathcal{X}/B,\tau_i)$ of punctured maps to the relative Artin stack $\mathcal{X}/B$ and their various cousins $\mathfrak{M}'(\mathcal{X}/B,\tau)$, $\widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau)$ etc., by providing a lift of the underlying evaluations to X. Note that such enriched stacks of maps to $\mathcal{X}$ have already been considered at the beginning of Section 4.2 in the context of obstruction theories with imposed point conditions.

For this discussion we mostly work with the stacks $\mathfrak{M}(\mathcal{X}/B,\tau)$ of marked maps (Definition 3.8), except in the analogue Corollary 5.15 of Theorem 5.8, which requires stacks $\widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau)$ with weak markings and sections (Section 5.2.2). All other results also hold in the weakly marked and decorated contexts.

We continue to assume that $X \to B$ is a morphism of fs logarithmic schemes over $\Bbbk$ fulfilling the assumptions stated at the beginning of Chapter 3.

Definition 5.14. Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ be a global type of punctured maps to X and $\mathbf{S} \subseteq E(G) \cup L(G)$ a subset of edges and legs. The *evaluation stack* of $\mathfrak{M}(\mathcal{X}/B,\tau)$ with respect to $\mathbf{S}$ is the fiber product

$$\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau) = \mathfrak{M}(\mathcal{X}/B,\tau) \times_{\prod_{S\in\mathbf{S}} \underline{\mathcal{X}}} \prod_{S\in\mathbf{S}} \underline{X}$$

of $\prod_{S\in\mathbf{S}} \underline{X} \to \prod_{S\in\mathbf{S}} \underline{\mathcal{X}}$ with the evaluation map

$$\underline{\mathrm{ev}}_{\mathbf{S}} : \mathfrak{M}(\mathcal{X}/B,\tau) \to \prod_{S\in\mathbf{S}} \underline{\mathcal{X}}, \quad (C^\circ/W, \mathbf{p}, f) \mapsto (\underline{f} \circ s_S)_{S\in\mathbf{S}},$$

evaluating at the punctured and nodal sections $s_S : \underline{W} \to \underline{C}^\circ$ for $S \in \mathbf{S}$.

Analogous definitions apply in the weakly marked and decorated contexts as in Definition 3.8, or for the stacks $\widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau)$ of Section 5.2.2.

Note that $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau)$ of course depends on the logarithmic scheme X, but we suppress this in the notation as $\mathcal{X}$ always denotes its relative Artin fan. We also suppress $\mathbf{S}$ in the notation of the evaluation stacks and rather specify this subset whenever not clear from the context.

As indicated in the definition, we endow $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau)$ with the log structure making the projection to $\mathfrak{M}(\mathcal{X}/B,\tau)$ strict, to obtain the sequence of morphisms of log stacks

$$\mathcal{M}(X/B,\tau) \xrightarrow{\varepsilon} \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau) \to \mathfrak{M}(\mathcal{X}/B,\tau)$$

as in (4.14). Recall that the obstruction theory for this sequence of morphisms has been worked out in Section 4.2. It was noted that, as the morphisms are strict, this coincides with the obstruction theory for the underlying stacks. We further saw that the obstruction theory of $\underline{\mathcal{M}}(X/B,\tau)$ over $\underline{\mathfrak{M}}(\mathcal{X}/B,\tau)$ is the composition of an obstruction theory for ε with the trivial obstruction theory in pure degree 0 of the smooth morphism $\underline{\mathfrak{M}^{\mathrm{ev}}}(\mathcal{X}/B,\tau) \to \underline{\mathfrak{M}}(\mathcal{X}/B,\tau)$ of relative dimension $(\dim X - \dim B)\cdot|\mathbf{S}|$.

We now adopt the setup of Section 5.2 and split τ at a subset $\mathbf{E} \subseteq E(G)$ of edges with $\mathbf{E} \subseteq \mathbf{S}$ to obtain global types $\tau_i = (G_i, \mathbf{g}_i, \boldsymbol{\sigma}_i, \bar{\mathbf{u}}_i)$. For the following corollary of Theorem 5.8 for evaluation stacks, we write $\widetilde{\mathfrak{M}}^{\mathrm{ev}}(\mathcal{X}/B,\tau)$ for the evaluation stack of $\widetilde{\mathfrak{M}}(\mathcal{X}/B,\tau)$ with evaluations at all nodes specified by $\mathbf{E}$, thus by Proposition 5.7 having the same underlying stack as $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau)$, but with the enlarged log structure admitting a logarithmic evaluation map analogous to (5.4). Similarly, we obtain evaluation stack analogues of the evaluation morphism for the τ_i (5.5), still denoted $\mathrm{ev}_{\mathbf{L}}$, and the splitting morphism (5.6), now denoted δ^{ev}.

Corollary 5.15. *In the situation of Theorem 5.8, the commutative diagram*

$$\begin{tikzcd}
\widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B,\tau) \arrow[r, "\delta^{\mathrm{ev}}"] \arrow[d, "\mathrm{ev}_{\mathbf{E}}"'] & \prod_{i=1}^{r} \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B,\tau_i) \arrow[d, "\mathrm{ev}_{\mathbf{L}}"] \\
\prod_{E\in\mathbf{E}} X \arrow[r, "\Delta"] & \prod_{E\in\mathbf{E}} X\times X
\end{tikzcd}$$

with arrows defined by the above adaptations to the evaluation stacks for $\mathbf{S} \subseteq E(G)\cup L(G)$ *with* $\mathbf{E}\subseteq\mathbf{S}$*, is cartesian in the category of fs log stacks.*

In particular, the splitting morphism δ^{ev} *is finite and representable.*

Proof. The stated commutative square is the front face of the commutative box

$$\begin{tikzcd}
\widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B,\tau) \arrow[rrrr, "\delta^{\mathrm{ev}}"] \arrow[dr, dotted] \arrow[ddd, "\mathrm{ev}_{\mathbf{E}}"'] & & & & \prod_{i=1}^{r} \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B,\tau_i) \arrow[dl, dotted] \arrow[ddd, "\mathrm{ev}_{\mathbf{L}}"] \\
& \widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau) \arrow[rr, dotted] \arrow[d, dotted] & & \prod_{i=1}^{r} \widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau_i) \arrow[d, dotted] & \\
& \prod_{E\in\mathbf{E}} \mathcal{X} \arrow[rr, dotted] & & \prod_{E\in\mathbf{E}} \mathcal{X}\times\mathcal{X} & \\
\prod_{E\in\mathbf{E}} X \arrow[rrrr, "\Delta"] \arrow[ur, dotted] & & & & \prod_{E\in\mathbf{E}} X\times X \arrow[ul, dotted]
\end{tikzcd}$$

with back face the cartesian square from Theorem 5.8 and the sides cartesian squares defining the evaluation stacks. Hence the stated diagram is cartesian.

The claimed properties of the splitting morphism δ^{ev} follow since an fs fiber product is the saturation and integralization of the ordinary fiber product. ∎

Remark 5.16. For systematic reasons we work in the category of log schemes over B in this section, and thus all products in the statement of Corollary 5.15 are fiber products over B. For explicit computations this leads to fibered sums of lattices, which sometimes require an extra treatment of multiplicities due to saturation issues. This additional step can be avoided by observing that the statement of Corollary 5.15 holds unchanged when interpreting the products as absolute products rather than as products over B, but still with $\mathcal{X}$ the relative Artin fan of X/B.

This statement is not a formal consequence of general properties of fiber products, but is due to the connectedness of the graph G given by τ, as in the argument in the proof of Proposition 5.13. To explain this let $\prod_B$ denote the relative fiber product and $\prod$ the absolute one. To check the universal property of the commutative square in Corollary 5.15 with absolute products, let be given a morphism $W \to \prod_i \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)$, $i = 1, \ldots, r$, such that the composition with $\mathrm{ev}_{\mathbf{L}}$ factors over Δ. For each leg $L = (E, v) \in L(G_i)$ we obtain an evaluation map $f_i \circ p_L : W \to \mathcal{X}$, and by composing with $\mathcal{X} \to B$ a map $b_L : W \to B$. This map is independent of the choice of $L \in L(G_i)$ since the i-th component of $W \to \prod_i \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)$ defines a punctured map over B, but a priori may vary with i. Now the factorization of ev_L over Δ implies that if the i-th and j-th vertex of G are connnected by an edge then the maps $W \to B$ obtained for i and j coincide. Since G is connected we conclude that all these maps agree. Hence the map $W \to \prod_i \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)$ factors over $(\prod_B)_i \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)$, and in turn the composition with $\mathrm{ev}_{\mathbf{L}}$ factors over $(\prod_B)_E X$. We are then in position to apply Corollary 5.15 in the stated form to obtain the unique lift to $\widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \tau)$.

By the corollary, we obtain a proper push-forward homomorphism in Chow theory for algebraic stacks, as defined by Kresch [45], for the evaluation stacks:

$$\delta_*^{\mathrm{ev}} : A_*(\underline{\mathfrak{M}}^{\mathrm{ev}}(\mathcal{X}/B, \tau)) \to A_*\Big(\prod_i \underline{\mathfrak{M}}^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)\Big), \quad \alpha \mapsto \delta_*(\alpha). \tag{5.14}$$

Note that we can work with markings or weak markings here because the corresponding stacks have the same reductions (Proposition 3.33).

It remains to relate δ^{ev} with the splitting morphism for moduli spaces of punctured maps to X rather than $\mathcal{X}$ and to show compatibility with the obstruction theory. Note that these results use the unenhanced, basic log structures on the moduli stacks.

Proposition 5.17. *Let $X \to B$ be a morphism of fs logarithmic schemes over $\Bbbk$ fulfilling the assumptions stated at the beginning of Chapter 3. Let $\tau_1, \ldots, \tau_r$ be the global types of punctured maps (Definition 2.44) obtained by splitting a global type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ along a subset of edges $\mathbf{E}$. Then there is a cartesian diagram*

$$\begin{tikzcd} \mathcal{M}(X/B,\tau) \arrow[r, "\delta"] \arrow[d, "\varepsilon"'] & \prod_{i=1}^r \mathcal{M}(X/B,\tau_i) \arrow[d, "\widehat{\varepsilon}=\prod_i \varepsilon_i"] \\ \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau) \arrow[r, "\delta^{\mathrm{ev}}"] & \prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau_i) \end{tikzcd} \tag{5.15}$$

with horizontal arrows the splitting maps from Proposition 5.4, finite and representable by Corollary 5.15, and the vertical arrows the canonical strict morphisms. Here we assume the set of edges and legs $\mathbf{S} \subseteq E(G) \cup L(G)$ used in the definition of the evaluation stacks (Definition 5.14) contains the set $\mathbf{E} \subseteq E(G)$ of splitting edges.

Analogous statements hold for decorated and for weakly marked versions of the moduli stacks (Definition 3.8).

Proof. We argue by spelling out the definitions of the various stacks. Indeed, a pair of morphisms from an fs log scheme W to $\prod_{i=1}^r \mathcal{M}(X/B, \tau_i)$ and to $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau)$ together with an isomorphism of their images in $\prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)$ is equivalent to (1) an ordinary stable map $(\underline{C}/\underline{W}, \underline{p}, \underline{f})$ to $\underline{X}$ marked by the genus-decorated graph $(G, \mathbf{g})$ given by τ, and (2) a punctured map $(C^\circ/W, \mathbf{p}, f_{\mathcal{X}})$ to $\mathcal{X}$ producing the morphism $W \to \prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)$ by splitting at the nodes labeled by $\mathbf{E} \subseteq E(G)$. Note that (1) is obtained by the schematic matching condition at the paired marked points provided by the evaluation stacks. Since $X \to \mathcal{X}$ is strict, $\underline{f}$ and $f_{\mathcal{X}}$ together are the same as a log morphism $f : C^\circ \to X$. Moreover, a marking by τ is equivalent to markings by τ_i of the punctured maps $(C_i^\circ/W, \mathbf{p}_i, f_i)$ obtained by splitting. The correspondence is also easily seen to be functorial. Thus the fiber categories over W of the cartesian product and of $\mathcal{M}(X/B, \tau)$ are equivalent. ∎

5.3.1 Notation for obstruction theories

To bring in the perfect obstruction theories discussed in Chapter 4, we now in addition to $\tau, \tau_i, \mathbf{E}, \mathbf{S}$ as in Proposition 5.17 assume $X \to B$ to be log smooth. To analyze the obstruction theories in (5.15), we introduce the following short-hand notation:[5]

$$\begin{array}{lll}
\mathcal{M}_{\mathrm{gl}} := \mathcal{M}(X/B, \tau), & \mathcal{M}_i := \mathcal{M}(X/B, \tau_i), & \mathcal{M}_{\mathrm{spl}} := \prod_{i=1}^r \mathcal{M}_i \\
\mathfrak{M}_{\mathrm{gl}} := \mathfrak{M}(\mathcal{X}/B, \tau), & \mathfrak{M}_i := \mathfrak{M}(\mathcal{X}/B, \tau_i), & \mathfrak{M}_{\mathrm{spl}} := \prod_{i=1}^r \mathfrak{M}_i \\
\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}} := \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau), & \mathfrak{M}^{\mathrm{ev}}_i := \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau_i), & \mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}} := \prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}}_i
\end{array} \tag{5.16}$$

Denote further by $\overline{C}_i^\circ \to \mathcal{M}_i$ and by $C^\circ \to \mathcal{M}_{\mathrm{gl}}$ the universal curves over $\mathcal{M}_i$ and $\mathcal{M}_{\mathrm{gl}}$, respectively, by $C_i^\circ \to \mathcal{M}_{\mathrm{spl}}$ the pullback of $\overline{C}_i^\circ$ under the projection from the product $\mathcal{M}_{\mathrm{spl}} \to \mathcal{M}_i$, and write $\pi_{\mathrm{spl}} : C^\circ_{\mathrm{spl}} = \coprod_i C_i^\circ \to \mathcal{M}_{\mathrm{spl}}$. We also have universal morphisms $f : C^\circ \to X$, $f_{\mathrm{spl}} : C^\circ_{\mathrm{spl}} \to X$, and the subspaces of special points to be considered

[5] For the sake of being specific we work with the marked versions here. Analogous results also hold for the weakly marked cases.

$\iota : Z \to C^\circ$, $\iota_{\mathrm{spl}} : Z_{\mathrm{spl}} \to C^\circ_{\mathrm{spl}}$ with projections $p = \pi \circ \iota$ and $p_{\mathrm{spl}} = \pi_{\mathrm{spl}} \circ \iota_{\mathrm{spl}}$ to $\mathcal{M}_{\mathrm{gl}}$ and $\mathcal{M}_{\mathrm{spl}}$, respectively. Here Z is the union of the images of the punctured and nodal sections labeled by $\mathbf{S} \subseteq E(G) \cup L(G)$, while Z_{spl} is the union of punctured and nodal sections given by $\mathbf{S}_i \subseteq E(G_i) \cup L(G_i)$, $i = 1, \ldots, r$, obtained from $\mathbf{S}$ by splitting, both endowed with the induced log structures making ι, ι_{spl} strict.

5.3.2 The fundamental diagram

We consider the following commutative diagram:

$$\begin{tikzcd}
\widetilde{C}^\circ \arrow[r] \arrow[d, "\kappa"] \arrow[dd, bend right, "\widetilde{\pi}"'] \arrow[rr, bend left, "\tilde{f}"] & C^\circ_{\mathrm{spl}} = \coprod_i C_i^\circ \arrow[r, "f_{\mathrm{spl}}"] \arrow[dd, "\pi_{\mathrm{spl}}"] & X \\
C^\circ \arrow[urr, "f"'] \arrow[d, "\pi"] & & \\
\mathcal{M}_{\mathrm{gl}} \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & \mathcal{M}_{\mathrm{spl}} = \prod_i \mathcal{M}_i \arrow[d, "\widehat{\varepsilon} = \prod_i \varepsilon_i"] & \\
\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}} \arrow[r, "\delta^{\mathrm{ev}}"] & \mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}} = \prod_i \mathfrak{M}^{\mathrm{ev}}_i . &
\end{tikzcd} \tag{5.17}$$

The lower square is the cartesian square from Proposition 5.17 with strict vertical arrows.

The strict map $\kappa : \widetilde{C}^\circ \to C^\circ$ is the map induced by splitting the nodal sections of $C^\circ \to \mathcal{M}_{\mathrm{gl}}$ given by $\mathbf{E} \subseteq \mathbf{S}$ according to Proposition 5.4. The underlying morphism $\underline{\kappa}$ of ordinary stacks is therefore the corresponding partial normalization from Definition 5.1.

The upper square thus identifies the pullback of C°_{spl} with the pre-stabilization of $\widetilde{C}^\circ$ (Definition 2.6). This part of the diagram is a pullback of nodal curves, cartesian only in the category of stacks, because of the pre-stabilization.

The morphism $\tilde{f}$ is as defined by the diagram. There is also the closed substack $\widetilde{Z} = \kappa^{-1}(Z) \to \widetilde{C}^\circ$ of special points on $\widetilde{C}^\circ$ with projection $\tilde{p} : \widetilde{Z} \to \mathcal{M}_{\mathrm{gl}}$, endowed with the log structure making $\widetilde{Z} \to \widetilde{C}^\circ$ strict.

5.3.3 An obstruction theory for ε and $\widehat{\varepsilon}$

The discussion in Section 4.2 provides obstruction theories $\mathbb{G} \to \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}}$ for $\varepsilon : \mathcal{M}_{\mathrm{gl}} \to \mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}$ and $\mathbb{G}_{\mathrm{spl}} \to \mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}}$ for $\widehat{\varepsilon} : \mathcal{M}_{\mathrm{spl}} \to \mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}$ with

$$\mathbb{G} = R\widetilde{\pi}_*\big(\tilde{f}^*\Omega_{X/B} \otimes \omega_{\widetilde{\pi}}(\widetilde{Z})\big), \quad \mathbb{G}_{\mathrm{spl}} = R\pi_{\mathrm{spl}*}\big(f^*_{\mathrm{spl}}\Omega_{X/B} \otimes \omega_{\pi_{\mathrm{spl}}}(Z_{\mathrm{spl}})\big). \tag{5.18}$$

Recall that this obstruction theory is obtained by taking the cone of a morphism of perfect obstruction theories provided by Proposition 4.3:

$$\begin{array}{ccc} L\widehat{\varepsilon}^*\mathbb{F}_{\mathrm{spl}} & \longrightarrow & \mathbb{E}_{\mathrm{spl}} \\ {\scriptstyle L\widehat{\varepsilon}^*\Psi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ L\widehat{\varepsilon}^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}}} & \longrightarrow & \mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}}} \end{array}$$

5.3.4 A justice of obstructions[6]

We now have four deformation/obstruction situations with corresponding perfect obstruction theories. Given $T \to \mathcal{M}_{\mathrm{gl}}$ a morphism from an affine scheme and $f_T : C^\circ_T \to X$, $h_T : Z_T \to X$, $\tilde{f}_T : \widetilde{C}^\circ_T \to X$, $\tilde{h}_T : \widetilde{Z}_T \to X$ the respective base-changes to T of the universal morphisms from the universal curve and universal sections, pulled back to $\mathcal{M}_{\mathrm{gl}}$ in the last two instances, these are as follows. All deformation situations are relative $\mathfrak{M}_{\mathrm{gl}}$, with the last two pulled back from a deformation situation relative $\mathfrak{M}_{\mathrm{spl}}$.

$(\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}})$ Deforming $f_T : C^\circ_T \to X$:

$$\mathbb{E} = R\pi_*(f^*\Omega_{X/B} \otimes \omega_\pi) \to \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}.$$

$(\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}})$ Deforming $h_T : Z_T \to X$:

$$L\varepsilon^*\mathbb{F} = p_*(h^*\Omega_{X/B}) \to L\varepsilon^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}.$$

$(\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}})$ Deforming $\tilde{f}_T : \widetilde{C}^\circ_T \to X$:

$$L\delta^*\mathbb{E}_{\mathrm{spl}} = R\tilde{\pi}_*(\tilde{f}^*\Omega_{X/B} \otimes \omega_{\tilde{\pi}}) \to L\delta^*\mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}}}.$$

$(\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}})$ Deforming $\tilde{h}_T : \widetilde{Z}_T \to X$:

$$L\delta^*L\widehat{\varepsilon}^*\mathbb{F}_{\mathrm{spl}} = \tilde{p}_*(\tilde{h}^*\Omega_{X/B}) \to L\delta^*L\widehat{\varepsilon}^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}}}.$$

Lemma 5.18. *There is a morphism of distinguished triangles*

$$\begin{array}{ccccccc} L\delta^*L\widehat{\varepsilon}^*\mathbb{F}_{\mathrm{spl}} & \longrightarrow & L\delta^*\mathbb{E}_{\mathrm{spl}} & \longrightarrow & \mathbb{G} & \longrightarrow & L\delta^*L\widehat{\varepsilon}^*\mathbb{F}_{\mathrm{spl}}[1] \\ \downarrow & & \downarrow & & \| & & \downarrow \\ L\varepsilon^*\mathbb{F} & \longrightarrow & \mathbb{E} & \longrightarrow & \mathbb{G} & \longrightarrow & L\varepsilon^*\mathbb{F}[1] \end{array}$$

with $\mathbb{G} = L\delta^*\mathbb{G}_{\mathrm{spl}} = R\tilde{\pi}_*(\tilde{f}^*\Omega_{X/B} \otimes \omega_{\tilde{\pi}}(\widetilde{Z}))$.

[6] Our Babel of coauthors proposes this collective noun for a system of compatible obstructions.

Proof. The lower row in the claimed diagram was produced in (4.17) in the proof of Proposition 4.5 by applying $R\pi_*$ to (4.16) tensored with $f^*\Omega_{X/B}$. We claim that (4.16) appears as the lower row in the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \kappa_*\omega_{\tilde{\pi}} \arrow[r] \arrow[d] & \kappa_*(\omega_{\tilde{\pi}}(\tilde{Z})) \arrow[r] \arrow[d, equal] & \kappa_*\tilde{\iota}_*\mathcal{O}_{\tilde{Z}}[1] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \omega_\pi \arrow[r] & \kappa_*(\omega_{\tilde{\pi}}(\tilde{Z})) \arrow[r] & \iota_*\mathcal{O}_Z[1] \arrow[r] & 0. \end{tikzcd} \tag{5.19}$$

Away from the nodal locus $Z'' \subset Z$ the upper and lower rows are identical, and this identification defines the diagram there. Étale locally near a node, the arrow $\kappa_*(\omega_{\tilde{\pi}}(\tilde{Z})) \to \iota_*\mathcal{O}_Z[1]$ takes the difference of the residues of a differential with at most simple poles along the two components of $\tilde{Z}'' \simeq Z'' \amalg Z''$ defined by the two branches at the node. This map factors as κ_* of the residue map

$$\rho \colon \omega_{\tilde{\pi}}(\tilde{Z}'') \to \tilde{\iota}_*\mathcal{O}_{\tilde{Z}''}[1]$$

and the [1]-twist of the difference map

$$\kappa_*\tilde{\iota}_*\mathcal{O}_{\tilde{Z}''} = \iota_*\mathcal{O}_{Z''} \oplus \iota_*\mathcal{O}_{Z''} \to \iota_*\mathcal{O}_{Z''}, \quad (a,b) \mapsto a-b.$$

The kernel of $\kappa_*\rho$ selects differentials without poles, that is, $\kappa_*\omega_{\tilde{\pi}}$. This extends the construction of Diagram (5.19) over the nodal locus.

To produce the morphism of triangles in the statement it remains to show that tensoring the upper row of (5.19) with $f^*\Omega_{X/B}$ and applying $R\pi_*$ leads to the upper row of the claimed diagram. From Proposition 4.5 we already know that the middle term leads to $\mathbb{G}$:

$$\mathbb{G} \overset{(5.18)}{=} R\tilde{\pi}_*\big(\tilde{f}^*\Omega_{X/B} \otimes \omega_{\tilde{\pi}}(\tilde{Z})\big) = R\pi_*\big(f^*\Omega_{X/B} \otimes \kappa_*(\omega_{\tilde{\pi}}(\tilde{Z}))\big).$$

The other two terms are readily obtained by the projection formula for κ using $\tilde{\pi} = \pi \circ \kappa$, $\tilde{f} = f \circ \kappa$, $\tilde{h} = \tilde{f} \circ \tilde{\iota}$, $\tilde{p} = \tilde{\pi} \circ \tilde{\iota}$:

$$\begin{aligned} L\delta^*\mathbb{E}_{\mathrm{spl}} &= R\tilde{\pi}_*\big(\tilde{f}^*\Omega_{X/B} \otimes \omega_{\tilde{\pi}}\big) = R\pi_*(f^*\Omega_{X/B} \otimes \kappa_*\omega_{\tilde{\pi}}) \\ L\delta^* L\hat{\varepsilon}^*\mathbb{F}_{\mathrm{spl}} &= \tilde{p}_*(\tilde{h}^*\Omega_{X/B}) = R\tilde{\pi}_*\big(\tilde{f}^*\Omega_{X/B} \otimes \tilde{\iota}_*\mathcal{O}_{\tilde{Z}}\big) \\ &= R\pi_*(f^*\Omega_{X/B} \otimes \kappa_*\tilde{\iota}_*\mathcal{O}_{\tilde{Z}}). \end{aligned}$$

■

Theorem 5.19. *Let $X \to B$ be a log smooth morphism of fs logarithmic schemes over $\Bbbk$ fulfilling the assumptions stated at the beginning of Chapter 3, and τ, τ_i,* **E**, **S**

as in Proposition 5.17. Then with the notation of (5.16), we have

(1) *The obstruction theory $\mathbb{G} \to \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}}$ for*

$$\mathcal{M}_{\mathrm{gl}} = \mathcal{M}_{\mathrm{gl}}(X/B,\tau) \to \mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}} = \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau)$$

coincides with the pullback of one of the obstruction theories $\mathbb{G}_{\mathrm{spl}} \to \mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}}$ (Remark 4.4) for

$$\mathcal{M}_{\mathrm{spl}} = \prod_i \mathcal{M}(X/B,\tau_i) \to \mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}} = \prod_i \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau_i)$$

described in Section 5.3.3.

(2) *If $\widehat{\varepsilon}^!$ and $\varepsilon^!$ denote Manolache's virtual pullback defined using the two given obstruction theories for the vertical arrows in diagram (5.15), then for $\alpha \in A_*(\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau))$, we have the identity*

$$\widehat{\varepsilon}^! \delta^{\mathrm{ev}}_*(\alpha) = \delta_* \varepsilon^!(\alpha).$$

Proof. (1) The morphism between the obstruction theories in question appear as the joint middle square in the following diagram of two adjacent cubes:

$$\begin{tikzcd}
L\delta^*\mathbb{E}_{\mathrm{spl}} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbb{G} \arrow[rr] \arrow[dr, dashed] \arrow[dd, equal] & & L\delta^* L\widehat{\varepsilon}^*\mathbb{F}_{\mathrm{spl}}[1] \arrow[dr] \arrow[dd] & \\
& L\delta^*\mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}}} \arrow[rr] \arrow[dd] & & L\delta^*\mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}} \arrow[rr] \arrow[dd] & & L\delta^* L\widehat{\varepsilon}^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}}}[1] \arrow[dd] \\
\mathbb{E} \arrow[rr] \arrow[dr] & & \mathbb{G} \arrow[rr] \arrow[dr, dashed] & & L\varepsilon^*\mathbb{F}[1] \arrow[dr] & \\
& \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}} \arrow[rr] & & \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}} \arrow[rr] & & L\varepsilon^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}[1]
\end{tikzcd}$$

The back face is the morphism of triangles from Lemma 5.18. The bottom face is commutative by the construction of the obstruction theory with point conditions $\mathbb{G} \to \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}}$ in (4.15) based on Proposition 4.3. Similarly, the top face is commutative as the pullback by δ of the corresponding diagram for $\mathbb{G}_{\mathrm{spl}} \to \mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}}$. The front face of the diagram is the morphism of distinguished triangles of cotangent complexes for the compositions $\mathcal{M}_{\mathrm{gl}} \to \mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}} \to \mathfrak{M}_{\mathrm{gl}}$ and $\mathcal{M}_{\mathrm{spl}} \to \mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}} \to \mathfrak{M}_{\mathrm{spl}}$, and hence is commutative as well.

For commutativity of the left face we argue in two steps. First apply functoriality of obstruction theories, Lemma 4.1, to compare the pulled-back obstruction theory $(\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}})$ for f_{spl} with the obstruction theory for $\tilde{f}$, both relative $\mathfrak{M}_{\mathrm{spl}}$, to obtain the commutative square

$$\begin{tikzcd}
L\delta^*\mathbb{E}_{\mathrm{spl}} \arrow[r] \arrow[d] & L\delta^*\mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}}} \arrow[d] \\
\widetilde{\mathbb{E}} \arrow[r] & \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}.
\end{tikzcd} \tag{5.20}$$

Here $\widetilde{\mathbb{E}} = R\tilde{\pi}_*(\tilde{f}^*\Omega_{X/B} \otimes \omega_{\tilde{\pi}})$ and we replaced the lower right-hand corner $\mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{spl}}}$ by $\mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}$ using functoriality of the cotangent complex. Note also that the proof of Lemma 4.1 did not use the general assumption in Section 4.1 that M is an open substack of the stack of diagrams described in (4.1), so does apply to the non-universal family over $\mathcal{M}_{\mathrm{gl}}$ given by $\tilde{f}$.

We are then in the situation of Section 4.1.7 with $Y \to S$ the universal curve over $\mathfrak{M}_{\mathrm{gl}}$, Z the partial normalization of this curve, and $M = N = \mathcal{M}_{\mathrm{gl}}$. Thus Proposition 4.3 provides the commutative square

$$\begin{tikzcd} \widetilde{\mathbb{E}} \arrow[r] \arrow[d] & \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}} \arrow[d] \\ \mathbb{E} \arrow[r] & \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}. \end{tikzcd} \tag{5.21}$$

Again, this result did not use universality of the family of maps over $\mathcal{M}_{\mathrm{gl}}$ given by $\tilde{f}$. Composing the two squares (5.20) and (5.21) proves commutativity of the left face of our big diagram of adjacent cubes.

An analogous argument for the nodal locus Z and its pullback $\widetilde{Z} \subset \widetilde{C}^\circ$ instead of C° and $\widetilde{C}^\circ$ also shows commutativity of the right face.

Thus the whole diagram is commutative except possibly the middle, separating square that describes the morphism of interest from the pullback of the obstruction theory for $(\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}})$ to $(\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}})$.

However, chasing the diagram, we see that the two morphisms from $\mathbb{G}$ to the front right corner $L\varepsilon^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}[1]$, one via the top dashed arrow, the other via the bottom dashed arrow, agree. Their difference factors over a homomorphism

$$\mathbb{G} \to \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}.$$

The set of such homomorphisms acts transitively on the set of dashed arrows on the bottom face defining the obstruction theory for $(\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}})$ as discussed in Remark 4.4 Thus there is a choice of dashed bottom arrow making the separating middle square of the diagram commutative, as claimed.

(2) This follows from the morphism δ^{ev} being finite and representable, hence projective, and the push-pull formula of [50, Theorem 4.1 (iii)]. ∎

5.3.5 Gluing by the numbers

We now achieve a numerical gluing formula for Gromov–Witten invariants for classes in $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau)$ whose push-forward to $\prod \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)$ decomposes as a sum of products of classes. This is for example the case for point classes in $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau)$, or if all gluing strata are toric [71].

Corollary 5.20. *In the situation of Theorem 5.19 let $\alpha \in A_*(\mathfrak{M}(\mathcal{X}/B,\tau))$ and assume that there exists $\alpha_{i,\mu} \in A_*(\mathfrak{M}(\mathcal{X}/B,\tau_i))$, $i = 1,\ldots,r$, $\mu = 1,\ldots,m$, with*

$$\delta_*^{\mathrm{ev}}(\alpha) = \sum_{\mu=1}^{m} \alpha_{1,\mu} \times \cdots \times \alpha_{r,\mu}.$$

Then writing $\varepsilon_i : \mathcal{M}(X/B,\tau_i) \to \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau_i)$ for the canonical map, the following equality of associated virtual classes holds in $A_*(\prod_i \mathcal{M}(X/B,\tau_i))$:

$$\delta_*\varepsilon^!(\alpha) = \sum_{\mu=1}^{m} \varepsilon_1^!(\alpha_{1,\mu}) \times \cdots \times \varepsilon_r^!(\alpha_{r,\mu}).$$

Proof. The claimed formula follows readily from Theorem 5.19 (2) by observing that

$$\widehat{\varepsilon}^!(\alpha_{1,\mu} \times \cdots \times \alpha_{r,\mu}) = \varepsilon_1^!(\alpha_{1,\mu}) \times \cdots \times \varepsilon_r^!(\alpha_{r,\mu}). \qquad \blacksquare$$

5.3.6 Compatibility with contractions of types

We end this section by noting that the relative obstruction theories are also compatible with contraction morphisms relating different global types (Definition 2.44 (1)).

Proposition 5.21. *Let $X \to B$ be as in Theorem 5.19 and assume $\tau' \to \tau$ is a contraction morphism of global types. Then the commutative diagram*

$$\begin{array}{ccc} \mathcal{M}(X/B,\tau') & \longrightarrow & \mathcal{M}(X/B,\tau) \\ \Big\downarrow{\scriptstyle \varepsilon'} & & \Big\downarrow{\scriptstyle \varepsilon} \\ \mathfrak{M}(\mathcal{X}/B,\tau') & \longrightarrow & \mathfrak{M}(\mathcal{X}/B,\tau) \end{array}$$

is cartesian, and the relative obstruction theory for ε pulls back to the relative obstruction theory for ε'. Taking curve classes into consideration, if $\boldsymbol{\tau} = (\tau,\mathbf{A})$, the commutative diagram

$$\begin{array}{ccc} \coprod_{\boldsymbol{\tau}'=(\tau',\mathbf{A}')} \mathcal{M}(X/B,\boldsymbol{\tau}') & \longrightarrow & \mathcal{M}(X/B,\boldsymbol{\tau}) \\ \Big\downarrow{\scriptstyle \varepsilon'} & & \Big\downarrow{\scriptstyle \varepsilon} \\ \mathfrak{M}(\mathcal{X}/B,\tau') & \longrightarrow & \mathfrak{M}(\mathcal{X}/B,\tau) \end{array} \tag{5.22}$$

is cartesian, and the same statement on relative obstruction theories holds. Here, the disjoint union is over all decorations $\boldsymbol{\tau}'$ of τ' such that the contraction morphism $\tau' \to \tau$ induces a contraction morphism $\boldsymbol{\tau}' \to \boldsymbol{\tau}$.

Analogous results hold for weakly marked versions of the stacks (Definition 3.8), and for evaluation stacks on the bottom (Definition 5.14).

Proof. That the diagrams are Cartesian follows from the definition of markings and decorated markings of punctured maps (Definition 3.8).

The statement about obstruction theories then follows from the functoriality statement Lemma 4.1 and the construction in Section 4.2 of the relative obstruction theory for $\mathcal{M}(X/B,\tau) \to \mathfrak{M}(\mathcal{X}/B,\tau)$. ■

Remark 5.22. The formalism for gluing presented here was found after many futile attempts leading to practically useless gluing procedures. With hindsight compatibility with the virtual formalism provides the strongest guiding principle that rules out many alternative approaches. From this point of view one discovers the imperative that one work with obstruction theories relative to a class of unobstructed base stacks that induce the gluing.

A first attempt would work with moduli stacks $\mathfrak{M}(\mathcal{X}/B)$ of punctured maps to the relative Artin fan $\mathcal{X} \to B$. This approach does indeed work, but it is often problematic for practical applications because the gluing map $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau}) \to \prod_i \mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau}_i)$ is neither representable nor proper, hence does not allow pushing forward of cycles.

The key insight is to use evaluation stacks to add just enough information to get rid of the stacky nature of the gluing in $\mathfrak{M}(\mathcal{X}/B)$, thus leading to a finite and representable splitting map δ^{ev}. In addition, δ^{ev} fits into the expected gluing diagram stated in Corollary 5.15 thus providing a practical path to explicit computations.

5.4 Gluing in the degeneration setup

We now apply our gluing theorems to the degeneration situation previously studied in [3]. In this case B is a smooth affine curve over $\operatorname{Spec} \Bbbk$ with log structure trivial except at a marked point $b_0 \in B$, and $\mathcal{A}_X$ is assumed Zariski. Base change to b_0 produces a log smooth space X_0 over the standard log point $\operatorname{Spec}(\mathbb{N} \to \Bbbk)$. Let $\beta = (g,\bar{\mathbf{u}},A)$ be a class of punctured maps to X. Note that $\Sigma(X_0) = \Sigma(X)$, so we can view β also as a class of punctured maps to X_0. The fiber of the tropicalization $\Sigma(X_0) \to \Sigma(b_0) = \mathbb{R}_{\geq 0}$ of the projection $X_0 \to b_0$ over $1 \in \mathbb{R}_{\geq 0}$ defines a polyhedral complex $\Delta(X_0) = \Delta(X)$. Restricting to this fiber turns our cone complexes into the polyhedral complexes of traditional tropical geometry.

The main result of [3] gives the following decomposition of the virtual fundamental class of $\mathcal{M}(X_0/b_0,\beta)$ in terms of rigid tropical maps to $\Delta(X_0)$. We emphasize that this result uses the marked rather than weakly marked versions of the moduli stacks.

Theorem 5.23. *Let β be a class of stable logarithmic maps to X_0/b_0. Then we have the following equality of Chow classes on $\mathcal{M}(X_0/b_0,\beta)$:*

$$[\mathcal{M}(X_0/b_0,\beta)]^{\mathrm{virt}} = \sum_{\boldsymbol{\tau}=(\tau,\mathbf{A})} \frac{m_\tau}{|\operatorname{Aut}(\boldsymbol{\tau})|} j_{\boldsymbol{\tau}*}[\mathcal{M}(X_0/b_0,\boldsymbol{\tau})]^{\mathrm{virt}}.$$

The sum runs over representatives of isomorphism classes of realizable global types $\boldsymbol{\tau}$ of punctured maps to X_0 over b_0 of total class $(g, \bar{\mathbf{u}}, A)$ and with basic monoid $Q_\tau \simeq \mathbb{N}$. The multiplicity m_τ is the index of the image of the homomorphism $\mathbb{N} \to Q_\tau$ given by the map $\mathcal{M}(X_0/b_0, \beta) \to b_0$. The morphism $j_{\boldsymbol{\tau}} : \mathcal{M}(X_0/b_0, \boldsymbol{\tau}) \to \mathcal{M}(X_0/b_0, \beta)$ is induced by the contraction morphism $\tau \to \beta$. Finally, $\mathrm{Aut}(\boldsymbol{\tau})$ *denotes the group of automorphisms of the decorated type $\boldsymbol{\tau}$, i.e., automorphisms of the underlying graph G preserving* $\mathbf{g}$, $\boldsymbol{\sigma}$, $\mathbf{u}$ *and* $\mathbf{A}$.

5.4.1 Degenerate types

Theorem 5.25 below is an analogous result in the punctured case, which also provides a stratified version in the case without punctures. Before stating this result we need some preparations concerning types in degeneration situations. Since one works with log spaces over b_0 and $\Sigma(b_0) = \mathbb{R}_{\geq 0}$, all tropical objects come with a map to $\mathbb{R}_{\geq 0}$. We denote all these maps by p in the following. Assuming $X_0 \to b_0$ is the fiber over the unique marked point $b_0 \to B$ in a log smooth curve B over the trivial log point, the tropicalization of a punctured map over the generic point $\eta \in \underline{B}$ maps to $0 \in \Sigma(B) = \Sigma(b_0)$ under p. Degenerations of families of punctured maps over η to b_0 then provide a contraction morphism of the associated types (Definition 2.44 (1)). This motivates the following definition.

Definition 5.24. Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ be a realizable global type (Definition 2.44 (2)) of punctured maps to X_0/b_0 (Definition 3.28) and Q_τ the associated basic monoid.

(1) We call τ *generic* if $(Q_\tau^\vee)_\mathbb{R}$ and $\boldsymbol{\sigma}(x)$ for each $x \in V(G) \cup E(G) \cup L(G)$ map to $\{0\} \subset \mathbb{R}_{\geq 0}$ under p.

(2) A *degeneration* of a realizable global type τ is a contraction morphism $\tau' \to \tau$ between realizable global types with $p : Q_{\tau'}^\vee \to \mathbb{N}$ non-constant. The *codimension* of $\tau' \to \tau$ is defined as $\mathrm{rk}\, Q_{\tau'}^{\mathrm{gp}} - \mathrm{rk}\, Q_\tau^{\mathrm{gp}}$. In the case of codimension one we define the *multiplicity* $m_{\tau'}$ as the index of $p^{\mathrm{gp}}(Q_{\tau'}^*)$ in $\mathbb{Z}$. Finally, $\mathrm{Aut}(\tau'/\tau)$ denotes the group of automorphisms of τ' commuting with $\tau' \to \tau$.

Analogous notions are used in the decorated case (Definition 3.8).

5.4.2 Degenerate types decompose

Let now $\boldsymbol{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{A})$ be a generic realizable decorated global type for X/B. By the assumption $p(\boldsymbol{\sigma}(x)) = 0$ we can view $\boldsymbol{\tau}$ also as a decorated global type for X_b/b for $b \neq b_0$. The analogue to the main results of [3] is:

Theorem 5.25. *In the above situation, additionally assuming X is simple, the following holds.*

(1) *For any point $j_b : \{b\} \hookrightarrow B$, one has* $j_b^![\mathcal{M}(X/B, \boldsymbol{\tau})]^{\mathrm{virt}} = [\mathcal{M}(X_b/b, \boldsymbol{\tau})]^{\mathrm{virt}}$.

(2) *The following equation holds:*

$$[\mathcal{M}(X_0/b_0,\boldsymbol{\tau})]^{\mathrm{virt}} = \sum_{\boldsymbol{\tau}'=(\tau',\mathbf{A}')} \frac{m_{\tau'}}{|\operatorname{Aut}(\boldsymbol{\tau}'/\boldsymbol{\tau})|} j_{\boldsymbol{\tau}'*}[\mathcal{M}(X_0/b_0,\boldsymbol{\tau}')]^{\mathrm{virt}} \tag{5.23}$$

The sum runs over representatives of isomorphism classes of degenerations $\boldsymbol{\tau}'=(\tau',\mathbf{A}')\to\boldsymbol{\tau}$ *of realizable global types of punctured maps to* X/B *of codimension one, with* $m_{\tau'}$ *its multiplicity.*

Proof. By Proposition 3.29, $\boldsymbol{\tau}$ can be viewed both as a type realizable over B and as a type realizable over $b\in B$ for $b\neq b_0$. Thus $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau})$ is non-empty and $\mathfrak{M}(\mathcal{X}_b/b,\boldsymbol{\tau})=\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau})\times_B b$ is non-empty for $b\in B\setminus\{b_0\}$. By Proposition 3.30, $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau})$ is pure-dimensional. Further, by the same proposition, every irreducible component of $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau})$ contains a point whose corresponding punctured map has tropical type $\boldsymbol{\tau}$, as all other strata are of lower dimension. By genericity of the type $\boldsymbol{\tau}$, the stratum of $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau})$ of points with type $\boldsymbol{\tau}$ maps to the open stratum of B. Thus the restriction of $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau})\to B$ to each irreducible component is dominant. There are no embedded components by the local description in Remark 3.27. We conclude that the structure map $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau})\to B$ is flat.

(1) then follows immediately from general properties of virtual pull-backs.

For (2), as in the proof of [3, Theorem 3.11], we begin by showing the corresponding decomposition as Chow classes

$$[\mathfrak{M}(\mathcal{X}_0/b_0,\tau)] = \sum_{\tau'\to\tau} \frac{m_{\tau'}}{|\operatorname{Aut}(\tau'/\tau)|}\iota_{\tau'*}[\mathfrak{M}(\mathcal{X}_0/b_0,\tau')]. \tag{5.24}$$

Here τ is the underlying global type of $\boldsymbol{\tau}$, and $\tau'\to\tau$ runs over all contraction morphisms as in the statement of the theorem (without the decoration). Finally, $\iota_{\tau'}:\mathfrak{M}(\mathcal{X}_0/b_0,\tau')\to\mathfrak{M}(\mathcal{X}_0/b_0,\tau)$ is the natural morphism. However, using the smooth local description of $\mathfrak{M}(\mathcal{X}/B,\tau)$ given in Remark 3.27 and the fact that $|\operatorname{Aut}(\tau'/\tau)|$ is the degree of the finite map $\iota_{\tau'}$ onto its image, we easily obtain the result using standard toric geometry. We leave the details to the reader.

We now make use of the diagram (5.22) for a given choice of contraction $\tau'\to\tau$, and we see by the push-pull result of [50, Theorem 4.1] that

$$\begin{aligned} \varepsilon^!\iota_{\tau'*}[\mathfrak{M}(\mathcal{X}_0/b_0,\tau')] &= \sum_{\boldsymbol{\tau}'=(\tau',\mathbf{A}')} j_{\tau'*}(\varepsilon')^![\mathfrak{M}(\mathcal{X}_0/b_0,\boldsymbol{\tau}')] \\ &= \sum_{\boldsymbol{\tau}'=(\tau',\mathbf{A}')} j_{\tau'*}[\mathcal{M}(X_0/b_0,\boldsymbol{\tau}')]^{\mathrm{virt}}, \end{aligned} \tag{5.25}$$

where the sum is over all choices of decorations $\boldsymbol{\tau}'$ of τ' giving a contraction morphism $\boldsymbol{\tau}'\to\boldsymbol{\tau}$ compatible with $\tau'\to\tau$. On the other hand, $\operatorname{Aut}(\tau'/\tau)$ acts on the set of all such decorations, with the orbit of a decoration $\boldsymbol{\tau}'$ having stabilizer $\mathrm{Aut}(\boldsymbol{\tau}'/\boldsymbol{\tau})$. Thus we may rewrite the last summation of (5.25) as

$$\sum_{\boldsymbol{\tau}'=(\tau',\mathbf{A}')} j_{\tau'*}[\mathcal{M}(X_0/b_0,\boldsymbol{\tau}')]^{\mathrm{virt}} \frac{|\mathrm{Aut}(\tau'/\tau)|}{|\mathrm{Aut}(\boldsymbol{\tau}'/\boldsymbol{\tau})|},$$

where now the sum is over a set of representatives of isomorphism classes of type $\boldsymbol{\tau}'$ with a contraction morphism $\boldsymbol{\tau}' \to \boldsymbol{\tau}$. Combining this with the relation (5.24) then gives the desired result. ∎

5.4.3 Splitting and factoring decomposed degenerate types

As a corollary of Theorem 5.19 we now obtain a formula for the computation of each summand $[\mathcal{M}(X_0/b_0,\boldsymbol{\tau}')]^{\mathrm{virt}}$ in (5.23) in terms of punctured Gromov–Witten theory of the strata. For the statement note that if τ_v is a global type with only one vertex, with associated stratum $\sigma \in \Sigma(X)$, then a τ_v-marked punctured map $(C^\circ/W,\mathbf{p},f)$ to X has a factorization

$$f : C^\circ \xrightarrow{f_\sigma} X_\sigma \to X,$$

where the stratum X_σ is now endowed with the log structure making the embedding $X_\sigma \to X$ strict. The composition with this strict closed embedding in fact induces an isomorphism

$$\mathcal{M}(X_0/b_0,\tau_v) \xrightarrow{\simeq} \mathcal{M}(X_\sigma/b_0,\tau_v).$$

Similarly, we obtain

$$\mathfrak{M}(\mathcal{X}_0/b_0,\tau_v) \simeq \mathfrak{M}(\mathcal{X}_\sigma/b_0,\tau_v) \quad \text{and} \quad \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}_0/b_0,\tau_v) \simeq \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}_\sigma/b_0,\tau_v).$$

Note also that $X_\sigma \to \mathcal{X}_\sigma$ is strict and smooth despite $\mathcal{X}_\sigma$ being only idealized log smooth over b_0 (see Proposition 2.48). Thus the obstruction theory developed in Section 4.2 still applies with target $X_\sigma \to \mathcal{X}_\sigma \to b_0$ and yields the same result as with $X_0 \to \mathcal{X}_0 \to b_0$. Theorem 5.19 applied to our degeneration situation can therefore be stated as follows.

Corollary 5.26. *Let $(G,\mathbf{g},\boldsymbol{\sigma},\mathbf{u},\mathbf{A})$ be a decorated type of punctured maps with basic monoid $Q_\tau \simeq \mathbb{N}$ and $\boldsymbol{\tau} = (G,\mathbf{g},\boldsymbol{\sigma},\bar{\mathbf{u}},\mathbf{A})$ the associated decorated global type. Denote by $\boldsymbol{\tau}_v$, $v \in V(G)$, the decorated global types obtained by splitting $\boldsymbol{\tau}$ at all edges, that is, for $\mathbf{E} = E(G)$. Then the diagram*

$$\begin{array}{ccc}
\mathcal{M}(X_0/b_0,\boldsymbol{\tau}) & \xrightarrow{\delta} & \prod_{v\in V(G)} \mathcal{M}(X_{\boldsymbol{\sigma}(v)}/b_0,\boldsymbol{\tau}_v) \\
\Big\downarrow{\scriptstyle \varepsilon} & & \Big\downarrow{\scriptstyle \widehat{\varepsilon}=\prod_{v\in V(G)}\varepsilon_v} \\
\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}_0/b_0,\boldsymbol{\tau}) & \xrightarrow{\delta^{\mathrm{ev}}} & \prod_{v\in V(G)} \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}_{\boldsymbol{\sigma}}(v)/b_0,\boldsymbol{\tau}_v)
\end{array}$$

with horizontal arrows the splitting maps from Proposition 5.4 finite and representable, is cartesian, and it holds

$$\delta_*[\mathcal{M}(X_{\mathbf{0}}/b_{\mathbf{0}}, \boldsymbol{\tau})]^{\mathrm{virt}} = \widehat{\varepsilon}^! \delta_*^{\mathrm{ev}}[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}_{\mathbf{0}}/b_{\mathbf{0}}, \boldsymbol{\tau})].$$

As in Corollary 5.20, a numerical formula in terms of punctured Gromov–Witten invariants of the strata X_σ of X can be derived assuming $\delta_*^{\mathrm{ev}}[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}_{\mathbf{0}}/b_{\mathbf{0}}, \boldsymbol{\tau})]$ decomposes into a sum of products. This is the case for example if all gluing strata $X_{\boldsymbol{\sigma}(E)}$, $E \in E(G)$, are toric, as proved in [71] based on Corollary 5.15.

Appendix A

Contact orders

Here we give a somewhat more sophisticated universal view on contact orders. This was the point of view we originally planned to give, but for most current applications, the simpler approach exposited in Section 2.4 suffices. Nevertheless, that approach obscures some of the subtleties of contact orders, and at times it may be worth having this more precise point of view.

For a target X with fs log structure, consider the following étale sheaves over $\underline{X}$:

$$\overline{\mathcal{M}}_X^\vee = \mathcal{H}om(\overline{\mathcal{M}}_X, \mathbb{N}) \quad \text{and} \quad \overline{\mathcal{M}}_X^* = \mathcal{H}om(\overline{\mathcal{M}}_X, \mathbb{Z}) \cong \mathcal{H}om\big(\overline{\mathcal{M}}_X^{\mathrm{gp}}, \mathbb{Z}\big).$$

Definition A.1. A *family of contact orders of* X consists of a strict morphism $Z \to X$ and a section $\mathbf{u} \in \Gamma(Z, \overline{\mathcal{M}}_Z^*)$ satisfying the following condition. Let $u : \mathcal{M}_Z \to \overline{\mathcal{M}}_Z \xrightarrow{\mathbf{u}} \mathbb{Z}$ be the composite homomorphism associated to $\mathbf{u}$. Then the map $\alpha : \mathcal{M}_Z \to \mathcal{O}_Z$ sends $u^{-1}(\mathbb{Z} \setminus \{0\})$ to 0.

We call the ideal $\mathcal{I}_\mathbf{u} \subset \mathcal{M}_Z$ generated by $u^{-1}(\mathbb{Z} \setminus \{0\})$ the *contact log-ideal* associated to $\mathbf{u}$, and denote by $\overline{\mathcal{I}}_\mathbf{u}$ the corresponding *contact ideal* in $\overline{\mathcal{M}}_Z$. These are coherent sheaves of ideals.

The family of contact orders is said to be *connected* if Z is connected.

For simplicity, we will refer to $\mathbf{u}$ as the contact order when there is no confusion about the strict morphism $Z \to X$. Given a family of contact orders $\mathbf{u} \in \Gamma(Z, \overline{\mathcal{M}}_Z^*)$ of X, the *pullback* of $\mathbf{u}$ along a strict morphism $Z' \to Z$ defines a family of contact orders $\mathbf{u}' \in \Gamma(Z', \overline{\mathcal{M}}_{Z'}^*)$.

Example A.2. To motivate this definition, consider a punctured map $f : C^\circ \to X$ over W, and a punctured section $p \in \mathbf{p}$. Take $\underline{Z} := \underline{W}$, and give $\underline{Z}$ the log structure given by pullback of $\mathcal{M}_X$ via $\underline{f} \circ p$, so that $Z \to X$ is strict. Let $\mathbf{u}$ be the following composition

$$\overline{\mathcal{M}}_Z \xrightarrow{\bar{f}^\flat} p^*\overline{\mathcal{M}}_{C^\circ} \to \overline{\mathcal{M}}_W \oplus \mathbb{Z} \to \mathbb{Z}, \tag{A.1}$$

where the middle arrow is the inclusion and the last arrow is the projection to the second factor.

We claim that $\mathbf{u}$ defines a family of contact orders of X. Indeed, let $\delta \in \mathcal{M}_Z$ and represent $f^\flat(\delta) = (e_\delta, \sigma^{u_p(\delta)})$, where σ is the element of $\mathcal{M}_C$ corresponding to a local defining equation of the section p.

If $u_p(\delta) > 0$ then

$$\alpha_Z(\delta) = p^*\alpha_C(f^\flat(\delta)) = p^*\alpha_C(e_\delta) \cdot p^*\alpha_C\big(\sigma^{u_p(\delta)}\big) = 0$$

since $p^*\alpha_C(\sigma) = 0$.

If $u_p(\delta) < 0$ then $f^\flat(\delta) \notin \mathcal{M}_C$ and hence, by Definition 2.1 (2) we have $\alpha_Z(\delta) = 0$.

The goal now is to define a universal family of contact orders for the Artin fan $\mathcal{A}_X$ of X.

A.1 Family of contact orders of Artin cones

Let $(Z \to X, \mathbf{u} \in \Gamma(Z, \bar{\mathcal{M}}_Z^*))$ be a family of contact orders of X. For any strict morphism $X \to Y$, $\mathbf{u}$ is naturally a family of contact orders of Y via the composition $Z \to X \to Y$. Conversely, we can pull back a contact order on Y to X by base-change: if we denote by $\mathcal{Z}_X$ and $\mathcal{Z}_Y$ the sheaves over **Sch** of families of contact orders on X and Y, respectively, and if we denote by $\mathcal{Z}_X \to X$ the map forgetting the section $\mathbf{u}$, then $\mathcal{Z}_X = \mathcal{Z}_Y \times_Y X$. Thus we may parameterize contact orders of the Artin fan $\mathcal{A}_X$ instead of X: pulling back such parametrization gives a parametrization of contact orders on X. This is the approach taken here, which is achieved in Proposition A.8 and Definition A.9. We first study the local case.

Consider a toric monoid P with $\sigma = \mathrm{Hom}(P, \mathbb{R}_{\geq 0})$, $N_\sigma = \mathrm{Hom}(P, \mathbb{Z}) = P^*$. This gives the toric variety $\mathbb{A}_\sigma = \mathrm{Spec}(P \xrightarrow{\alpha} \Bbbk[P])$, torus $T_\sigma := \mathrm{Spec}(\Bbbk[P^{\mathrm{gp}}])$ and Artin cone

$$\mathcal{A}_\sigma = [\mathbb{A}_\sigma / T_\sigma]. \tag{A.2}$$

Choose an integral vector $u \in N_\sigma$, which we view as $u \in \mathrm{Hom}(P, \mathbb{Z})$. Let I_u be the ideal of P generated by $u^{-1}(\mathbb{Z} \setminus \{0\})$. This generates a T_σ-invariant ideal in $\Bbbk[P]$, defining an invariant closed subscheme $Z_{u,\sigma} \subseteq \mathbb{A}_\sigma$ with quotient a closed substack $\mathcal{Z}_{u,\sigma} \subseteq \mathcal{A}_\sigma$. We proceed to construct a family of contact orders parametrized by $\mathcal{Z}_{u,\sigma}$.

For each face $\tau \prec \sigma$ (where $\prec$ denotes an inclusion of faces) consider the prime ideal $\mathcal{K}_{\tau \prec \sigma} = P \setminus \tau^\perp$. It defines a toric stratum $Z_{\tau \prec \sigma} := V(\alpha(\mathcal{K}_{\tau \prec \sigma})) \subseteq \mathbb{A}_\sigma$ where the duals of the stalks of $\bar{\mathcal{M}}_{Z_{\tau \prec \sigma}}$ are identified with the faces of σ containing τ. Note that the torus T_σ acts on $Z_{\tau \prec \sigma}$. Denote by $\mathcal{Z}_{\tau \prec \sigma} := [Z_{\tau \prec \sigma} / \mathrm{Spec}(\Bbbk[P^{\mathrm{gp}}])] \subseteq \mathcal{A}_\sigma$.

Lemma A.3. *We have* $(\mathcal{Z}_{u,\sigma})_{\mathrm{red}} = \bigcup_{\tau^{\mathrm{gp}} \ni u} \mathcal{Z}_{\tau \prec \sigma} \subseteq \mathcal{A}_\sigma$.

Proof. The ideal $\sqrt{I_u}$ defines some union of strata and we identify those strata $\mathcal{Z}_{\tau \prec \sigma}$ on which it vanishes. If $u \notin \tau^{\mathrm{gp}}$ there is an element $p \in \tau^\perp \cap P$ such that $u(p) \neq 0$. Therefore $p \in I_u$ but the monomial z^p does not vanish at the generic point of $\mathcal{Z}_{\tau \prec \sigma}$. Hence $(\mathcal{Z}_{u,\sigma})_{\mathrm{red}}$ is contained in the given union of strata. Conversely, if $u \in \tau^{\mathrm{gp}}$, and if $p \in u^{-1}(\mathbb{Z} \setminus \{0\})$, then $p \notin \tau^\perp \cap P$, hence z^p vanishes along $\mathcal{Z}_{\tau \prec \sigma}$. Thus $\mathcal{Z}_{\tau \prec \sigma}$ is contained in $(\mathcal{Z}_{u,\sigma})_{\mathrm{red}}$, proving the result. ■

Since $\bar{\mathcal{M}}^*_{(\mathcal{Z}_{u,\sigma})_{\mathrm{red}}}$ is the pullback of $\bar{\mathcal{M}}^*_{\mathcal{Z}_{u,\sigma}}$ under the reduction $(\mathcal{Z}_{u,\sigma})_{\mathrm{red}} \to \mathcal{Z}_{u,\sigma}$, and reduction induces an isomorphism of étale sites, we have

$$\Gamma\big(\mathcal{Z}_{u,\sigma}, \bar{\mathcal{M}}^*_{\mathcal{Z}_{u,\sigma}}\big) = \Gamma\big((\mathcal{Z}_{u,\sigma})_{\mathrm{red}}, \bar{\mathcal{M}}^*_{(\mathcal{Z}_{u,\sigma})_{\mathrm{red}}}\big).$$

We define an element $\mathbf{u}_{u,\sigma}$ of this group by defining it on stalks in a manner compatible with generization. For a point z in the dense stratum of $\mathcal{Z}_{\tau\prec\sigma}$, with $F_\tau = P \cap \tau^\perp$, we have $\overline{\mathcal{M}}_{\mathcal{Z}_{u,\sigma},z} = (P + F_\tau^{\mathrm{gp}})/F_\tau^{\mathrm{gp}}$. Thus the condition $u \in \tau^{\mathrm{gp}}$ guarantees that $u : P \to \mathbb{Z}$ descends to $u : \overline{\mathcal{M}}_{\mathcal{Z}_{u,\sigma},z} \to \mathbb{Z}$. Being induced by the same element u, this is compatible with generization. Note that the scheme $\mathcal{Z}_{u,\sigma}$ was defined in such a way so that $\alpha_{\mathcal{Z}_{u,\sigma}}(\mathcal{I}_{\mathbf{u}_{u,\sigma}}) = 0$, so that $\mathcal{Z}_{u,\sigma}$ acquires the structure of an idealized log stack.

Thus u defines a family of contact orders of $\mathcal{A}_\sigma$

$$\mathbf{u}_{u,\sigma} \in \Gamma\big(\mathcal{Z}_{u,\sigma}, \overline{\mathcal{M}}^*_{\mathcal{Z}_{u,\sigma}}\big). \tag{A.3}$$

It is connected since the most degenerate stratum $\mathcal{Z}_{\sigma\prec\sigma}$ is contained in the closure of $\mathcal{Z}_{\tau\prec\sigma}$ for each face τ.

Lemma A.4. *For any connected family of contact orders $\mathbf{u} \in \Gamma(Z, \overline{\mathcal{M}}_Z^*)$ of $\mathcal{A}_\sigma$, there exists a unique $u \in N_\sigma$ such that $\psi : Z \to \mathcal{A}_\sigma$ factors uniquely through $\mathcal{Z}_{u,\sigma}$, and $\mathbf{u}_{u,\sigma}$ pulls back to $\mathbf{u}$.*

Proof. The global chart $P \to \overline{\mathcal{M}}_{\mathcal{A}_\sigma}$ over $\mathcal{A}_\sigma$ pulls back to a global chart $P \to \overline{\mathcal{M}}_Z$ over Z. The composition $P \to \overline{\mathcal{M}}_Z \xrightarrow{\mathbf{u}} \mathbb{Z}$ defines an integral vector $u \in N_\sigma$. Consider the sheaf of monoid ideals $\mathcal{J}_u \subset \mathcal{M}_{\mathcal{A}_\sigma}$ generated by I_u. By definition, the contact log-ideal $\mathcal{I}_{\mathbf{u}}$ is generated by $\psi^{-1}\mathcal{J}_u$. Since $\alpha_Z(\mathcal{I}_{\mathbf{u}}) = 0$ and since $\mathcal{J}_u$ defines $\mathcal{Z}_{u,\sigma} \subseteq \mathcal{A}_\sigma$, we have the factorization $Z \to \mathcal{Z}_{u,\sigma}$ of ψ, with $\mathbf{u}$ the pullback of $\mathbf{u}_{u,\sigma}$. ∎

We can now assemble all the $\mathcal{Z}_{u,\sigma}$ by defining

$$\mathcal{Z}_\sigma = \coprod_{u \in N_\sigma} \mathcal{Z}_{u,\sigma},$$

and write $\psi_\sigma : \mathcal{Z}_\sigma \to \mathcal{A}_\sigma$ for the morphism which restricts to the closed embedding $\mathcal{Z}_{u,\sigma} \hookrightarrow \mathcal{A}_\sigma$ on each connected component $\mathcal{Z}_{u,\sigma}$ of $\mathcal{Z}_\sigma$. Then the $\mathbf{u}_{u,\sigma}$ yield a section $\mathbf{u}_\sigma \in \Gamma(\mathcal{Z}_\sigma, \overline{\mathcal{M}}^*_{\mathcal{Z}_\sigma})$, giving the universal family, over $\mathcal{Z}_\sigma$, of contact orders of $\mathcal{A}_\sigma$. This follows immediately from Lemma A.4 by restricting to connected components.

Proposition A.5. *Assume Z is locally connected. For any family of contact orders $\mathbf{u} \in \Gamma(Z, \overline{\mathcal{M}}_Z^*)$ of $\mathcal{A}_\sigma$, $\psi : Z \to \mathcal{A}_\sigma$ factors uniquely through $\mathcal{Z}_\sigma$, and $\mathbf{u}_\sigma$ pulls back to $\mathbf{u}$.*

Corollary A.6. *If τ is a face of σ, viewing $\mathcal{A}_\tau$ naturally as an open substack of $\mathcal{A}_\sigma$ we then have $\mathcal{Z}_\tau \cong \psi_\sigma^{-1}(\mathcal{A}_\tau)$, and the section $\mathbf{u}_\sigma \in \Gamma(\mathcal{Z}_\sigma, \overline{\mathcal{M}}^*_{\mathcal{Z}_\sigma})$ pulls back to the section $\mathbf{u}_\tau \in \Gamma(\mathcal{Z}_\tau, \overline{\mathcal{M}}^*_{\mathcal{Z}_\tau})$.*

Proof. The statement is immediate from the universal property stated in Proposition A.5. ∎

A.2 Family of contact orders of Zariski Artin fans

We now consider the case of an Artin fan $\mathcal{A}_X$. Recall that $\mathcal{A}_X$ has an étale cover by Artin cones. It was constructed in [5, Proposition 3.1.1] as a colimit of Artin cones $\mathcal{A}_\sigma$, viewed as sheaves over Log.

Definition A.7. We say that the Artin fan $\mathcal{A}_X$ is *Zariski* if it admits a Zariski cover by Artin cones.

A sufficient condition for $\mathcal{A}_X$ to be Zariski is that X is simple, because then $\mathcal{A}_X$ is the Artin fan associated to the ordinary cone complex $\Sigma(X)$ [14, Theorem 6.11]. Proposition C.11 shows that X is simple provided X has Zariski log structure and is log smooth over a simple B. The case B a trivial log point has previously been treated in [3, Lemma 2.6].

Fix a Zariski Artin fan $\mathcal{A}_X$. Let $\mathcal{Z}$ be the colimit of the $\mathcal{Z}_\sigma$ viewed as sheaves over $\mathcal{A}_X$. Note that $\mathcal{Z}$ is obtained by gluing together the local model $\mathcal{Z}_\sigma$ for each Zariski open $\mathcal{A}_\sigma \subseteq \mathcal{A}_X$ via the canonical identification given by Corollary A.6.[1]

The following proposition classifies contact orders on $\mathcal{A}_X$ by globalizing Proposition A.5.

Proposition A.8. *There is a section* $\mathbf{u}_\mathcal{X} \in \Gamma(\mathcal{Z}, \bar{\mathcal{M}}^*_\mathcal{Z})$ *making* $\mathcal{Z}$ *into a family of contact orders for* $\mathcal{A}_X$*. This family of contact orders is universal in the sense that for any family of contact orders* $\mathbf{u} \in \Gamma(Z, \bar{\mathcal{M}}^*_Z)$ *of* $\mathcal{A}_X$*,* $\psi : Z \to \mathcal{A}_X$*, there is a unique factorization of* ψ *through* $\mathcal{Z} \to \mathcal{A}_X$ *such that* $\mathbf{u}$ *is the pullback of* $\mathbf{u}_\mathcal{X}$*.*

Proof. If $\mathcal{A}_\sigma \to \mathcal{A}_X$ is a Zariski open set, then by the construction of $\mathcal{Z}$,

$$\mathcal{Z} \times_{\mathcal{A}_X} \mathcal{A}_\sigma = \mathcal{Z}_\sigma.$$

By Corollary A.6, the sections $\mathbf{u}_\sigma$ glue to give a section $\mathbf{u}_\mathcal{X} \in \Gamma(\mathcal{Z}, \bar{\mathcal{M}}^*_\mathcal{Z})$, yielding a family of contact orders in $\mathcal{A}_X$.

Consider a family of contact orders $Z \to \mathcal{A}_X$, $\mathbf{u}$. To show the desired factorization, it suffices to prove the existence and uniqueness locally on each Zariski open subset $\mathcal{A}_\sigma \to \mathcal{A}_X$, which follows from Proposition A.5. ∎

Definition A.9. A *connected contact order* for X is a choice of connected component of $\mathcal{Z}$.

We end this discussion with a couple of properties of the space $\mathcal{Z}$ of contact orders.

[1] It should be possible to carry this process out for more general Artin fans.

Proposition A.10. *Suppose that the Artin fan $\mathcal{A}_X$ of X is Zariski. There is a one-to-one correspondence between irreducible components of $\mathcal{Z}$ and pairs (u,σ) where $\sigma \in \Sigma(X)$ is a minimal cone such that $u \in \sigma^{\mathrm{gp}}$.*

Proof. Since $\mathcal{Z}_\sigma \subseteq \mathcal{Z}$ is Zariski open, an irreducible component of $\mathcal{Z}$ is the closure of an irreducible component of some $\mathcal{Z}_\sigma$, so we may assume $\mathcal{A}_X = \mathcal{A}_\sigma$. Then the statement follows from the description of $\mathcal{Z}_{u,\sigma}$ in Lemma A.3. ■

Remark A.11. Note that if $u \in N_\sigma$ with $u \in \sigma$ or $-u \in \sigma$, then $\mathcal{Z}_{u,\sigma}$ is irreducible and reduced. In fact, topologically $\mathcal{Z}_{u,\sigma}$ is the closure of the stratum $\mathcal{Z}_{\tau\prec\sigma}$ where $\tau \subseteq \sigma$ is the minimal face containing u. Further, the ideal generated by $u^{-1}(\mathbb{Z}\setminus\{0\})$ is precisely $P \setminus F_\tau$, so that $\mathcal{Z}_{u,\sigma}$ is reduced. In the case that $u \in \sigma$, it is the contact orders associated to ordinary marked points, as developed in [2, 15, 30].

For a simple non-reduced example let $P = \sigma_{\mathbb{Z}}^\vee$ be the submonoid of $\mathbb{N}^2$ generated by $(e,0)$, $(0,e)$, $(1,1)$ and $u : P \to \mathbb{Z}$ given by $u(a,b) = a - b$. Then I_u is generated by $(e,0)$, $(0,e)$, and $\Bbbk[P]/I_u \simeq \Bbbk[t]/(t^e)$ is non-reduced for $e > 1$.

Thus the situation for more general contact orders associated to punctures may be more complex than that for marked points.

Example A.12. Even in the Zariski case, there may be monodromy which creates a difference between the point of view taken on contact orders in this appendix and that taken in Section 2.4. See Example 2.38 for a simple example with monodromy. There, taking $u = (0,1,0)$ as a tangent vector to any of the top-dimensional cones of $\Sigma(X)$, the corresponding connected contact order is a double cover of a one-dimensional closed subscheme of X. Explicitly, X contains ℓ strata isomorphic to $\mathbb{P}^1$, forming a cycle, i.e., a nodal elliptic curve. Then u induces a family of contact orders $Z \to X$ which is a double cover of this elliptic curve. This curve has 2ℓ irreducible components, in one-to-one correspondence with the set of pairs of the form (u,σ) and $(-u,\sigma)$ for σ running over two-dimensional cones of $\Sigma(X)$ tangent to u.

Example 2.39 provides an X with $\mathcal{A}_X$ Zariski where a similar monodromy produces connected contact orders with an infinite number of irreducible components. In this case one sees connected components of moduli spaces of punctured maps with an infinity of irreducible components.

By the discussion in Remark A.11 above, additional hypotheses are usually needed to obtain good control of moduli spaces of punctured maps. Here is a simple criterion that often suffices in practice.

Proposition A.13. *Suppose $\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by its global sections. Assume further X quasi-compact, with locally connected logarithmic strata. Then every connected component of contact orders of $\mathcal{A}_X$ has finitely many irreducible components.*

Proof. Let $V = \Gamma(\underline{X}, \overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R})$, so that the induced map $|\Sigma(X)| \to V^*$ is injective on each $\sigma \in \Sigma(X)$ as in Proposition 3.13. Suppose $\mathbf{u} \in \Gamma(\mathcal{Z}, \overline{\mathcal{M}}_{\mathcal{Z}}^*)$ is a connected component of contact orders of $\mathcal{A}_X$. Denote the composition $V \to \overline{\mathcal{M}}_{\mathcal{Z}}^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\mathbf{u}} \mathbb{R}$ by $v \in V^*$. For each irreducible component of $\mathcal{Z}$, its corresponding vector u as in Proposition A.10 is then uniquely determined by v. By Proposition A.10 again $\mathcal{Z}$ has finitely many irreducible components, as $\Sigma(X)$ has finitely many cones by quasi-compactness. ∎

A.3 Connection with the global contact orders of Section 2.4

We continue to work with a Zariski X. For simplicity in this discussion, let us also assume that X is log smooth over $\operatorname{Spec} \Bbbk$, so in particular associated to any $\sigma \in \Sigma(X)$ is a closed stratum $X_\sigma \subseteq X$ such that the dual cone of the stalk of $\overline{\mathcal{M}}_X$ at the generic point of X_σ is σ.

In this case, $\mathcal{A}_X$ is Zariski, and $\mathcal{A}_{X_\sigma}$, the Artin fan of X_σ, is also Zariski. Let $\mathcal{Z}^\sigma \to \mathcal{A}_{X_\sigma}$ be the universal family of contact orders for $\mathcal{A}_{X_\sigma}$. Further, write $\mathcal{X}_\sigma$ for the reduced closed stratum of $\mathcal{A}_{X_\sigma}$ corresponding to σ. In this situation, we have:

Proposition A.14. *There is a one-to-one correspondence between $\mathfrak{C}_\sigma(X)$ and the set of connected components of $\mathcal{Z}^\sigma \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$.*

Proof. By the construction of the colimit of sets, $\mathfrak{C}_\sigma(X)$ is the quotient of the set $\coprod_{\sigma \prec \sigma' \in \Sigma(X)} N_{\sigma'}$ by the equivalence relation $\sim$ generated by the following set of relations. Whenever given inclusions of faces $\sigma \prec \sigma' \prec \sigma''$ in $\Sigma(X)$, one obtains an induced map $\iota_{\sigma'\sigma''} : N_{\sigma'} \to N_{\sigma''}$. Then for $x \in N_{\sigma'}$, we have $x \sim \iota_{\sigma'\sigma''}(x)$.

On the other hand, we may cover $\mathcal{A}_{X_\sigma}$ with Zariski open sets $\mathcal{A}_{\sigma'}$ with σ' running over $\sigma' \in \Sigma(X)$ with $\sigma \prec \sigma'$. Note that by the construction of the universal contact order of $\mathcal{A}_{\sigma'}$, there is a one-to-one correspondence between $N_{\sigma'}$ and the set of connected components of $\mathcal{Z}_{\sigma'} = \mathcal{Z}^\sigma \times_{\mathcal{A}_{X_\sigma}} \mathcal{A}_{\sigma'}$, with $u \in N_{\sigma'}$ corresponding to $\mathcal{Z}_{u,\sigma'}$. Note the same is then true of the set of connected components of $\mathcal{Z}_{\sigma'} \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$, with $u \in N_{\sigma'}$ corresponding to $\mathcal{Z}_{u,\sigma'} \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$.

Define another equivalence relation $\approx$ on $\coprod_{\sigma \prec \sigma'} N_{\sigma'}$ as follows. Suppose $u' \in N_{\sigma'}$, $u'' \in N_{\sigma''}$. Then $u' \approx u''$ if $\mathcal{Z}_{u',\sigma'} \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$ and $\mathcal{Z}_{u'',\sigma''} \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$ are open substacks of the same connected component of $\mathcal{Z}^\sigma \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$. The statement follows once we show that the two equivalence relations $\sim$ and $\approx$ are equal.

Note that $\mathcal{A}_{\sigma'} \cap \mathcal{A}_{\sigma''} \cap \mathcal{X}_\sigma$ may be covered with sets $\mathcal{A}_\tau \cap \mathcal{X}_\sigma$ with $\tau \in \Sigma(X)$ running over those τ with $\sigma \prec \tau \prec \sigma' \cap \sigma''$. This makes it clear that $\approx$ is also generated by the following relations. Suppose given $\sigma \prec \sigma' \prec \sigma''$ with $u' \in N_{\sigma'}$, $u'' \in N_{\sigma''}$. Then because of the inclusion $\mathcal{Z}_{\sigma'} \subseteq \mathcal{Z}_{\sigma''}$ of Corollary A.6, $\mathcal{Z}_{\sigma',u'} \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$ and $\mathcal{Z}_{\sigma'',u''} \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$ may be both viewed as open substacks of $\mathcal{Z}^\sigma \times_{\mathcal{A}_{X_\sigma}} (\mathcal{X}_\sigma \times_{\mathcal{A}_{X_\sigma}}$

$\mathscr{A}_{\sigma''}$). If these two open substacks are not disjoint, then $u' \approx u''$. However, it follows from Proposition A.5 that this is the case precisely when $\iota_{\sigma'\sigma''}(u') = u''$. Thus $\sim$ and $\approx$ are the same equivalence relation, since they are generated by the same set of relations. ∎

Appendix B

Charts for morphisms of log stacks

We discuss here properties of charts of morphisms of algebraic log stacks, due to a lack of a good reference. There are many standard results involving existence and properties of charts for morphisms between fs log schemes étale locally, e.g., [52, Section II.2], as well as local descriptions of log smooth or étale morphisms, e.g., [52, Section IV.3.3]. However, to apply these results to morphisms of log stacks, one would need to pass to smooth neighborhoods, which destroys any discussion of the more delicate condition of being log étale. Thus it is far more convenient to think of charts as being given by maps to toric stacks rather than toric varieties. The results of this appendix are used in Section 3.5 to describe local models for our moduli spaces of punctured maps to Artin fans, but are also used extensively elsewhere, e.g., in [33].

Here we fix a ground field $\Bbbk$ of characteristic 0, as usual, and all schemes and stacks are defined over $\operatorname{Spec}\Bbbk$. We define, given P a fine monoid and $K \subseteq P$ a monoid ideal,

$$\mathcal{A}_P := [\operatorname{Spec}\Bbbk[P]/\operatorname{Spec}\Bbbk[P^{\mathrm{gp}}]], \quad \mathcal{A}_{P,K} = [(\operatorname{Spec}\Bbbk[P]/K)/\operatorname{Spec}\Bbbk[P^{\mathrm{gp}}]]. \tag{B.1}$$

Here both stacks carry a canonical log structure coming from P, and the second stack carries a canonical idealized log structure induced by the monoid ideal K, see [52, Section III.1.3].

Remark B.1. If P is a fine monoid, define $\overline{P} = P/P^\times$. Then $\mathcal{A}_P \cong \mathcal{A}_{\overline{P}}$.

Proposition B.2. *Let $f : X \to Y$ be a morphism of (idealized) fs log stacks over* $\operatorname{Spec}\Bbbk$, *with coherent sheaves of ideals $\mathcal{K}_X$ and $\mathcal{K}_Y$ in the idealized case. Let $\bar{x}$ be a geometric point of $\underline{X}$, $\bar{y} = f(\bar{x})$, $P = \overline{\mathcal{M}}_{X,\bar{x}}$, $Q = \overline{\mathcal{M}}_{Y,\bar{y}}$, ($K = \overline{\mathcal{K}}_{X,\bar{x}}$, $J = \overline{\mathcal{K}}_{Y,\bar{y}}$ in the idealized case). Then in the two cases, there are strict étale neighborhoods X' and Y' of $\bar{x}$ and $\bar{y}$ respectively and commutative diagrams*

$$\begin{array}{ccc} X' & \longrightarrow & \mathcal{A}_P \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & \mathcal{A}_Q \end{array} \qquad\qquad \begin{array}{ccc} X' & \longrightarrow & \mathcal{A}_{P,K} \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & \mathcal{A}_{Q,J} \end{array}$$

with horizontal arrows (idealized) strict. If further Y is already equipped with a strict morphism $Y \to \mathcal{A}_Q$, we may take $Y = Y'$.

Proof. If Y is equipped with a strict morphism $Y \to \mathcal{A}_Q$, we take $Y = Y'$. Otherwise, there is a tautological morphism $Y \to \mathrm{Log}_{\Bbbk}$. By [53, Corollary 5.25], $\mathrm{Log}_{\Bbbk}$ has a strict étale cover by stacks of the form $\mathcal{A}_Q$ for various monoids Q. Thus we may choose a strict étale morphism $\mathcal{A}_Q \to \mathrm{Log}_{\Bbbk}$ whose image contains the image of $\bar{y}$, and take Y' to be the étale neighborhood $Y \times_{\mathrm{Log}_{\Bbbk}} \mathcal{A}_Q$ of $\bar{y}$. If the image of $\bar{y}$ in $\mathcal{A}_Q$ is not the deepest stratum of $\mathcal{A}_Q$, we may replace $\mathcal{A}_Q$ with a Zariski open subset of the form $\mathcal{A}_{Q'}$ where Q' is a localization of Q along some face and such that $\bar{y}$ maps to the deepest stratum of $\mathcal{A}_{Q'}$. By Remark B.1, $\mathcal{A}_{Q'} \cong \mathcal{A}_{Q'/(Q')^\times}$, so we may assume that $Q = \overline{\mathcal{M}}_{Y,\bar{y}}$.

Let $X'' = X \times_Y Y'$ be the corresponding étale neighborhood of $\bar{x}$. Similarly, we have a tautological strict morphism $X'' \to \mathrm{Log}_{Y'}$. Now $\mathrm{Log}_{Y'}$ can be covered by strict étale morphisms of the form $\mathcal{A}_P \times_{\mathcal{A}_Q} Y' \to \mathrm{Log}_{Y'}$ for various P such that the projection to $\mathcal{A}_P$ is strict, again by [53, Corollary 5.25]. Here we range over fs monoids P and morphisms $\theta : Q \to P$. Take $X' = X'' \times_{\mathrm{Log}_{Y'}} (\mathcal{A}_P \times_{\mathcal{A}_Q} Y')$ for a suitable choice of P and $\theta : Q \to P$ so that X' is an étale neighborhood of $\bar{x}$. The projection of this stack to $\mathcal{A}_P$ then yields the desired strict morphism $X' \to \mathcal{A}_P$ making the diagram commutative. As before, we can pass to a Zariski open substack of $\mathcal{A}_P$ to be able to assume that $P = \overline{\mathcal{M}}_{X,\bar{x}}$.

In the idealized case, the morphisms $X' \to \mathcal{A}_P$, $Y' \to \mathcal{A}_Q$ factor through $\mathcal{A}_{P,K}$ and $\mathcal{A}_{Q,J}$ respectively, and the factored morphisms are idealized strict. ■

Lemma B.3. *Let $\theta : Q \to P$ be a morphism of fs monoids, $J \subseteq Q$, $K \subseteq P$ with $\theta(J) \subseteq K$. Then the induced morphism $\theta : \mathcal{A}_{P,K} \to \mathcal{A}_{Q,J}$ is idealized log étale.*

Proof. The morphism θ is clearly locally of finite presentation, so we need only verify the formal lifting criterion. Suppose given a diagram

$$\begin{tikzcd} T_0 \arrow[r, "g_0"] \arrow[d, "i"'] & \mathcal{A}_{P,K} \arrow[r] \arrow[d, "\theta"] & \mathcal{A}_P \arrow[d, "\theta"] \\ T \arrow[r, "g"'] \arrow[ur, dashed, "g'"] & \mathcal{A}_{Q,J} \arrow[r] & \mathcal{A}_Q \end{tikzcd}$$

Here i is a strict and idealized strict closed immersion with ideal sheaf having square zero, and the right-hand horizontal arrows are strict, but not idealized strict, closed immersions, with the right-hand square commutative. We wish to show there is a unique g' making the diagram commute.

By [53, Corollary 5.23], $\mathcal{A}_P \to \mathcal{A}_Q$ is log étale. Thus forgetting the idealized structure on T, we obtain a unique morphism $h : T \to \mathcal{A}_P$ in the above diagram making everything commute. Let $\mathcal{K}_{T_0}$, $\mathcal{K}_T$ be the coherent sheaves of monoid ideals for T_0 and T respectively giving the idealized structure. Note $\mathcal{K}_{T_0}$ is the pullback of a sheaf of ideals $\overline{\mathcal{K}}_{T_0} \subseteq \overline{\mathcal{M}}_{T_0}$ under the projection $\mathcal{M}_{T_0} \to \overline{\mathcal{M}}_{T_0}$, and similarly for T. By strictness, $\overline{\mathcal{K}}_{T_0} = \overline{\mathcal{K}}_T$. Since $\bar{g}_0^\flat(K) \subseteq \overline{\mathcal{K}}_{T_0}$, as g_0 is an idealized morphism, we have by commutativity that $\bar{h}^\flat(K) \subseteq \overline{\mathcal{K}}_T$. Hence α_T vanishes on any lift to $\mathcal{M}_T$ of an element $\bar{h}^\flat(k)$ for $k \in K$. It then follows that h factors through the closed immersion $\mathcal{A}_{P,K} \to \mathcal{A}_P$, and this factorization yields the unique lifting g'. ∎

Proposition B.4. *With the hypotheses of Proposition B.2, suppose in addition that f is log smooth (resp. log étale, idealized log smooth, idealized log étale). Then in the non-idealized case, the induced morphism $X' \to Y' \times_{\mathcal{A}_Q} \mathcal{A}_P$ is smooth (resp. étale) and in the idealized case, the induced morphism $X' \to Y' \times_{\mathcal{A}_{Q,J}} \mathcal{A}_{P,K}$ is smooth (resp. étale).*

Proof. In the non-idealized case, [53, Theorem 4.6], shows that $X \to Y$ is log smooth (étale) if and only if the tautological morphism $\underline{X} \to \mathrm{Log}_Y$ is smooth (étale). It follows immediately by base-change from the construction of the proof of Proposition B.2 that, if f is log smooth (étale), the morphism $X'' \to Y'$ is log smooth (étale) and hence $\underline{X}'' \to \mathrm{Log}_{Y'}$ is smooth (étale). Thus by another base-change, we see that the projection $X' \to Y' \times_{\mathcal{A}_Q} \mathcal{A}_P$ is smooth (étale).

The idealized case requires a little bit more work because the analogous statement for idealized log smooth (étale) morphisms does not seem to appear in the literature. First, by [53, Lemma 4.8], $X'' \to \mathrm{Log}_{Y'}$ is locally of finite presentation, as $X \to Y$ is locally of finite presentation, being idealized log étale, and thus $X' \to Y' \times_{\mathcal{A}_Q} \mathcal{A}_P$ is locally of finite presentation. However, as in the proof of Proposition B.2, $X' \to Y' \times_{\mathcal{A}_Q} \mathcal{A}_P$ factors through the closed immersion $Y' \times_{\mathcal{A}_{Q,J}} \mathcal{A}_{P,K} \hookrightarrow Y' \times_{\mathcal{A}_Q} \mathcal{A}_P$, and thus $X' \to Y' \times_{\mathcal{A}_{Q,J}} \mathcal{A}_{P,K}$ is also locally of finite presentation. So we just need to show the formal lifting criterion, i.e., given a diagram

$$\begin{tikzcd}
\underline{T}_0 \arrow[r, "g_0"] \arrow[d, "\underline{i}"'] & \underline{X}' \arrow[d, "\underline{f}'"] \\
\underline{T} \arrow[r, "\underline{g}"'] \arrow[ur, dashed] & \underline{Y' \times_{\mathcal{A}_{Q,J}} \mathcal{A}_{P,K}}
\end{tikzcd} \tag{B.2}$$

where i is a closed immersion with ideal of square zero, there is, étale locally on $\underline{T}_0$, a dotted line as indicated, unique in the étale case. Give $\underline{T}_0$ and $\underline{T}$ the idealized log structure making all arrows in the above square strict and idealized strict. Via composition of g with the projection to Y', we obtain a diagram

$$\begin{tikzcd}
T_0 \arrow[r, "g_0"] \arrow[d, "i"'] & X' \arrow[d] \\
T \arrow[r, "g''"'] \arrow[ur, dashed, "g'"] & Y'
\end{tikzcd}$$

Formal idealized log smoothness (idealized log étaleness) then implies, étale locally, a (unique) lift g'. It is then sufficient to show that $f' \circ g'$ coincides with g in (B.2).

However, by Lemma B.3, the projection $Y' \times_{\mathcal{A}_{Q,J}} \mathcal{A}_{P,K} \to Y'$ is idealized log étale, and hence by uniqueness in the formal lifting criterion for idealized log étale morphisms, $f' \circ g' = g$. ■

Appendix C

Functorial tropicalization and the category of points

Various definitions of tropicalization in logarithmic geometry are available in the literature [1,3,14,30,42,69]. The purpose of this appendix is to spell out the construction of tropicalization as a functor from the category of fine log algebraic stacks to the category of generalized cone complexes generalizing [69, Proposition 6.3] to cases with monodromy, and closer in spirit to [30, Appendix B]. This refines the discussion in [3, Section 2.1].

We adopt the definition from [43, Section II.1], [64, Section 2], [1, Section 2.2] of a *generalized cone complex* Σ as a topological space $|\Sigma|$ together with a *presentation* given by a homeomorphism with the colimit in the category of topological spaces of a diagram in **Cones** with all arrows face morphisms. Here we use the topology induced by embedding a cone σ into its vector space $N_\sigma \otimes_{\mathbb{Z}} \mathbb{R}$. For any cone σ in a presentation we always include all face embeddings $\tau \to \sigma$ in the diagram. The *strata* of $|\Sigma|$ are the images of the interiors of cones from the presentation. We consider generalized cone complexes up to equivalence generated by adding more cones to a presentation. A morphism of cone complexes $\Sigma \to \Sigma'$ is given by a continuous map $|\Sigma| \to |\Sigma'|$ that locally lifts to a morphism of diagrams of presentations. Unlike the cited references, we do not impose any finiteness conditions since we want to admit situations with infinitely many strata.

C.1 Tropicalization of fine log schemes

We begin by recalling the definition of the category of geometric points $\mathbf{Pt}(X)$ of a scheme X with arrows defined by specialization, following [9, Section VIII.7], see also [67, Section 0GJ2]. An object in $\mathbf{Pt}(X)$ is a morphism $\bar{x} : \operatorname{Spec} \kappa \to X$ with $\kappa = \kappa(\bar{x})$ an algebraically closed field. Given $\bar{x}$ we have the associated local scheme $X(\bar{x}) = \operatorname{Spec} \mathcal{O}_{X,\bar{x}}$. A *specialization* arrow $\bar{x} \to \bar{y}$ is an X-morphism $\operatorname{Spec} \kappa(\bar{x}) \to X(\bar{y})$ or, equivalently by [9, Section VIII.7, Proposition 7.4], an X-morphism $X(\bar{x}) \to X(\bar{y})$.

Composition with a morphism $f : X \to Y$ defines a functor

$$f_* : \mathbf{Pt}(X) \to \mathbf{Pt}(Y)$$

compatible with composition, so $\mathbf{Pt}$ is a functor from the category of schemes to the category of categories $\mathbf{Cat}$.

For each étale sheaf of sets $\mathcal{F}$ on X, a specialization arrow $\bar{x} \to \bar{y}$ in $\mathbf{Pt}(X)$ induces a generization map[1]

$$\mathcal{F}_{\bar{y}} \to \mathcal{F}_{\bar{x}}. \tag{C.1}$$

This assignment is compatible with morphisms of sheaves. Thus if $\mathbf{Sh}(X_{\text{ét}})$ denotes the étale topos of X, we obtain a functor

$$\text{Stalks} : \mathbf{Sh}(X_{\text{ét}}) \to \text{Func}(\mathbf{Pt}(X)^{\text{op}}, \mathbf{Sets}) \tag{C.2}$$

associating to an étale sheaf its functor of stalks, a diagram in **Sets** indexed by $\mathbf{Pt}(X)^{\text{op}}$. We emphasize that the generization homomorphism (C.1) does not only depend on $\bar{x}, \bar{y}$, but on the choice of specialization arrow $\bar{x} \to \bar{y}$.

Example C.1. Let C be the nodal cubic. If $\overline{\eta}$ denotes a geometric generic point and $\bar{x}$ a geometric point over the node, there are two different C-morphisms

$$\operatorname{Spec} \kappa(\overline{\eta}) \to X(\bar{x})$$

that reflect the specialization along the two branches of C at $\bar{x}$. This statement can most easily be seen by going over to the usual two-fold étale cover $\pi : \widetilde{C} \to C$, and observing that each of the two lifts $\tilde{\bar{x}} \in \mathbf{Pt}(\widetilde{C})$ of $\bar{x}$ has generization homomorphisms to both lifts of $\overline{\eta}$.

Charts for the log structure define a locally finite stratification of $\underline{X}$ with a *stratum* a maximal connected locally closed subset $Z \subseteq |\underline{X}|$ with $\overline{\mathcal{M}}_X|_Z$ locally constant. Denote by $\text{Strata}(X)$ the set of strata of X. For each $Z \in \text{Strata}(X)$ choose a geometric point $\bar{x} = \bar{x}_Z$ of Z and define

$$\sigma_Z = \operatorname{Hom}\big(\overline{\mathcal{M}}_{X,\bar{x}}, \mathbb{R}_{\geq 0}\big) \in \mathbf{Cones}. \tag{C.3}$$

Different choices of $\bar{x}$ lead to isomorphic σ_Z, but the isomorphism is only unique up to the monodromy action of the étale fundamental group $\pi_1(Z, \bar{x})$ of the stratum on $\overline{\mathcal{M}}_{X,\bar{x}}$. More precisely, since the automorphism group of a fine monoid is finite, arguing with [67, Lemma 0DV5] shows the following. There exists a finite connected étale Galois cover

$$f : \widetilde{Z} \to Z$$

with $f^{-1}\overline{\mathcal{M}}_X$ a constant sheaf. Lifting $\bar{x}$ to $\widetilde{Z}$ yields an isomorphism

$$\Gamma\big(\widetilde{Z}, f^{-1}\overline{\mathcal{M}}_X\big) \xrightarrow{\simeq} \overline{\mathcal{M}}_{X,\bar{x}} \tag{C.4}$$

[1] We prefer "generization map" over the common "specialization map" in this context since the map goes from the stalk at the more special point to the stalk at the more generic point.

by restriction. Now by definition, $\pi_1(Z, \bar{x})$ acts on f, and the induced action on $\Gamma(\widetilde{Z}, f^{-1}\bar{\mathcal{M}}_X)$ by pullback corresponds to the action of $\pi_1(Z, \bar{x})$ on $\bar{\mathcal{M}}_{X,\bar{x}}$ via (C.4). The minimal choice of f with $f^{-1}\bar{\mathcal{M}}_X$ a constant sheaf has connected $\widetilde{Z}$ and is a Galois cover. Moreover, in the minimal case, the action of $\pi_1(Z, \bar{x})$ on $\bar{\mathcal{M}}_{X,\bar{x}}$ factors over a faithful action of the Galois group $\mathrm{Aut}(\widetilde{Z}/Z)$.

For each stratum Z with chosen geometric point $\bar{x} = \bar{x}_Z$ denote by

$$G_Z \subseteq \mathrm{Aut}(\bar{\mathcal{M}}_{X,\bar{x}}) \tag{C.5}$$

the image of the monodromy action of $\pi_1(Z, \bar{x})$ on $\bar{\mathcal{M}}_{X,\bar{x}}$. By the previous discussion, $G_Z \simeq \mathrm{Aut}(\widetilde{Z}/Z)$ for any minimal connected Galois cover $f : \widetilde{Z} \to Z$ with $f^{-1}\bar{\mathcal{M}}_X$ a constant sheaf.

Now if $W \in \mathrm{Strata}(X)$ is another stratum, and $\bar{w}$ is a geometric point of $W \cap \mathrm{cl}(Z)$, there exists a geometric point $\bar{\eta}$ of Z and a specialization arrow $\chi : \bar{\eta} \to \bar{w}$ [67, Section 0BUP], hence a generization homomorphism $\bar{\mathcal{M}}_{X,\bar{w}} \to \bar{\mathcal{M}}_{X,\bar{\eta}}$. Since $\bar{\mathcal{M}}_X$ is locally constant on the strata there are also isomorphisms

$$\bar{\mathcal{M}}_{X,\bar{w}} \xrightarrow{\simeq} \bar{\mathcal{M}}_{\bar{x}_W}, \quad \bar{\mathcal{M}}_{X,\bar{\eta}} \xrightarrow{\simeq} \bar{\mathcal{M}}_{\bar{x}_Z}, \tag{C.6}$$

for $\bar{x}_Z$, $\bar{x}_W$ the chosen reference points for the two strata. These isomorphisms are unique up to composing with elements of G_W and G_Z, respectively. We call any morphism

$$\iota : \sigma_Z \to \sigma_W \tag{C.7}$$

obtained by applying $\mathrm{Hom}(\cdot, \mathbb{R}_{\geq 0})$ to any of the compositions

$$\bar{\mathcal{M}}_{\bar{x}_W} \xrightarrow{\simeq} \bar{\mathcal{M}}_{X,\bar{w}} \xrightarrow{\chi} \bar{\mathcal{M}}_{X,\bar{\eta}} \xrightarrow{\simeq} \bar{\mathcal{M}}_{\bar{x}_Z}$$

a *specialization morphism* or *specialization arrow*. Note that ι also depends on the choice of $\bar{w}$, and hence the actions of G_Z and G_W on the set of specialization arrows may not be transitive. For $Z = W$ the set of specialization arrows equals $G_Z = G_W$.

If $f : X \to Y$ is a morphism of fine log schemes, $Z \in \mathrm{Strata}(X)$ and $f(\bar{x}_Z)$ a geometric point of $Z' \in \mathrm{Strata}(Y)$, then $f^\flat : f^{-1}\bar{\mathcal{M}}_Y \to \bar{\mathcal{M}}_X$ together with a choice of isomorphism $\bar{\mathcal{M}}_{Y,f(\bar{x}_Z)} \simeq \bar{\mathcal{M}}_{Y,\bar{x}_{Z'}}$ in (C.6) defines a morphism

$$\varphi : \sigma_Z \to \sigma_{Z'} \tag{C.8}$$

in **Cones** by the composition

$$\mathrm{Hom}\big(\bar{\mathcal{M}}_{X,\bar{x}_Z}, \mathbb{R}_{\geq 0}\big) \to \mathrm{Hom}\big(\bar{\mathcal{M}}_{Y,f(\bar{x}_Z)}, \mathbb{R}_{\geq 0}\big) \xrightarrow{\simeq} \mathrm{Hom}\big(\bar{\mathcal{M}}_{Y,\bar{x}_{Z'}}, \mathbb{R}_{\geq 0}\big).$$

Note such φ are not in general face morphisms. The set of all such arrows is compatible with specialization in the sense that if $\iota : \sigma_Z \to \sigma_W$ is a specialization morphism (C.7) in X then there exists a specialization morphism $\iota' : \sigma_{Z'} \to \sigma_{W'}$ in Y and morphisms $\varphi : \sigma_Z \to \sigma_{Z'}$, $\psi : \sigma_W \to \sigma_{W'}$ as in (C.8) making the following diagram commute:

$$\begin{tikzcd} \sigma_Z \arrow[r, "\iota"] \arrow[d, "\varphi"] & \sigma_W \arrow[d, "\psi"] \\ \sigma_{Z'} \arrow[r, "\iota'"] & \sigma_{W'}. \end{tikzcd} \tag{C.9}$$

We are then in position to define the tropicalization of X as a generalized cone complex.

Definition C.2. Let $X = (\underline{X}, \mathcal{M}_X)$ be a fine log scheme. The *tropicalization* $\Sigma(X)$ of X is the generalized cone complex defined by the diagram in **Cones** with one object σ_Z from (C.3) for each stratum $Z \subset X$ and face morphisms $\sigma_Z \to \sigma_W$ the set of specialization morphisms from (C.7).

A morphism $f : X \to Y$ of fine log schemes induces the morphism

$$\Sigma(f) : \Sigma(X) \to \Sigma(Y)$$

defined by all arrows $\varphi : \sigma_Z \to \sigma_{Z'}$ as in (C.8).

Note that diagrams of specialization arrows as in (C.9) show that the map of topological spaces $|\Sigma(X)| \to |\Sigma(Y)|$ is well defined and continuous, and that it lifts locally to a morphism of presentations. Thus $\Sigma(f)$ indeed is a morphism of generalized cone complexes.

We need to check that our definition of tropicalization does not depend on the choices of a geometric point $\bar{x}_Z$ for each stratum Z of X.

Lemma C.3. *The definition of tropicalization in Definition C.2 is independent of choices.*

Proof. Let Z be a logarithmic stratum of X and $\bar{x}'_Z$ another choice of geometric point. Since $\bar{\mathcal{M}}_X|_Z$ is locally constant there exists an isomorphism

$$\varphi : \sigma_Z = \mathrm{Hom}\big(\bar{\mathcal{M}}_{X,\bar{x}_Z}, \mathbb{R}_{\geq 0}\big) \xrightarrow{\simeq} \sigma'_Z = \mathrm{Hom}\big(\bar{\mathcal{M}}_{X,\bar{x}'_Z}, \mathbb{R}_{\geq 0}\big)$$

that is unique up to the action of $\pi_1(Z)$ on σ_Z. Replacing σ_Z by σ'_Z and all arrows involving σ_Z by composition with φ or φ^{-1} as appropriate, gives an alternative presentation of $|\Sigma(X)|$ as a colimit of a diagram in **Cones**. By construction, both diagrams are locally isomorphic, and hence they lead to the same generalized cone complex. This argument is local to each geometric point, thus also applies to any two different sets of choices of geometric points. ∎

We finally check functoriality of this notion of tropicalization.

Proposition C.4. *If $f : X \to Y$ and $g : Y \to Z$ are morphisms of fine log schemes then $\Sigma(g \circ f) = \Sigma(g) \circ \Sigma(f)$.*

Proof. Given a specialization morphism $\iota : Z \to W$ of strata of X there exist two commutative diagrams of the form (C.9) with horizontal arrows specialization morphisms $\iota' : Z' \to W'$ and $\iota'' : Z'' \to W''$ of strata in Y and Z, respectively. The two small commutative squares now define the local liftings of $\Sigma(f)$ and $\Sigma(g)$ to presentations, while their composition defines the lifting of $\Sigma(g \circ f)$. The result is now obvious. ∎

Remark C.5. A canonical and obviously functorial definition of $\Sigma(X)$ runs as follows. The composition of the functor Stalks in (C.2) with $\mathrm{Hom}(\cdot, \mathbb{R}_{\geq 0})$ defines a diagram

$$\mathbf{Pt}(X)^{\mathrm{op}} \to \mathbf{Cones} \tag{C.10}$$

with all morphisms face inclusions. The reasoning in the proof of Lemma C.3 shows that the associated generalized cone complex is canonically isomorphic to $\Sigma(X)$. We preferred to base our definition on the more explicit treatment with one cone for each stratum.

Remark C.6. One might think that a slightly refined definition could also give a functorial notion of tropicalization as a diagram of cones associated to strata. This is, however, not the case. The problem appears already with locally constant sheaves in the étale topology, which can not be described by groupoids of sets obtained from the associated representations of the étale fundamental group. The étale fundamental group of a scheme X depends on the choice of a geometric point and is otherwise only defined up to non-unique isomorphism. Thus a functorial definition would have to involve at least a skeleton of $\mathbf{Pt}(X)$, and hence completely loses the combinatorial flavor of tropicalization.

C.2 Tropicalization of fine log algebraic stacks

Now let X be a fine log algebraic stack, with $\mathcal{M}_X$ and $\bar{\mathcal{M}}_X$ sheaves in the lisse-étale topology. To define the tropicalization $\Sigma(X)$ let

$$h : U \to X$$

be a strict smooth surjection from a log scheme. Then $U \times_X U$ is a scheme that comes with two projections to U. Tropicalizing defines a double arrow of generalized cone complexes

$$\Sigma(U \times_X U) \rightrightarrows \Sigma(U). \tag{C.11}$$

For a geometric point $\bar{x}$ of $U \times_X U$, composition with the two projections defines two geometric points $\bar{x}_1, \bar{x}_2$ of U. Since both projections $U \times_X U \to U$ are strict, we have two isomorphisms

$$\bar{\mathcal{M}}_{U,\bar{x}_i} \to \bar{\mathcal{M}}_{U\times_X U,\bar{x}}, \quad i = 1, 2. \tag{C.12}$$

These isomorphisms induce an equivalence relation on Strata(U), and provide isomorphisms between stalks of $\overline{\mathcal{M}}_U$ at pairs of geometric points in equivalent strata. The quotient Strata$(U)/\sim$ can easily be seen to be independent of the choice of smooth cover $U \to X$, and in fact defines the set Strata(X) of strata of the log algebraic stack X.

To define the tropicalization $\Sigma(X)$, we add $\mathrm{Hom}(\cdot, \mathbb{R}_{\geq 0})$ of the isomorphisms in (C.12) to the set of arrows in the diagram defining $\Sigma(U)$.

Definition C.7. The tropicalization $\Sigma(X)$ of the fine log algebraic stack X is the generalized cone complex defined by the diagram of $\Sigma(U)$ with the added isomorphisms induced by the tropicalization of (C.12).

Restricting the diagram defining $\Sigma(X)$ to one cone for each stratum of X gives an alternative presentation with index category Strata(X).

We need to check independence of our definition of $\Sigma(X)$ from choices.

Lemma C.8. *The definition of $\Sigma(X)$ is independent of the choice of strict smooth cover $U \to X$.*

Proof. It suffices to consider the composition of $U \to X$ with a strict smooth surjection $V \to U$. We obtain the following commutative diagram of strict smooth surjections of log schemes:

$$\begin{tikzcd} V \times_X V \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & U \times_X U \arrow[d, shift left] \arrow[d, shift right] \\ V \arrow[r] & U \end{tikzcd} \tag{C.13}$$

Now all arrows are surjective on geometric points. Since smooth maps are open, all arrows are also surjective on the set of generizations. Thus each cone and arrow of $\Sigma(V)$ maps isomorphically to a cone or arrow of $\Sigma(U)$, and each cone or arrow of $\Sigma(U)$ arises as an image. Moreover, if two cones σ_1, σ_2 in $\Sigma(U)$ belong to the same stratum in X, that is, are isomorphic images of a cone σ in $\Sigma(U \times_X U)$ appearing from a geometric point in $U \times_X U$, then lifting this geometric point to $V \times_X V$ provides a cone $\widetilde{\sigma}$ in $\Sigma(V \times_X V)$ mapping to cones $\widetilde{\sigma}_1, \widetilde{\sigma}_2$ in $\Sigma(V)$. The tropicalization of (C.13) now shows that the diagram of cones

$$\begin{tikzcd} \widetilde{\sigma}_1 \arrow[d] & \widetilde{\sigma} \arrow[l] \arrow[r] \arrow[d] & \widetilde{\sigma}_2 \arrow[d] \\ \sigma_1 & \sigma \arrow[l] \arrow[r] & \sigma_2 \end{tikzcd}$$

commutes up to composing the lower horizontal arrows with isomorphisms in $\Sigma(U)$.

Taken together we see that the diagram defining $\Sigma(X)$ from $V \to X$ just adds a number of isomorphic cones to the diagram defining $\Sigma(X)$ from $U \to X$. Thus the corresponding generalized cone complexes are equivalent. ■

The proof of functoriality of this notion of tropicalization now follows by local lifting to a presentation as in Proposition C.4. We omit the details.

Proposition C.9. *If $f : X \to Y$ and $g : Y \to Z$ are morphisms of fine log algebraic stacks then $\Sigma(g \circ f) = \Sigma(g) \circ \Sigma(f)$.*

C.3 Tropicalization in the log smooth case

We end this section with some facts on logarithmic strata and tropicalization in the Zariski log smooth case.

Lemma C.10. *Let $f : X \to B$ be a log smooth morphism of fine log schemes. Assume that B is locally noetherian with geometrically unibranch logarithmic strata. Then the logarithmic strata of X are irreducible and geometrically unibranch.*

Proof. First note that X is locally noetherian since B is locally noetherian and f is locally of finite presentation by the definition of log smoothness. Thus a locally irreducible connected subset of $|X|$ is irreducible. It thus suffices to show the stronger statement that each logarithmic stratum Z of X is geometrically unibranch.

Let $z \in |Z|$ and $Z_B \subseteq B$ the logarithmic stratum containing $\underline{f}(z)$. Being geometrically unibranch is a local property that is stable under étale morphisms. By [52, Theorem IV.3.3.1] we may thus replace X and B by étale neighborhoods of z and $\underline{f}(z)$ to obtain a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f"'] & B \times_{A_Q} A_P \arrow[r, "k"] \arrow[d] & A_P \arrow[d, "A_\theta"] \\ & B \arrow[r, "h"] & A_Q \end{tikzcd}$$

with $A_P = \operatorname{Spec} \mathbb{Z}[P]$, $A_Q = \operatorname{Spec} \mathbb{Z}[Q]$, A_θ the morphism induced by a homomorphism $\theta : Q \to P$ of fine monoids, all horizontal arrrows strict, the square cartesian, g étale, and h a neat chart at z. Thus $Z_B = (h^{-1}(O))_{\mathrm{red}}$, where $O \subseteq A_Q$ is the closed torus orbit defined by the monoid ideal $Q \setminus \{0\}$.

Since $k \circ g$ is strict, the composition $Z \to B \times_{A_Q} A_P \to A_P$ factors over the inclusion of a logarithmic stratum $Z_P \subseteq A_P$. Now toric morphisms respect the decomposition into logarithmic strata. Thus $\underline{A_\theta}(Z_P)$ is contained in a logarithmic stratum of A_Q. But $\underline{h}(\underline{f}(z)) \in \underline{A_\theta}(Z_P)$, so this latter stratum is the closed stratum $O \subseteq A_Q$.

This shows that $g(Z)$ is contained in

$$Z_B \times_{A_Q} A_P = Z_B \times_{A_Q} Z_P = Z_B \times_O Z_P = Z_B \times_{\mathbb{Z}} Z_P.$$

Since $Z_B \times_{\mathbb{Z}} Z_P$ has constant ghost sheaf $\bar{\mathcal{M}}$ it follows that $Z = g^{-1}(Z_B \times_{\mathbb{Z}} Z_P)$, and hence Z is étale over $Z_B \times_{\mathbb{Z}} Z_P$. Here we are using that the preimage of a reduced subscheme under an étale morphism remains reduced [67, Proposition 0250]. Finally, $Z_B \times_{\mathbb{Z}} Z_P$ is geometrically unibranch by the assumption on the strata of B. This shows that Z is geometrically unibranch at z. ∎

Proposition C.11. *Let $f : X \to B$ be a log smooth morphism of fine log schemes with B locally noetherian and with geometrically unibranch logarithmic strata. Assume that B is simple, that is, $\Sigma(B)$ is a cone complex rather than a generalized cone complex, and that the log structure of X is defined in the Zariski topology. Then X is simple as well, and the logarithmic strata of X are irreducible and geometrically unibranch.*

Proof. Lemma C.10 shows the statement on the log strata of X. Thus each logarithmic stratum Z has a unique generic point η_Z. It is then obvious that there is an arrow $\sigma_Z \to \sigma_W$ if and only if $\eta_W \in \mathrm{cl}(\eta_Z)$. Moreover, since $\mathcal{M}_X$ is a sheaf on the Zariski site, $\sigma_Z \to \sigma_W$ must then be the dual of the generization homomorphism $\mathcal{M}_{X,\eta_W} \to \mathcal{M}_{X,\eta_Z}$. Thus there is at most one such arrow, and hence $\Sigma(X)$ is a cone complex. ∎

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MEMOIRS OF THE EUROPEAN MATHEMATICAL SOCIETY

Dan Abramovich, Qile Chen, Mark Gross, Bernd Siebert

Punctured Logarithmic Maps

We introduce a variant of stable logarithmic maps, which we call *punctured logarithmic maps*. They allow an extension of logarithmic Gromov–Witten theory in which marked points have a negative order of tangency with boundary divisors.

As a main application we develop a gluing formalism which reconstructs stable logarithmic maps and their virtual cycles without expansions of the target, with tropical geometry providing the underlying combinatorics.

Punctured Gromov–Witten invariants also play a pivotal role in the intrinsic construction of mirror partners by the last two authors, conjecturally relating to symplectic cohomology, and in the logarithmic gauged linear sigma model in work of Qile Chen, Felix Janda and Yongbin Ruan.

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