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Seiberg–Witten Floer Spectra for $b_1 > 0$

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Abstract

The Seiberg–Witten Floer spectrum is a stable homotopy refinement of the monopole Floer homology of Kronheimer and Mrowka. The Seiberg–Witten Floer spectrum was defined by Manolescu for closed, spin^c 3-manifolds with $b_1 = 0$ in an S^1-equivariant stable homotopy category and has been producing interesting topological applications. Lidman and Manolescu showed that the S^1-equivariant homology of the spectrum is isomorphic to the monopole Floer homology.

For closed spin^c 3-manifolds Y with $b_1(Y) > 0$, there are analytic and homotopy-theoretic difficulties in defining the Seiberg–Witten Floer spectrum. In this memoir, we address the difficulties and construct the Seiberg–Witten Floer spectrum for Y, provided that the first Chern class of the spin^c structure is torsion and that the triple-cup product on $H^1(Y;\mathbb{Z})$ vanishes. We conjecture that its S^1-equivariant homology is isomorphic to the monopole Floer homology.

For a 4-dimensional spin^c cobordism X between Y_0 and Y_1, we define the Bauer–Furuta map on these new spectra of Y_0 and Y_1, which is conjecturally a refinement of the relative Seiberg–Witten invariant of X. As an application, for a compact spin 4-manifold X with boundary Y, we prove a $\frac{10}{8}$-type inequality for X which is written in terms of the intersection form of X and an invariant $\kappa(Y)$ of Y.

In addition, we compute the Seiberg–Witten Floer spectrum for some 3-manifolds.

Keywords. Seiberg–Witten equations, Floer theory, stable homotopy, Conley index, spectral sections

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Chapter 1

Introduction

1.1 Background

The Seiberg–Witten equations [51] have been an important tool in the study of 4-manifolds since their introduction. Soon after these equations appeared, Kronheimer–Mrowka [28] extended their study to define the monopole Floer homology of 3-manifolds, and established its relationship with the 4-manifold invariant; the resulting theory has since had many applications in low-dimensional topology.

In both gauge theory and symplectic geometry, certain Floer homology theories have since been shown to arise as the homology of well-defined *Floer spectra* as envisioned by Cohen, Jones and Segal [11], and some invariants, obtained by counting solutions of certain PDEs, are now either known or conjectured to come from the degree of certain maps between spectra. One of the first examples of such a construction is the Bauer–Furuta invariant [8,21], which associates an element in stable homotopy $\pi^{\mathrm{st}}(S^0)$ to certain 4-manifolds, refining the ordinary Seiberg–Witten invariant. Building on the finite-dimensional approximation technique introduced by Furuta, Manolescu [35] constructed an S^1-equivariant stable homotopy type $SWF(Y, \mathfrak{s})$ associated to rational homology 3-spheres with spin^c structure $(Y, \mathfrak{s})$.

It is natural to want to extend Manolescu's construction to 3-manifolds with $b_1(Y) > 0$. In the case $b_1(Y) = 1$, Kronheimer–Manolescu [30] constructed a *periodic pro-spectrum* for pairs $(Y, \mathfrak{s})$. Later, together with T. Khandhawit and J. Lin, the first author constructed the *unfolded* Seiberg–Witten Floer spectrum for arbitrary closed, oriented $(Y, \mathfrak{s})$ in [24, 25].

The *unfolded* spectrum comes in multiple flavors. For now, we consider only the type-A unfolded invariant $\underline{\mathrm{swf}}^A(Y, \mathfrak{s})$, which depends on $(Y, \mathfrak{s})$ as well as some additional data we suppress. This invariant is a directed system in the S^1-equivariant stable homotopy category. In particular, it is not per se a spectrum, and tends to be very large.

Khandhawit, Lin and the first author [25] showed that the unfolded invariant allowed for gluing formulas, in a very general setting, for the calculation of the Bauer–Furuta invariant of a 4-manifold cut along 3-manifolds with $b_1 > 0$. In particular, this enables one to prove vanishing formulas for the Bauer–Furuta invariant in many situations.

However, the invariant $\underline{\mathrm{swf}}^A(Y, \mathfrak{s})$ is not expected to recover the monopole Floer homology, but is instead expected to recover a version of monopole Floer homology with fully twisted coefficients.

Here we construct a new Seiberg–Witten Floer spectrum $\mathbf{SWF}(Y, \mathfrak{s})$ for $b_1(Y) > 0$, as follows.

Theorem 1.1.1. *Let $(Y, \mathfrak{s})$ be a closed,* spin^c *3-manifold which satisfies that the first Chern class $c_1(\mathfrak{s}) \in H^2(Y;\mathbb{Z})$ is torsion, and so that the triple-cup product on $H^1(Y;\mathbb{Z})$ vanishes. Associated to a* Floer framing $\mathfrak{P}$ *(see Section 3.5 for this notation), there is a well-defined parameterized, S^1-equivariant stable homotopy type $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P})$, over the Picard torus* $\mathrm{Pic}(Y) = H^1(Y;\mathbb{R})/H^1(Y;\mathbb{Z})$*, called the* Seiberg–Witten Floer stable homotopy type *of $(Y, \mathfrak{s}, \mathfrak{P})$. Moreover, there is a well-defined (unparameterized) S^1-equivariant connected simple system of spectra $\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P})$, the* Seiberg–Witten Floer spectrum.

If $\mathfrak{s}$ is self-conjugate and $\mathfrak{P}$ is a Pin(2)*-equivariant Floer framing, then the equivariant, parameterized stable homotopy type $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P})$ naturally comes with the structure of a parameterized* Pin(2)*-equivariant stable homotopy type, where the Picard torus has a* Pin(2)*-action factoring through $\pi_0(\mathrm{Pin}(2))$ by conjugation. Similarly, $\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P})$ has an underlying (unparameterized)* Pin(2)*-equivariant spectrum, $\mathbf{SWF}^{u,\mathrm{Pin}(2)}(Y, \mathfrak{s}, \mathfrak{P})$.*

The homotopy type $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P})$, viewed without its parameterization, has the homotopy type of a finite S^1 (respectively Pin(2)*)-CW complex. The Seiberg–Witten Floer spectrum $\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P})$ (respectively $\mathbf{SWF}^{u,\mathrm{Pin}(2)}(Y, \mathfrak{s}, \mathfrak{P})$) has the homotopy type of a finite S^1 (respectively* Pin(2)*) CW-spectrum.*

If $b_1(Y) = 0$, $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P})$ agrees with the invariant $SWF(Y, \mathfrak{s})$ in [35]*, in that*

$$\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P}) \simeq \Sigma^{n\mathbb{C}} SWF(Y, \mathfrak{s}),$$

for some $n \in \mathbb{Q}$, depending only on $\mathfrak{P}$.

For the notion of the parameterized spaces that we use, ex-spaces, we refer to Appendix A, as well as for the notion of a connected simple system. In particular, see Definition A.1.9 for the notion of a parameterized equivariant stable homotopy type.

The collection of Floer framings of $(Y, \mathfrak{s})$, should any exist, is an affine space over $K(\mathrm{Pic}(Y)) \cong \mathbb{Z}^{2^{b_1(Y)-1}}$. Moreover, there is an explicit relationship between the Floer spectra constructed for different spectral sections; see Corollary 3.6.3.

In order to explain the context of Theorem 1.1.1, and its apparent difference from the unfolded invariant, we review below the process of finite-dimensional approximation, introduced by Furuta, and used by Manolescu [35] in his construction of the 3-manifold invariant for rational homology 3-sphere input, as well as in [24, 25, 30].

1.2 Finite-dimensional approximation

There are two main approaches to refining the construction of Floer-theoretic invariants from homology theories to generalized homology theories (and, in some instances, spectra). There is the approach by constructing *framed flow categories* (or variations on this type of category) as envisioned originally by [11]. A very general version of this has just been accomplished in [1] (while the present work was in its final stages of preparation). There is also the method of finite-dimensional approximation, mentioned above, which we now summarize.

Manolescu's construction of $SWF(Y, \mathfrak{s})$ takes place inside the *Coulomb gauge slice* of the Seiberg–Witten equations. All that matters for this introduction is that, roughly speaking, the Coulomb slice is some Hilbert space on which the Seiberg–Witten equations admit a particularly simple form, as a linear operator plus a compact perturbation. For certain linear subspaces of the Coulomb slice (adapted to the linear part of the Seiberg–Witten equations), Manolescu considers an approximation of the formal L^2-gradient flow of the Seiberg–Witten equations. The approximations tend to stabilize as larger and larger finite-dimensional subspaces are chosen. Associated to suitable flows on suitable topological spaces, there is a convenient invariant, the *Conley index*, which is a well-defined homotopy type associated to the flow (along with some extra data). The invariant $SWF(Y, \mathfrak{s})$ is then taken as the Conley index of these approximated flows.

The most pressing difficulty facing finite-dimensional approximation to other equations of gauge theory or symplectic geometry is that the configuration space in these other situations is usually not linear, so that it is not obvious which finite-dimensional submanifolds one should consider "approximations" on.

For $b_1(Y) > 0$ the gauge slice of the Seiberg–Witten equations is no longer linear, but Kronheimer–Manolescu [30], and the authors of [24, 25], avoided the problem of having a more general configuration space by considering the Seiberg–Witten equations on the universal cover (which is once again a Hilbert space) of a gauge slice to the Seiberg–Witten equations, where finite-dimensional approximation is still possible, but where the usual compactness of the space of Seiberg–Witten trajectories is lost. The loss of compactness leads to the resulting invariant $\underline{\mathrm{swf}}^A$ not being a single spectrum, but rather a system of them.

The problem of performing finite-dimensional approximation in nonlinear situations has been open for some time (though see [27]). In this memoir our objective is to resolve it in one (relatively simple) case, for the Seiberg–Witten equations. We hope that this method may be useful in other situations where one would like to apply finite-dimensional approximation for topologically complicated configuration spaces.

The main work of the present memoir is showing that there exist families of submanifolds of the configuration space of the Seiberg–Witten equations (for $b_1(Y) > 0$) on which the Seiberg–Witten equations can be approximated very accurately. This comes down to carefully controlling spectral sections of the Dirac operator, in the sense of Melrose–Piazza [40], and in particular relies on some control of spectra of Dirac operators. Once the submanifolds are constructed, there also remains the problem of showing that the approximate Seiberg–Witten equations thereon are sufficiently accurate; for this we use a refined version of the original argument of Manolescu which requires weaker assumptions than the original, but does not yield the same strength of convergence as in Manolescu's case.

A word is also in order about the hypotheses on the input in Theorem 1.1.1. Cohen–Jones–Segal conjectured that Floer spectra should exist for many of the familiar Floer homology theories – but only in the event that the *polarization* is trivial. The hypotheses in the theorem are necessary for the vanishing of the polarization (indeed, a Floer framing is the same thing as a trivialization of the polarization), as observed in [30].

However, in spite of usually having a dependence on the Floer framing, we can consider generalized homology theories applied to $\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P})$ that are insensitive to the framing. In the following theorem, $n(Y, \mathfrak{s}, \mathfrak{P})$ is a certain numerical invariant of a Floer framing, introduced in Chapter 6, and MU and MU_{S^1} denote, respectively, complex cobordism and S^1-equivariant complex cobordism. For the notion of an equivariant complex orientation, see Section 3.6 (and for more detail, [12]).

Theorem 1.2.1. *Let E be a (possibly S^1-equivariant) complex-oriented (resp. S^1-equivariantly complex oriented) cohomology theory. Then*

$$E^{*-2n(Y,\mathfrak{s},\mathfrak{P})}(\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P}))$$

is (canonically) independent of $\mathfrak{P}$.

In particular, the complex-cobordism theories

$$FMU^*(Y, \mathfrak{s}) = \widetilde{MU}^{*-2n(Y,\mathfrak{s},g,\mathfrak{P})}(\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P})),$$
$$FMU^*_{S^1}(Y, \mathfrak{s}) = \widetilde{MU}^{*-2n(Y,\mathfrak{s},g,\mathfrak{P})}_{S^1}(\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P})),$$

are invariants of the pair $(Y, \mathfrak{s})$, which we call the Floer (equivariant) complex cobordism *of $(Y, \mathfrak{s})$.*

As MU, MU_{S^1} are the universal complex-oriented cohomology theories, in some sense $FMU^*(Y, \mathfrak{s})$ and $FMU^*_{S^1}(Y, \mathfrak{s})$ might be interpreted as the universal monopole Floer-type invariants that are independent of the framing.

More speculatively, we remark that the independence of FMU^* on the framing suggests that its definition could be extended to pairs $(Y, \mathfrak{s})$ which do not admit a Floer framing. We plan to pursue this in future work.

It would also be desirable to relate the (generalized) homology theories of the Seiberg–Witten Floer spectrum $\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P})$ to the monopole-Floer homology of Kronheimer–Mrowka. In particular, we conjecture the following.

Conjecture 1.2.2. For $(Y, \mathfrak{s})$ a pair as in Theorem 1.1.1,

$$\begin{aligned} H^{S^1}_{\bullet-2n(Y,\mathfrak{s},\mathfrak{P})}(\mathbf{SWF}^u(Y,\mathfrak{s},\mathfrak{P})) &\cong HM_\bullet(Y,\mathfrak{s}), \\ cH^{S^1}_{\bullet-2n(Y,\mathfrak{s},\mathfrak{P})}(\mathbf{SWF}^u(Y,\mathfrak{s},\mathfrak{P})) &\cong \widehat{HM}_\bullet(Y,\mathfrak{s}), \\ tH^{S^1}_{\bullet-2n(Y,\mathfrak{s},\mathfrak{P})}(\mathbf{SWF}^u(Y,\mathfrak{s},\mathfrak{P})) &\cong \overline{HM}(Y,\mathfrak{s}), \\ H_{\bullet-2n(Y,\mathfrak{s},\mathfrak{P})}(\mathbf{SWF}^u(Y,\mathfrak{s},\mathfrak{P})) &\cong \widetilde{HM}_\bullet(Y,\mathfrak{s}). \end{aligned}$$

Note that ordinary homology is (equivariantly) complex-orientable, and so the homology theories on each left-hand side are independent of the choice of spectral section (and we have been somewhat imprecise about the gradings on the right). Here, H^{S^1}, cH^{S^1}, tH^{S^1} are, respectively, Borel, coBorel and Tate homology.

This conjecture has already been established by Lidman–Manolescu in the case that Y is a rational-homology sphere [32].

We note that there is a natural generalization of Conjecture 1.2.2 to include the case of local coefficient systems on monopole Floer homology HM°; this involves using other parameterized cohomology theories (as in [39, Section 20.3]) applied to $\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{P})$. There is also a further generalization of the conjecture to relate the Pin(2)-equivariant cohomology of $\mathbf{SWF}^u(Y,\mathfrak{s},\mathfrak{P})$, for $(Y,\mathfrak{s})$ admitting a Pin(2)-equivariant Floer framing, to the equivariant monopole Floer homology defined by Lin [34].

We remark that Theorem 1.1.1 should yield a well-defined connected simple system $\mathbf{SWF}(Y,\mathfrak{s},\mathfrak{P})$ of equivariant, parameterized spectra. Indeed, this would follow if the *parameterized* Conley index of a dynamical system were known to be well defined as a connected simple system (rather than as a homotopy type; the ordinary Conley index is known [47] to be a connected simple system). We hope to return to this point, and other improvements to naturality, in future work.

1.3 Four-manifolds

In this memoir we also define a Bauer–Furuta invariant associated to a spin^c 4-manifold with boundary.

Let $(Y,\mathfrak{s})$ be a closed spin^c 3-manifold and $\mathfrak{P}$ be a Floer framing of $(Y,\mathfrak{s})$. Recall that, in the parameterized setting, we only define the ex-space $\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{P})$ up to stable homotopy equivalence. To fix notation, define a *map class* of maps $P \to Q$ between two spaces P, Q, themselves only well defined up to homotopy equivalence, to mean just a homotopy class, up to the action of self-homotopy-equivalences on P or Q.

For an S^1-equivariant virtual vector bundle V over a base B, let S_B^V denote the corresponding sphere bundle over B. We then construct a Bauer–Furuta invariant $\mathcal{BF}$ as follows.

Theorem 1.3.1. *Let $(X, \mathfrak{t})$ be a smooth, compact,* spin^c *4-manifold with boundary $(Y, \mathfrak{s})$, and fix a Floer framing $\mathfrak{P}$ of $(Y, \mathfrak{s})$. Then there is a well-defined (parameterized, S^1-equivariant, stable) map class*

$$\mathcal{BF}(X, \mathfrak{t})\colon S_{\mathrm{Pic}(Y)}^{\mathrm{ind}(D_X, \mathfrak{P})} \to \mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P}).$$

For the definition of the index $\mathrm{ind}(D_X, \mathfrak{P})$, see Chapter 5. There is also a version of Theorem 1.3.1 at the spectrum level, which is more complicated to state; see Corollary 5.2.7.

As a by-product of our proof of well-definedness of $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P})$, we also obtain an invariant of families.

Theorem 1.3.2. *Let $\mathcal{F}$ be a Floer-framed family of* spin^c *3-manifolds, with compact base B and fibers denoted by $\mathcal{F}_b$ for $b \in B$. Let* $\mathrm{Pic}(\mathcal{F})$ *denote the bundle over B with fiber* $\mathrm{Pic}(\mathcal{F}_b)$*. There is a well-defined parameterized, S^1-equivariant stable-homotopy type $\mathcal{SWF}(\mathcal{F})$, which is parameterized over* $\mathrm{Pic}(\mathcal{F})$.

A similar families invariant exists for the Bauer–Furuta invariant, but we omit its discussion, as we do not have need of it in the present memoir.

As an application of our construction, we construct Frøyshov-type invariants associated to the Seiberg–Witten Floer stable homotopy type. In particular, we define a generalization of Manolescu's κ-invariant, from Pin(2)-equivariant K-theory of 3-manifolds with $b_1(Y) = 0$, to Y with $b_1(Y) > 0$. We show the following theorem.

Theorem 1.3.3. *Let $(X, \mathfrak{t})$ be a compact, spin 4-manifold with boundary $-Y_0 \coprod Y_1$. Assume that Y_0 is a rational homology 3-sphere and the index* $\mathrm{ind}\, D$ *for $(Y_1, \mathfrak{t}|_{Y_1})$ is zero in $KQ^1(\mathrm{Pic}(Y_1))$. Here, KQ^1 stands for the quaternionic K-theory. (See* [19,33]*.) Then we have*

$$-\frac{\sigma(X)}{8} + \kappa(Y_0, \mathfrak{t}|_{Y_0}) - 1 \le b^+(X) + \kappa(Y_1, \mathfrak{t}|_{Y_1}).$$

See Remark 6.2.13 for the reason why we assume $b_1(Y_0) = 0$ in this theorem.

We also define invariants associated to the S^1-equivariant monopole Floer homology, corresponding roughly to the generalized d-invariants introduced by Levine–Ruberman [31] in Heegaard Floer homology.

We also calculate the Seiberg–Witten Floer homotopy-type invariant in some relatively simple situations; see Chapter 4.

1.4 Further directions

We do not prove any gluing theorems for the Bauer–Furuta invariant, or for its families analog, and this is a natural point of departure, remaining within Seiberg–Witten theory. In this direction, we expect the surgery exact triangles [28, Section 42] (and variations) to hold for homology theories other than ordinary homology. For this, it would be particularly desirable to obtain a description of the map on FMU^* induced by the Bauer–Furuta invariant, independent of choices like the Floer framing. It is also natural to ask how the unfolded spectrum $\underline{\mathrm{swf}}^A(Y,\mathfrak{s})$ is related to the folded spectrum $\mathcal{SWF}(Y,\mathfrak{s})$.

A technical problem that may make the invariant $\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{P})$ more wieldy is to establish a natural topological description (on Y) of the set of Floer framings. We hope to address some of these points in the future.

Furthermore, we expect that it should be possible to consider more detailed applications to the question of when a family of 3-manifolds extends to a family of 4-manifolds with boundary. Compare with recent work by Konno–Taniguchi [26] in the case that the boundary family of 3-manifolds is the trivial family of a rational homology sphere.

Finally, given an extension of $FMU^*(Y,\mathfrak{s})$ to 3-manifolds that do not admit a Floer framing, it seems likely that the excision argument of [29] should apply, in which case we would expect there to exist generalizations of sutured monopole Floer homology to various generalized homology theories.

1.5 Organization

This memoir is organized as follows. We first construct special families of spectral sections to the Dirac operator in Chapter 2, and show that certain subsets of the (approximate) Seiberg–Witten configuration space are isolating neighborhoods in the sense of Conley index theory. In Chapter 3 we show that the resulting invariant is well defined, as a consequence of this process we establish a Seiberg–Witten Floer homotopy type for families. This consists of showing that all of the possible choices for different approximations to the Seiberg–Witten equations are compatible. In Chapter 4 we give various example calculations of $\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{P})$. In Chapter 5 we construct a relative Bauer–Furuta invariant, and show that it is well defined. Finally, in Chapter 6 we establish various Frøyshov-type inequalities that are a consequence of the existence of the new relative Bauer–Furuta invariant.

There is one appendix, Appendix A, on homotopy-theoretic background, as well as an afterword on potential further applications outside of Seiberg–Witten theory.

Chapter 2

Finite-dimensional approximation on 3-manifolds

2.1 Spectral sections

In order to define Seiberg–Witten Floer spectra, we will make use of spectral sections of a family of Dirac operators introduced by Melrose–Piazza [40]. We will recall definitions and basic things on spectral sections in this section.

Suppose that we have a closed, oriented $(2n-1)$-manifold Y and that we have a fiber bundle

$$\mathcal{Y} \to B$$

with fiber Y. Here, B is a compact Hausdorff space. Also suppose that we are given a finite-dimensional vector bundle

$$F_Y \to \mathcal{Y}$$

with metric. We consider an infinite-dimensional vector bundle on B defined by

$$\mathcal{E}_{Y,\infty} := \bigcup_{z \in B} \Gamma(F_Y|_{\mathcal{Y}_z}).$$

Let

$$D_Y \colon \mathcal{E}_{Y,\infty} \to \mathcal{E}_{Y,\infty}$$

be a family of first-order elliptic, self-adjoint differential operators. That is, D_Y preserves the fibers of $\mathcal{E}_{Y,\infty}$ and for each $z \in B$,

$$D_{Y,z} \colon \mathcal{E}_{Y,\infty,z} \to \mathcal{E}_{Y,\infty,z}$$

is a first-order, elliptic, self-adjoint differential operator. Here, $\mathcal{E}_{Y,\infty,z}$ is the fiber of $\mathcal{E}_{Y,\infty}$ over z.

We assume that for each $z \in B$, there is an open neighborhood U of z such that we have a trivialization

$$F_Y|_{\mathcal{Y}_U} \cong U \times F_{Y,z}, \tag{2.1.1}$$

where $\mathcal{Y}_U$ is the restriction of the bundle $\mathcal{Y}$ to U, and we can write

$$D_{Y,w} = D_{Y,z} + A_{Y,w}$$

for $w \in U$ through the isomorphism $\mathcal{E}_{Y,\infty,z} \cong \mathcal{E}_{Y,\infty,w}$ induced by (2.1.1). Here, $A_{Y,w}$ is the operator acting on $\mathcal{E}_{Y,\infty,w}$ induced by a fiberwise linear bundle map $F_Y|_{\mathcal{Y}_w} \to F_Y|_{\mathcal{Y}_w}$ which continuously depends on w.

For $k \geq 0$, define the L^2_k-inner product on $\mathcal{E}_{Y,\infty}$ by

$$\langle \phi_1, \phi_2 \rangle_k = \int_{\mathcal{Y}_z} \langle \phi_1, \phi_2 \rangle + \langle |D_{Y,z}|^k \phi_1, |D_{Y,z}|^k \phi_2 \rangle \, d\mu.$$

Here, $|D_{Y,z}|$ denotes the absolute value of $D_{Y,z}$ defined as in [46, Chapter VIII, §9]. We write $\mathcal{E}_{Y,k}$ for the completions with respect to the L^2_k-norm. The operator D_Y extends to a bounded operator

$$D_Y : \mathcal{E}_{Y,k} \to \mathcal{E}_{Y,k-1}.$$

For $w \in U$, the algebraic operator $A_{Y,w}$ extends to a bounded operator $\mathcal{E}_{Y,k,w} \to \mathcal{E}_{Y,k,w}$ which continuously depends on w with respect to the operator norm, and $D_{Y,w} = D_{Y,z} + A_{Y,w}$ as operators $\mathcal{E}_{Y,k,w} \to \mathcal{E}_{Y,k-1,w}$ through the local trivialization (2.1.1).

We now recall the definition of a spectral section from [40].

Definition 2.1.1 ([40]). A *spectral section* for $D_Y : \mathcal{E}_{Y,k} \to \mathcal{E}_{Y,k-1}$ over a compact base B is a family of self-adjoint projections $P : \mathcal{E}_{Y,0} \to \mathcal{E}_{Y,0}$ so that there is a constant $C > 0$ such that the following holds. Suppose that $z \in B$, $u \in \mathcal{E}_{Y,\infty,z}$, $D_{Y,z} u = \lambda u$ for some $\lambda \in \mathbb{R}$. Then $P_z u = u$ if $\lambda > C$ and $P_z u = 0$ if $\lambda < -C$. Here, a *family* is meant to be a continuous family in the L^2-operator norm topology, parameterized by B.

We note that the condition that P be continuous families in the L^2-norm topology is equivalent to P being continuous families in any L^2_k-norm topology with $k > 0$, using the interaction of P with the spectrum of D_Y. Also note that since P is self-adjoint, P is an orthogonal projection onto its image with respect to the L^2-inner product. In fact, for $\phi_1, \phi_2 \in \mathcal{E}_{Y,\infty,z}$, we have

$$\langle P\phi_1, (1-P)\phi_2 \rangle_0 = \langle \phi_1, P(1-P)\phi_2 \rangle_0 = 0.$$

Here we have used the fact that P is self-adjoint and $P^2 = P$.

Melrose and Piazza proved the following about the existence of a spectral section.

Theorem 2.1.2 ([40, Proposition 1]). *There exists a spectral section of D_Y if and only if the index* $\operatorname{ind} D_Y$ *is zero in $K^1(B)$. Here,* $\operatorname{ind} D_Y$ *is the index defined in* [6].

Using a spectral section, we can define the Atiyah–Patodi–Singer index for a family of differential operators on a manifold with boundary. Let X be a compact, oriented $2n$-manifold with boundary Y. Suppose that we have a fiber bundle

$$\mathcal{X} \to B$$

with fiber X, such that the family obtained by taking the boundary of each fiber of $\mathcal{X}$ is $\mathcal{Y}$. Also suppose that we have finite-dimensional vector bundles

$$F^0_X, F^1_X \to \mathcal{X}$$

and that isomorphisms

$$F_X^0|_{\mathcal{Y}} \cong F_X^1|_{\mathcal{Y}} \cong F_Y$$

are given. Define infinite-dimensional vector bundles over B by

$$\mathcal{E}_{X,\infty}^0 = \bigcup_{z \in B} \Gamma(F_X^0|_{\mathcal{X}_z}), \quad \mathcal{E}_{X,\infty}^1 = \bigcup_{z \in B} \Gamma(F_X^1|_{\mathcal{X}_z}).$$

We consider a family of first-order elliptic differential operators

$$D_X \colon \mathcal{E}_{X,\infty}^0 \to \mathcal{E}_{X,\infty}^1$$

such that

$$D_X = \frac{\partial}{\partial t} + D_Y$$

near the boundary $\mathcal{Y}$. Here, t is the coordinate of the first component of a neighborhood of $\mathcal{Y}$ in $\mathcal{X}$ which is diffeomorphic to $[0,1] \times \mathcal{Y}$. As before, we assume that for $z \in B$, there is an open neighborhood U of z and we can write $D_{X,w} = D_{X,z} + A_{X,w}$ for $w \in U$ through local trivializations of F_X^0, F_X^1. Here, $A_{X,w}$ is an algebraic operator induced by a linear bundle map $F_X^0|_{\mathcal{X}_w} \to F_X^1|_{\mathcal{X}_w}$ depending on w continuously.

We define Hilbert bundles $\mathcal{E}_{X,k}^0$, $\mathcal{E}_{X,k}^1$ over B for $k \geq 0$ using D_X as before. Note that $\operatorname{ind} D_Y = 0$ in $K^1(B)$ because of the cobordism invariance of the index. Hence there is a spectral section of D_Y.

Let $(\mathcal{E}_{Y,k-\frac{1}{2}})_{-\infty}^0$ be the subspace spanned by nonpositive eigenvectors of D_Y and p^0 be the $L_{k-\frac{1}{2}}^2$-orthogonal projection onto $(\mathcal{E}_{Y,k-\frac{1}{2}})_{-\infty}^0$. Let us consider the family of operators with the APS boundary condition. That is, we consider the family of operators

$$(D_X, p^0 \circ r) \colon \mathcal{E}_{X,k}^0 \to \mathcal{E}_{X,k-1}^1 \oplus (\mathcal{E}_{Y,k-\frac{1}{2}})_{-\infty}^0.$$

Here, r is the restriction to $\mathcal{Y}$. Note that this family is not continuous because of the spectral flow of D_Y. Hence we cannot use this family to define the index. A spectral section enables us to avoid this issue. Since our sign convention is different from that of [40], taking a spectral section of $-D_Y$ rather than D_Y is more convenient for us.

Proposition 2.1.3. *Fix $k \geq 1$. Let P be a spectral section of $-D_Y$. We also denote by P the image of P in $\mathcal{E}_{Y,0}$, which is a Hilbert subbundle. Let π_P be the $L_{k-\frac{1}{2}}^2$-projection onto $P \cap \mathcal{E}_{Y,k-\frac{1}{2}}$. Then*

$$(D_X, \pi_P \circ r) \colon \mathcal{E}_{X,k}^0 \to \mathcal{E}_{X,k-1}^1 \oplus (P \cap \mathcal{E}_{Y,k-\frac{1}{2}})$$

is a continuous family of Fredholm operators and we can define the index $\operatorname{ind}(D_X, P) \in K(B)$. *The index* $\operatorname{ind}(D_X, P)$ *is independent of the choice of k.*

Let P be a spectral section of $-D_Y$. We write P for the image of P in $\mathcal{E}_{Y,0}$ too. Then we can take other spectral sections Q, R of $-D_Y$ such that

$$Q \subset P \subset R.$$

See our construction of spectral sections in Section 2.4. Define a family of operators

$$D'_Y := QD_Y Q + (1-R)D_Y(1-R) - (1-Q)P + R(1-P).$$

We can see that D'_Y is injective and that P is equal to the subspace spanned by negative eigenvectors of D'_Y. Also we see that the operator $\mathbb{A} = D'_Y - D_Y$ is a family of smoothing operators acting on $\mathcal{E}_{Y,k}$. In fact, the image of $\mathbb{A}$ is included in the subspace spanned by finitely many eigenvectors of D_Y.

Take a smooth function $f\colon \mathcal{X} \to [0,1]$ such that

$$f(x) = \begin{cases} 1 & \text{for } x \in [\frac{1}{2}, 1] \times \mathcal{Y}, \\ 0 & \text{for } x \in \mathcal{X} \setminus ([0,1] \times \mathcal{Y}). \end{cases}$$

Define $D'_X\colon \mathcal{E}^0_{X,k} \to \mathcal{E}^1_{X,k-1}$ by

$$D'_X = D_X + f\mathbb{A}.$$

Then

$$D'_X = \frac{\partial}{\partial t} + D'_Y$$

near $\mathcal{Y}$ and there is no spectral flow of D'_Y. Therefore, the family of operators D'_X with the APS boundary condition defines the index $\operatorname{ind} D'_X \in K(B)$, and

$$\operatorname{ind} D'_X = \operatorname{ind}(D_X, P).$$

2.2 Connections on Hilbert bundles

Since we will consider a connection on a Hilbert bundle later, we give the definition of a connection on a Hilbert bundle.

Let M be a connected, smooth n-manifold and H be a Hilbert space. We write $\operatorname{Aut} H$ and $\operatorname{End} H$ for the group of bounded linear isomorphisms $H \to H$ and the ring of bounded operators $H \to H$ respectively.

Take a coordinate chart (U, φ) of M. For a map

$$f\colon U \to H,$$

we define the partial derivative $\frac{\partial f}{\partial x^i}(x)$ at $x \in U$ by

$$\frac{\partial f}{\partial x^i}(x) = \lim_{h \to 0} \frac{1}{h}\big(f \circ \varphi^{-1}(\varphi(x) + he_i) - f(x)\big)$$

if the limit exists in H. Here, e_i is the ith standard basis of $\mathbb{R}^n$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$, we define $\frac{\partial^\alpha f}{\partial x^\alpha}$ to be $\left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n} f$. We say that f is smooth if the derivatives $\frac{\partial^\alpha f}{\partial x^\alpha}$ exist and are continuous on U for all $\alpha \in (\mathbb{Z}_{\geq 0})^n$.

Let $p\colon \mathcal{E} \to M$ be a smooth Hilbert bundle on M with fiber H. By a smooth Hilbert bundle we mean that for each small open set U in M, we have a local trivialization

$$\psi\colon \mathcal{E}|_U \to U \times H$$

such that if $\psi'\colon \mathcal{E}|_{U'} \to U' \times H$ is another local trivialization with $U \cap U' \neq \emptyset$, we can write

$$\psi' \circ \psi^{-1}(x, v) = (x, g(x)v)$$

for $x \in U \cap U'$ and $v \in H$, and g is a map $U \cap U' \to \operatorname{Aut} H$ which is smooth with respect to the operator norm. We always assume that Hilbert bundles are smooth.

A section $s\colon M \to \mathcal{E}$ is said to be smooth if for each local trivialization $\psi\colon \mathcal{E}|_U \to U \times H$, the map

$$\psi \circ s|_U\colon U \to U \times H$$

is smooth. We denote by $\Gamma(\mathcal{E})$ the space of smooth sections of $\mathcal{E}$.

A connection ∇ on $\mathcal{E}$ is defined to be a map

$$\nabla\colon \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E})$$

having the following properties:

(i) For any sections $s_1, s_2 \in \Gamma(\mathcal{E})$,

$$\nabla(s_1 + s_2) = \nabla s_1 + \nabla s_2.$$

(ii) For any section $s \in \Gamma(\mathcal{E})$, vector fields $X_1, X_2 \in \Gamma(TM)$ and smooth functions $f_1, f_2 \in C^\infty(M)$,

$$\nabla_{f_1 X_1 + f_2 X_2} s = f_1 \nabla_{X_1} s + f_2 \nabla_{X_2} s.$$

(iii) For any section $s \in \Gamma(\mathcal{E})$ and function $f \in C^\infty(M)$,

$$\nabla(fs) = df \otimes s + f \nabla s.$$

We define a connection ∇ on the dual Hilbert bundle $\mathcal{E}^*$ by

$$(\nabla_X \alpha)(s) := X(\alpha(s)) - \alpha(\nabla_X s).$$

Here, $s \in \Gamma(\mathcal{E})$, $\alpha \in \Gamma(\mathcal{E}^*)$, $X \in \Gamma(TM)$.

For connections ∇_1, ∇_2 on Hilbert bundles $\mathcal{E}_1, \mathcal{E}_2$ over M, we define connections $\nabla_1 \oplus \nabla_2$, $\nabla_1 \otimes \nabla_2$ on $\mathcal{E}_1 \oplus \mathcal{E}_2$, $\mathcal{E}_1 \otimes \mathcal{E}_2$ by

$$\begin{aligned}
(\nabla_1 \oplus \nabla_2)(s_1 \oplus s_2) &:= (\nabla_1 s_1) \oplus (\nabla_2 s_2), \\
(\nabla_1 \otimes \nabla_2)(s_1 \otimes s_2) &:= (\nabla_1 s_1) \otimes s_2 + s_1 \otimes (\nabla_2 s_2).
\end{aligned}$$

Write $\Omega^i(M;\mathcal{E})$ for the space of i-forms on M with values in $\mathcal{E}$:

$$\Omega^i(M;\mathcal{E}) := \Gamma(\Lambda^i T^*M \otimes \mathcal{E}).$$

For a connection ∇ on $\mathcal{E}$, we have the exterior derivative

$$d_\nabla \colon \Omega^i(M;\mathcal{E}) \to \Omega^{i+1}(M;\mathcal{E})$$

defined by

$$\begin{aligned} d_\nabla(\eta s) &= (d\eta)s + \eta \wedge (\nabla s), \\ d_\nabla(\eta_1 + \eta_2) &= d_\nabla \eta_1 + d_\nabla \eta_2. \end{aligned}$$

Here, $s \in \Gamma(\mathcal{E})$, $\eta \in \Omega^i(M)$, $\eta_1, \eta_2 \in \Omega^i(M;\mathcal{E})$.

We will make an assumption on the smoothness of ∇. Take a local trivialization $\psi \colon \mathcal{E}|_U \to U \times H$. We can write

$$\psi \nabla_X s = X(\psi s) + \omega(X)(\psi s) \tag{2.2.1}$$

for $s \in \Gamma(\mathcal{E}|_U)$ and $X \in \Gamma(TU)$. Here, for each $x \in U$ and $X \in T_x U$, $\omega(X)$ is a linear map $H \to H$. The assumption is that $\omega(X)$ is bounded and the map $\omega \colon TU \to \operatorname{End} H$ is smooth with respect to the operator norm. In particular, for a compact set K in U, the restriction $\omega(X)|_K$ is a Lipschitz continuous map $K \to \operatorname{End} H$.

Under the above assumption, for any smooth curve $c \colon [-\varepsilon, \varepsilon] \to U$ and $e \in \mathcal{E}_{c(0)}$, where $\varepsilon > 0$, we have a unique smooth section s of $\mathcal{E}$ along c which solves the ordinary differential equation in the Hilbert space:

$$\frac{d}{dt}\psi(s(t)) + \omega\Big(\frac{dc}{dt}(t)\Big)(\psi s(t)) = 0, \quad s(0) = e.$$

We call s a parallel section of $\mathcal{E}$ along c or a horizontal lift of c. See [18] for the existence and uniqueness of solutions to the equation.

Take $x \in U$ and let $x^1, \dots, x^n$ be local coordinates around x. For $i = 1, \dots, n$, let c_i be a smooth curve $[-\varepsilon, \varepsilon] \to U$ such that

$$c_i(0) = x, \quad \frac{dc_i}{dt}(0) = \frac{\partial}{\partial x^i}.$$

For $e \in \mathcal{E}_x$, we define the horizontal component $(T_e\mathcal{E})_H$ of $T_e\mathcal{E}$ to be the subspace spanned by $\{ds_i(\frac{\partial}{\partial t})\}_{i=1,\dots,n}$. Here, s_i is the parallel section of $\mathcal{E}$ along c_i with $s_i(0) = e$. We can show that $(T_e\mathcal{E})_H$ is independent of the choice of the local coordinates $x^1, \dots, x^n$. The connection ∇ defines a decomposition

$$T\mathcal{E} = (T\mathcal{E})_H \oplus p^*\mathcal{E}.$$

Note that we have a natural isomorphism

$$(T\mathcal{E})_H \cong p^*TM.$$

As usual, there is a unique 2-form $F_\nabla \in \Omega^2(M; \operatorname{End}\mathcal{E})$ such that

$$d_\nabla \circ d_\nabla \eta = F_\nabla \wedge \eta$$

for $\eta \in \Omega^i(M; \mathcal{E})$. We can write

$$\psi F_\nabla = d\omega + \omega \wedge \omega$$

on U, where ω is the 1-form with values in $\operatorname{End} H$ in (2.2.1). We call F_∇ the curvature of ∇. We say that ∇ is flat if $F_\nabla = 0$.

We can associate a flat connection to a representation

$$\rho\colon \pi_1(M) \to \operatorname{Aut}(H)$$

in the usual way. Let $\mathcal{E}$ be the Hilbert bundle on M defined by

$$\mathcal{E} := \widetilde{M} \times_\rho H,$$

where $\widetilde{M}$ is the universal cover of M. A smooth section $s\colon M \to \mathcal{E}$ corresponds to a smooth map $\tilde{s}\colon \widetilde{M} \to H$ such that

$$\tilde{s}(\gamma \cdot x) = \rho(\gamma)\tilde{s}(x)$$

for $x \in \widetilde{M}$, $\gamma \in \pi_1(M)$. Taking the exterior derivative, we have

$$d\tilde{s}(\gamma \cdot x) = \rho(\gamma)\, d\tilde{s}(x)$$

and hence $d\tilde{s}$ descends to a section of $T^*M \otimes \mathcal{E}$, which we denote by ∇s. We can show that the map

$$\nabla\colon \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E})$$

is a flat connection on $\mathcal{E}$.

2.3 Notation and main statements

Let Y be a connected, closed, oriented 3-manifold and take a Riemannian metric g and spin^c structure $\mathfrak{s}$ with $c_1(\mathfrak{s})$ torsion on Y. We denote the spinor bundle over Y by $\mathbb{S}$. Fix a spin^c connection A_0 on Y with $F_{A_0} = 0$. For a 1-form $a \in \Omega^1(Y)$, we write D_a for the Dirac operator $D_{A_0 + ia}$ which acts on the space $C^\infty(\mathbb{S})$ of smooth sections of $\mathbb{S}$. The family $\{D_a\}_{a \in \mathcal{H}^1(Y)}$ parameterized by the harmonic 1-forms on Y induces an operator D acting on the vector bundle

$$\mathcal{E}_\infty = \mathcal{H}^1(Y) \times_{H^1(Y;\mathbb{Z})} C^\infty(\mathbb{S})$$

over the Picard torus $\mathrm{Pic}(Y) = H^1(Y;\mathbb{R})/H^1(Y;\mathbb{Z})$. The action of $H^1(Y;\mathbb{Z})$ is defined by

$$h(a,\phi) = (a-h, u_h\phi)$$

for $h \in H^1(Y;\mathbb{Z})$, $a \in \mathcal{H}^1(Y)$, $\phi \in C^\infty(\mathbb{S})$, where u_h is the harmonic gauge transformation $Y \to U(1)$ with $-iu_h^{-1}\,du_h = h$ in $\mathcal{H}^1(Y)$.

For $k \in \mathbb{R}_{\geq 0}$, define a Hilbert bundle on $\mathrm{Pic}(Y)$ by

$$\mathcal{E}_k := \mathcal{H}^1(Y) \times_{H^1(Y;\mathbb{Z})} L^2_k(\mathbb{S}).$$

For $k \geq 1$, the operator D on $\mathcal{E}_\infty$ extends to a bounded operator

$$D\colon \mathcal{E}_k \to \mathcal{E}_{k-1}.$$

We have a canonical flat connection ∇ on $\mathcal{E}_k$ corresponding to the representation

$$\begin{aligned}\pi_1(B) = H^1(Y;\mathbb{Z}) &\to \mathrm{Aut}(L^2_k(\mathbb{S})),\\ h &\mapsto u_h,\end{aligned}$$

where $B = \mathrm{Pic}(Y)$, $\mathrm{Aut}(L^2_k(\mathbb{S}))$ is the group of bounded linear automorphisms on $L^2_k(\mathbb{S})$. See Section 2.2.

A smooth section $s\colon B \to \mathcal{E}_k$ can be considered to be a smooth map

$$\tilde{s}\colon \mathcal{H}^1(Y) \to L^2_k(\mathbb{S})$$

such that

$$\tilde{s}(a-h) = u_h\tilde{s}(a)$$

for $h \in \mathrm{im}(H^1(Y;\mathbb{Z}) \to \mathcal{H}^1(Y))$. The covariant derivative ∇s corresponds to the usual exterior derivative $d\tilde{s}$ of $\tilde{s}$.

Denote by $\langle\cdot,\cdot\rangle_{a,k}$ the L^2_k-inner product with respect to D_a:

$$\langle\phi_1,\phi_2\rangle_{a,k} = \langle\phi_1,\phi_2\rangle_0 + \langle|D_a|^k\phi_1, |D_a|^k\phi_2\rangle_0,$$

where $\langle\cdot,\cdot\rangle_0$ is the $L^2(Y)$-inner product. Here we write $|D_a|$ for the absolute value of the Dirac operator D_a, defined using the spectral theorem (see e.g. [46, Chapter VIII, §9]). Then the family $\{\langle\cdot,\cdot\rangle_{a,k}\}_{a\in\mathcal{H}^1(Y)}$ of L^2_k-inner products induces a fiberwise inner product $\langle\cdot,\cdot\rangle_k$ on $\mathcal{E}_k$. To see this, take sections $s_1, s_2\colon B \to \mathcal{E}_k$ and $h \in \mathrm{im}(H^1(Y;\mathbb{Z}) \to \mathcal{H}^1(Y))$. Let $\tilde{s}_1, \tilde{s}_2\colon \mathcal{H}^1(Y) \to L^2_k(\mathbb{S})$ be the maps corresponding to s_1, s_2. Note that

$$\tilde{s}_i(a-h) = u_h\tilde{s}_i(a), \quad D_{a-h} = u_h D_a u_h^{-1}.$$

Therefore,

$$\begin{aligned}
&\langle \tilde{s}_1(a-h), \tilde{s}_2(a-h)\rangle_{a-h,k} \\
&\qquad = \langle \tilde{s}_1(a-h), \tilde{s}_2(a-h)\rangle_0 + \langle |D_{a-h}|^k \tilde{s}_1(a-h), |D_{a-h}|^k \tilde{s}_2(a-h)\rangle_0 \\
&\qquad = \langle u_h\tilde{s}_1(a), u_h\tilde{s}_2(a)\rangle_0 + \langle (u_h|D_a|^k u_h^{-1})u_h\tilde{s}_1(a), (u_h|D_a|^k u_h^{-1})u_h\tilde{s}_2(a)\rangle_0 \\
&\qquad = \langle u_h\tilde{s}_1(a), u_h\tilde{s}_2(a)\rangle_0 + \langle u_h|D_a|^k \tilde{s}_1(a), u_h|D_a|^k \tilde{s}_2(a)\rangle_0 \\
&\qquad = \langle \tilde{s}_1(a), \tilde{s}_2(a)\rangle_{a,k}.
\end{aligned}$$

This implies that the family $\{\langle\cdot,\cdot\rangle_{a,k}\}_{a\in\mathcal{H}^1(Y)}$ descends to a fiberwise inner product $\langle\cdot,\cdot\rangle_k$ on $\mathcal{E}_k$. We write $\|\cdot\|_k$ for the fiberwise norm on $\mathcal{E}_k$ induced by $\langle\cdot,\cdot\rangle_k$.

The flat connection ∇, with respect to $k=0$, defines a decomposition

$$T\mathcal{E}_0 = p^*TB \oplus p^*\mathcal{E}_0, \tag{2.3.1}$$

where $p\colon \mathcal{E}_0 \to B$ is the projection, p^*TB is the horizontal component and $p^*\mathcal{E}_0$ is the vertical component. See Section 2.2. Note that the flat connection ∇ is not compatible with the inner product $\langle\cdot,\cdot\rangle_k$ on $\mathcal{E}_k$ for $k>0$.

Put

$$\mathcal{W}_k = B \times L^2_k(\operatorname{im} d^*),$$

where $d^*\colon i\Omega^2(Y) \to i\Omega^1(Y)$ is the adjoint of the exterior derivative. We consider $\mathcal{W}_k$ to be a trivial Hilbert bundle on B. The Seiberg–Witten equations on $Y\times\mathbb{R}$ are equations for $\gamma=(\phi,a,\omega)\colon \mathbb{R}\to L^2_k(\mathbb{S})\times\mathcal{H}^1(Y)\times L^2_k(\operatorname{im} d^*)$, written as

$$\begin{aligned}
\frac{d\phi}{dt} &= -D_a\phi(t) - c_1(\gamma(t)),\\
\frac{da}{dt} &= -X_H(\phi),\\
\frac{d\omega}{dt} &= -*d\omega - c_2(\gamma(t)).
\end{aligned} \tag{2.3.2}$$

The terms $X_H(\phi)$, $c_1(\gamma(t))$, $c_2(\gamma(t))$ are defined by

$$\begin{aligned}
q(\phi) &= \rho^{-1}\Big(\phi\otimes\phi^* - \frac{1}{2}|\phi|^2\mathrm{id}\Big) \in \Omega^1(Y),\\
X_H(\phi) &= q(\phi)_{\mathcal{H}} \in \mathcal{H}^1(Y),\\
c_1(\gamma(t)) &= \big(\rho(\omega(t)) - i\xi(\phi(t))\big)\phi(t),\\
c_2(\gamma(t)) &= \pi_{\operatorname{im} d^*}\big(q(\phi(t))\big),
\end{aligned} \tag{2.3.3}$$

where ρ is the Clifford multiplication which defines an isomorphism

$$T^*Y\otimes\mathbb{C} \to \mathfrak{sl}(\mathbb{S}),$$

$q(\phi)_{\mathcal{H}}$ is the harmonic component of $q(\phi)$, $\pi_{\mathrm{im}\, d^*}$ is the L^2-projection on $\mathcal{W}_k$ and $\xi(\phi)$ is the function $Y \to \mathbb{R}$ satisfying

$$d\xi(\phi) = i\pi_{\mathrm{im}\, d}(q(\phi)), \qquad \int_Y \xi(\phi)\,\mathrm{vol} = 0.$$

The equations (2.3.2) do not correspond to the Seiberg–Witten equations in Coulomb gauge in $Y \times \mathbb{R}$ (that is, solutions of the equations are not Seiberg–Witten trajectories in Coulomb gauge). Instead, we use the *pseudo-temporal gauge* of [32, Definition 5.2.1] (see also [35, Section 3]). The correspondence between solutions of (2.3.2) and the Seiberg–Witten equations modulo gauge is given by [32, Proposition 5.4.2]. Note that Lidman–Manolescu work in the setting of $b_1 = 0$; however, the argument is local in the configuration space and passes over without change to the $b_1 > 0$ case. We will, however, call solutions of (2.3.2) *Seiberg–Witten trajectories*.

The equations descend to equations for $\gamma = (\phi, \omega)\colon \mathbb{R} \to \mathcal{E}_k \oplus \mathcal{W}_k$:

$$\begin{aligned} \left(\frac{d\phi}{dt}(t)\right)_V &= -D\phi(t) - c_1(\gamma(t)), \\ \left(\frac{d\phi}{dt}(t)\right)_H &= -X_H(\phi(t)), \\ \frac{d\omega}{dt}(t) &= -*d\omega(t) - c_2(\gamma(t)). \end{aligned} \tag{2.3.4}$$

Here, $(\frac{d\phi}{dt})_V$, $(\frac{d\phi}{dt})_H$ are the vertical component and horizontal component of $\frac{d\phi}{dt}$ respectively, and we have suppressed the subscript from D.

Assume that the family index of the family of Dirac operators D over $\mathrm{Pic}(Y)$ vanishes, that is,

$$\mathrm{ind}\, D = 0 \in K^1(B).$$

Then we can choose a spectral section P_0 of $-D$, and using P_0, we can define a self-adjoint (with respect to the L^2) operator

$$\mathbb{A}\colon C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$$

such that the image of $\mathbb{A}$ is included in a subspace spanned by finitely many eigenvectors of D, and so that $\ker(D + \mathbb{A}) = 0$. Put $D' = D + \mathbb{A}$. The L^2-closure of the subspace spanned by the negative eigenvectors of D' is exactly the image of P_0, acting on L^2 (see [40] and Section 2.1 for all of these assertions). In the future, for a spectral section P, we will also often write P to refer to the image of P. We have a decomposition

$$\mathcal{E}_\infty = \mathcal{E}_\infty^+ \oplus \mathcal{E}_\infty^-,$$

where $\mathcal{E}_\infty^+$ and $\mathcal{E}_\infty^-$ are the subbundles of $\mathcal{E}$ spanned by positive eigenvectors and negative eigenvectors of D'.

For positive numbers k_+, k_- and $s_1, s_2 \in C^\infty(\mathbb{S})$, we define an inner product $\langle s_1, s_2\rangle_{a,k_+,k_-}$ by

$$\langle s_1, s_2\rangle_{a,k_+,k_-} := \langle |D'_a|^{k_+} s_1^+, |D'_a|^{k_+} s_2^+\rangle_0 + \langle |D'_a|^{k_-} s_1^-, |D'_a|^{k_-} s_2^-\rangle_0, \tag{2.3.5}$$

where $s_j = s_j^+ + s_j^-$ and $s_j^+ \in \mathcal{E}_\infty^+$, $s_j^- \in \mathcal{E}_\infty^-$. Note that we do not need the term $\langle s_1, s_2\rangle_0$, since the kernel of D'_a is zero. We call this inner product the $L^2_{k_+,k_-}$-inner product.

As before, the family $\{\langle\cdot,\cdot\rangle_{a,k_+,k_-}\}_{a\in\mathcal{H}^1(Y)}$ induces a fiberwise inner product on $\mathcal{E}_\infty$ and we denote by $\mathcal{E}_{k_+,k_-}$ the completion of $\mathcal{E}_\infty$ with respect to the norm $\|\cdot\|_{k_+,k_-}$.

On the space $\operatorname{im} d^* \cap \Omega^1(Y)$, we define an inner product $\langle\cdot,\cdot\rangle_{k_+,k_-}$ by

$$\langle\omega_1, \omega_2\rangle_{k_+,k_-} = \langle |*d|^{k_+}\omega_1^+, |*d|^{k_+}\omega_2^+\rangle_0 + \langle |*d|^{k_-}\omega_1^-, |*d|^{k_-}\omega_2^-\rangle_0,$$

where $\omega_j = \omega_j^+ + \omega_j^-$ and ω_j^+ is in the subspace spanned by positive eigenvectors of the operator $*d$ and ω_j^- is in the negative one. We denote by $\mathcal{W}_{k_+,k_-}$ the completion of the vector bundle $B \times \operatorname{im} d^*$ over B with respect to $\|\cdot\|_{k_+,k_-}$. We will use the $L^2_{k-\frac{1}{2},k}$-norm in Chapter 5 to define the relative Bauer–Furuta invariant. See Remark 5.1.4 for the reason why we use the $L^2_{k-\frac{1}{2},k}$-norm.

We recall the definition of finite-type trajectories (from e.g. [35, Definition 1]).

Definition 2.3.1. A Seiberg–Witten trajectory $\gamma(t) = (\phi(t), a(t), \omega(t))$ is *finite-type* if $CSD(\gamma(t))$ and $\|\phi(t)\|_{C^0}$ are bounded functions of t, where CSD is the Chern–Simons–Dirac functional.

The following is a direct consequence of a standard argument in Seiberg–Witten theory; see e.g. [35, Proposition 1].

Proposition 2.3.2. *For positive numbers $k_+, k_- > 0$, there is a positive constant $R_{k_+,k_-} > 0$ such that for any finite-type solution $\gamma\colon \mathbb{R} \to \mathcal{E}_2 \times \mathcal{W}_2$ to (2.3.4), we have*

$$\|\gamma(t)\|_{k_+,k_-} \le R_{k_+,k_-}$$

for all $t \in \mathbb{R}$.

Write $\mathcal{E}_0(D)_{b'}^{b}$ for the span of eigenvectors of D with eigenvalue in $(b', b]$, as a space over $\mathcal{H}^1(Y)$ (note that it will not usually be a bundle). For a spectral section P of D, we also write P for the image of the projection P. By Theorem 2.4.1 below, we can take sequences of smooth spectral sections P_n, Q_n, of $-D$, D, respectively, such that

$$\begin{aligned} (\mathcal{E}_0(D))_{-\infty}^{\mu_{n,-}} \subset P_n \subset (\mathcal{E}_0(D))_{-\infty}^{\mu_{n,+}}, \\ (\mathcal{E}_0(D))_{\lambda_{n,+}}^{\infty} \subset Q_n \subset (\mathcal{E}_0(D))_{\lambda_{n,-}}^{\infty}, \end{aligned} \tag{2.3.6}$$

with

$$\begin{aligned}&\mu_{n,-} + 10 < \mu_{n,+} < \mu_{n,+} + 10 < \mu_{n+1,-},\\ &\lambda_{n+1,+} < \lambda_{n,-} - 10 < \lambda_{n,-} < \lambda_{n,+} - 10,\\ &\mu_{n,+} - \mu_{n,-} < \delta,\\ &\lambda_{n,+} - \lambda_{n,-} < \delta.\end{aligned} \tag{2.3.7}$$

Here, $\delta > 0$ is a positive constant independent of n, and a smooth spectral section means a spectral section which depends smoothly on the base space B.

We define a finite rank subbundle F_n in $\mathcal{E}_\infty$ by

$$F_n = P_n \cap Q_n.$$

Define a connection ∇_{F_n} on F_n by

$$\nabla_{F_n} = \pi_{F_n} \nabla,$$

where π_{F_n} is the $L^2_{k_+,k_-}$-projection on F_n. The connection ∇_{F_n} defines a decomposition

$$TF_n = (TF_n)_{H,\nabla_{F_n}} \oplus (TF_n)_V \cong p^*TB \oplus p^*F_n. \tag{2.3.8}$$

A calculation shows that the horizontal component $(T_\phi F_n)_{H,\nabla_{F_n}}$ of TF_n at $\phi \in F_n$ is given by

$$\big\{(v, (\nabla_v \pi_{F_n})\phi) : v \in T_a B\big\} \subset (p^*TB \oplus p^*\mathcal{E}_0)_\phi = T_\phi \mathcal{E}_0. \tag{2.3.9}$$

Here, $a = p(\phi) \in B$.

Let W_n be the finite-dimensional subbundle of the Hilbert bundle $\mathcal{W}_k$ spanned by the eigenvectors of the operator $*d$ whose eigenvalues are in the interval $(\lambda_{n,-}, \mu_{n,+}]$:

$$W_n = (\mathcal{W}_k)^{\mu_{n,+}}_{\lambda_{n,-}} = B \times L^2_k(\operatorname{im} d^*)^{\mu_{n,+}}_{\lambda_{n,-}}.$$

Fix a positive number R' with $R' \geq 100R_{k_+,k_-}$ and a smooth function

$$\chi\colon \mathcal{E}_{k_+,k_-} \oplus \mathcal{W}_{k_+,k_-} \to [0,1]$$

with compact support such that $\chi(\phi,\omega) = 1$ if $\|(\phi,\omega)\|_{k_+,k_-} \leq R'$. We consider the following equations for $\gamma = (\phi,\omega)\colon \mathbb{R} \to F_n \oplus W_n$, which we call the *finite-dimensional approximation* of (2.3.4):

$$\begin{aligned}\left(\frac{d\phi}{dt}(t)\right)_V &= -\chi\big\{(\nabla_{X_H}\pi_{F_n})\phi(t) + \pi_{F_n}\big(D\phi(t) + c_1(\gamma(t))\big)\big\},\\ \left(\frac{d\phi}{dt}(t)\right)_H &= -\chi X_H(\phi(t)),\\ \frac{d\omega}{dt}(t) &= -\chi\big\{*d\omega(t) + \pi_{W_n} c_2(\gamma(t))\big\}.\end{aligned} \tag{2.3.10}$$

Here, $\left(\frac{d\phi}{dt}\right)_V$, $\left(\frac{d\phi}{dt}\right)_H$ are the vertical component and the horizontal component with respect to the fixed decomposition (2.3.1) rather than (2.3.8). It follows from (2.3.9) that the right-hand side of (2.3.10) is a tangent vector on $F_n \oplus W_n$. Hence, equations (2.3.10) define a flow

$$\varphi_n = \varphi_{n,k_+,k_-} \colon (F_n \oplus W_n) \times \mathbb{R} \to F_n \oplus W_n.$$

(This flow depends on k_+, k_- because π_{F_n} does.)

We have decompositions

$$F_n = F_n^+ \oplus F_n^-, \quad W_n = W_n^+ \oplus W_n^-,$$

where F_n^+, W_n^+ are the positive eigenvalue components of D', $*d$, and F_n^-, W_n^- are the negative eigenvalue components. In the remainder of Chapter 2, we will prove the following.

Theorem 2.3.3. *Let k_+, k_- be half-integers (that is, $k_+, k_- \in \frac{1}{2}\mathbb{Z}$) with $k_+, k_- > 5$ and with $|k_+ - k_-| \leq \frac{1}{2}$. Fix a positive number R with $R_{k_+,k_-} < R < \frac{1}{10}R'$, where R_{k_+,k_-} is the constant of Proposition 2.3.2. Then*

$$(B_{k_+}(F_n^+; R) \times_B B_{k_-}(F_n^-; R)) \times_B (B_{k_+}(W_n^+; R) \times_B B_{k_-}(W_n^-; R))$$

is an isolating neighborhood of the flow φ_{n,k_+,k_-} for $n \gg 0$. Here, $B_{k_\pm}(F_n^\pm; R)$ are the disk bundle of $F_n^\pm$ of radius R in $L^2_{k_\pm}$ and $B_{k_+}(F_n^+; R) \times_B B_{k_-}(F_n^-; R)$ is the fiberwise product. Similarly for $B_{k_\pm}(W_n^\pm; R)$.

The general strategy to prove Theorem 2.3.3 is as follows: once we have Theorem 2.4.1 in hand, we must control the gradient term $(\nabla_{X_H} \pi_{F_n})\phi(t)$ appearing in the approximate Seiberg–Witten equations (2.3.10); a number of bounds for this are obtained in Sections 2.5 and 2.6. The proof proper is in Section 2.7, where Theorem 2.3.3 follows from establishing that, for sufficiently large approximations, the linear term in the approximate Seiberg–Witten equations (2.3.10) tends to dominate the other terms with respect to appropriate norms.

We also note that the total space $B_{n,R}$ appearing in Theorem 2.3.3 is an ex-space over $B = \mathrm{Pic}(Y)$ in the sense of Appendix A.1, with projection given by restricting $p \colon \mathcal{E}_k \to B$ to $B_{n,R}$, and with a section $s_B \colon \mathrm{Pic}(Y) \to B_{n,R}$ given by the zero-section.

2.4 Construction of spectral sections

We will prove the following.

Theorem 2.4.1. *Assume that* ind $D = 0$ *in $K^1(B)$. Take a sequence μ_n of positive numbers $\mu_n \ll \mu_{n+1}$, where $\mu_n \to \infty$ as $n \to \infty$. There is a sequence of spectral sections P_n of $-D$ with the following properties:*

(i) *We have*

$$\mathcal{E}_0(D)^{\mu_n}_{-\infty} \subset P_n \subset \mathcal{E}_0(D)^{\mu_n+\delta}_{-\infty},$$

where δ is a positive constant independent of n.

(ii) *We can write*

$$P_{n+1} = P_n \oplus \langle f_1^{(n)}, \dots, f_{r_n}^{(n)} \rangle,$$

where $\{f_1^{(n)}, \dots, f_{r_n}^{(n)}\}$ is a frame of $P_n^\perp$ (where $P_n^\perp$ is the L^2-orthogonal complement of P_n inside of P_{n+1}). In particular,

$$P_{n+1} \cong P_n \oplus \underline{\mathbb{C}}^{r_n},$$

where $\underline{\mathbb{C}}^{r_n}$ is the trivial vector bundle over B.

Before we start proving Theorem 2.4.1, we will show the following.

Proposition 2.4.2. *Take any nonnegative numbers k, l. Let P_n be a sequence of spectral sections of $-D$ having property* (i) *of Theorem* 2.4.1. *Let $\pi_n\colon \mathcal{E}_k \to P_n \cap \mathcal{E}_k$ be the L^2_k-projection.*

(1) *The commutators*

$$[D, \pi_n]\colon \mathcal{E}_\infty \to \mathcal{E}_\infty$$

extend to bounded operators

$$[D, \pi_n]\colon \mathcal{E}_l \to \mathcal{E}_l$$

and we have

$$\|[D, \pi_n]\colon \mathcal{E}_l \to \mathcal{E}_l\| < C,$$

where C is a positive constant independent of n. Moreover, for any $l > 0$, $\varepsilon > 0$ with $0 < \varepsilon \leq l$,

$$\sup_{a \in B} \|[D_a, \pi_{n,a}]\colon L^2_l(\mathbb{S}) \to L^2_{l-\varepsilon}(\mathbb{S})\| \to 0$$

as $n \to \infty$.

(2) *The operator $\pi_n\colon \mathcal{E}_\infty \to \mathcal{E}_\infty$ extends to a bounded operator $\mathcal{E}_l \to \mathcal{E}_l$ for each nonnegative real number l. Moreover, there is a positive constant C independent of n such that*

$$\|\pi_n\colon \mathcal{E}_l \to \mathcal{E}_l\| < C.$$

Proof. Take $a \in B$ and let $\{e_j\}_j$ be an orthonormal basis of $L^2(\mathbb{S})$ with

$$D_a e_j = \eta_j e_j,$$

where $\eta_j \in \mathbb{R}$.

Let $P_{n,a}$ be the fiber of P_n over a. Take $\phi \in \mathcal{E}_\infty \cap P_{n,a}$. We can write

$$\phi = \sum_{\eta_j \le \mu_n + \delta} c_j e_j,$$

where $c_j \in \mathbb{C}$. Note that

$$\sum_{\eta_j \le \mu_n} c_j e_j \in \mathcal{E}_\infty \cap P_{n,a}, \qquad \sum_{\mu_n < \eta_j \le \mu_n + \delta} c_j e_j \in \mathcal{E}_\infty \cap P_{n,a}.$$

We have

$$\begin{aligned}
[D_a, \pi_{n,a}]\phi &= (D_a \pi_{n,a} - \pi_{n,a} D)\phi \\
&= \sum_{\eta_j \le \mu_n + \delta} \eta_j c_j e_j - \pi_{n,a} \sum_{\eta_j \le \mu_n + \delta} \eta_j c_j e_j \\
&= (1 - \pi_{n,a}) \sum_{\mu_n < \eta_j \le \mu_n + \delta} \eta_j c_j e_j \\
&= (1 - \pi_{n,a}) \Bigg\{ \sum_{\mu_n < \eta_j \le \mu_n + \delta} (\eta_j - \mu_n) c_j e_j + \mu_n \sum_{\mu_n < \eta_j \le \mu_n + \delta} c_j e_j \Bigg\} \\
&= \sum_{\mu_n < \eta_j \le \mu_n + \delta} (\eta_j - \mu_n) c_j (1 - \pi_{n,a}) e_j. \qquad (2.4.1)
\end{aligned}$$

Since

$$\pi_n = \pi_{-\infty}^{\mu_n} + \pi_{P_n \cap \mathcal{E}(D)_{\mu_n}^{\mu_n + \delta}},$$

for j with $\mu_n < \eta_j \le \mu_n + \delta$, we have

$$(1 - \pi_{n,a}) e_j \in \mathcal{E}_0(D_a)_{\mu_n}^{\mu_n + \delta}.$$

Hence we can write

$$(1 - \pi_{n,a}) e_j = \sum_{\mu_n < \eta_p \le \mu_n + \delta} \alpha_{jp} e_p \qquad (2.4.2)$$

for j with $\mu_n < \eta_j \le \mu_n + \delta$. Here, $\alpha_{jp} \in \mathbb{C}$. Since

$$\|(1 - \pi_{n,a})\colon L_k^2 \to L_k^2\| = 1, \quad \|e_j\|_k = (1 + |\eta_j|^{2k})^{\frac{1}{2}},$$

we have

$$\|(1 - \pi_{n,a}) e_j\|_k^2 = \sum_{\mu_n < \eta_p \le \mu_n + \delta} |\alpha_{jp}|^2 (1 + |\eta_p|^{2k}) \le (1 + |\eta_j|^{2k}).$$

For j with $\mu_n < \eta_j \le \mu_n + \delta$,

$$\begin{aligned}
\sum_{\mu_n<\eta_p\le\mu_n+\delta} |\alpha_{jp}|^2 &= \sum_{\mu_n<\eta_p\le\mu_n+\delta} |\alpha_{jp}|^2(1+|\eta_p|^{2k})\frac{1}{1+|\eta_p|^{2k}} \\
&\le \frac{C_1}{1+(\mu_n+\delta)^{2k}} \sum_{\mu_n<\eta_p\le\mu_n+\delta} |\alpha_{jp}|^2(1+|\eta_p|^{2k}) \\
&\le \frac{C_1(1+|\eta_j|^{2k})}{1+(\mu_n+\delta)^{2k}} \\
&\le C_1,
\end{aligned} \tag{2.4.3}$$

where C_1 is a positive constant independent of j, n.

By (2.4.1), (2.4.2) and (2.4.3),

$$\begin{aligned}
\|[D_a,\pi_{n,a}]\phi\|_l^2 &= \sum_{\mu_n<\eta_j\le\mu_n+\delta}\ \sum_{\mu_n<\eta_p\le\mu_n+\delta} |\eta_j-\mu_n|^2(1+|\eta_p|^{2l})|c_j|^2|\alpha_{jp}|^2 \\
&\le \delta^2 \sum_{\mu_n<\eta_j\le\mu_n+\delta}\ \sum_{\mu_n<\eta_p\le\mu_n+\delta} (1+|\eta_j|^{2l})|c_j|^2|\alpha_{jp}|^2\cdot\frac{1+|\eta_p|^{2l}}{1+|\eta_j|^{2l}} \\
&\le C_2 \sum_{\mu_n<\eta_j\le\mu_n+\delta} (1+|\eta_j|^{2l})|c_j|^2\Bigg(\sum_{\mu_n<\eta_p\le\mu_n+\delta}|\alpha_{jp}|^2\Bigg) \\
&\le C_3 \sum_{\mu_n<\eta_j\le\mu_n+\delta} (1+|\eta_j|^{2l})|c_j|^2 \\
&\le C_3\|\phi\|_l^2.
\end{aligned}$$

Here, $C_2, C_3 > 0$ are positive constants independent of n, ϕ, a. Also we have

$$\begin{aligned}
&\|[D_a,\pi_{n,a}]\phi\|_{l-\varepsilon}^2 \\
&= \sum_{\mu_n<\eta_j\le\mu_n+\delta}\ \sum_{\mu_n<\eta_p\le\mu_n+\delta} |\eta_j-\mu_n|^2(1+|\eta_p|^{2(l-\varepsilon)})|c_j|^2|\alpha_{jp}|^2 \\
&\le \delta^2 \sum_{\mu_n<\eta_j\le\mu_n+\delta}\ \sum_{\mu_n<\eta_p\le\mu_n+\delta} (1+|\eta_j|^{2(l-\varepsilon)})|c_j|^2|\alpha_{jp}|^2\cdot\frac{1+|\eta_p|^{2(l-\varepsilon)}}{1+|\eta_j|^{2(l-\varepsilon)}} \\
&\le C_4 \sum_{\mu_n<\eta_j\le\mu_n+\delta} (1+|\eta_j|^{2(l-\varepsilon)})|c_j|^2\Bigg(\sum_{\mu_n<\eta_p\le\mu_n+\delta}|\alpha_{jp}|^2\Bigg) \\
&\le C_5 \sum_{\mu_n<\eta_j\le\mu_n+\delta} (1+|\eta_j|^{2(l-\varepsilon)})|c_j|^2 \\
&\le C_6(\mu_n^{-2l}+\mu_n^{-2\varepsilon})\|\phi\|_l^2.
\end{aligned}$$

Here, C_4, C_5, C_6 are positive constants independent of n, ϕ, a.

On the other hand, consider $\phi \in \mathcal{E}_\infty \cap P_{n,a}^{\perp_k}$, where $P_{n,a}^{\perp_k}$ is the L_k^2-orthogonal complement of $P_{n,a} \cap L_k^2(\mathbb{S})$ in $L_k^2(\mathbb{S})$. We can write

$$\phi = \sum_{\eta_j > \mu_n} c_j e_j .$$

Note that

$$\sum_{\eta_j > \mu_n + \delta} c_j e_j \in \mathcal{E}_\infty \cap P_{n,a}^{\perp_k}, \qquad \sum_{\mu_n < \eta_j \le \mu_n + \delta} c_j e_j \in \mathcal{E}_\infty \cap P_{n,a}^{\perp_k}.$$

We have

$$\begin{aligned}
[D_a, \pi_{n,a}]\phi &= \pi_{n,a} \sum_{\mu_n < \eta_j \le \mu_n + \delta} \eta_j c_j e_j \\
&= \pi_{n,a}\Bigg(\sum_{\mu_n < \eta_j \le \mu_n + \delta} (\eta_j - \mu_n) c_j e_j + \mu_n \sum_{\mu_n < \eta_j \le \mu_n + \delta} c_j e_j \Bigg) \\
&= \sum_{\mu_n < \eta_j \le \mu_n + \delta} (\eta_j - \mu_n) c_j \pi_{n,a} e_j .
\end{aligned}$$

As before, using this equality, we can show that

$$\|[D_a, \pi_{n,a}]\phi\|_l \le C_7 \|\phi\|_l, \quad \|[D_a, \pi_{n,a}]\phi\|_{l-\varepsilon} \le C_8 \mu_n^{-\varepsilon} \|\phi\|_l$$

for some positive constants C_7, C_8 independent of n, ϕ, a.

Therefore $[D_a, \pi_{n,a}]$ extend to bounded maps $L_l^2 \to L_l^2$ with

$$\|[D_a, \pi_{n,a}] \colon L_l^2 \to L_l^2\| \le C_9,$$

for some constant C_9 independent of n, a. Also

$$\sup_{a \in B} \|[D_a, \pi_{n,a}] \colon L_l^2(\mathbb{S}) \to L_{l-\varepsilon}^2(\mathbb{S})\| \to 0$$

as $n \to \infty$. We have proved (1).

We will prove (2). It is easy to see that if $\mu_n < \eta_j \le \mu_n + \delta$, we have

$$\pi_n e_j \in (\mathcal{E}_l)_{\mu_n}^{\mu_n + \delta}.$$

So we can write

$$\pi_n e_j = \sum_{\mu_n < \eta_p \le \mu_n + \delta} \alpha_{jp} e_p .$$

Because the operator norm of $\pi_n \colon L_k^2 \to L_k^2$ is 1 and $\|e_j\|_k^2 = 1 + |\eta_j|^{2k}$, we have

$$|\mu_n|^{2k} \sum_{\mu_n < \eta_p \le \mu_n + \delta} |\alpha_{jp}|^2 \le \sum_{\mu_n < \eta_p \le \mu_n + \delta} |\alpha_{jp}|^2 (1 + |\eta_p|^{2k}) \le 1 + |\eta_j|^{2k}.$$

Therefore, for j with $\mu_n < \eta_j \leq \mu_n + \delta$,

$$\sum_{\mu_n < \eta_p \leq \mu_n + \delta} |\alpha_{jp}|^2 \leq \frac{1 + |\eta_j|^{2k}}{|\mu_n|^{2k}} \leq C_9. \tag{2.4.4}$$

Here, $C_9 > 0$ is a constant independent of n, j. Take $\phi \in \mathcal{E}_\infty$. We can write

$$\phi = \sum_{\eta_j \leq \mu_n} c_j e_j + \sum_{\mu_n < \eta_j \leq \mu_n + \delta} c_j e_j + \sum_{\mu_n + \delta < \eta_j} c_j e_j.$$

Then

$$\pi_n \phi = \sum_{\eta_j \leq \mu_n} c_j e_j + \sum_{\substack{\mu_n < \eta_j \leq \mu_n + \delta \\ \mu_n < \eta_p \leq \mu_n + \delta}} c_j \alpha_{jp} e_p.$$

Hence we obtain

$$\begin{aligned}
\|\pi_n \phi\|_l^2 &= \sum_{\eta_j \leq \mu_n} |c_j|^2 (1 + |\eta_j|^{2l}) + \sum_{\substack{\mu_n < \eta_j \leq \mu_n + \delta \\ \mu_n < \eta_p \leq \mu_n + \delta}} |c_j|^2 |\alpha_{jp}|^2 (1 + |\eta_p|^{2l}) \\
&\leq C_{10} \Bigg(\sum_{\eta_j \leq \mu_n} |c_j|^2 (1 + |\eta_j|^{2l}) + (1 + |\mu_n|^{2l}) \sum_{\substack{\mu_n < \eta_j \leq \mu_n + \delta \\ \mu_n < \eta_p \leq \mu_n + \delta}} |c_j|^2 |\alpha_{jp}|^2 \Bigg) \\
&\leq C_{11} \Bigg(\sum_{\eta_j \leq \mu_n} |c_j|^2 (1 + |\eta_j|^{2l}) + (1 + |\mu_n|^{2l}) \sum_{\mu_n < \eta_j \leq \mu_n + \delta} |c_j|^2 \Bigg) \\
&\leq C_{12} \|\phi_n\|_l^2,
\end{aligned}$$

where we have used (2.4.4) and C_{10}, C_{11}, C_{12} are constant independent of n. Therefore $\|\pi_n \colon L_l^2 \to L_l^2\| \leq C_{12}$. ∎

To prove Theorem 2.4.1, we need the following theorem and lemma.

Theorem 2.4.3 ([4, Theorem 1*]). *Let W be a closed, spin manifold of odd dimension. Then there is $C_* > 0$ such that each interval of length C_* contains an eigenvalue of D_A. Here, A is a connection on a complex vector bundle V over W and $D_A \colon C^\infty(\mathbb{S} \otimes V) \to C^\infty(\mathbb{S} \otimes V)$ is the twisted Dirac operator.*

Assume that ind $D = 0$. By [40], we have a spectral section P_0 of $-D$. By [40, Lemma 8], using P_0, we can construct a smoothing operator $\mathbb{A} \colon \mathcal{E}_0 \to \mathcal{E}_\infty$ whose image is included in the space spanned by finitely many eigenvectors of D such that $\ker D' = 0$ and

$$\mathcal{E}_0(D')^0_{-\infty} = P_0,$$

where $D' = D + \mathbb{A}$. Moreover, there is $\nu_0 \gg 0$ such that $\mathbb{A} = 0$ on $\mathcal{E}_0(D)_{-\infty}^{-\nu_0}$ and $\mathcal{E}_0(D)_{\nu_0}^{\infty}$. From the construction of $\mathbb{A}$ in the proof of [40, Lemma 8], it is easy to see that for $\lambda \ll 0$ and $\mu \gg 0$,

$$\mathcal{E}_0(D)_{-\infty}^{\mu} = \mathcal{E}_0(D')_{-\infty}^{\mu}, \quad \mathcal{E}_0(D)_{\lambda}^{\infty} = \mathcal{E}_0(D')_{\lambda}^{\infty}, \quad \mathcal{E}_0(D)_{\lambda}^{\mu} = \mathcal{E}_0(D')_{\lambda}^{\mu}.$$

Lemma 2.4.4. *There is a constant $\delta > 0$ such that for any $\mu > 0$ and $a, a' \in B$,*

$$\dim \mathcal{E}_0(D'_a)_0^{\mu} \le \dim \mathcal{E}_0(D'_{a'})_0^{\mu+\delta}.$$

Proof. Put

$$M = \max\big\{\|\nabla_v D' \colon L^2(\mathbb{S}) \to L^2(\mathbb{S})\| : v \in TB,\ \|v\| = 1\big\}.$$

Take a smooth path $\{a_t\}_{t=0}^{\ell}$ in B from a to a' with $\|\frac{d}{dt}a_t\| = 1$. Here, ℓ is the length of the path. Since B is compact, we may assume that there is a constant $C > 0$ independent of a, a' such that $\ell \le C$. Put

$$I = \big\{t \in [0,\ell] : \forall s \le t,\ \dim \mathcal{E}_0(D'_a)_0^{\mu} \le \dim \mathcal{E}_0(D'_{a_s})_0^{\mu+sM}\big\}.$$

Note that $0 \in I$ and that I is closed in $[0,\ell]$ by the continuity of the eigenvalues of D'_{a_s}. It is sufficient to prove that $\sup I = \ell$.

Put $t_0 = \sup I$ and assume that $t_0 < \ell$. Choose $t_+ \in (t_0, \ell]$ with

$$t_+ - t_0 \ll 1.$$

Let $\nu_1(t), \dots, \nu_m(t)$ be the eigenvalues of D'_{a_t} with

$$0 < \nu_1(t_0) \le \dots \le \nu_m(t_0) \le \mu + t_0 M$$

such that $\nu_j(t)$ are continuous in $t \in [t_0, t_+]$ and $\dim \mathcal{E}(D'_{a_{t_0}})_0^{\mu+t_0M} = m$. Note that $t_0 \in I$ since I is closed in $[0,\ell]$ and that

$$\dim \mathcal{E}_0(D'_a)_0^{\mu} \le m$$

by the definition of I. Let ν' be the smallest eigenvalue of $D'_{a_{t_0}}$ with $\nu' > \nu_m(t_0)$. We may assume that

$$M(t_+ - t_0) \ll \nu' - \nu_m(t_0). \tag{2.4.5}$$

By [22, Theorem 4.10, p. 291], we have

$$\operatorname{dist}(\nu_j(t), \Sigma(D'_{a_{t_0}})) \le M(t - t_0)$$

for $t \in [t_0, t_+]$. Here, $\Sigma(D'_{a_{t_0}})$ is the set of eigenvalues of $D'_{a_{t_0}}$. It follows from this inequality and (2.4.5) that

$$0 < \nu_j(t) \le \nu_m(t_0) + M(t - t_0) \le \mu + Mt$$

for $t \in [t_0, t_+]$ and $j \in \{1, \dots, m\}$. This implies that

$$\dim \mathcal{E}_0(D'_a)_0^{\mu} \leq m \leq \dim \mathcal{E}_0(D'_{a_t})_0^{\mu+tM}$$

for $t \in [t_0, t_+]$. This is a contradiction and we obtain $t_0 = \ell$. ∎

Proof of Theorem 2.4.1. For some $\mu \gg 0$, to construct a spectral section P between $\mathcal{E}(D)_{-\infty}^{\mu}$ and $\mathcal{E}(D)_{-\infty}^{\mu+\delta}$, it is sufficient to find a frame $\{f_1, \dots, f_r\}$ in $\mathcal{E}_0(D')_0^{\mu+\delta}$ such that

$$\mathcal{E}_0(D')_0^{\mu} \subset \operatorname{span}\{f_1, \dots, f_r\} \subset \mathcal{E}_0(D')_0^{\mu+\delta}, \tag{2.4.6}$$

because the direct sum $\mathcal{E}_0(D')_{-\infty}^{0} \oplus \operatorname{span}\{f_1, \dots, f_r\}$ is a spectral section between $\mathcal{E}_0(D)_{-\infty}^{\mu}$ and $\mathcal{E}_0(D)_{-\infty}^{\mu+\delta}$.

Put $d = \dim B$. Fix an integer N with $N \gg d$. By Theorem 2.4.3, there is $\delta_0 > 0$ such that

$$\dim(\mathcal{E}_0(D'_a))_{\mu}^{\mu+\delta_0} \geq N \tag{2.4.7}$$

for all $a \in B$ and $\mu \in \mathbb{R}$. By Lemma 2.4.4, we may assume that

$$\dim \mathcal{E}_0(D'_{a'})_0^{\mu-\delta_0} \leq \dim \mathcal{E}_0(D'_a)_0^{\mu} \leq \dim \mathcal{E}_0(D'_{a'})_0^{\mu+\delta_0} \tag{2.4.8}$$

for all $a, a' \in B$ and $\mu \in \mathbb{R}$ with $\mu > \delta_0$.

Fix a positive number δ with $\delta > 10\delta_0$. Take $\mu \in \mathbb{R}$ with $\mu \gg 0$. For $j \in \{0, 1, \dots, d\}$, choose positive numbers

$$\mu < a_j^- < b_j^- < c^- < c^+ < a_j^+ < b_j^+ < \mu + \delta$$

such that

$$\begin{gathered} b_{j+1}^- < a_j^-, \quad b_j^+ < a_{j+1}^+, \\ b_j^- < c^- - 2\delta_0, \quad c^+ + 2\delta_0 < a_j^+. \end{gathered}$$

Take a CW complex structure of B such that for each j-dimensional cell e there are real numbers $\mu^-(e), \mu^+(e)$ such that $\mu^-(e), \mu^+(e)$ are spectral gaps of D'_a for $a \in e$ with

$$a_j^- < \mu^-(e) < b_j^-, \quad a_j^+ < \mu^+(e) < b_j^+.$$

Choose a 0-dimensional cell e_0 (= 1 pt) and $\mu_0 \in (c^-, c^+)$, and then put $r := \dim \mathcal{E}_0(D'_{e_0})_0^{\mu_0}$.

Lemma 2.4.5. *For any cell e and $a \in e$, we have*

$$\dim \mathcal{E}_0(D'_a)_0^{\mu^-(e)} + N \leq r \leq \dim \mathcal{E}_0(D'_a)_0^{\mu^+(e)} - N.$$

Proof. Because $\mu_0 + 2\delta_0 < \mu^+(e)$, by (2.4.7) and (2.4.8), we have

$$\begin{aligned}\dim \mathcal{E}_0(D'_a)_0^{\mu^+(e)} &\geq \dim \mathcal{E}_0(D'_a)_0^{\mu_0+2\delta_0} \\ &= \dim \mathcal{E}_0(D'_a)_0^{\mu_0+\delta_0} + \dim \mathcal{E}_0(D'_a)_{\mu_0+\delta_0}^{\mu_0+2\delta_0} \\ &\geq \dim \mathcal{E}_0(D'_{e_0})_0^{\mu_0} + N \\ &= r + N.\end{aligned}$$

Hence

$$r \leq \dim \mathcal{E}_0(D'_a)_0^{\mu^+(e)} - N.$$

The proof of the inequality $\dim \mathcal{E}_0(D'_a)_0^{\mu^-(e)} + N \leq r$ is similar. ∎

By Lemma 2.4.5, for each 0-dimensional cell e, we can take a frame (meaning a linearly independent collection) $\{f_1, \dots, f_r\}$ of $\mathcal{E}_0(D'_e)_0^{\mu^+(e)}$ such that

$$\mathcal{E}_0(D'_e)_0^{\mu^-(e)} \subset \langle f_1, \dots, f_r \rangle \subset \mathcal{E}_0(D'_e)_0^{\mu^+(e)}.$$

Assume that we have a frame $\{f_1, \dots, f_r\}$ in $\mathcal{E}_0(D')_0^{\infty}$ on the $(j-1)$-dimensional skeleton of B such that

$$\mathcal{E}_0(D'_a)_0^{\mu^-(e)} \subset \langle f_{1,a}, \dots, f_{r,a} \rangle \subset \mathcal{E}_0(D'_a)_0^{\mu^+(e)}$$

for each cell e with $\dim e \leq j-1$ and $a \in e$.

Take a cell e' of B with $\dim e' = j$. Note that $\mathcal{E}_0(D')_0^{\mu^+(e')}$, $\mathcal{E}_0(D')_0^{\mu^-(e')}$ are vector bundles over e'. We denote by $\mathcal{F}$ the bundle

$$\bigcup_{a \in e'} \{\text{frames of rank } r \text{ in } \mathcal{E}_0(D'_a)_0^{\mu^+(e')}\}$$

over e'.

Note that $\mu^+(e) \leq \mu^+(e')$ for any cell e with $\dim e \leq j-1$. Hence the frame $\{f_1, \dots, f_r\}$ defines a section of $\mathcal{F}$ on the boundary $\partial e'$.

We have a homeomorphism

$$\mathcal{F}_a \cong GL(m; \mathbb{C})/GL(m-r; \mathbb{C}),$$

where $a \in e'$, $\mathcal{F}_a$ is the fiber of $\mathcal{F}$ over a and $m = \dim \mathcal{E}_0(D'_a)_0^{\mu^+(e')}$. By Lemma 2.4.5,

$$m = \dim \mathcal{E}_0(D'_a)_0^{\mu^+(e')} \geq r + N.$$

Because $N \gg d$, we have

$$m, m-r \gg d.$$

By the homotopy exact sequence,

$$\pi_i(GL(m;\mathbb{C})/GL(m-r;\mathbb{C})) = 0$$

for $i = 0, 1, \dots, d$. Therefore we can extend $\{f_1, \dots, f_r\}$ to a frame in $\mathcal{E}_0(D')_0^{\mu^+(e')}$ over e'. We will denote the extended frame on e' by the same notation $\{f_1, \dots, f_r\}$. We will modify $\{f_1, \dots, f_r\}$ on the interior $\mathrm{Int}\, e'$ of e' to get a frame $\{f'_1, \dots, f'_r\}$ such that

$$\mathcal{E}_0(D')_0^{\mu^-(e')} \subset \langle f'_1, \dots, f'_r \rangle \subset \mathcal{E}_0(D')_0^{\mu^+(e')}$$

on e'. Since $\mu^-(e') \le \mu^-(e)$, on $\partial e'$ we have

$$\mathcal{E}_0(D')_0^{\mu^-(e')} \subset \mathcal{E}_0(D')_0^{\mu^-(e)} \subset \mathrm{span}\{f_1, \dots, f_r\}.$$

As mentioned before, $\mathcal{E}_0(D')_0^{\mu^-(e')}$ and $\mathcal{E}_0(D')_0^{\mu^+(e')}$ are vector bundles over e'. Let

$$p \colon \mathcal{E}_0(D')_0^{\mu^+(e')}\big|_{e'} \to \mathcal{E}_0(D')_0^{\mu^-(e')}\big|_{e'}$$

be the orthogonal projection.

Lemma 2.4.6. *We can perturb $f_1, \dots, f_r$ slightly on* $\mathrm{Int}\, e'$ *such that*

$$\mathcal{E}_0(D')_0^{\mu^-(e')} = p(\langle f_1, \dots, f_r \rangle)$$

on e'. Here, $\mathrm{Int}\, e'$ *is the interior of e'.*

Proof. We may suppose that

$$\mathcal{E}_0(D')_0^{\mu^+(e')}\big|_{e'} = e' \times (\mathbb{C}^n \oplus \mathbb{C}^{n'}), \quad \mathcal{E}'_0(D')_0^{\mu^-(e')}\big|_{e'} = e' \times (\mathbb{C}^n \oplus \{0\}).$$

For each $a \in e'$, we can write

$$f_{j,a} = g_{j,a} \oplus g'_{j,a},$$

where

$$g_{j,a} \in \mathbb{C}^n, \quad g'_{j,a} \in \mathbb{C}^{n'}.$$

Note that

$$\mathbb{C}^n = p(\langle f_{1,a}, \dots, f_{r,a} \rangle)$$

if and only if the $(n \times r)$-matrix $(g_{1,a} \dots g_{r,a})$ is of rank n. Let M be the set of $(n \times r)$-complex matrices, which is naturally a smooth manifold of dimension $2nr$. We denote by R_l the set of $(n \times r)$-matrices of rank l. Then R_l is a smooth submanifold of M of codimension $2(n-l)(r-l)$. If $l \le n-1$ we have

$$\mathrm{codim}_{\mathbb{R}}(R_l \subset M) = 2(n-l)(r-l) \ge 2(r-n+1) \ge 2(N+1) \gg d.$$

Here we have used

$$n = \dim \mathcal{E}_0(D'_a)_0^{\mu^-(e')} \leq r - N.$$

See Lemma 2.4.5. So we can slightly perturb $(g_1 \dots g_r)$ on $\operatorname{Int} e'$ such that for all $a \in e'$ and $l \in \{0, 1, \dots, n-1\}$,

$$(g_{1,a} \dots g_{r,a}) \notin R_l.$$

Hence the rank of $(g_{1,a} \dots g_{r,a})$ is n. Therefore $\mathbb{C}^n = p(\langle f_{1,a}, \dots, f_{r,a} \rangle)$ for all $a \in e'$. We can assume that the perturbation is small enough such that after the perturbation, $f_1, \dots, f_r$ is still linearly independent. ∎

By this lemma, we may suppose that

$$\mathcal{E}_0(D')_0^{\mu^-(e')} = p(\langle f_1, \dots, f_r \rangle)$$

on e'. For $a \in e'$, define $F_a : \mathbb{C}^r \to \mathcal{E}_0(D'_a)_0^{\mu^+(e')}$ by

$$F_a(c_1, \dots, c_r) = c_1 f_{1,a} + \dots + c_r f_{r,a}.$$

We have

$$\mathcal{E}_0(D'_a)_0^{\mu^-(e')} = \operatorname{im}(p \circ F_a).$$

Put

$$K := \bigcup_{a \in e'} \ker(p \circ F_a).$$

Then K is a subbundle of the trivial bundle $\underline{\mathbb{C}}^r$ on e'. We have the orthogonal decomposition

$$\underline{\mathbb{C}}^r = K \oplus K^\perp.$$

We define

$$F' : \underline{\mathbb{C}}^r \to \mathcal{E}(D')_0^{\mu^+(e')}\big|_{e'}$$

by

$$F' = F|_K + p \circ F|_{K^\perp}.$$

Then

$$\mathcal{E}(D')_0^{\mu^-(e')}\big|_{e'} \subset \operatorname{im} F'.$$

Lemma 2.4.7. *The following statements hold:*

(1) *$F = F'$ on $\partial e'$.*

(2) *The map F' is injective on e'.*

Proof. (1) Take $a \in \partial e'$. It is sufficient to show that $F_a|_{K^\perp} = F'_a|_{K^\perp}$. Recall that

$$\mathcal{E}_0(D'_a)_0^{\mu^-(e')} \subset \operatorname{im} F_a.$$

Since $\operatorname{im} F_a|_{K_a} \subset (\mathcal{E}_0(D'_a)_0^{\mu^-(e')})^\perp$ and $\dim \mathcal{E}_0(D'_a)_0^{\mu^-(e')} = \dim K_a^\perp$, we have

$$\operatorname{im}(F_a|_{K^\perp}) = \mathcal{E}_0(D'_a)_0^{\mu^-(e')}.$$

Therefore, for $v \in K_a^\perp$, $F'_v(v) = pF_a(v) = F_a(v)$.

(2) Suppose that

$$F'(v, v') = 0$$

for $v \in K, v' \in K^\perp$. Then

$$F(v) + pF(v') = 0.$$

So we have

$$pF(v) + p^2 F(v') = 0.$$

Since $v \in K = \ker p \circ F$ and $p^2 = p$,

$$pF(v') = 0.$$

Because $p \circ F$ is an isomorphism on $K^\perp$, we have

$$v' = 0.$$

Hence

$$F(v) = 0,$$

which implies that $v = 0$ because F is injective. ∎

Put

$$f'_{1,a} := F'_a(e_1), \dots, f'_{r,a} := F'_a(e_r)$$

for $a \in e'$. Here, $e_1, \dots, e_r$ is the standard basis of $\mathbb{C}^r$. Then the frame $\{f'_1, \dots, f'_r\}$ of $\mathcal{E}_0(D')_0^{\mu^+(e')}$ on e', which is an extension of the frame on $\partial e'$, has the property that

$$\mathcal{E}(D')_0^{\mu^-(e')} \subset \langle f'_1, \dots, f'_r \rangle \subset \mathcal{E}(D')_0^{\mu^+(e')}.$$

We have obtained a frame $f_1, \dots, f_r$ satisfying (2.4.6). Putting

$$P = \mathcal{E}_0(D')_{-\infty}^0 \oplus \langle f_1, \dots, f_r \rangle,$$

we obtain a spectral section with

$$\mathcal{E}_0(D)_{-\infty}^{\mu} \subset P \subset \mathcal{E}_0(D)_{-\infty}^{\mu+\delta},$$

where $\delta > 0$ is a constant independent of μ.

Take another positive number $\tilde{\mu}$ with $\mu \ll \tilde{\mu}$. Doing this procedure one more time, we get a frame $\{\tilde{f}_1, \dots, \tilde{f}_s\}$ of $P^{\perp} \cap \mathcal{E}(D')_0^{\tilde{\mu}+\delta}$ such that

$$\mathcal{E}_0(D)_{-\infty}^{\tilde{\mu}} \subset P \oplus \langle \tilde{f}_1, \dots, \tilde{f}_s \rangle \subset \mathcal{E}_0(D)_{-\infty}^{\tilde{\mu}+\delta}.$$

Repeating this, we get a sequence of spectral sections satisfying the conditions of Theorem 2.4.1. ∎

We will state a Pin(2)-equivariant version of Theorem 2.4.1. If $\mathfrak{s}$ is a self-conjugate spin^c structure of Y, we have an action of Pin(2) on $\mathcal{E}_k$. The action is induced by the action of Pin(2) on $\mathcal{H}^1(Y) \times L_k^2(\mathbb{S})$, which is an extension of the S^1-action, defined by

$$j(a, \phi) = (-a, j\phi).$$

The Dirac operator D is Pin(2)-equivariant and we have the index

$$\operatorname{ind} D \in KQ^1(B).$$

Here, $KQ^1(B)$ is the quaternionic K-theory defined in [19], which is used in [33].

Theorem 2.4.8. *If $\mathfrak{s}$ is a self-conjugate* spin^c *structure of Y and* $\operatorname{ind} D = 0$ *in $KQ^1(B)$, then we have a sequence P_n of* Pin(2)*-equivariant spectral sections having the properties of Theorem* 2.4.1.

Proof. We will show an outline of the proof. Since $\operatorname{ind} D = 0$ in $KQ^1(B)$, it follows from the arguments in [33, Section 1] that the family D of Dirac operators is Pin(2)-equivariantly homotopic to a constant family. Hence we can apply the proof of [40, Proposition 1] to show that there exists a Pin(2)-equivariant spectral section P_0 of $-D$.

Choose a CW complex structure of B such that for each cell e, $(-1) \cdot e$ is also a cell. Note that

$$\pi_i(Sp(m)/Sp(m-r)) = 0$$

for $i = 1, \dots, d$, provided that $m, m-r \gg d$. Hence for $\mu \gg 0$, we can construct a Pin(2)-equivariant frame $f_1, \dots, f_r$ of $P_0^{\perp}$ with

$$\mathcal{E}_0(D')_0^{\mu} \subset \langle f_1, \dots, f_r \rangle \subset \mathcal{E}_0(D')_0^{\mu+\delta}$$

as in the proof of Theorem 2.4.1. Here, δ is the positive constant from the proof of Theorem 2.4.1. Then

$$P_0 \oplus \langle f_1, \dots, f_r \rangle$$

is a Pin(2)-equivariant spectral section between $\mathcal{E}_0(D)_{-\infty}^{\mu}$ and $\mathcal{E}_0(D)_{-\infty}^{\mu+\delta}$. Repeating this construction, we obtain the desired sequence P_n. ∎

2.5 Derivative of projections

Let $D\colon \mathcal{E}_k \to \mathcal{E}_{k-1}$ be the original Dirac operator. Recall that we have a canonical flat connection ∇ on $\mathcal{E}_k$. See Section 2.3. Note that for $a \in B$, $v \in T_a B = \mathcal{H}^1(Y)$, we have

$$\nabla_v D = \frac{d}{dt}\Big|_{t=0} D_{a+tv} = \frac{d}{dt}\Big|_{t=0} (D_a + t\rho(v)) = \rho(v).$$

Here, $\rho(v)$ is the Clifford multiplication. Since v is a harmonic (and hence smooth) 1-form, we have $\|v\|_k < \infty$ for any $k \geq 0$. Therefore $\nabla_v D$ is a bounded operator from $L^2_k(\mathbb{S})$ to $L^2_k(\mathbb{S})$ for each $k \geq 0$.

Take $\mu \in \mathbb{R}$. We write $\pi^{\mu}_{-\infty}$ for the L^2-projection on $\mathcal{E}_0(D)^{\mu}_{-\infty}$. Similarly, π^{μ}_{λ} is the L^2-projection on $\mathcal{E}_0(D)^{\mu}_{\lambda}$. We have the following proposition.

Proposition 2.5.1. *Fix $a \in B$. Let $\{e_i\}_{i=-\infty}^{\infty}$ be an L^2-orthonormal basis of $L^2(\mathbb{S})$ such that*

$$D_a e_i = \eta_i e_i.$$

Here, η_i are the eigenvalues of D_a. Take $\lambda, \mu \in \mathbb{R}$ with $\lambda < \mu$. Suppose that λ, μ are not eigenvalues of D_a. For $v \in T_a B = \mathcal{H}^1(Y)$,

$$\begin{aligned} &\langle (\nabla_v \pi^{\mu}_{\lambda}) e_i, e_j \rangle_0 \\ &= \begin{cases} \dfrac{\langle \rho(v) e_i, e_j \rangle_0}{\eta_i - \eta_j} & \text{if } \eta_i < \lambda < \eta_j < \mu \text{ or } \lambda < \eta_j < \mu < \eta_i, \\ \dfrac{\langle \rho(v) e_i, e_j \rangle_0}{\eta_j - \eta_i} & \text{if } \eta_j < \lambda < \eta_i < \mu \text{ or } \lambda < \eta_i < \mu < \eta_j, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{2.5.1}$$

and

$$\langle (\nabla_v \pi^{\mu}_{-\infty}) e_i, e_j \rangle_0 = \begin{cases} \dfrac{\langle \rho(v) e_i, e_j \rangle_0}{\eta_i - \eta_j} & \text{if } \eta_j < \mu < \eta_i, \\ \dfrac{\langle \rho(v) e_i, e_j \rangle_0}{\eta_j - \eta_i} & \text{if } \eta_i < \mu < \eta_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.2}$$

Here, $\rho(v)$ is the Clifford multiplication by v.

Proof. Since the connection ∇ is induced by the trivial connection on $\mathcal{H}^1(Y) \times C^{\infty}(\mathbb{S})$, to compute $\nabla_v \pi^{\mu}_{\lambda}$, $\nabla_v \pi^{\mu}_{-\infty}$, we can do computations over $\mathcal{H}^1(Y)$ where we have the canonical trivialization, and the covariant derivative is equal to the usual exterior derivative.

Take a loop Γ^{μ}_{λ} in $\mathbb{C}$ defined by

$$\begin{aligned} \Gamma^{\mu}_{\lambda} = &\left\{x - i\varepsilon : \lambda \leq x \leq \mu\right\} \cup \left\{\mu + iy : -\varepsilon \leq y \leq \varepsilon\right\} \\ &\cup \left\{x + i\varepsilon : \lambda \leq x \leq \mu\right\} \cup \left\{\lambda + iy : -\varepsilon \leq y \leq \varepsilon\right\} \end{aligned}$$

for some $\varepsilon > 0$. We orient Γ_λ^μ counterclockwise. We will show that for $\phi \in C^\infty(\mathbb{S})$,

$$(\pi_a)_\lambda^\mu \phi = \frac{1}{2\pi i} \int_{\Gamma_\lambda^\mu} (z - D_a)^{-1} \phi \, dz.$$

See also [22, Chapter II, Section 4]. We can write

$$\phi = \sum_{i=-\infty}^{\infty} c_i e_i$$

for some $c_i \in \mathbb{C}$ with

$$\sum_{i=-\infty}^{\infty} |c_i|^2 (1 + |\eta_i|^{2k}) < \infty$$

for any $k \geq 0$. For $z \in \mathbb{C}$ which is not an eigenvalue of D_a, the operator $z - D_a$ is invertible and

$$(z - D_a)^{-1} \phi = \sum_{i=-\infty}^{\infty} \frac{c_i}{z - \eta_i} e_i. \tag{2.5.3}$$

Note that the sum in (2.5.3) converges uniformly on Γ_λ^μ in the L_k^2-norm for any $k \geq 0$ since

$$\left| \frac{c_i}{z - \eta_i} \right| \leq |c_i| \quad (z \in \Gamma_\lambda^\mu)$$

if $|i| \gg 0$. Hence, by the residue formula,

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\Gamma_\lambda^\mu} (z - D_a)^{-1} (\phi) \, dz &= \sum_{i=-\infty}^{\infty} \frac{1}{2\pi i} \left(\int_{\Gamma_\lambda^\mu} \frac{c_i}{z - \eta_i} \, dz \right) e_i \\
&= \sum_{\lambda < \eta_i < \mu} c_i e_i \\
&= (\pi_a)_\lambda^\mu \phi.
\end{aligned}$$

Here we have used the fact that we are allowed to take the term-by-term integration because of the uniform convergence.

Take $v \in T_a B = \mathcal{H}^1(Y)$. Then, by the above formula for π_λ^μ, we have

$$\begin{aligned}
(\nabla_v \pi_\lambda^\mu) e_i &= -\frac{1}{2\pi i} \int_{\Gamma_\lambda^\mu} (z - D_a)^{-1} (\nabla_v D)(z - D_a)^{-1} e_i \, dz \\
&= -\frac{1}{2\pi i} \int_{\Gamma_\lambda^\mu} (z - D_a)^{-1} \rho(v)(z - \eta_i)^{-1} e_i \, dz \\
&= -\frac{1}{2\pi i} \int_{\Gamma_\lambda^\mu} (z - \eta_i)^{-1} (z - D_a)^{-1} \rho(v) e_i \, dz.
\end{aligned}$$

Therefore

$$\begin{aligned}\langle(\nabla_v \pi_\lambda^\mu)e_i, e_j\rangle_0 &= -\frac{1}{2\pi i}\int_{\Gamma_\lambda^\mu}(z-\eta_i)^{-1}\langle\rho(v)e_i, (\bar{z}-D_a)^{-1}e_j\rangle_0\,dz\\ &= -\frac{1}{2\pi i}\int_{\Gamma_\lambda^\mu}(z-\eta_i)^{-1}\langle\rho(v)e_i, (\bar{z}-\eta_j)^{-1}e_j\rangle_0\,dz\\ &= -\frac{\langle\rho(v)e_i, e_j\rangle_0}{2\pi i}\int_{\Gamma_\lambda^\mu}(z-\eta_i)^{-1}(z-\eta_j)^{-1}\,dz.\end{aligned}$$

From this, we obtain the formula (2.5.1) for $\langle(\nabla_v\pi_\lambda^\mu)e_i, e_j\rangle_0$.

Note that since $\rho(v)$ defines a bounded operator $L^2 \to L^2$, we can see that the operators $(T_a)_\lambda^\mu$, $(T_a)_{-\infty}^\mu$ defined by the right-hand side of (2.5.1) and (2.5.2) are bounded from L^2 to L^2. Moreover, for each compact set K in $\mathscr{H}^1(Y)$, $(T_a)_\lambda^\mu$ converges to $(T_a)_{-\infty}^\mu$ on K uniformly as $\lambda \to -\infty$. We have

$$\begin{aligned}\langle(\pi_{a+tv})_\lambda^\mu(e_i), e_j\rangle_0 - \langle(\pi_a)_\lambda^\mu(e_i), e_j\rangle_0 &= \int_0^t \frac{d}{ds}\langle\pi_{a+sv}e_i, e_j\rangle_0\,ds\\ &= \int_0^t \langle(\nabla_v\pi_{a+sv}e_i), e_j\rangle_0\,ds\\ &= \int_0^t \langle(T_{a+sv})_\lambda^\mu(e_i), e_j\rangle_0\,ds.\end{aligned}$$

Taking the limit as $\lambda \to -\infty$, we obtain

$$\langle(\pi_{a+tv})_{-\infty}^\mu(e_i), e_j\rangle_0 - \langle(\pi_a)_{-\infty}^\mu e_i\rangle_0 = \int_0^t \langle(T_{a+sv})_{-\infty}^\mu(e_i), e_j\rangle_0\,ds.$$

Therefore

$$\langle(\nabla_v\pi_{-\infty}^\mu)e_i, e_j\rangle_0 = \frac{d}{dt}\Big|_{t=0}\langle(\pi_{a+tv})_{-\infty}^\mu(e_i), e_j\rangle_0 = \langle(T_a)_{-\infty}^\mu(e_i), e_j\rangle_0.$$

We have obtained (2.5.2). ∎

Corollary 2.5.2. *Suppose that μ is not an eigenvalue of D_a. Then for each $v \in TB$ and nonnegative k,*

$$\nabla_v\pi_{-\infty}^\mu\colon L_k^2(\mathbb{S}) \to L_{k+1}^2(\mathbb{S})$$

is a bounded operator. Moreover, if $|\mu| \geq 2$, $\alpha < k$ and if there is no eigenvalue of D_a in the interval $[\mu-\mu^{-\alpha}, \mu+\mu^{-\alpha}]$, for $v \in T_aB$ with $\|v\| \leq 1$,

$$\|\nabla_v\pi_{-\infty}^\mu : L_k^2(\mathbb{S}) \to L_{k-\alpha}^2(\mathbb{S})\| \leq C.$$

Here, $C > 0$ is a constant independent of v, μ. Similar statements hold for $\nabla_v\pi_\lambda^\mu$, $\nabla_v\pi_\mu^\infty$.

Proof. Let e_i, η_i be as in Proposition 2.5.1. Take $v \in T_a B = \mathscr{H}^1(Y)$. Put

$$\rho_{ij} := \langle \rho(v) e_i, e_j \rangle_0.$$

Take $\phi = \sum_i c_i e_i \in C^\infty(\mathbb{S})$ with $\|\phi\|_k = 1$. Since $\rho(v)$ is a bounded operator from L^2_k to L^2_k we have

$$\|\rho(v)\phi\|_k^2 = \left\| \sum_{i,j} c_i \rho_{ij} e_j \right\|_k^2 = \sum_j \left| \sum_i c_i \rho_{ij} \right|^2 (1 + |\eta_j|^{2k}) \le C_1,$$

where $C_1 > 0$ is a constant independent of ϕ.

By Proposition 2.5.1, we have

$$\begin{aligned}
&\|(\nabla_v \pi^\mu_{-\infty})\phi\|^2_{k+1} \\
&\quad = \left\| \sum_{\eta_i < \mu < \eta_j} \frac{c_i \rho_{ij}}{\eta_j - \eta_i} e_j + \sum_{\eta_j < \mu < \eta_i} \frac{c_i \rho_{ij}}{\eta_i - \eta_j} e_j \right\|^2_{k+1} \\
&\quad = \sum_{\mu < \eta_j} \left| \sum_{\eta_i < \mu} \frac{c_i \rho_{ij}}{\eta_j - \eta_i} \right|^2 (1 + |\eta_j|^{2k+2}) + \sum_{\eta_j < \mu} \left| \sum_{\mu < \eta_i} \frac{c_i \rho_{ij}}{\eta_i - \eta_j} \right|^2 (1 + |\eta_j|^{2k+2}).
\end{aligned}$$

Note that there is a constant $C_2 > 0$ independent of i, j such that

$$\frac{1 + |\eta_j|^{2k+2}}{|\eta_j - \eta_i|^2} \le C_2 (1 + |\eta_j|^{2k})$$

for i, j with $\eta_i < \mu < \eta_j$ or $\eta_j < \mu < \eta_i$. Hence

$$\|(\nabla_v \pi^\mu_{-\infty})\phi\|^2_{k+1} \le C_2 \sum_j \left| \sum_i c_i \rho_{ij} \right|^2 (1 + |\eta_j|^{2k}) \le C_1 C_2.$$

Therefore $\nabla_v \pi^\mu_{-\infty}$ extends to a bounded operator $L^2_k \to L^2_{k+1}$.

Next assume that there is no eigenvalue of D_a in the interval $[\mu - \mu^{-\alpha}, \mu + \mu^{-\alpha}]$. Take $v \in T_a B$ with $\|v\| = 1$. It is easy to see that if $\eta_i < \mu < \eta_j$ or $\eta_j < \mu < \eta_i$ we have

$$\frac{1 + |\eta_j|^{2k-2\alpha}}{|\eta_i - \eta_j|^2} \le C_3 (1 + |\eta_j|^{2k}),$$

where $C_3 > 0$ is independent of i, j. It follows from this and Proposition 2.5.1 that

$$\|\nabla_v \pi^\mu_{-\infty} \colon L^2_k \to L^2_{k-\alpha}\| \le C_4,$$

where $C_4 > 0$ is a constant independent of μ and v. ∎

Lemma 2.5.3. *Fix positive numbers α, β with $\alpha + 3 < \beta$ and $a \in \mathscr{H}^1(Y)$. For $\mu \in \mathbb{R}$ with $|\mu| \gg 0$, there exists $\mu' \in (\mu - |\mu|^{-\alpha}, \mu + |\mu|^{-\alpha}]$ such that there is no eigenvalue of D_a in the interval $(\mu' - |\mu|^{-\beta}, \mu' + |\mu|^{-\beta}]$.*

Proof. Suppose that the statement is not true. Then there is a sequence μ_n with $|\mu_n| \to \infty$ such that for any $\mu' \in (\mu_n - |\mu_n|^{-\alpha}, \mu_n + |\mu_n|^{-\alpha}]$ there is an eigenvalue of D_a in $(\mu' - |\mu_n|^{-\beta}, \mu' + |\mu_n|^{-\beta}]$. Therefore, for each integer m with $1 \le m \le |\mu_n|^{\beta-\alpha}$, there is an eigenvalue of D_a in the interval $(\mu_n + (m-1)|\mu_n|^{-\beta}, \mu_n + m|\mu_n|^{-\beta}]$. This implies that

$$\dim(\mathcal{E}_0(D_a))_{\mu_n - |\mu_n|^{-\alpha}}^{\mu_n + |\mu_n|^{-\alpha}} \ge |\mu_n|^{\beta-\alpha} - 1.$$

On the other hand, by the Weyl law,

$$\dim(\mathcal{E}(D_a))_{\mu_n - |\mu_n|^{-\alpha}}^{\mu_n + |\mu_n|^{-\alpha}} \le C|\mu_n|^3.$$

We have obtained a contradiction. ∎

Corollary 2.5.4. *For $\mu \in \mathbb{R}$ with $|\mu| \gg 0$, there is $\mu' \in [\mu, \mu+1]$, such that for $v \in TB$ with $\|v\| = 1$,*

$$\|\nabla_v \pi_{-\infty}^{\mu'} : L_k^2(\mathbb{S}) \to L_{k-4}^2(\mathbb{S})\| \le C.$$

Here, $C > 0$ is a constant independent of v, μ. Similar statements hold for π_λ^∞, π_λ^μ.

Proof. This is a direct consequence of Corollary 2.5.2 and Lemma 2.5.3. ∎

Proposition 2.5.5. *Take a nonnegative real number m and a smooth spectral section P of $-D$ with*

$$(\mathcal{E}_0(D))_{-\infty}^{\mu_-} \subset P \subset (\mathcal{E}_0(D))_{-\infty}^{\mu_+}.$$

Let π_P be the L^2-projection onto P. Then for each $v \in TB$, $\nabla_v \pi_P$ is a bounded operator from $L_m^2(\mathbb{S})$ to $L_{m+1}^2(\mathbb{S})$.

Proof. We can take an open covering $\{U_i\}_{i=1}^N$ of B such that there are real numbers λ_i, ν_i with $\lambda_i < \mu_-$, $\mu_+ < \nu_i$, which are not eigenvalues of D_a for $a \in U_i$. Also we may assume that for each i, we have a trivialization

$$\mathcal{E}_0|_{U_i} \cong U_i \times L^2(\mathbb{S})$$

such that the flat connection ∇ is equal to the exterior derivative d through this trivialization. Also for each i, we have smooth L^2-orthonormal frames $f_{i,1}, \dots, f_{i,r_i}$ of the normal bundle of $(\mathcal{E}_0)_{-\infty}^{\lambda_i}|_{U_i}$ in $P|_{U_i}$. We can write

$$\pi_P = \pi_{-\infty}^{\lambda_i} + \sum_{l=1}^{r_i} f_{i,l}^* \otimes f_{i,l}$$

over U_i. We have

$$\nabla_v \pi_P = \nabla_v \pi_{-\infty}^{\lambda_i} + \sum_{l=1}^{r_i} (\nabla_v f_{i,l}^* \otimes f_{i,l} + f_{i,l}^* \otimes \nabla_v f_{i,l}).$$

By Corollary 2.5.2, $\nabla_v \pi_{-\infty}^{\lambda_i}$ is a bounded operator from L^2_m to L^2_{m+1}. Also we have

$$\nabla_v f_{i,l} = \nabla_v(\pi_{\lambda_i}^{\nu_i} f_{i,l}) = (\nabla_v \pi_{\lambda_i}^{\nu_i}) f_{i,l} + \pi_{\lambda_i}^{\nu_i}(\nabla_v f_{i,l}).$$

Since $f_{i,l}(b) \in C^\infty(\mathbb{S})$ for $b \in U_i$, and $\nabla_v \pi_{\lambda_i}^{\nu_i}$ is a bounded operator $L^2_m \to L^2_{m+1}$, we have

$$\nabla_v f_{i,l}(b) \in C^\infty(\mathbb{S})$$

for $b \in U_i$. Also we have

$$|f_{i,l}^*(\phi)| = |\langle f_{i,l}, \phi\rangle_0| \leq \|\phi\|_0$$

for $\phi \in C^\infty(\mathbb{S})$. Therefore

$$\sum_{l=1}^{r_i} f_{i,l}^* \otimes \nabla_v f_{i,l} \colon L^2_m \to L^2_{m+1}$$

is bounded.

Take $\phi \in C^\infty(\mathbb{S})$. We have

$$(\nabla_v f_{i,l}^*)(\phi) = \langle \phi, \nabla_v f_{i,l}\rangle_0.$$

Note that $\nabla_v f_{i,l}(b) \in C^\infty(\mathbb{S})$ for $b \in U_i$. Hence

$$\|(\nabla_v f_{i,l}^* \otimes f_{i,l})(\phi)\|_{m+1} = |(\nabla_v f_{i,l}^*)(\phi) \cdot f_{i,l}|_{m+1} \leq C\|\phi\|_0.$$

Therefore

$$\sum_{l=1}^{r_i} \nabla_v f_{i,l}^* \otimes f_{i,l} \colon L^2_m \to L^2_{m+1}$$

is bounded. ■

Corollary 2.5.6. *Suppose that* ind $D = 0$ *in* $K^1(B)$ *and let* P_0 *be a spectral section of* $-D$. *Then there is a family of smoothing operators* $\mathbb{A}$ *acting on* $\mathcal{E}_0$ *such that the kernel of* $D' = D + \mathbb{A}$ *is trivial and*

$$P_0 = \mathcal{E}_0(D')^0_{-\infty}.$$

Moreover, for each positive number k *and* $v \in TB$,

$$\nabla_v D' \colon L^2_k(\mathbb{S}) \to L^2_k(\mathbb{S})$$

is bounded.

Proof. The operator $\mathbb{A}$ is obtained as follows. (See the proof of [40, Lemma 8].) We can take smooth spectral sections Q, R of D and a positive number s with

$$(\mathcal{E}_0)^{-s}_{-\infty} \subset P_0 \subset (\mathcal{E}_0)^{s}_{-\infty}, \quad (\mathcal{E}_0)^{-2s}_{-\infty} \subset Q \subset (\mathcal{E}_0)^{-s}_{-\infty}, \quad (\mathcal{E}_0)^{s}_{-\infty} \subset R \subset (\mathcal{E}_0)^{2s}_{-\infty}.$$

Put

$$D' = \pi_Q D \pi_Q - s\pi_{P_0}(1-\pi_Q) + (1-\pi_R)D(1-\pi_R) + s(1-\pi_{P_0})\pi_R,$$

where π_{P_0}, π_Q, π_R are the L^2-projections. Then $\ker D' = 0$. The operator $\mathbb{A}$ is given by

$$\mathbb{A} = D' - D.$$

The image of $\mathbb{A}$ is included in the subspace spanned by finitely many eigenvectors of D. By Proposition 2.5.10, $\nabla_v \pi_{P_0}$, $\nabla_v \pi_Q$, $\nabla_v \pi_R$ are bounded operators from $L^2_k(\mathbb{S})$ to $L^2_{k+1}(\mathbb{S})$. Note that $\nabla_v D$ is the Clifford multiplication of the harmonic 1-form v. Hence $\nabla_v D$ is a bounded operator $L^2_k(\mathbb{S}) \to L^2_k(\mathbb{S})$. Therefore $\nabla_v D'$ is a bounded operator from L^2_k to L^2_k. ∎

Proposition 2.5.7. *The statements of Proposition 2.5.1, Corollary 2.5.2 and Corollary 2.5.4 hold for the perturbed Dirac operator D', replacing $\rho(v)$ with $\nabla_v D'$.*

Proof. By Corollary 2.5.6, for any nonnegative number k,

$$\nabla_v D' \colon L^2_k(\mathbb{S}) \to L^2_k(\mathbb{S})$$

is bounded and we can do the same computations as those done for the original Dirac operator D. ∎

Lemma 2.5.8. *For a positive integer k, a positive number l with $l \geq k-1$ and $v \in T_a B$, the expression*

$$\nabla_v |D'|^k \colon L^2_l \to L^2_{l-k+1}$$

is bounded.

Proof. Note that

$$|D'|^k = (D')^k(1-\pi_{P_0}) + (-1)^k (D')^k \pi_{P_0}.$$

Here, π_{P_0} is the L^2-projection on P_0. We have

$$\nabla_v (D')^k = (\nabla_v D')(D')^{k-1} + D'(\nabla_v D')(D')^{k-2} + \cdots + (D')^{k-1}\nabla_v D',$$

which implies that $\nabla_v (D')^k$ is a bounded operator $L^2_l \to L^2_{l-k+1}$ by Corollary 2.5.6. Also $\nabla_v \pi_{P_0}$ is a bounded operator $L^2_l \to L^2_{l+1}$ by Proposition 2.5.5. ∎

Remark 2.5.9. So far the authors have not been able to prove Lemma 2.5.8 in the case when k is not an integer, though there is an explicit formula

$$|D'|^k = \sum_j |\eta_j|^k \pi_j .$$

Here, π_j is the projection onto the jth eigenspace which can be written as

$$\pi_j = \frac{1}{2\pi i} \int_{\Gamma_j} (z - D)^{-1}\, dz.$$

Suppose that $\operatorname{ind} D = 0$ in $K^1(B)$ and fix a spectral section P_0 and recall the definition of the $L^2_{k_+,k_-}$-inner product $\langle \cdot, \cdot \rangle_{k_+,k_-}$ defined by using the perturbed Dirac operator $D' = D + \mathbb{A}$ of Corollary 2.5.6. (See (2.3.5).) Let $\mathcal{E}_{k_+,k_-}$ be the completion of $\mathcal{E}_\infty$ with respect to $\langle \cdot, \cdot \rangle_{k_+,k_-}$.

We will prove a generalization of Proposition 2.5.5.

Proposition 2.5.10. *Take nonnegative half-integers k_+, k_- and a smooth spectral section P of $-D$ with*

$$(\mathcal{E}_0)^{\mu_-}_{-\infty} \subset P \subset (\mathcal{E}_0)^{\mu_+}_{-\infty}.$$

Let π_P be the $L^2_{k_+,k_-}$-projection on P. Then for each nonnegative real number m, $v \in TB$, $\nabla_v \pi_P$ is a bounded operator from $L^2_m(\mathbb{S})$ to $L^2_{m+1}(\mathbb{S})$.

Proof. Let U_i, λ_i, ν_i be as in the proof of Proposition 2.5.5 and $f_{i,1}, \dots, f_{i,r_i}$ are smooth $L^2_{k_+,k_-}$-orthonormal frames of the normal bundle of $(\mathcal{E}_0)^{\lambda_i}_{-\infty}|_{U_i}$ in P. We can write

$$\pi_P = \pi^{\lambda_i}_{-\infty} + \sum_{l=1}^{r_i} f^*_{i,l} \otimes f_{i,l}$$

on U_i. Here,

$$f^*_{i,l}(\phi) = \langle \pi_{P_0}\phi, |D'|^{2k_-} f_{i,l} \rangle_0 + \langle (1 - \pi_{P_0})\phi, |D'|^{2k_+} f_{i,l} \rangle_0,$$

P_0 is the fixed spectral section used to define the $L^2_{k_+,k_-}$-norm, and π_{P_0} is the L^2-projection onto P_0. We have

$$\nabla_v \pi_P = \nabla_v \pi^{\lambda_i}_{-\infty} + \sum_{l=1}^{r_i} (\nabla_v f^*_{i,l} \otimes f_{i,l} + f^*_{i,l} \otimes \nabla_v f_{i,l}).$$

As stated in the proof of Proposition 2.5.5, $\nabla_v \pi^{\lambda_i}$ and $f^*_{i,l} \otimes \nabla_v f_{i,l}$ are bounded operators from L^2_m to L^2_{m+1}.

For $\phi \in C^\infty(\mathbb{S})$,

$$
\begin{aligned}
(\nabla_v f_{i,l}^*)(\phi) = {} & \langle (\nabla_v \pi_{P_0})\phi, |D'|^{2k_-} f_{i,+}\rangle_0 + \langle \pi_{P_0}\phi, (\nabla_v |D'|^{2k_-}) f_{i,l}\rangle_0 \\
& + \langle \pi_{P_0}\phi, |D'|^{2k_-}(\nabla_v f_{i,l})\rangle_0 - \langle (\nabla_v \pi_{P_0})\phi, |D'|^{2k_+} f_{i,;}\rangle_0 \\
& + \langle (1-\pi_{P_0})\phi, (\nabla_v |D'|^{2k_+}) f_{i,l}\rangle_0 \\
& + \langle (1-\pi_{P_0})\phi, |D'|^{2k_+}(\nabla_v f_{i,l})\rangle_0 .
\end{aligned}
$$

Note that $2k_\pm$ are nonnegative integers. By Proposition 2.5.5 and Lemma 2.5.8,

$$
\|(\nabla_v f_{i,l}^* \otimes f_{i,l})(\phi)\|_{m+1} = \|(\nabla_v f_{i,l}^*)(\phi) \cdot f_{i,l}\|_{m+1} \le C\|\phi\|_0 .
$$

Hence $\nabla_v f_{i,l}^* \otimes f_{i,l}$ are bounded operators from L_m^2 to L_{m+1}^2. ∎

Lemma 2.5.11. *Let ∇ be a connection on $\mathcal{E}_{k_+,k_-}$ (which is not necessarily the flat connection defined in Section 2.3). Let F be a subbundle in $\mathcal{E}_{k_+,k_-}$ of finite rank and $\pi_F : \mathcal{E}_{k_+,k_-} \to F$ be the $L^2_{k_+,k_-}$-projection. For $a \in B$, $\phi, \psi \in F_a$ and $v \in T_a B$, we have*

$$
\langle (\nabla_v \pi_F)\phi, \psi\rangle_{k_+,k_-} = 0.
$$

Similarly, for $\phi', \psi' \in F_a^\perp$, we have

$$
\langle (\nabla_v \pi_F)\phi', \psi'\rangle_{k_+,k_-} = 0.
$$

Proof. Since

$$
\pi_F \pi_F = \pi_F ,
$$

we have

$$
(\nabla_v \pi_F)\pi_F + \pi_F(\nabla_v \pi_F) = \nabla_v \pi_F .
$$

Hence

$$
(\nabla_v \pi_F)\phi + \pi_F(\nabla_v \pi_F)\phi = (\nabla_v \pi_F)\phi .
$$

Here we have used $\pi_F \phi = \phi$. Therefore

$$
\pi_F(\nabla_v \pi_F)\phi = 0,
$$

which implies that

$$
\langle (\nabla_v \pi_F)\phi, \psi\rangle_{k_+,k_-} = 0.
$$

The proof of the other equality is similar. ∎

2.6 Weighted Sobolev space

Assume that ind $D = 0$ and fix a spectral section P_0 of $-D$. Let $D' = D + \mathbb{A}$ be the perturbed Dirac operator as in Corollary 2.5.6.

From now on, for $k > 0$, we consider the norm defined by

$$\|\phi\|_k = \| |D'|^k \phi\|_0.$$

Note that this norm is equivalent to the original L^2_k-norm since $\ker D' = 0$. That is, there is a constant $C > 1$ such that

$$C^{-1}\|(1 + |D|^k)\phi\|_0 \le \| |D'|^k \phi\|_0 \le C\|(1 + |D|^k)\phi\|_0.$$

Hence we can apply Corollary 2.5.2, Corollary 2.5.4, Proposition 2.5.7 to the Sobolev norms with respect to D'.

Let P_n, Q_n be spectral sections of $-D$, D with

$$(\mathcal{E}_0(D))_{-\infty}^{\mu_{n,-}} \subset P_n \subset (\mathcal{E}_0(D))_{-\infty}^{\mu_{n,+}},$$
$$(\mathcal{E}_0(D))^{\infty}_{\lambda_{n,+}} \subset Q_n \subset (\mathcal{E}_0(D))^{\infty}_{\lambda_{n,-}}.$$

We may suppose that

$$\mu_{n,-} + 10 < \mu_{n,+} < \mu_{n+1,-} - 10,$$
$$\lambda_{n+1,+} + 10 < \lambda_{n,-} < \lambda_{n,+} - 10,$$
$$\mu_{n,+} - \mu_{n,-} < \delta, \quad \lambda_{n,+} - \lambda_{n,-} < \delta$$

for some positive number δ independent of n. See Theorem 2.4.1. By the definition of $D' = D + \mathbb{A}$ in the proof of Corollary 2.5.6, we have

$$\mathcal{E}_0(D)_{-\infty}^{\mu_{n,\pm}} = \mathcal{E}_0(D')_{-\infty}^{\mu_{n,\pm}},$$
$$\mathcal{E}_0(D)^{\infty}_{\lambda_{n,\pm}} = \mathcal{E}_0(D')^{\infty}_{\lambda_{n,\pm}}$$

for $n \gg 0$. Fix half-integers $k_+, k_- > 5$. Put $\ell = \min\{k_+, k_-\}$. Let π_{P_n}, π_{Q_n} be the $L^2_{k_+,k_-}$-projections on P_n, Q_n. By Proposition 2.5.10, we can assume that for each n, there is $C_n > 0$ such that for $v \in TB$ with $\|v\| \le 1$,

$$\|\nabla_v \pi_{P_n} \colon L^2_{k_+,k_-} \to L^2_{\ell+1}\| \le C_n, \quad \|\nabla_v \pi_{Q_n} \colon L^2_{k_+,k_-} \to L^2_{\ell+1}\| \le C_n. \tag{2.6.1}$$

Define a finite-dimensional subbundle F_n of $\mathcal{E}_\infty$ by

$$F_n = P_n \cap Q_n \subset (\mathcal{E}_0)^{\mu_{n,+}}_{\lambda_{n,-}}.$$

We will next introduce weighted Sobolev spaces. Take positive numbers ε_n with

$$C_n \varepsilon_n \le \frac{1}{n}, \tag{2.6.2}$$

where C_n are the constants from (2.6.1). Fix a smooth function

$$w\colon \mathbb{R} \to \mathbb{R}$$

with

$$\begin{aligned} &0 < w(x) \leq 1 && \text{for all } x \in \mathbb{R}, \\ &w(x) = \varepsilon_n && \text{if } x \in [\lambda_{n,-} - 3, \lambda_{n,+} + 3] \cup [\mu_{n,-} - 3, \mu_{n,+} + 3] \text{ for some } n. \end{aligned}$$

Take $a \in \mathcal{H}^1(Y)$. Let $\{e_j\}_j$ be an orthonormal basis of $L^2(\mathbb{S})$ with

$$D'_a e_j = \eta_j e_j,$$

where η_j are the eigenvalues of D'_a.

For a positive number k and $\phi = \sum_j c_j e_j \in C^\infty(\mathbb{S})$, we define a weighted Sobolev norm $\|\phi\|_{a,k,w}$ by

$$\|\phi\|_{a,k,w} := \left(\sum_j |c_j|^2 |\eta_j|^{2k} w(\eta_j)^2 \right)^{\frac{1}{2}}.$$

Denote by $L^2_{a,k,w}(\mathbb{S})$ the completion of $C^\infty(\mathbb{S})$ with respect to $\|\cdot\|_{a,k,w}$. The family $\{\|\cdot\|_{a,k,w}\}_{a \in \mathcal{H}^1(Y)}$ of norms induces a fiberwise norm $\|\cdot\|_{k,w}$ on $\mathcal{E}_\infty$. We denote the completion of $\mathcal{E}_\infty$ with respect to $\|\cdot\|_{k,w}$ by $\mathcal{E}_{k,w}$. Note that

$$\|\phi\|_{k,w} \leq \|\phi\|_k.$$

Proposition 2.6.1. *Let k_+, k_- be half-integers with $k_+, k_- > 5$ and put $\ell = \min\{k_+, k_-\}$. Then*

$$\sup_{v \in B(TB;1)} \|\nabla_v \pi_{P_n} \colon L^2_{k_+,k_-} \to L^2_{\ell-5,w}\| \to 0.$$

A similar statement holds for π_{Q_n}.

Proof. For $\lambda, \mu \in \mathbb{R}$, let π^μ_λ be the L^2-projection to $(\mathcal{E}_0(D'))^\mu_\lambda$. Take $a \in B$ and $v \in T_a B$ with $\|v\| \leq 1$. By Corollary 2.5.4 and Proposition 2.5.7, for $n \gg 0$, we can take

$$\nu_{n,-} \in [\mu_{n,-} - 2, \mu_{n,-} - 1], \quad \nu_{n,+} \in [\mu_{n,+} + 1, \mu_{n,+} + 2]$$

such that

$$\begin{aligned} \|\nabla_v \pi^{\nu_{n,-}}_{-\infty} \colon L^2_{\ell-1} \to L^2_{\ell-5}\| \leq C, \\ \|(\nabla_v \pi^{\nu_{n,+}}_{\nu_{n,-}}) \colon L^2_{\ell-1} \to L^2_{\ell-5}\| \leq C, \end{aligned}$$

where $C > 0$ is a constant independent of n. Note that

$$\begin{aligned}\pi_{P_n} &= \mathrm{id}_{\mathcal{E}_0} \circ \pi_{P_n}\\ &= (\pi_{-\infty}^{\nu_{n,-}} + \pi_{\nu_{n,-}}^{\nu_{n,+}} + \pi_{\nu_{n,+}}^{\infty}) \circ \pi_{P_n}\\ &= \pi_{-\infty}^{\nu_{n,-}} + \pi_{\nu_{n,-}}^{\nu_{n,+}} \circ \pi_{P_n}.\end{aligned}$$

Hence

$$\nabla_v \pi_{P_n} = \nabla_v \pi_{-\infty}^{\nu_{n,-}} + (\nabla_v \pi_{\nu_{n,-}}^{\nu_{n,+}}) \pi_{P_n} + \pi_{\nu_{n,-}}^{\nu_{n,+}} (\nabla_v \pi_{P_n}). \tag{2.6.3}$$

For $\varepsilon > 0$, take a positive number β with $\beta > \frac{1}{\varepsilon}$. Then for any $\phi \in \mathcal{E}_{k_+,k_-}$ with $\|\phi\|_{k_+,k_-} \le 1$, we have

$$\|\pi_\beta^\infty \phi\|_{\ell-1} < \varepsilon.$$

By Proposition 2.5.1 and Corollary 2.5.4, for $n \gg 0$ with $\beta < \nu_{n,-}$,

$$\begin{aligned}\|(\nabla_v \pi_{-\infty}^{\nu_{n,-}})\phi\|_{\ell-5} &= \|(\nabla_v \pi_{-\infty}^{\nu_{n,-}})(\pi_{-\infty}^{\beta}\phi + \pi_\beta^\infty \phi)\|_{\ell-5}\\ &\le C'\Big(\frac{1}{|\beta - \nu_{n,-}|} + \varepsilon\Big).\end{aligned} \tag{2.6.4}$$

Here, $C' > 0$ is independent of n. Similarly,

$$\|(\nabla_v \pi_{\nu_{n,-}}^{\nu_{n,+}}) \pi_{P_n} \phi\|_{\ell-5} \le C''\Big(\frac{1}{\min\{|\beta - \nu_{n,+}|, |\beta - \nu_{n,-}|\}} + \varepsilon\Big) \tag{2.6.5}$$

for $n \gg 0$, where $C'' > 0$ is a constant independent of n. By the definition of the weighted Sobolev norm $\|\cdot\|_{\ell,w}$ and (2.6.2),

$$\|\pi_{\nu_{n,-}}^{\nu_{n,+}} (\nabla_v \pi_{P_n}) \phi\|_{\ell,w} \le C_n \varepsilon_n \|\phi\|_{k_+,k_-} \le \frac{1}{n}. \tag{2.6.6}$$

The statement follows from (2.6.3), (2.6.4), (2.6.5), (2.6.6). ∎

Lemma 2.6.2. *Let K be a compact set in $\mathcal{H}^1(Y)$. There is a norm $\|\cdot\|_{K,k,w}$ on $C^\infty(\mathbb{S})$ such that for any $a \in K$ and $\phi \in C^\infty(\mathbb{S})$ we have*

$$\|\phi\|_{K,k,w} \le \|\phi\|_{a,k,w}.$$

Let $L^2_{K,k,w}$ be the completion of $C^\infty(\mathbb{S})$ with respect to $\|\cdot\|_{K,k,w}$. For $l \ge k$, the natural map $L^2_l \to L^2_{K,k,w}$ is injective.

Proof. Take a compact set K in $\mathcal{H}^1(Y)$ and fix $a_0 \in K$. Choose $a \in K$. Put

$$\begin{aligned}a_t &= (1-t)a_0 + ta,\\ r &= \|a_0 - a\|,\\ \delta &:= \max\big\{\|\nabla_v D' \colon L^2 \to L^2\| : t \in [0,1],\ v \in T_{a_t}\mathcal{H}^1(Y),\ \|v\| = 1\big\}.\end{aligned}$$

Let $\tilde{\mathcal{E}}_0$ be the trivial bundle $\mathcal{H}^1(Y) \times L^2(\mathbb{S})$ over $\mathcal{H}^1(Y)$, which is the pullback of $\mathcal{E}_0$ by the projection $\mathcal{H}^1(Y) \to B$. Also take a sequence $\{\lambda_l\}_{l=-\infty}^{\infty}$ of real numbers with

$$\lambda_l + r\delta \ll \lambda_{l+1}.$$

We will prove that for each l, there is a constant $c_l(a) > 0$ such that for $\phi \in \tilde{\mathcal{E}}_0(D'_{a_0})_{\lambda_l}^{\lambda_{l+1}}$, we have

$$c_l(a)\|\phi\|_0 \leq \|(\pi_a)_{\lambda_l - r\delta}^{\lambda_{l+1}+r\delta}\phi\|_0. \tag{2.6.7}$$

Fix an integer l. We consider the following set:

$$I = \Big\{t \in [0,1] : \forall s \in [0,1],\ s \leq t,\ \exists c(s) > 0,\ \forall \phi \in \tilde{\mathcal{E}}_0(D'_{a_0})_{\lambda_l}^{\lambda_{l+1}},$$
$$c(s)\|\phi\|_0 \leq \|(\pi_{a_s})_{\lambda_l - sr\delta}^{\lambda_{l+1}+sr\delta}\phi\|_0\Big\}.$$

Note that $0 \in I$. To prove (2.6.7), it is sufficient to show that $\sup I = 1$. Put $t_0 = \sup I$ and assume that $t_0 < 1$.

Then take $t_+ \in (t_0, 1]$ with $|t_+ - t_0|$ sufficiently small. For $t \in [t_0, t_+]$, let

$$\nu_1(t), \dots, \nu_m(t)$$

be the eigenvalues of D'_{a_t} which are continuous in t such that

$$\lambda_l - t_0 r\delta < \nu_1(t_0), \nu_2(t_0), \dots, \nu_m(t_0) \leq \lambda_{l+1} + t_0 r\delta,$$
$$\dim \tilde{\mathcal{E}}_0(D'_{a_t})_{\lambda_l - t_0 r\delta}^{\lambda_{l+1} t_0 r\delta} = m.$$

Take real numbers λ_-, λ_+ sufficiently close to $\lambda_l - t_0 r\delta$, $\lambda_{l+1} + t_0 r\delta$, which are not eigenvalues of D'_{a_t} for $t \in [t_0, t_+]$, such that

$$\tilde{\mathcal{E}}_0(D'_{a_{t_0}})_{\lambda_-}^{\lambda_+} = \tilde{\mathcal{E}}_0(D'_{a_{t_0}})_{\lambda_l - t_0 r\delta}^{\lambda_{l+1}+t_0 r\delta}.$$

By [22, Theorem 4.10, p. 291], for $t \in [t_0, t_+]$,

$$\lambda_l - tr\delta < \nu_1(t), \dots, \nu_m(t) \leq \lambda_{l+1} + tr\delta$$

which implies that

$$\tilde{\mathcal{E}}_0(D'_{a_t})_{\lambda_-}^{\lambda_+} = \tilde{\mathcal{E}}_0(D'_{a_t})_{\lambda_l - tr\delta}^{\lambda_{l+1}+tr\delta}.$$

So we have

$$\|(\pi_{a_t})_{\lambda_-}^{\lambda_+}\phi\|_0 = \|(\pi_{a_t})_{\lambda_l - tr\delta}^{\lambda_{l+1}+tr\delta}\phi\|_0.$$

From the equality

$$\frac{d}{dt}\|(\pi_{a_t})_{\lambda_-}^{\lambda_+}\phi\|_0^2 = 2\,\mathrm{Re}\langle(\nabla_v(\pi_{a_t})_{\lambda_-}^{\lambda_+})\phi, \phi\rangle_0,$$

for $t \in [t_0, t_+]$ and $\phi \in \tilde{\mathscr{E}}_0(D'_{a_{t_-}})^{\lambda_{l+1}+t_-r\delta}_{\lambda_l-t_-r\delta}$, we have

$$\{1 - 2M(t - t_0)\}\|\phi\|_0 \leq \|(\pi_{a_t})^{\lambda_+}_{\lambda_-}\phi\|_0 = \|(\pi_{a_t})^{\lambda_{l+1}+tr\delta}_{\lambda_l-tr\delta}\phi\|_0,$$

where

$$M = \max\bigl\{\|\nabla_v(\pi_t)^{\lambda_+}_{\lambda_-} \colon L^2 \to L^2\| : t \in [t_0, t_+]\bigr\}$$

and $v = a - a_0$. Taking t_+ sufficiently close to t_0, we have

$$2M|t_+ - t_0| < 1.$$

This implies that

$$t_+ \in I$$

and we get a contradiction. We have obtained (2.6.7).

Take a sufficiently small open neighborhood $U_{l,a}$ of a in $\mathscr{H}^1(Y)$. Then for all $a' \in U_{l,a}$ we have

$$\frac{1}{2}c_l(a)\|\phi\|_0 \leq \|(\pi_{a'})^{\lambda_{l+1}+r\delta+1}_{\lambda_l-r\delta-1}\phi\|_0$$

for $\phi \in \tilde{\mathscr{E}}_0(D'_{a_0})^{\lambda_{l+1}}_{\lambda_l}$. Since K is compact, there exist $a_{l,1}, \dots, a_{l,N_l} \in K$ such that

$$K \subset U_{l,a_1} \cup \cdots \cup U_{l,a_{N_l}}.$$

Take a small positive number $\varepsilon > 0$ such that there are no eigenvalues of D'_a in $[-\varepsilon, \varepsilon]$ for $a \in K$. Put

$$\begin{aligned} c_l &= \min\bigl\{c_l(a_{l,1}), \dots, c_l(a_{l,N_l})\bigr\}, \\ \underline{w}(l) &:= \min\bigl\{|x|^k w(x) : x \notin [-\varepsilon, \varepsilon],\ x \in [\lambda_{l-1}, \lambda_{l+2}]\bigr\}. \end{aligned}$$

For $\phi \in C^\infty(\mathbb{S})$, define

$$\|\phi\|_{K,k,w} = \left\{\sum_l \left(\frac{1}{10}c_l\underline{w}(l)\|(\pi_{a_0})^{\lambda_{l+1}}_{\lambda_l}\phi\|_0\right)^2\right\}^{\frac{1}{2}}. \tag{2.6.8}$$

Then

$$\|\phi\|_{K,k,w} \leq \|\phi\|_{a,k,w}$$

for all $a \in K$ and $\phi \in C^\infty(\mathbb{S})$. From definition (2.6.8) of $\|\cdot\|_{K,k,w}$, we have that the natural map $L^2_l \to L^2_{K,k,w}$ is injective for $l \geq k$. ∎

Proposition 2.6.3. *Let W be a closed, oriented, smooth manifold and E be a vector bundle on W. Let k be a positive number with $k \geq 1$, I be a compact interval in $\mathbb{R}$ and $\|\cdot\|$ be any norm on $C^\infty(E)$ such that $\|\phi\| \leq \|\phi\|_{k-1}$ for all $\phi \in C^\infty(E)$. Assume that the natural map $L^2_l(E) \to \overline{C^\infty(E)}$ is injective for $l \geq k-1$. Here, $\overline{C^\infty(E)}$ is* *the completion with respect to the norm $\|\cdot\|$. We consider $L^2_l(E)$ to be a subspace of $\overline{C^\infty(E)}$ through this map.*

Suppose that we have a sequence $\gamma_n\colon I \to C^\infty(E)$ such that γ_n are equicontinuous in $\|\cdot\|$ and uniformly bounded in L^2_k. Then after passing to a subsequence, γ_n converges uniformly in L^2_{k-1} to a continuous

$$\gamma\colon I \to L^2_{k-1}(E).$$

Proof. Let $q_1, q_2, \ldots,$ be the rational numbers in I. Since γ_n are uniformly bounded in L^2_k, it follows from the Rellich lemma and the diagonal argument that there is a subsequence $n(i)$ such that $\gamma_{n(i)}(q_m)$ converges in L^2_{k-1} (and hence in $\|\cdot\|$) as $i \to \infty$ for each m. Since γ_n are equicontinuous in $\|\cdot\|$, for any $\varepsilon > 0$ and $t \in I$, we can find q_m which is independent of i, with

$$\|\gamma_{n(i)}(t) - \gamma_{n(i)}(q_m)\| < \varepsilon.$$

So we have, for any t,

$$\begin{aligned}
&\|\gamma_{n(i)}(t) - \gamma_{n(j)}(t)\| \\
&\qquad \le \|\gamma_{n(i)}(t) - \gamma_{n(i)}(q_m)\| + \|\gamma_{n(i)}(q_m) - \gamma_{n(j)}(q_m)\| + \|\gamma_{n(j)}(q_m) - \gamma_{n(j)}(t)\| \\
&\qquad \le \|\gamma_{n(i)}(q_m) - \gamma_{n(j)}(q_m)\| + 2\varepsilon.
\end{aligned}$$

This implies that for each $t \in I$, $\gamma_{n(i)}(t)$ is a Cauchy sequence in $\|\cdot\|$, and hence $\gamma_{n(i)}$ has a pointwise limit $\gamma\colon I \to \overline{C^\infty(E)}$, where $\overline{C^\infty(E)}$ is the completion with respect to $\|\cdot\|$.

Since γ_n are equicontinuous in $\|\cdot\|$, for any $\varepsilon > 0$ there is $\delta > 0$ such that for $t, t' \in I$ with $|t - t'| < \delta$ we have $\|\gamma_n(t) - \gamma_n(t')\| < \varepsilon$. Taking the limit, we have $\|\gamma(t) - \gamma(t')\| \le \varepsilon$. We can choose finitely many rational numbers $q_1, \ldots, q_N$ in I such that for all $t \in I$ there is q_l with $l \in \{1, \ldots, N\}$ such that $|t - q_l| < \delta$. If i_0 is large enough, for $i > i_0$ we have $\|\gamma_{n(i)}(q_m) - \gamma(q_m)\| < \varepsilon$ for all $m \in \{1, \ldots, N\}$. Therefore, for $i > i_0$,

$$\begin{aligned}
\|\gamma_{n(i)}(t) - \gamma(t)\| &\le \|\gamma_{n(i)}(t) - \gamma_{n(i)}(q_l)\| + \|\gamma_{n(i)}(q_l) - \gamma(q_l)\| + \|\gamma(q_l) - \gamma(t)\| \\
&< 3\varepsilon.
\end{aligned}$$

Hence $\gamma_{n(i)}$ converges uniformly to γ in $\|\cdot\|$.

We first show that the limit γ defined above in fact lies in $L^2_{k-\frac{1}{2}}$. Indeed, for any fixed t_∞ and any sequence $t_i \to t_\infty$ in I, we have that $\gamma_{n(i)}(t_i)$ converges, in $(k - \frac{1}{2})$-norm, after extracting a subsequence, to some δ. However, as above, $\gamma_{n(i)}(t_i)$ also converges in $\|\cdot\|$-norm to $\gamma(t_\infty)$. Recall that $L^2_{k-\frac{1}{2}}$ is a subspace of $\overline{C^\infty(E)}$, so $\delta \in \overline{C^\infty(E)}$, and we have

$$\begin{aligned}
\|\gamma(t_\infty) - \delta\| &\le \|\gamma(t_\infty) - \gamma_{n(i)}(t_i)\| + \|\gamma_{n(i)}(t_i) - \delta\| \\
&\le \|\gamma(t_\infty) - \gamma_{n(i)}(t_i)\| + \|\gamma_{n(i)}(t_i) - \delta\|_{k-\frac{1}{2}}.
\end{aligned}$$

It follows that $\delta = \gamma(t_\infty)$. This establishes that γ is defined as a function $I \to L^2_{k-\frac{1}{2}}$, but not that it is continuous, nor that the $\{\gamma_{n(i)}\}$ converges pointwise in $(k-\frac{1}{2})$-norm. Note that, since $\|\gamma_n(t)\|_k \leq C$ for a positive constant C independent of n, t by assumption, we have $\|\gamma(t)\|_{k-\frac{1}{2}} \leq C$ for all $t \in I$.

Assume that $\gamma_{n(i)}$ does not converge uniformly in L^2_{k-1}. Then after passing to a subsequence, there is $\varepsilon_0 > 0$ such that for any i we have $t_i \in I$ with

$$\|\gamma_{n(i)}(t_i) - \gamma(t_i)\|_{k-1} \geq \varepsilon_0.$$

After passing to a subsequence, t_i converges to some $t_\infty \in I$. Then $\gamma_{n(i)}(t_i)$ converges to $\gamma(t_\infty)$ in $\|\cdot\|$. Since $\gamma_{n(i)}(t_i)$ are uniformly bounded in L^2_k, by the Rellich lemma, after passing to a subsequence $\gamma_{n(i)}(t_i)$ converges to some δ in L^2_{k-1}; by the argument to show that $\gamma(t_\infty) \in L^2_{k-\frac{1}{2}}$ above, we see that $\delta = \gamma(t_\infty)$. Similarly, since $\|\gamma(t_i)\|_{k-\frac{1}{2}} \leq C$ for all i, after passing to a subsequence, $\gamma(t_i)$ converges to some δ' in L^2_{k-1}. Since $\gamma(t_i) \to \gamma(t_\infty)$ in $\overline{C^\infty(E)}$, the previous argument gives that $\delta' = \gamma(t_\infty)$.

Therefore, after passing to a subsequence,

$$\|\gamma_{n(i)}(t_i) - \gamma(t_i)\|_{k-1} \to 0$$

as $i \to \infty$. This is a contradiction. Thus $\gamma_{n(i)}$ converges to γ in L^2_{k-1} uniformly. Since the convergence is uniform in L^2_{k-1}, γ is continuous in L^2_{k-1}. ∎

2.7 Proof of Theorem 2.3.3

Take half-integers k_+, k_- with $k_+, k_- > 5$ and with $|k_+ - k_-| \leq \frac{1}{2}$. We put $\ell = \min\{k_+, k_-\}$ and

$$A_n := (B_{k_+}(F_n^+; R) \times_B B_{k_-}(F_n^-; R)) \times_B (B_{k_+}(W_n^+; R) \times_B B_{k_-}(W_n^-; R)).$$

We want to prove that A_n are isolating neighborhoods for $\varphi_{n,k_+,k_-} = \varphi_n$ for n large. If this is not true, after passing to a subsequence,

$$\operatorname{inv} A_n \cap \partial A_n \neq \emptyset$$

for all n. Then we can take

$$y_{n,0} = (\phi_{n,0}, \omega_{n,0}) \in \operatorname{inv} A_n \cap \partial A_n.$$

After passing to a subsequence, we may suppose that one of the following cases holds for all n:

(i) $\phi^+_{n,0} \in S_{k_+}(F_n^+; R)$

(ii) $\phi^-_{n,0} \in S_{k_-}(F_n^-; R)$,

(iii) $\omega_{n,0}^+ \in S_{k_+}(W_n^+; R)$,

(iv) $\omega_{n,0}^- \in S_{k_-}(W_n^-; R)$.

Let $\gamma_n = (\phi_n, \omega_n)\colon \mathbb{R} \to F_n \oplus W_n$ be the solution to (2.3.10) with $\gamma_n(0) = y_{n,0}$:

$$\begin{aligned}
\Big(\frac{d\phi_n}{dt}(t)\Big)_V &= -(\nabla_{X_H}\pi_{F_n})\phi_n(t) - \pi_{F_n}\big(D\phi_n(t) + c_1(\gamma_n(t))\big),\\
\Big(\frac{d\phi_n}{dt}(t)\Big)_H &= -X_H(\phi_n(t)),\\
\frac{d\omega_n}{dt}(t) &= - * d\omega_n(t) - \pi_{W_n} c_2(\gamma_n(t)).
\end{aligned}\tag{2.7.1}$$

We have

$$\|\phi_n^+(t)\|_{k_+} \le R, \quad \|\phi_n^-(t)\|_{k_-} \le R, \quad \|\omega_n^+(t)\|_{k_+} \le R, \quad \|\omega_n^-(t)\|_{k_-} \le R \tag{2.7.2}$$

for all $t \in \mathbb{R}$. By the Sobolev multiplication theorem,

$$\begin{aligned}
\|c_1(\gamma_n(t))\|_\ell &\le C\|\gamma_n(t)\|_\ell^2 \le CR^2,\\
\|c_2(\gamma_n(t))\|_\ell &\le C\|\gamma_n(t)\|_\ell^2 \le CR^2,\\
\|X_H(\phi(t))\|_\ell &\le C\|\gamma_n(t)\|_\ell^2 \le CR^2.
\end{aligned}$$

Let $\Delta \subset \mathcal{H}^1(Y)$ be a fundamental domain of the action of $H^1(Y;\mathbb{Z})$ on $\mathcal{H}^1(Y)$, which is a bounded set. By the path lifting property of the covering space $\mathcal{H}^1(Y) \times L^2_{k_+,k_-}(\mathbb{S}) \to \mathcal{E}_{k_+,k_-}$, we have a lift

$$\tilde{\gamma}_n = (\tilde{\phi}_n, \omega_n)\colon \mathbb{R} \to \mathcal{H}^1(Y) \times L^2_{k_+,k_-}(\mathbb{S}) \times L^2_{k_+,k_-}(\operatorname{im} d^*)$$

of γ_n with

$$p_{\mathcal{H}}(\tilde{\gamma}_n(0)) \in \Delta. \tag{2.7.3}$$

By (2.7.1), we have

$$\Big\|\Big(\frac{d\phi_n}{dt}(t)\Big)_H\Big\| \le CR^2. \tag{2.7.4}$$

Fix $T > 0$. It follows from (2.7.3) and (2.7.4) that we can take a compact set K_T of $\mathcal{H}^1(Y)$ such that for any n and $t \in [-T, T]$ we have

$$p_{\mathcal{H}}(\tilde{\gamma}_n(t)) \in K_T.$$

Note that $\frac{d\tilde{\phi}_n}{dt}$ is uniformly bounded on $[-T, T]$ in $\|\cdot\|_{K_T,\ell-5,w}$ by (2.7.1), Proposition 2.6.1 and Lemma 2.6.2, which implies that $\tilde{\phi}_n$ are equicontinuous in $L^2_{K_T,\ell-5,w}$ on $[-T, T]$. The ω_n are also equicontinuous in $L^2_{\ell-1}$. By Proposition 2.6.3, after passing to a subsequence, $\tilde{\gamma}_n|_{[-T,T]}$ converges to a map

$$\tilde{\gamma}^{(T)} = (\tilde{\phi}^{(T)}, \omega^{(T)})\colon [-T, T] \to \mathcal{H}^1(Y) \times L^2_{\ell-1}(\mathbb{S}) \times L^2_{\ell-1}(\operatorname{im} d^*)$$

uniformly in $L^2_{\ell-1}$. By the diagonal argument, we can show that there is a continuous map

$$\tilde{\gamma} = (\tilde{\phi}, \omega) : \mathbb{R} \to \mathcal{H}^1(Y) \times L^2_{\ell-1}(\mathbb{S}) \times L^2_{\ell-1}(\operatorname{im} d^*)$$

such that, after passing to a subsequence, $\tilde{\gamma}_n$ converges to $\tilde{\gamma}$ uniformly in $L^2_{\ell-1}$ on each compact set in $\mathbb{R}$.

Lemma 2.7.1. *The limit $\tilde{\gamma}$ is a solution to the Seiberg–Witten equations over $Y \times \mathbb{R}$.*

Proof. Fix $T > 0$. For $t \in [-T, T]$, we have

$$\begin{aligned}
&\tilde{\phi}_n(t) - \tilde{\phi}_n(0) \\
&\quad = \int_0^t \frac{d\tilde{\phi}_n}{ds}(s)\, ds \\
&\quad = -\int_0^t (\nabla_{X_H} \pi_{\tilde{F}_n})\tilde{\phi}_n(s) + \pi_{\tilde{F}_n}\big(D\tilde{\phi}_n(t) + c_1(\tilde{\gamma}_n(t))\big) + X_H(\phi_n(s))\, ds. \quad (2.7.5)
\end{aligned}$$

We have that $p_{\mathcal{H}}(\tilde{\gamma}_n(t)) \in K_T$ for any n and $t \in [-T, T]$. Note that we have no estimate on $(\nabla_{X_H} \pi_{F_n})\tilde{\phi}_n$ in *any* L^2_j-norm and that we just have control on it in the auxiliary space $L^2_{K_T,\ell-5,w}$. By Proposition 2.6.1 and Lemma 2.6.2,

$$(\nabla_{X_H} \pi_{\tilde{F}_n})\tilde{\phi}_n(s) \to 0$$

uniformly in $L^2_{K_T,\ell-5,w}$ as $n \to \infty$. Recall that $\tilde{\phi}_n$, ω_n converge in $L^2_{\ell-1}$ uniformly on $[-T, T]$. It follows from Proposition 2.4.2 and the inequality

$$\begin{aligned}
\|\pi_{F_n} D\tilde{\phi}_n - D\tilde{\phi}\|_{\ell-2} &= \|\pi_{F_n} D\tilde{\phi}_n - D\tilde{\phi}_n + D\tilde{\phi}_n - D\tilde{\phi}\|_{\ell-2} \\
&\le \|[\pi_{F_n}, D]\tilde{\phi}_n\|_{\ell-2} + \|D\tilde{\phi}_n - D\tilde{\phi}\|_{\ell-2}
\end{aligned}$$

that $\pi_{F_n} D\tilde{\phi}_n$ converges to $D\tilde{\phi}$ uniformly in $L^2_{\ell-2}$ on $[-T, T]$.

Taking the limit with $n \to \infty$ in (2.7.5), we obtain

$$\tilde{\phi}(t) - \tilde{\phi}(0) = -\int_0^t \big(D\tilde{\gamma}(t) + c_1(\tilde{\gamma}(t))\big) + X_H(\tilde{\phi}(s))\, ds.$$

Hence, by the fundamental theorem of calculus,

$$\frac{d\tilde{\phi}}{dt}(t) = -\big(D\tilde{\phi}(t) + c_1(\tilde{\gamma}(t))\big) - X_H(\tilde{\phi}(t)).$$

A priori, the left-hand side $\frac{d\tilde{\phi}}{dt}(t)$ only lives in the auxiliary space $L^2_{K_T,\ell-5,w}$. However, since $L^2_{\ell-2}$ is a subspace of $L^2_{K_T,\ell-2,w}$ and the right-hand side is in $L^2_{\ell-2}$, $\frac{d\tilde{\phi}}{dt}(t)$ is in $L^2_{\ell-2}$ and both sides are equal to each other as elements of $L^2_{\ell-2}$.

Similarly, we can show that

$$\frac{d\omega}{dt}(t) = - * d\omega(t) - c_2(\tilde{\gamma}(t)).$$

Therefore $\tilde{\gamma}$ is a solution to the Seiberg–Witten equations (2.3.4) and the ordinary theory of elliptic regularity shows that $\tilde{\gamma}$ is in C^∞ as a section on any compact set in $Y \times (-T, T)$. ∎

Composing $\tilde{\gamma}\colon \mathbb{R} \to \mathcal{H}^1(Y) \times L^2_{\ell-1}(\mathbb{S}) \times L^2_{\ell-1}(\operatorname{im} d^*)$ with the projection

$$\mathcal{H}^1(Y) \times L^2_{\ell-1}(\mathbb{S}) \times L^2_{\ell-1}(\operatorname{im} d^*) \to \mathcal{E}_{\ell-1} \oplus \mathcal{W}_{\ell-1},$$

we get a Seiberg–Witten trajectory

$$\gamma\colon \mathbb{R} \to \mathcal{E}_{\ell-1} \oplus \mathcal{W}_{\ell-1}.$$

Since $\|\gamma(t)\|_{\ell-1} \le R$ for all $t \in \mathbb{R}$, γ has finite energy. By Proposition 2.3.2,

$$\|\gamma(t)\|_{k_+,k_-} \le R_{k_+,k_-}, \tag{2.7.6}$$

for all $t \in \mathbb{R}$.

Assume that case (i) holds for all n. We have

$$\|\phi_n^+(0)\|_{k_+} = R.$$

Lemma 2.7.2. *There is a constant $C > 0$ such that for all n,*

$$\|\phi_n^+(0)\|_{k_+ + \frac{1}{2}} < C.$$

Proof. Note that

$$\frac{d}{dt}\Big|_{t=0} \|\phi_n^+(t)\|^2_{k_+} = 0.$$

Let us consider the case when $k_+ \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.

Let π^+ be the $L^2_{k_+,k_-}$-projection onto $\mathcal{E}^+_{k_+,k_-}$. (That is, $\pi^+ = 1 - \pi_{P_0}$.) Then we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big|_{t=0} \|\phi_n^+(t)\|^2_{k_+} &= \frac{1}{2}\frac{d}{dt}\Big|_{t=0} \langle |D'|^{k_+ + \frac{1}{2}} \pi^+ \phi_n(t), |D'|^{k_+ - \frac{1}{2}} \pi^+ \phi_n(t) \rangle_0 \\
&= \langle (\nabla_{X_H} |D'|^{k_+ + \frac{1}{2}}) \phi_n^+(0), |D'|^{k_+ - \frac{1}{2}} \phi_n^+(0) \rangle_0 \\
&\quad + \langle |D'|^{k_+ + \frac{1}{2}} \phi_n^+(0), (\nabla_{X_H} |D'|^{k_+ - \frac{1}{2}}) \phi_n^+(0) \rangle_0 \\
&\quad + \operatorname{Re}\langle (\nabla_{X_H} \pi^+) \phi_n(0), \phi_n^+(0) \rangle_{k_+} + \operatorname{Re}\Big\langle \frac{d\phi_n}{dt}(0), \phi_n^+(0) \Big\rangle_{k_+}.
\end{aligned}$$

Note that $k_+ + \frac{1}{2}$ and $k_+ - \frac{1}{2}$ are integers. By Lemma 2.5.8,

$$\begin{aligned}
\left|\langle (\nabla_{X_H} |D'|^{k_+ + \frac{1}{2}})\phi_n^+(0), |D'|^{k_+ - \frac{1}{2}}\phi_n^+(0)\rangle_0\right| &\le C\|\phi_n^+(0)\|^2_{k_+ - \frac{1}{2}} \le CR^2, \\
\left|\langle |D'|^{k_+ + \frac{1}{2}}\phi_n^+(0), (\nabla_{X_H} |D'|^{k_+ - \frac{1}{2}})\phi_n^+(0)\rangle_0\right| &\le C\|\phi_n^+(0)\|_{k_+ + \frac{1}{2}} \|\phi_n^+(0)\|_{k_+ - \frac{1}{2}} \\
&\le CR\|\phi_n^+(0)\|_{k_+ + \frac{1}{2}}.
\end{aligned}$$

By Proposition 2.5.10,

$$\begin{aligned}
\left|\langle (\nabla_{X_H} \pi^+)\phi_n(0), \phi_n^+(0)\rangle_{k_+}\right| &\le \|(\nabla_{X_H}\pi^+)\phi_n(0)\|_{k_+}\|\phi_n^+(0)\|_{k_+} \\
&\le C\|\phi_n(0)\|_{k_+ - 1}\|\phi_n^+(0)\|_{k_+} \\
&\le C\|\phi_n(0)\|_{\ell}\|\phi_n^+(0)\|_{k_+} \\
&\le CR^2.
\end{aligned}$$

We have

$$\begin{aligned}
&\left\langle \frac{d\phi_n}{dt}(0), \phi_n^+(0)\right\rangle_{k_+} \\
&\qquad = -\left\langle (\nabla_{X_H}\pi_{F_n})\phi_n(0) + \pi_{F_n}\left(D'\phi_n(0) - \mathbb{A}\phi_n(0) + c_1(\gamma_n(0))\right), \phi_n^+(0)\right\rangle_{k_+}.
\end{aligned}$$

By Lemma 2.5.11,

$$\langle (\nabla_{X_H}\pi_{F_n})\phi_n(0), \phi_n^+(0)\rangle_{k_+} = \langle (\nabla_{X_H}\pi_{F_n})\phi_n(0), \phi_n^+(0)\rangle_{k_+, k_-} = 0.$$

We have

$$\begin{aligned}
\langle \pi_{F_n} D'\phi_n(0), \phi_n^+(0)\rangle_{k_+} &= \langle D'\phi_n(0), \pi_{F_n}\phi_n^+(0)\rangle_{k_+} \\
&= \langle D'\phi_n(0), \phi_n^+(0)\rangle_{k_+} \\
&= \|\phi_n^+(0)\|^2_{k_+ + \frac{1}{2}}.
\end{aligned}$$

Since $\mathbb{A}$ is a smoothing operator,

$$|\langle \pi_{F_n}\mathbb{A}\phi_n(0), \phi_n^+(0)\rangle_{k_+}| \le C\|\phi_n(0)\|_0\|\phi_n(0)\|_{k_+} \le CR^2.$$

Since D' is self-adjoint,

$$\begin{aligned}
|\langle \pi_{F_n} c_1(\gamma_n(0)), \phi_n^+(0)\rangle_{k_+}| &= |\langle c_1(\gamma_n(0)), \phi_n^+(0)\rangle_{k_+}| \\
&= |\langle |D'|^{k_+} c_1(\gamma_n(0)), |D'|^{k_+}\phi_n^+(0)\rangle_0| \\
&= |\langle |D'|^{k_+ - \frac{1}{2}} c_1(\gamma_n(0)), |D'|^{k_+ + \frac{1}{2}}\phi_n^+(0)\rangle_0| \\
&\le \|c_1(\gamma_n(0))\|_{k_+ - \frac{1}{2}}\|\phi_n^+(0)\|_{k_+ + \frac{1}{2}} \\
&\le C\|c_1(\gamma_n(0))\|_{\ell}\|\phi_n^+(0)\|_{k_+ + \frac{1}{2}} \quad (\ell = \min\{k_+, k_-\}) \\
&\le CR^2\|\phi_n^+(0)\|_{k_+ + \frac{1}{2}}.
\end{aligned}$$

Therefore

$$0=\frac{1}{2}\frac{d}{dt}\Big|_{t=0}\|\phi_n^+(t)\|_{k_+}^2 \leq -\|\phi_n^+(0)\|_{k_++\frac{1}{2}}^2 + CR^2\|\phi_n^+(0)\|_{k_++\frac{1}{2}} + CR^2.$$

This inequality implies that the sequence $\|\phi_n^+(0)\|_{k_++\frac{1}{2}}$ is bounded.

The proof in the case $k_+ \in \mathbb{Z}$ is similar. ∎

It follows from Lemma 2.7.2 and the Rellich lemma that after passing to a subsequence, $\phi_n^+(0)$ converges to $\phi^+(0)$ in $L^2_{k_+}$ strongly. By the assumption, $\|\phi_n^+(0)\|_{k_+} = R$ for all n. Hence,

$$\|\gamma(0)\|_{k_+,k_-} \geq \|\phi^+(0)\|_{k_+,k_-} = R.$$

This contradicts (2.7.6).

Let us consider case (ii). In this case, we have

$$\|\phi_n^-(0)\|_{k_-} = R.$$

Lemma 2.7.3. *There is a constant $C > 0$ such that for all n,*

$$\|\phi_n^-(0)\|_{k_-+\frac{1}{2}} < C.$$

Proof. Note that

$$\langle D'\phi_n(0), \phi_n^-(0)\rangle_{k_-} = -\|\phi_n^-(0)\|_{k_-+\frac{1}{2}}^2.$$

As in the proof of Lemma 2.7.2, we can show that

$$0=\frac{d}{dt}\Big|_{t=0}\|\phi_n^-(t)\|_{k_-}^2 \geq \|\phi_n^-(0)\|_{k_-+\frac{1}{2}}^2 - CR^2\|\phi_n^+(0)\|_{k_-+\frac{1}{2}} - CR^2.$$

This implies that the sequence $\|\phi_n^-(0)\|_{k_-+\frac{1}{2}}$ is bounded. ∎

By the Rellich lemma, $\phi_n^-(0)$ converges to $\phi^-(0)$ in $L^2_{k_-}$ strongly. Hence

$$\|\gamma(0)\|_{k_+,k_-} \geq \|\phi^-(0)\|_{k_-} = R.$$

We get a contradiction.

In the other cases (iii), (iv) where $y_{n,0}$ is in the other components of ∂A_n, we similarly have a contradiction.

Definition 2.7.4. For this definition we refer to some notions from parameterized homotopy theory and parameterized Conley index theory; refer to Sections A.1 and A.2, respectively. For notation as in Theorem 2.3.3, let $\mathcal{SWF}_{[n]}(Y,\mathfrak{s})$ be the parameterized Conley index of the flow φ_{n,k_+,k_-} on the isolated invariant set A_n. We call $\mathcal{SWF}_{[n]}(Y,\mathfrak{s})$ the *pre-Seiberg–Witten Floer invariant* of $(Y,\mathfrak{s})$ (for short, the pre-SWF invariant of $(Y, \mathfrak{s})$). The object $\mathcal{SWF}_{[n]}(Y, \mathfrak{s})$ is an (equivariant) topological space, depending on a number of choices (which are not all reflected in its notation). First, $\mathcal{SWF}_{[n]}(Y, \mathfrak{s})$ depends on the choice of an index pair, but its (equivariant, parameterized) homotopy type is independent of the choice of index pair – we will abuse notation and also write $\mathcal{SWF}_{[n]}(Y, \mathfrak{s})$ for its (equivariant, parameterized) homotopy type. It also depends on a choice of metric on Y, as well as spectral sections P_n, Q_n and subspaces $W_n^{\pm}$, as in the preliminaries to Theorem 2.3.3.

The projection used in the parameterized Conley index is from the ex-space $B_{n,R}$ over $\mathrm{Pic}(Y)$, as explained in the discussion after Theorem 2.3.3.

We write $\mathcal{SWF}^u_{[n]}(Y, \mathfrak{s})$ to refer to the Conley index with trivial parameterization. By Lemma A.2.7, $\nu_! \mathcal{SWF}_{[n]}(Y, \mathfrak{s}) = \mathcal{SWF}^u_{[n]}(Y, \mathfrak{s})$, where $\nu\colon B \to *$ is the map collapsing the Picard torus to a point, and $\nu_!$ is as defined in Appendix A.1.

If $\mathfrak{s}$ is a self-conjugate spin^c structure, the bundle $L^2_k(\mathbb{S}) \times \mathcal{H}^1(Y) \times L^2_k(\operatorname{im} d^*)$ admits a Pin(2)-action extending the S^1-action on spinors, by

$$j(\phi, v, \omega) = (j\phi, -v, -\omega).$$

In the event that the spectral sections P_n, Q_n are preserved by the Pin(2)-action, then the approximate flow on $F_n \oplus W_n$ will be Pin(2)-equivariant, and we define $\mathcal{SWF}^{\mathrm{Pin}(2)}_{[n]}(Y, \mathfrak{s})$ to be the Pin(2)-equivariant parameterized Conley index, so that its underlying S^1-space is $\mathcal{SWF}_{[n]}(Y, \mathfrak{s})$. We similarly define $\mathcal{SWF}^{u,\mathrm{Pin}(2)}_{[n]}(Y, \mathfrak{s})$ (and we will occasionally write $\mathcal{SWF}^{u,S^1}_{[n]}(Y, \mathfrak{s})$ to distinguish what equivariance is meant). See Theorem 2.4.8 for the existence of Pin(2)-equivariant spectral sections.

Chapter 3

Well-definedness

Here we show how changing the choices in the construction above affect the resulting space output.

3.1 Variation of approximations

First, we consider the change due to passing between different approximations. For this section, we fix a 3-manifold with spin^c structure $(Y, \mathfrak{s})$.

As before, let P_n, Q_n be spectral sections of $-D$, D with

$$\begin{aligned}(\mathcal{E}_0(D))_{-\infty}^{\mu_{n,-}} \subset P_n \subset (\mathcal{E}_0(D))_{-\infty}^{\mu_{n,+}},\\ (\mathcal{E}_0(D))_{\lambda_{n,+}}^{\infty} \subset Q_n \subset (\mathcal{E}_0(D))_{\lambda_{n,-}}^{\infty}.\end{aligned}$$

We may assume that $|\mu_{n,+} - \mu_{n,-}|$ and $|\lambda_{n,+} - \lambda_{n,-}|$ are bounded. We call any such sequence of spectral sections a *good sequence of spectral sections*.

Fix half-integers $k_+, k_- > 5$. Put $\ell = \min\{k_+, k_-\}$.

Let $F_n = P_n \cap Q_n \subset (\mathcal{E}_0)_{\lambda_{n,-}}^{\mu_{n,+}}$, as before. Fix $\mathbb{H}$ to be the quaternion representation of Pin(2), and let $B = \mathrm{Pic}(Y)$ denote the Picard torus of Y. We write $I(\varphi, S)$ for the (parameterized) Conley index of a flow φ and isolated invariant set S; we will usually suppress S from the notation, and $I^u(\varphi, S)$ for the unparameterized version; see Appendix A.2. Finally, a further bit of notation for the statement of the following theorem. Let $\mathrm{Th}(E, Z)$, for a vector bundle $\pi\colon E \to Z$, denote the Thom construction of π.

Theorem 3.1.1. *Let $\eta_n^P\colon P_{n+1} \to P_n \oplus \mathbb{C}^{k_{P,n}}$ and $\eta_n^Q\colon Q_{n+1} \to Q_n \oplus \mathbb{C}^{k_{Q,n}}$ be vector-bundle isometries (with respect to the $k_\pm$-metric), where $\mathbb{C}^{k_{P,n}}$ and $\mathbb{C}^{k_{Q,n}}$ are the trivial bundles over B of rank $k_{P,n}$ and $k_{Q,n}$. Let $\eta_n^{W,+}\colon W_{n+1}^+ \to W_n^+ \oplus \mathbb{R}^{k_{W,+,n}}$ and $\eta_n^{W,-}\colon W_{n+1}^- \to W_n^- \oplus \mathbb{R}^{k_{W,-,n}}$ be another pair of isometries. Then there is an S^1-equivariant parameterized homotopy equivalence of Conley indices*

$$\eta_*\colon I(\varphi_{n+1}) \to \Sigma_B^{\mathbb{C}^{k_{Q,n}} \oplus \mathbb{R}^{k_{W,-,n}}} I(\varphi_n),$$

which is well defined up to homotopy for the induced map

$$\nu_! \eta_*\colon I^u(\varphi_{n+1}) \to \Sigma^{\mathbb{C}^{k_{Q,n}} \oplus \mathbb{R}^{k_{W,-,n}}} I^u(\varphi_n).$$

Furthermore, if $\mathfrak{s}$ is a self-conjugate spin^c *structure and instead $\eta_n^P\colon P_{n+1} \to P_n \oplus \mathbb{H}^{k_{\mathbb{H},P,n}}$ and $\eta_n^Q\colon Q_{n+1} \to Q_n \oplus \mathbb{H}^{k_{\mathbb{H},Q,n}}$, and the maps $\eta^{W,\pm}$ above are equivariant* *with respect to the C_2-action on W_{n+1}, W_n and $\widetilde{\mathbb{R}}^{k_{W,\pm,n}}$, then there is a well-defined, up to equivariant homotopy,* $\mathrm{Pin}(2)$*-equivariant homotopy equivalence*

$$\nu_!\eta_*\colon I^u(\varphi_{n+1}) \to \Sigma^{\mathbb{H}^{k_{\mathbb{H},Q,n}}\oplus\widetilde{\mathbb{R}}^{k_{W,-,n}}} I^u(\varphi_n),$$

and similarly for the parameterized version.

The restriction η_* to the S^1-fixed point set $I(\varphi_{n+1})^{S^1}$ is a fiber-preserving homotopy equivalence to $\Sigma_B^{\mathbb{R}^{k_{W,-,n}}} I_n(\varphi)^{S^1}$.

More generally, without a selection of maps η_n° as above, there is an S^1-equivariant parameterized homotopy equivalence of Conley indices

$$\eta_*\colon I(\varphi_{n+1}) \to \Sigma_B^{Q_{n+1}/Q_n}\Sigma_B^{W_{n+1}^-/W_n^-} I^u(\varphi_n),$$

so that the induced, unparameterized map

$$\nu_!\eta_*\colon I^u(\varphi_{n+1}) \to \mathrm{Th}(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, I^u(\varphi_n))$$

is well defined up to homotopy, as well as a similar statement for self-conjugate $\mathfrak{s}$.

Proof. By Lemma 3.1.2 below and invariance of the Conley index under deformations, there is a well-defined homotopy equivalence $\eta^1\colon I^u(\varphi_{n+1}) \to I^u(\varphi_{n+1}^{\mathrm{split}})$, where $\varphi_{n+1}^{\mathrm{split}}$ is defined in Lemma 3.1.2 (and similarly for the parameterized version). Using the invariance of the Conley index under homeomorphism, we have a well-defined homotopy equivalence

$$\eta^2\colon I(\varphi_{n+1}^{\mathrm{split}}) \to I(\varphi_{n+1}^{\mathrm{split},\eta}),$$

where $\varphi_{n+1}^{\mathrm{split},\eta}$ is defined in Lemma 3.1.9. Finally, by Lemma 3.1.9, the well-definedness of the Conley index (independent of a choice of index pair), and the definition of the Conley index (using our choice of index pair from Lemma 3.1.9), there is a well-defined homotopy equivalence

$$\eta^3\colon I(\varphi_{n+1}^{\mathrm{split},\eta}) \to \Sigma_B^{Q_{n+1}/Q_n}\Sigma_B^{W_{n+1}^-/W_n^-} I(\varphi_n).$$

In the case that we have fixed trivializations, as above, of W_{n+1}^-/W_n^- and Q_{n+1}^-/Q_n^-, the target of η^3 is identified with

$$\Sigma_B^{\mathbb{C}^{k_{Q,n}}\oplus\mathbb{R}^{k_{W,-,n}}} I(\varphi_n).$$

Since the flows used to define the homotopy equivalences preserve the fibers of the S^1-fixed point sets (that is, $X(\phi)_H = 0$ if $\phi = 0$), we can see from the formulas for the maps f, g, F_λ, G_λ in the proof of [43, Theorem 6.2] that the restrictions of η^1, η^2, η^3 to the S^1-fixed point sets preserve the fibers.

The argument adapts immediately to the case in which there is a spin structure, and the theorem follows. ∎

Let $\Sigma^{\pm}_{n+1}$ be the $L^2_{k_+,k_-}$-orthogonal complement to P_n in P_{n+1} (resp. Q_n in Q_{n+1}). Similarly, let $\Sigma^{W,\pm}_{n+1}$ be the $L^2_{k_+,k_-}$-orthogonal complement to $W^{\pm}_n$ in $W^{\pm}_{n+1}$. Let $\Sigma_{n+1} = \Sigma^+_{n+1} \oplus \Sigma^-_{n+1}$ and $\Sigma^W_{n+1} = \Sigma^{W,+}_{n+1} \oplus \Sigma^{W,-}_{n+1}$. Then $F_{n+1} = F_n \oplus \Sigma_{n+1}$ and $W_{n+1} = W_n \oplus \Sigma^W_{n+1}$. Write $\pi_{\Sigma_{n+1}}$ for the projection to Σ_{n+1} with respect to the $L^2_{k_+,k_-}$-norm. We also write $\pi_{\Sigma^W_{n+1}}$ for the projection Σ^W_{n+1} with respect to the $L^2_{k_+,k_-}$-norm.

Let $\mathcal{X}_n$ be the approximate Seiberg–Witten vector field on $F_n \oplus W_n$, for all n, as defined in (2.3.10). Let R be large enough as in Theorem 2.3.3.

For a path $\gamma(t)$ in the total space of $F_{n+1} \oplus W_{n+1}$, we write $\gamma(t) = (\phi^{(1)}(t) + \sigma(t)) \oplus (\omega^{(1)}(t) + \omega^{(2)}(t))$, as an element in the fiber over $b(t) = p(\gamma(t))$, where $\phi^{(1)}(t)$ is an element of $(F_n)_{b(t)}$, $\sigma(t) \in (\Sigma_n)_{b(t)}$, $\omega^{(1)}(t) \in (W_n)_{b(t)}$ and $\omega^{(2)}(t) \in (\Sigma^W_n)_{b(t)}$.

We then write $\gamma(t) = (\phi^{(1)}(t), \sigma(t), \omega^{(1)}(t), \omega^{(2)}(t), b(t))$ to describe γ in terms of these coordinates. We also write $\phi_{n+1}(t)$ to refer to the path in the total space of F_{n+1} determined by $(\phi^{(1)}_{n+1}(t), \sigma_{n+1}(t), b(t))$.

Lemma 3.1.2. *Let $\mathcal{X}^{\text{split}}_n$ be the vector field on the total space of $(F_n \oplus \Sigma_n) \oplus (W_n \oplus \Sigma^W_n)$ defined by (3.1.1), where*

$$\gamma_{n+1}(t) = (\phi^{(1)}_{n+1}(t), \sigma_{n+1}(t), \omega^{(1)}_{n+1}(t), \omega^{(2)}_{n+1}(t), b_{n+1}(t))$$

and $\hat{\gamma}_{n+1}(t)$ is the path obtained by (fiberwise) projecting $\gamma_{n+1}(t)$ to $(F_n \oplus W_n)_{b_{n+1}(t)}$:

$$\begin{aligned}
\frac{d\phi^{(1)}_{n+1}}{dt}(t) &= -\chi\big\{(\nabla_{X_H}\pi_{F_n})\phi^{(1)}_{n+1}(t) + \pi_{F_n}\big(D\phi^{(1)}_{n+1}(t) + c_1(\hat{\gamma}_{n+1}(t))\big)\big\},\\
\frac{d\sigma_{n+1}}{dt}(t) &= -\chi\big\{(\nabla_{X_H}\pi_{\Sigma_{n+1}})\sigma_{n+1}(t) + \pi_{\Sigma_{n+1}}(D\sigma_{n+1}(t))\big\},\\
\frac{db_{n+1}}{dt}(t) &= -\chi X_H(\phi^{(1)}_{n+1}(t)),\\
\frac{d\omega^{(1)}_{n+1}}{dt}(t) &= -\chi\big\{*d\omega^{(1)}_{n+1}(t) + \pi_{W_n}c_2(\hat{\gamma}_{n+1}(t))\big\},\\
\frac{d\omega^{(2)}_{n+1}}{dt}(t) &= -\chi * d\omega^{(2)}_{n+1}(t).
\end{aligned} \tag{3.1.1}$$

Here, χ is the cut-off function in (2.3.10). Then, for n sufficiently large, there is a continuous family of vector fields $\mathcal{X}^{\tau}_{n+1}$ on (the total space of) $F_{n+1} \oplus W_{n+1}$ between $\mathcal{X}_{n+1}$ and $\mathcal{X}^{\text{split}}_{n+1}$, with associated flows φ^{τ}_{n+1}, so that A_{n+1} is an isolating neighborhood for all τ, where

$$\begin{aligned}
A_{n+1} = A^o_n &\times_B B_{k_+}(\Sigma^+_{n+1}; R) \times_B B_{k_-}(\Sigma^-_{n+1}; R)\\
&\times_B B_{k_+}(\Sigma^{W,+}_{n+1}; R) \times_B B_{k_-}(\Sigma^{W,-}_{n+1}; R),
\end{aligned}$$

where A^o_n is as A_n in the proof of Theorem 2.3.3.

Proof. This is an immediate consequence of Lemmas 3.1.3, 3.1.7 and 3.1.8. ■

We construct the homotopy $\mathcal{X}^\tau_{n+1}$, with associated flow $\varphi^\tau_{n+1,k_+,k_-}$, in stages.

Lemma 3.1.3. *Let $\mathcal{X}^\tau_{n+1}$ for $\tau \in [0,1]$ be defined by*

$$\begin{aligned}
\frac{d\phi^{(1)}_{n+1}}{dt}(t) &= -\chi\big\{(\nabla_{X_H}\pi_{F_n})(\phi^{(1)}_{n+1}(t)+\sigma_{n+1}(t)) \\
&\qquad + (1-\tau)\pi_{F_n}\big(D\phi_{n+1}(t)+c_1(\gamma_{n+1}(t))\big) \\
&\qquad + \tau\pi_{F_n}\big(D(\phi^{(1)}_{n+1})+c_1(\hat{\gamma}_{n+1}(t))\big)\big\}, \\
\frac{d\sigma_{n+1}}{dt}(t) &= -\chi\big\{(\nabla_{X_H}\pi_{\Sigma_{n+1}})(\phi^{(1)}_{n+1}+\sigma_{n+1}(t)) \\
&\qquad + (1-\tau)\pi_{\Sigma_{n+1}}\big(D(\phi^{(1)}_{n+1}(t)+\sigma_{n+1}(t))+c_1(\gamma_{n+1}(t))\big) \\
&\qquad + \tau\pi_{\Sigma_{n+1}}D\sigma_{n+1}(t)\big\}, \\
\frac{db_{n+1}}{dt}(t) &= -\chi X_H(\phi_{n+1}(t)), \\
\frac{d\omega^{(1)}_{n+1}}{dt}(t) &= -\chi\big\{*d\omega^{(1)}_{n+1}(t)+\tau\pi_{W_n}c_2(\hat{\gamma}_{n+1}(t)) \\
&\qquad + (1-\tau)\pi_{W_n}c_2(\gamma_{n+1}(t))\big\}, \\
\frac{d\omega^{(2)}_{n+1}}{dt}(t) &= -\chi\big\{*d\omega^{(2)}_{n+1}(t)+(1-\tau)\pi_{\Sigma^W_{n+1}}c_2(\gamma_{n+1}(t))\big\}.
\end{aligned}$$

Here, χ is the cut-off function in (2.3.10). Then, for all $n \gg 0$, A_{n+1} is an isolating neighborhood of $\varphi^\tau_{n+1,k_+,k_-}$ for all $\tau \in [0,1]$.

Proof. The lemma is a consequence of Lemmas 3.1.4, 3.1.5 and 3.1.6. Indeed, let

$$A^o_n = (B_{k_+}(F^+_n;R)\times_B B_{k_-}(F^-_n;R))\times_B(B_{k_+}(W^+_n;R)\times_B B_{k_-}(W^-_n;R))$$

be as in the proof of Theorem 2.3.3. Suppose that

$$\operatorname{inv} A_{n+1} \not\subset \operatorname{int} A_{n+1},$$

for some $\tau_n \in [0,1]$, for all n. Then there is a sequence of finite-energy approximate trajectories $\gamma_{n+1}(t)$, for $\varphi^{\tau_{n+1}}_{n+1,k_+,k_-}$, so that $\gamma_{n+1}(0) \in \partial A_{n+1}$. There are four cases as in the proof of Theorem 2.3.3; we only treat the case that

$$\begin{aligned}
\gamma_{n+1}(0) \in (&S_{k_+}(F^+_{n+1};R)\times_B B_{k_-}(F^-_{n+1};R)) \\
&\times_B (B_{k_-}(W^+_{n+1};R)\times_B B_{k_-}(W^-_{n+1};R))
\end{aligned}$$

for all n, the other cases being similar.

As in the proof of Theorem 2.3.3, we have a lift

$$\tilde{\gamma}_{n+1} = (\tilde{\phi}_{n+1}, \omega_{n+1}) : \mathbb{R} \to \mathcal{H}^1(Y) \times L^2_{k_+,k_-}(\mathbb{S}) \times L^2_{k_+,k_-}(\operatorname{im} d^*)$$

with $p(\tilde{\gamma}_{n+1}(0)) \in \Delta$.

By Lemma 3.1.5 and Proposition 2.6.3, the sequence $\tilde{\gamma}$ has a subsequence converging, uniformly in $(\ell - 1)$-norm to some continuous map

$$\tilde{\gamma} : I \to \mathcal{H}^1(Y) \times L^2_{k_+-1,k_--1}(\mathbb{S}) \times L^2_{k_+-1,k_--1}(\operatorname{im} d^*).$$

By Lemma 3.1.6, $\tilde{\gamma}$ is a solution of the Seiberg–Witten equations. Finally, by Lemma 3.1.4, we obtain that the sequence $\tilde{\phi}_n^+(0)$ converged to $\tilde{\phi}^+(0)$ uniformly in $L^2_{k_+}$-norm, which is a contradiction. ∎

Lemma 3.1.4. *Assume that we have a sequence of trajectories $\tilde{\gamma}_{n+1}$ as in the proof of Lemma 3.1.2, with in particular*

$$\gamma_{n+1}(0) \in (S_{k_+}(F^+_{n+1}; R) \times_B B_{k_-}(F^-_{n+1}; R)) \\ \times_B (B_{k_-}(W^+_{n+1}; R) \times_B B_{k_-}(W^-_{n+1}; R)).$$

Then there is some R_1 so that

$$\|\phi^+_{n+1}(0)\|_{k_+ + \frac{1}{2}} < R_1,$$

for all n.

Proof. We emphasize only what must be changed from the proof of Lemma 2.7.2. We check the case where k_+ is an integer. We calculate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big|_{t=0} \|\phi^+_{n+1}(t)\|^2_{k_+} \\
&\quad = \operatorname{Re}\big(\langle (\nabla_{X_H}(D')^{k_+})\phi^+_{n+1}(0), (D')^{k_+}\phi^+_{n+1}(0)\rangle_0 \\
&\qquad + \langle (\nabla_{X_H}\pi^+)\phi_{n+1}(0), \phi^+_{n+1}(0)\rangle_{k_+} \\
&\qquad - \langle (\nabla_{X_H}\pi_{F_{n+1}})\phi_{n+1}(0), \phi^+_{n+1}(0)\rangle_{k_+} \\
&\qquad - \langle (1-\tau)\pi_{F_n} D'\phi_{n+1}(0), \phi^+_{n+1}(0)\rangle_{k_+} \\
&\qquad + \langle ((1-\tau)\mathbb{A} - (1-\tau)\pi_{F_{n+1}})c_1(\gamma_{n+1}(0)), \phi^+_{n+1}(0)\rangle_{k_+} \\
&\qquad - \tau\langle \pi_{F_n} D(\phi^{(1)}_{n+1}(0)), \phi^+_{n+1}(0)\rangle_{k_+} - \langle \tau\pi_{F_n} c_1(\hat{\gamma}_{n+1}(0)), \phi^+_{n+1}(0)\rangle_{k_+} \\
&\qquad - \langle \pi_{\Sigma_{n+1}} D\sigma_{n+1}(0), \phi^+_{n+1}(0)\rangle_{k_+} \\
&\qquad - (1-\tau)\big\langle \pi_{\Sigma_{n+1}} D\big(\phi^{(1)}_{n+1}(0) + c_1(\gamma_{n+1}(0))\big), \phi^+_{n+1}(0)\big\rangle_{k_+}\big).
\end{aligned}$$

Following the argument of Lemma 2.7.2, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big|_{t=0}\|\phi^+_{n+1}(t)\|^2_{k_+}&\\ \le CR^3\|\phi^+_{n+1}(0)\|_{k_++\frac{1}{2}} - \langle \pi_{F_{n+1}} D'\phi_{n+1}(0), \phi^+_{n+1}(0)\rangle_{k_+}&\\ + \tau\big(\langle \pi_{\Sigma_{n+1}} D\phi^{(1)}_{n+1}(0), \phi^+_{n+1}(0)\rangle_{k_+} + \langle \pi_{F_n} D\sigma_{n+1}(0), \phi^+_{n+1}(0)\rangle_{k_+}\big).&\end{aligned}$$

But

$$\langle \pi_{F_{n+1}} D'\phi_{n+1}(t), \phi^+_{n+1}(t)\rangle_{k_+} = \|\phi^+_{n+1}(t)\|^2_{k_++\frac{1}{2}}.$$

Since $[D', \pi_{\Sigma_{n+1}}]$ is uniformly bounded, we obtain

$$\tau\langle \pi_{\Sigma_{n+1}} D\phi^{(1)}_{n+1}, \phi^+_{n+1}\rangle_{k_+} \le CR^2$$

for some constant C independent of n.

A similar argument applies to $\langle \pi_{F_n} D\sigma_{n+1}, \phi^+_{n+1}\rangle_{k_+}$. The lemma then follows as did Lemma 2.7.2. ∎

Lemma 3.1.5. *The sequence $(\tilde{\phi}_n, \omega_n)$ is equicontinuous in $L^2_{K_T,\ell-5,w}$-norm.*

Proof. This follows exactly as in the proof of Theorem 2.3.3. ∎

By Proposition 2.6.3, any sequence which is equicontinuous in $L^2_{K_T,\ell-5,w}$-norm and bounded in ℓ-norm has a subsequence converging, uniformly in $\|\cdot\|_{\ell-1}$, to some continuous map $\tilde{\gamma}: I \to \mathcal{H}^1(Y) \times L^2_{\ell-1}(\mathbb{S}) \times L^2_{\ell-1}(\operatorname{im} d^*)$.

Lemma 3.1.6. *A limit $\tilde{\gamma}$ for the sequence (ϕ_n, ω_n) as above, is a solution of the Seiberg–Witten equations over $Y \times \mathbb{R}$.*

Proof. Take $T \in \mathbb{Z}_{>0}$ and $t \in [-T, T]$. We have

$$\begin{aligned}\tilde{\phi}_{n+1}(t) - \tilde{\phi}_{n+1}(0)&\\ = \int_0^t \frac{d\tilde{\phi}_{n+1}}{ds}(s)\,ds&\\ = -\int_0^t Z_1 + Z_2 + Z_3 + \pi_{\tilde{F}_{n+1}}\big(D(\tilde{\phi}^{(1)}_{n+1}(t) + \sigma_{n+1}(t)) + c_1(\tilde{\gamma}_n(t))\big)&\\ + X_H(\phi_{n+1}(s))\,ds,&\end{aligned}$$

where

$$\begin{aligned}Z_1 &= (\nabla_{X_H(\phi_{n+1}(t))}\pi_{F_{n+1}})\tilde{\phi}_{n+1},\\ Z_2 &= -\tau\pi_{\Sigma_{n+1}} D\phi^{(1)}_{n+1} - \tau\pi_{F_n} D\sigma_{n+1}(t),\\ Z_3 &= -\tau\big(\pi_{\Sigma_{n+1}} c_1(\tilde{\gamma}_n(t)) + \pi_{F_n} c_1(\tilde{\gamma}_n(t)) - \pi_{F_n} c_1(\tilde{\tilde{\gamma}}_n(t))\big).\end{aligned}$$

It suffices to show that the Z_i terms approach 0 uniformly in $L^2_{K_T,\ell-5,w}$, and that

$$\pi_{\tilde{F}_{n+1}}\big(D(\tilde{\phi}_{n+1}) + c_1(\tilde{\gamma}_{n+1}(t))\big) + X_H(\phi_{n+1}(t))$$
$$\to D(\tilde{\phi}(t)) + c_1(\tilde{\gamma}(t)) + X_H(\phi(t)),$$

also in $L^2_{K_T,\ell-5,w}$. Indeed, if that is the case, then the limit of integrals on the right-hand side is well defined, and

$$\tilde{\phi}(t) - \tilde{\phi}(0) = -\int_0^t \big(D\tilde{\phi} + c_1(\tilde{\gamma}(t)) + X_H(\phi(s))\big)\,ds, \tag{3.1.2}$$

giving the conclusion of the lemma.

Exactly as in the proof of Theorem 2.3.3, we obtain that Z_1 converges to 0 uniformly in $L^2_{K_T,\ell-5,w}$.

To show that $\pi_{F_n} D\sigma_{n+1}(t) \to 0$ in $L^2_{K_T,\ell-5,w}$, we use an elementary observation about projection with respect to different norms. That is, if V is a finite-dimensional vector space with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, then for a subspace $V' \subset V$ and projection Π_1 to V' with respect to $\|\cdot\|_1$, then $\|\Pi_1 x\|_2/\|x\|_2 \leq \rho_1\rho_2$ for $x \in V$, where $\rho_2 = \sup_{x\in V^*}\{\|x\|_2/\|x\|_1\}$ and $\rho_1 = \sup_{x\in V}\{\|x\|_1/\|x\|_2\}$.

We say a collection of finite-dimensional vector spaces V_i with norms $\|\cdot\|_{1,i}$ and $\|\cdot\|_{2,i}$ is *controlled* if $\rho_{1,i}\rho_{2,i}$ is bounded above.

We claim that the orthogonal complement of F_n in $(\mathcal{E}^{\mu_{n,+}}_{\lambda_{n,-}})_a$, call it $F_n^{\perp}$, with norms given by the restriction of $L^2_{k_+,k_-}$ and $L^2_{k_+-1,k_--1}$ (respectively), is controlled. Indeed, $F_n^{\perp}$ is a subspace of $(\mathcal{E}^{\mu_{n,+}}_{\mu_{n,-}})_a$. On $(\mathcal{E}^{\mu_{n,+}}_{\mu_{n,-}})_a$, by definition we have $\rho_1\rho_2 < \mu_{n,+}/\mu_{n,-}$. By our condition on the growth of the $\mu_{n,\pm}$, we then have that $\rho_{1,n}\rho_{2,n}$ is bounded as a function of n.

We claim that $\pi_{F_n} D\sigma_{n+1}(t) \to 0$ in $L^2_{k_+-2,k_--2}$. Indeed, $\sigma_{n+1}(t)$ converges to 0 weakly in $L^2_{k_+,k_-}$ by definition and $\sigma_{n+1}(t)$ converges strongly to 0 in $L^2_{k_+-1,k_--1}$. Then $D\sigma_{n+1}(t)$ converges to 0 in $L^2_{k_+-2,k_--2}$. Finally, π_{F_n} is a bounded family of operators in $L^2_{k_+-2,k_--2}$ by the above argument, giving the claim. As a consequence, we also have convergence in $L^2_{K_T,\ell-5,w}$.

To show that $\pi_{\Sigma_{n+1}} D\phi^{(1)}_{n+1}$ converges to 0, we note that by Proposition 2.4.2,

$$\|[D,\pi_{\Sigma_{n+1}}]\colon L^2_j \to L^2_j\| \leq C$$

for some constant C independent of n, for all half-integers $j \leq k_+$. Moreover, we have $\pi_{\Sigma_{n+1}}\phi^{(1)}_{n+1} = 0$, and so we need only show that the sequence $[\pi_{\Sigma_{n+1}}, D]\phi^{(1)}_{n+1}$ converges to zero. Given the bound on $\sigma D\phi^{(1)}_{n+1}$ from the bound on the commutator $[D,\pi_{\Sigma_{n+1}}]$ above, and using the definition of the norms involved, we see that $\pi_{\Sigma_{n+1}} D\phi^{(1)}_{n+1} \to 0$ in $L^2_{\ell-1}$-norm.

A very similar argument shows that $\pi_{\Sigma_{n+1}} c_1(\gamma_n(t)) \to 0$ in $L^2_{K_T,\ell-5,w}$, and also that $\pi_{F_n} c_1(\gamma_n(t))$ and $\pi_{F_n} c_1(\hat{\gamma}_n(t))$ converge to $c_1(\gamma(t))$ in $L^2_{K_T,\ell-5,w}$, so that $Z_3 \to 0$.

A similar argument also shows the convergence in (3.1.2), and the proof is complete. ■

For $\tau \in [1,2]$, define a flow $\varphi^{\tau}_{n+1,k_+,k_-}$ on $F_{n+1} \oplus W_{n+1}$ by

$$\begin{aligned}
\frac{d\phi^{(1)}_{n+1}}{dt}(t) &= -\chi\big\{(2-\tau)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{F_n})(\phi_{n+1}(t)) \\
&\quad + \big(\pi_{F_n} D\phi^{(1)}_{n+1}(t) + c_1(\hat{\gamma}_{n+1}(t))\big) \\
&\quad + (\tau-1)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{F_n})\phi^{(1)}_{n+1}(t)\big\}, \\
\frac{d\sigma_{n+1}}{dt}(t) &= -\chi\big\{(2-\tau)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{\Sigma_{n+1}})(\phi^{(1)}_{n+1}(t) + \sigma_{n+1}(t)) \\
&\quad + (\tau-1)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{\Sigma_{n+1}})\sigma_{n+1}(t) + \pi_{\Sigma_{n+1}} D\sigma_{n+1}(t)\big\},
\end{aligned}$$

with the other terms unchanged. Inspection shows that the total space of $F_{n+1} \oplus W_{n+1}$ is preserved by the flow.

Lemma 3.1.7. *For $n \gg 0$, for all $\tau \in [1,2]$, A_{n+1} is an isolating neighborhood for $\varphi^{\tau}_{n+1,k_+,k_-}$.*

Proof. We highlight only the difference in the argument compared to the proof of Lemma 3.1.3. We have a sequence of trajectories

$$\gamma_{n+1}(t) = (\phi^{(1)}_{n+1}(t), \sigma_{n+1}(t), \omega_{n+1}(t))$$

exactly as in that argument. We assume that

$$\begin{aligned}
\gamma_{n+1}(0) \in (&S_{k_+}(F^+_{n+1};R) \times_B B_{k_-}(F^-_{n+1};R)) \\
&\times_B (B_{k_-}(W^+_{n+1};R) \times_B B_{k_-}(W^-_{n+1};R))
\end{aligned}$$

for all n; the other cases are similar. The proofs of the analogs of Lemma 3.1.5 and Lemma 3.1.6 are unchanged, and we obtain that a lift $\tilde{\gamma}_n$ of γ_n to the universal covering converges in $L^2_{K_T,\ell-5,w}$-norm to a solution $\tilde{\gamma}(t)$ of the Seiberg–Witten equations. We need only prove an analog of Lemma 3.1.4, that $\|\phi^+_{n+1}\|_{k_++\frac{1}{2}}$ is bounded independent of τ, n. Suppose this is false, that is, that

$$\|\phi^{(1),+}_{n+1}(0) + \sigma^+_{n+1}(0)\|_{k_++\frac{1}{2}} \to \infty.$$

Then we study (for the case $k_+ \in \mathbb{Z}$, the other case being similar)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big|_{t=0} \|\phi^{(1),+}_{n+1}(t) + \sigma^+_{n+1}(t)\|^2_{k_+} \\
&\quad = \operatorname{Re}\big(\langle(\nabla_{X_H}\pi^+)\phi^{(1)}_{n+1}(0), \phi^{(1),+}_{n+1}(0)\rangle_{k_+} \\
&\qquad\quad + \langle(\nabla_{X_H}(D')^{k_+})\phi^{(1),+}_{n+1}(0), (D')^{k_+}\phi^{(1),+}_{n+1}(0)\rangle_0 \\
&\qquad\quad - \langle\pi_{F_{n+1}} D'\phi^{(1),+}_{n+1}(0), \phi^{(1),+}_{n+1}(t)\rangle_{k_+} \\
&\qquad\quad + \langle(\mathbb{A} - \pi_{F_{n+1}})c_1(\hat{\gamma}_{n+1}(0)), \phi^{(1),+}_{n+1}(0)\rangle_{k_+} \\
&\qquad\quad - \langle(\nabla_{X_H}\pi_{F_{n+1}})\phi^{(1),+}_{n+1}(0), \phi^{(1),+}_{n+1}(0)\rangle_{k_+} \\
&\qquad\quad - (2-\tau)\langle(\nabla_{X_H}\pi_{F_n})\sigma_{n+1}(0), \phi^{(1),+}_{n+1}(0)\rangle_{k_+} \\
&\qquad\quad + \langle(\nabla_{X_H}(D')^{k_+})\sigma_{n+1}(0), (D')^{k_+}\sigma^{(1),+}_{n+1}(0)\rangle_0 \\
&\qquad\quad + \langle(\nabla_{X_H}\pi^+)\sigma_{n+1}(0), \sigma^+_{n+1}(0)\rangle_{k_+} - \langle\pi_{\Sigma_{n+1}} D'\sigma_{n+1}(0), \sigma^+_{n+1}(t)\rangle_{k_+} \\
&\qquad\quad - \langle(\nabla_{X_H}\pi_{\Sigma_{n+1}})\sigma_{n+1}(0), \sigma^+_{n+1}(0)\rangle_{k_+} \\
&\qquad\quad - (2-\tau)\langle(\nabla_{X_H}\pi_{\Sigma_{n+1}})\phi^{(1),+}_{n+1}(0), \sigma^+_{n+1}(0)\rangle_{k_+}\big).
\end{aligned} \tag{3.1.3}$$

All of these terms can be dealt with as in the proof of Lemma 3.1.4, with the exception of

$$\begin{aligned}
&-(2-\tau)\operatorname{Re}\langle(\nabla_{X_H}\pi_{F_n})\sigma_{n+1}(0), \phi^{(1),+}_{n+1}(0)\rangle_{k_+} \\
&\qquad -(2-\tau)\operatorname{Re}\langle(\nabla_{X_H}\pi_{\Sigma_{n+1}})\phi^{(1)}_{n+1}(0), \sigma^+_{n+1}(0)\rangle_{k_+}.
\end{aligned}$$

To bound this term, consider the expression $\langle\phi^{(1),+}_{n+1}(t), \sigma^+_{n+1}(t)\rangle_{k_+}$ as a function of t. By definition, this is zero, but expanding its derivative gives

$$\begin{aligned}
0 &= \operatorname{Re}\langle(\nabla_{X_H}\pi^+)\phi^{(1)}_{n+1}(t), \sigma^+_{n+1}(t)\rangle_{k_+} \\
&\quad + \operatorname{Re}\langle(\nabla_{X_H}(D')^{k_+})\phi^{(1),+}_{n+1}(t), (D')^{k_+}\sigma^+_{n+1}(t)\rangle_0 \\
&\quad + \operatorname{Re}\langle\phi^{(1),+}_{n+1}(t), (\nabla_{X_H}\pi^+)\sigma_{n+1}(t)\rangle_{k_+} \\
&\quad + \operatorname{Re}\langle(D')^{k_+}\phi^{(1),+}_{n+1}(t), (\nabla_{X_H}(D')^{k_+})\sigma^+_{n+1}(t)\rangle_0 \\
&\quad + \operatorname{Re}\langle(\nabla_{X_H}\pi_{F_n})\phi^{(1)}_{n+1}(t), \sigma^+_{n+1}(t)\rangle_{k_+} \\
&\quad + \operatorname{Re}\langle\phi^{(1),+}_{n+1}(t), \nabla_{X_H}\pi_{\Sigma_{n+1}}\sigma_{n+1}(t)\rangle_{k_+}.
\end{aligned} \tag{3.1.4}$$

Recall that

$$\begin{aligned}
\pi_{\Sigma_{n+1}}(\nabla_{X_H}\pi_{F_n})\phi^{(1)}_{n+1} &= -\pi_{\Sigma_{n+1}}(\nabla_{X_H}\pi_{\Sigma_{n+1}})\phi^{(1)}_{n+1}, \\
\pi_{F_n}(\nabla_{X_H}\pi_{F_n})\sigma_{n+1} &= -\pi_{F_n}(\nabla_{X_H}\pi_{\Sigma_{n+1}})\sigma_{n+1}.
\end{aligned}$$

Then (3.1.4), also using the estimates from the proof of Lemma 2.7.2, becomes

$$\big|\langle(\nabla_{X_H}\pi_{F_n})\phi^{(1)}_{n+1}(t), \sigma^+_{n+1}(t)\rangle_{k_+} + \langle\phi^{(1),+}_{n+1}(t), (\nabla_{X_H}\pi_{\Sigma_{n+1}})\sigma_{n+1}(t)\rangle_{k_+}\big| \leq CR^2.$$

Then, using (3.1.3), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big|_{t=0}\|\phi^{(1),+}_{n+1}(t)+\sigma^{+}_{n+1}(t)\|^2_{k_+} \le{}& CR^3\|\phi^{(1),+}_{n+1}(0)\|_{k_+ + \frac{1}{2}}\\ &-\mathrm{Re}\langle \pi_{\Sigma_{n+1}} D'\sigma_{n+1}(0), \sigma^{+}_{n+1}(0)\rangle_{k_+}\\ &-\mathrm{Re}\langle \pi_{F_{n+1}} D'\phi^{(1)}_{n+1}(0), \phi^{(1),+}_{n+1}(t)\rangle_{k_+}+C.\end{aligned}$$

The argument from Lemma 2.7.2 gives

$$\begin{aligned}0 &= \frac{1}{2}\frac{d}{dt}\Big|_{t=0}\|\phi^{(1),+}_{n+1}(t)+\sigma^{+}_{n+1}(t)\|^2_{k_+}\\ &\le CR^3\|\phi^{(1),+}_{n+1}(0)\|_{k_+ + \frac{1}{2}} - \|\phi^{(1),+}_{n+1}(0)\|^2_{k+\frac{1}{2}} - \|\sigma^{+}_{n+1}(0)\|^2_{k+\frac{1}{2}}+C.\end{aligned}$$

Thus, $\|\phi^{(1),+}_{n+1}(0)+\sigma^{+}_{n+1}(0)\|_{k+\frac{1}{2}}$ is bounded. The proof of Lemma 3.1.7 then follows exactly as Theorem 2.3.3. ∎

Finally, for $\tau\in[2,3]$, set

$$\begin{aligned}\frac{d\phi^{(1)}_{n+1}(t)}{dt} &= -\chi\big\{(3-\tau)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{F_{n+1}})\phi^{(1)}_{n+1}(t)\\ &\qquad + (\tau-2)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{F_{n+1}})\phi^{(1)}_{n+1}(t)\\ &\qquad + \pi_{F_n}\big(D\phi^{(1)}_{n+1}(t)+c_1(\hat{\gamma}_{n+1}(t))\big)\\ &\qquad + (\tau-2)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{F_n})\phi^{(1)}_{n+1}(t)\big\},\\ \frac{d\sigma_{n+1}}{dt}(t) &= -\chi\big\{(3-\tau)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{\Sigma_{n+1}})\sigma_{n+1}(t)\\ &\qquad + (\tau-2)(\nabla_{X_H(\phi^{(1)}_{n+1}(t))}\pi_{\Sigma_{n+1}})\sigma_{n+1}(t)\big\},\\ \frac{db}{dt}(t) &= -\chi\big\{(3-\tau)X_H(\phi_{n+1}(t))+(\tau-2)X_H(\phi^{(1)}_{n+1}(t))\big\},\end{aligned}$$

with the other terms unchanged. Note that it is clear that these equations preserve the total space of $F_{n+1}\oplus W_{n+1}$.

Lemma 3.1.8. *For $n\gg 0$, for all $\tau\in[2,3]$, A_{n+1} is an isolating neighborhood for $\varphi^{\tau}_{n+1,k_+,k_-}$.*

Proof. This claim is a consequence of the arguments used in Lemma 3.1.3 and 3.1.7, and there are no new difficulties. ∎

Write $B(Q_{n+1}/Q_n, R)$ for the R-disk bundle of Q_{n+1}/Q_n over $\mathrm{Pic}(Y)$, etc.

Lemma 3.1.9. *Say that (A_n^o, L_n) is an index pair for $\mathcal{X}_n$, for some L_n, of $\mathcal{X}_n$ on $F_n \oplus W_n$. Then $(\tilde{A}_{n+1}, \tilde{L}_{n+1})$ is an index pair for $\mathcal{X}_{n+1}^{\text{split}}$, where*

$$\begin{aligned}\tilde{A}_{n+1} = A_n^o &\times_B (B(P_{n+1}/P_n \oplus W_{n+1}^+/W_n^+, R)) \\ &\times_B (B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R)),\end{aligned}$$

for some R sufficiently large, and

$$\begin{aligned}\tilde{L}_{n+1} = L_n^o &\times_B (B(P_{n+1}/P_n \oplus W_{n+1}^+/W_n^+, R)) \\ &\times_B (\partial B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R)).\end{aligned}$$

Proof. It follows from Lemma 3.1.2 that $\mathrm{inv}(\tilde{A}_n \setminus \tilde{L}_n) \subset \mathrm{int}(\tilde{A}_n \setminus \tilde{L}_n)$.

We next check that $\tilde{L}_n$ is positively invariant in $\tilde{A}_n$. Write

$$(\phi_{n+1}^{(1)}(t), \omega_{n+1}^{(1)}(t), \zeta_{n+1}(t))$$

in

$$\begin{aligned}(F_n \oplus W_n) &\times_B (B(P_{n+1}/P_n \oplus W_{n+1}^+/W_n^+, R)) \\ &\times_B (B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R))\end{aligned}$$

for a trajectory of $\varphi_{n+1,k_+,k_-}^{\text{split}}$. The flow on the $\mathcal{F}_n \times_B \mathcal{W}_n$-factor is independent of position on the $(B(P_{n+1}/P_n \oplus W_{n+1}^+/W_n^+, R)) \times_B (B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R))$ factor, and in particular, if $(\phi_{n+1}^{(1)}(T_0), \omega_{n+1}^{(1)}(T_0)) \in L_n$, then $(\phi_{n+1}^{(1)}(t), \omega_{n+1}^{(1)}(t)) \in L_n$ for all $t \geq T_0$, by our assumption on L_n.

We must then show that if $\zeta_{n+1}(T_0) \in \partial B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R)$, then

$$\zeta_{n+1}(t) \in \partial B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R_1),$$

or exits $\tilde{A}_{n+1}$, for all $t \geq T_0$, if n is large enough. We regard the path $(\phi_{n+1}^{(1)}(t), \omega_{n+1}^{(1)}(t))$ as fixed, and $\zeta_{n+1}(t)$ as a trajectory of a vector field on the boundary $\partial B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R_1)$.

Write $\zeta_{n+1}(t) = (b(t), \zeta_{n+1}^{(1),+}, \zeta_{n+1}^{(1),-}, \zeta_{n+1}^{(2),+}, \zeta_{n+1}^{(2),-})$, as a section of

$$V_n(R_1) = (B(P_{n+1}/P_n \oplus W_{n+1}^+/W_n^+, R_1)) \times_B (B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R_1)).$$

We may, and do, assume without loss of generality that $T_0 = 0$. Then if $(\zeta_{n+1}^{(1),-}, \zeta_{n+1}^{(2),-}) \in \partial B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R)$, either $\zeta_{n+1}^{(1),-}$ or $\zeta_{n+1}^{(2),-}$ has $\|\zeta_{n+1}^{(i),-}\|_{k_-} \geq R_1/2$. Assume $i = 1$, the other case being similar.

Recall that $(\phi_{n+1}^{(1)}(t), \omega_{n+1}^{(1)}(t), \zeta_{n+1}(t))$ is equivalent to a trajectory

$$\gamma_{n+1}(t) = (\phi_{n+1}^{(1)}(t), \sigma_{n+1}(t), \omega_{n+1}(t))$$

of $\mathcal{X}_{n+1}^{\text{split}}$ on $F_{n+1} \oplus W_{n+1}$.

We consider

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big|_{t=0}\|\zeta^{(1),-}_{n+1}(t)\|^2_{k_-}\\
&\quad=\frac{1}{2}\frac{d}{dt}\Big|_{t=0}\|\sigma^-_{n+1}(t)\|^2_{k_-}\\
&\quad=\langle-(\nabla_{X_H}\pi_{\Sigma_{n+1}})\sigma_{n+1}(0)-\pi_{\Sigma_{n+1}}D'\sigma_{n+1}(0),\sigma^-_{n+1}(0)\rangle_{k_-}\\
&\qquad-\langle\nabla_{X_H}(D')^{k_-}\sigma_{n+1}(0),(D')^{k_-}\sigma^-_{n+1}\rangle_0+\langle(\nabla_{X_H}\pi^-)\sigma_{n+1}(0),\sigma^-_{n+1}(0)\rangle_{k_-}\\
&\quad\le CR^2-\langle\pi_{\Sigma_{n+1}}D'\sigma_{n+1}(0),\sigma^-_{n+1}(0)\rangle_{k_-}\\
&\quad=CR^2-\|\sigma^-_{n+1}(0)\|^2_{k_-+\frac{1}{2}}.
\end{aligned}
$$

Note that we have used that n can be taken sufficiently large that Σ_{n+1} is perpendicular to the image of $\mathbb{A}$.

Now, by definition of Σ_{n+1}, we have

$$
\frac{\|\sigma^-_{n+1}(0)\|^2_{k_-+\frac{1}{2}}}{\|\sigma^-_{n+1}(0)\|^2_{k_-}}\to\infty
$$

as $n\to\infty$.

Thus, if $\|\sigma^-_{n+1}(0)\|_{k_-}\ge R/2$, we have that $\|\zeta^{(1),-}_{n+1}(t)\|_{k_-}$ is always increasing at $t=0$ (similarly, $\|\zeta^{(1),+}_{n+1}(t)\|_{k_+}$ is decreasing at $t=0$).

This shows that $\tilde{L}_{n+1}$ is positively invariant in $\tilde{A}_{n+1}$. It follows similarly that $\tilde{L}_{n+1}$ is an exit set. ∎

3.2 Spin^c structure for family of manifolds

Since we consider a family of spin^c 3-manifolds to show that the Conley index for the flow φ_n is independent of the choice of Riemannian metric of Y in Section 3.3, we will give the definition of spin^c structure for a family of Riemannian manifolds.

Take an n-dimensional real, oriented vector space V and an inner product g on V. We denote by $Fr(V,g)$ the space of orthonormal bases of (V,g) compatible with the orientation. Choose another inner product h on V. We define an isomorphism between $Fr(V,g)$ and $Fr(V,h)$. For $\{e_i\}_{i=1}^n\in Fr(V,g)$, put

$$
h_{ij}=h(e_i,e_j)\in\mathbb{R}.
$$

Then the matrix $H=(h_{ij})_{i,j=1,\dots,n}$ is symmetric and positive definite. We have the square root $\sqrt{H}$ of H defined as follows. Since H is symmetric and positive definite, we have the eigenspace decomposition

$$
\mathbb{R}^n=\bigoplus_{i=1}^{r}V_{\lambda_i},
$$

where $\lambda_i > 0$ are the distinct eigenvalues of H, and V_{λ_i} are the eigenspaces. Define $\sqrt{H}$ to be the matrix corresponding to the linear map $\mathbb{R}^n \to \mathbb{R}^n$ defined by $v \mapsto \sqrt{\lambda_i} v$ for $v \in V_{\lambda_i}$. Define a basis $f_1, \dots, f_n$ of V by

$$(f_1 \dots f_n) = (e_1 \dots e_n)\sqrt{H}^{-1}.$$

We can see that $f_1, \dots, f_n$ are an orthonormal basis with respect to h. So we get a map

$$Fr(V, g) \to Fr(V, h). \tag{3.2.1}$$

Take $G \in SO(n)$ and put

$$(e'_1 \dots e'_n) = (e_1 \dots e_n)G, \quad H' = (h(e'_i, e'_j))_{i,j=1,\dots,n}.$$

It is easy to see that

$$H' = G^{-1}HG, \quad \sqrt{H'} = G^{-1}\sqrt{H}G.$$

This implies that the map (3.2.1) is an $SO(n)$-equivariant isomorphism.

For an oriented smooth Riemannian n-manifold (X, g), let $P_{X,g}$ be the principal $SO(n)$-bundle of oriented, orthonormal frames in TX. Recall that a spin^c structure of (X, g) is a pair of a principal Spin$^c(n)$ bundle $\tilde{P}_X$ on X and a smooth map $\xi\colon \tilde{P}_X \to P_{X,g}$ such that the diagram

$$\begin{tikzcd} \tilde{P}_X \arrow[rr, "\xi"] \arrow[dr] & & P_{X,g} \arrow[dl] \\ & X & \end{tikzcd}$$

commutes, and for $p \in \tilde{P}_X$ and $s \in \mathrm{Spin}^c(n)$ we have

$$\xi(p \cdot s) = \xi(p) \cdot \pi(s).$$

Here, $\pi\colon \mathrm{Spin}^c(n) \to SO(n)$ is the projection.

Take another Riemannian metric h on X. The $SO(n)$-equivariant isomorphism (3.2.1) induces an isomorphism

$$P_{X,g} \cong P_{X,h} \tag{3.2.2}$$

of principal bundles. Hence a spin^c structure $(\tilde{P}_X, \xi)$ of (X, g) naturally defines a spin^c structure of (X, h).

A locally trivial family of spin^c manifolds over a topological space L is a tuple $(E, G, \tilde{P}_E, \xi)$. The first component E stands for a locally trivial fiber bundle

$$X \to E \to L$$

over L with fiber X. For each $\ell \in L$ we have an open neighborhood U_ℓ of ℓ and a trivialization

$$E|_{U_\ell} \cong U_\ell \times E_\ell.$$

Here, E_ℓ is the fiber of E over ℓ. The second component G is a fiberwise Riemannian metric of E. Let P_E be the principal $SO(n)$-bundle on E whose fiber over ℓ is the principal $SO(n)$-bundle of oriented, orthonormal frames in TE_ℓ. Note that the local trivialization of E on U_ℓ and the isomorphism (3.2.2) induce an isomorphism

$$P_E|_{U_\ell} \cong U_\ell \times P_{E_\ell}$$

of principal bundles. The third component $\tilde{P}_E$ is a principal $\mathrm{Spin}^c(n)$ bundle over E. The fourth component ξ is a smooth map

$$\tilde{P}_E \to P_E$$

such that the diagram

$$\begin{tikzcd} \tilde{P}_E \arrow[rr, "\xi"] \arrow[dr] & & P_E \arrow[dl] \\ & E & \end{tikzcd}$$

commutes and $\xi(p, \cdot s) = \xi(p) \cdot \pi(s)$ for $p \in \tilde{P}_E$ and $s \in \mathrm{Spin}^c(n)$. Moreover, we assume that $\tilde{P}_E$ is locally trivial. That is, for each $\ell \in L$ there is an isomorphism

$$\tilde{P}_E|_{U_\ell} \cong U_\ell \times (\tilde{P}_E|_{E_\ell})$$

of principal bundles such that the following diagram commutes:

$$\begin{tikzcd} \tilde{P}_E|_{U_\ell} \arrow[r, "\cong"] \arrow[d, "\xi"'] & U_\ell \times (\tilde{P}_E|_{E_\ell}) \arrow[d, "\mathrm{id}_{U_\ell} \times \xi"] \\ P_E|_{U_\ell} \arrow[r, "\cong"'] & U_\ell \times P_{E_\ell}. \end{tikzcd}$$

3.3 Independence of metric

In this section we prove that the approximate Seiberg–Witten flow defined in (2.3.10) varies continuously as we vary the 3-manifold.

To make this precise, let $\mathcal{F}$ be a locally trivial family of spin^c metrized 3-manifolds with compact base space L, so that L is a CW complex. See Section 3.2 for the definition of a locally trivial family of spin^c metrized manifolds. Note that associated to $\mathcal{F}$ there is also a bundle over L, $\mathrm{Pic}(\mathcal{F})$, whose fiber is the Picard-bundle at $\ell \in L$.

Suppose that we are given a sequence of continuously varying spectral sections $P_{n,\ell}$, $Q_{n,\ell}$ for $\ell \in L$ so that the $P_{n,\ell}$, $Q_{n,\ell}$ are good as at the beginning of Chapter 2, with $F_{n,\ell} = P_{n,\ell} \cap Q_{n,\ell}$ as a fiber bundle over (the total space of) $\tilde{L}$. Let φ_{n,ℓ,k_+,k_-} be the flow defined by projection onto $F_{n,\ell}$. Here, unlike in the case of a single 3-manifold, the flow preserves fibers of $F_{n,\ell}$ over L (though the flow can of course move over $\tilde{L}_\ell$, the fiber of $\tilde{L} \to L$).

There is one subtlety in that now the eigenvalues of $*d$ may vary in the family $\mathcal{F}$. In particular, we will assume the existence of increasing spectral sections $W_{P,n}$ for $-*d$, and increasing spectral sections $W_{Q,n}$ for $*d$, satisfying the analogs of (2.3.6)–(2.3.7), and set $W_n = W_{P,n} \cap W_{Q,n}$. With this notation fixed, we define W_n^+ and W_n^- as before.

Theorem 3.3.1. *Let $\mathcal{F}$, with compact base L, be a family of* spin^c *metrized 3-manifolds, with fiber $\mathcal{F}_b$ for $b \in L$. Let k_+, k_- be half-integers with $k_\pm > 5$ and with $|k_+ - k_-| \leq \frac{1}{2}$. Fix a positive number R with $R > R_{k_+,k_-}$ for some R_{k_+,k_-}. Then*

$$(B_{k_+}(F_n^+; R) \times_B B_{k_-}(F_n^-; R)) \times_B (B_{k_+}(W_n^+; R) \times_B B_{k_-}(W_n^-; R))$$

is an isolating neighborhood of the flow φ_{n,ℓ,k_+,k_-} for $n \gg 0$. Here, $B_{k_\pm}(F_n^\pm; R)$ are the disk bundle of $F_n^\pm$ of radius R in $L^2_{k_\pm}$ and $B_{k_+}(F_n^+; R) \times_B B_{k_-}(F_n^-; R)$ is the fiberwise product.

The proof of this theorem differs from the proof of Theorem 2.3.3 only in notation, so we will not write out the details.

In particular, we have the following corollary.

Corollary 3.3.2. *Let $(Y, \mathfrak{s})$ be a* spin^c *manifold, with metrics g_0, g_1, and fix a family of good spectral sections $P_{n,0}$, $Q_{n,0}$ over (Y, g_0). Choose a family of metrics g_t connecting g_0 to g_1. Then there exists a family of spectral sections $P_{n,t}$, $Q_{n,t}$ extending $P_{n,0}$, $Q_{n,0}$ and so that the flow $\varphi_{n,0,k_+,k_-}$ on $F_{n,0}$ extends to a continuously varying flow φ_{n,t,k_+,k_-} on $F_{n,t}$, so that*

$$(B_{k_+}(F_n^+; R) \times_B B_{k_-}(F_n^-; R)) \times_B (B_{k_+}(W_n^+; R) \times_B B_{k_-}(W_n^-; R))$$

is an isolating neighborhood of the flow φ_{n,t,k_+,k_-} for $n \gg 0$ and all $t \in [0, 1]$. In particular, $I(\varphi_{n,0,k_+,k_-})$ is canonically, up to homotopy equivalence, identified with $I(\varphi_{n,1,k_+,k_-})$.

Proof. The claim about the existence of the extended spectral sections follows from the homotopy description of spectral sections and the fact that $[0, 1]$ is contractible. The claim on isolating neighborhoods is a consequence of Theorem 3.3.1. The well-definedness of the Conley index follows from the continuity property of the Conley index. ∎

3.4 Variation of Sobolev norms

Proposition 3.4.1. *Let (k_+^1, k_-^1) and (k_+^2, k_-^2) be pairs of half-integers > 5, with $|k_+^i - k_-^i| \leq \frac{1}{2}$ for $i = 1, 2$. Fix R sufficiently large. Then there exists a family of flows φ_n^τ for $\tau \in [0, 1]$ so that*

$$(B_{g_+^\tau}(F_n^+; R) \times_B B_{g_-^\tau}(F_n^-; R)) \times_B (B_{g_+^\tau}(W_n^+; R) \times_B B_{g_-^\tau}(W_n^-; R))$$

is a family of isolating neighborhoods, where $g_\pm^\tau$ is the interpolated metric (defined below), and where $\varphi_n^0 = \varphi_{n,k_+^1,k_-^1}$ and $\varphi_n^1 = \varphi_{n,k_+^2,k_-^2}$. In particular, there is a homotopy equivalence

$$I(\varphi_{n,k_+^1,k_-^1}) \to I(\varphi_{n,k_+^2,k_-^2}),$$

suppressing the spectral section choices from the notation. The restriction to the S^1-fixed point set is a fiber-preserving homotopy equivalence.

Proof. Define the *interpolated metric* g^τ by

$$g^\tau(x, y) := \langle x, y\rangle_{k_\pm^\tau} := (1 - \tau)\langle x, y\rangle_{k_+^1,k_-^1} + \tau\langle x, y\rangle_{k_+^2,k_-^2}.$$

We abuse notation and also write g^τ for the restriction of g^τ to subbundles, including $F_n^\pm$ and $W_n^\pm$.

The equation (2.7.1) defines a flow φ_n^τ, with π_{F_n}, π_{W_n} replaced appropriately. Hypothesis (2.7.2) continues to hold, with the subscripts $k_\pm$ replaced with $k_\pm^\tau$. Write $\pi_{F_n}^\tau$ for projection with respect to g^τ.

As usual, we will assume for a contradiction that

$$y_{n,0}^{\tau_n} = (\phi_{n,0}^{\tau_n}, \omega_{n,0}^{\tau_n}) \in \mathrm{inv} A_n \cap \partial A_n.$$

Let us treat the case that

$$\phi_{n,0}^{\tau_n} \in S_{g_+^\tau}(F_n^+; R) \in \mathrm{inv} A_n \cap \partial A_n,$$

where $S_{g_+^\tau}(V, R)$, for V a vector bundle over B, is the R-sphere bundle.

Exactly as in the proof of Theorem 2.3.3, we can extract a sequence of approximate solutions $\tilde{\gamma}_n^{\tau_n} = (\tilde{\phi}_n^{\tau_n}, \omega_n^{\tau_n})$, for $t \in [-T, T]$, with T fixed. To see this, we need to control $\frac{d\tilde{\phi}_n^\tau}{dt}$ in $(K_T, \ell - 5, w)$-norm. This amounts to generalizing Proposition 2.6.1 to the following situation.

Proposition 3.4.2. *Let k_+, k_- be half-integers, with $k_\pm > 5$, and also set $\ell = \min_{i=1,2}\{k_+^i, k_-^i\}$. Then*

$$\sup_{v \in B(TB;1)} \|\nabla_v \pi_{P_n}^\tau : L^2_{k^\tau} \to L^2_{\ell-5,w}\| \to 0,$$

uniformly in τ.

This proposition holds because the natural modification of the estimate at the end of Corollary 2.5.2 holds.

Then the sequence $\tilde{\gamma}_n^{\tau_n}(t)$ converges to a map

$$\tilde{\gamma} \colon [-T, T] \to \mathcal{H}^1(Y) \times L^2_{\ell-1}(\mathbb{S}) \times L^2_{\ell-1}(\operatorname{im} d^*).$$

To verify that $\tilde{\gamma}$ solves the Seiberg–Witten equations, we observe that

$$(\nabla_{X_H} \pi^{\tau_n}_{F_n})\tilde{\phi}_n(s) \to 0$$

in $L^2_{K_T,\ell-5,w}$-norm, as follows from Proposition 3.4.2.

We have

$$\begin{aligned}\|\pi^{\tau_n}_{F_n} D\phi_n - D\phi_n\|_{\ell-2} &= \|\pi^{\tau_n}_{F_n} D\phi_n - D\phi_n + D\phi_n - D\phi_n\|_{\ell-2}\\ &\leq \|[\pi^{\tau_n}_{F_n}, D]\phi_n\|_{\ell-2} + \|D\phi_n - D\phi\|_{\ell-2}.\end{aligned}$$

The first term drops out, using the rule of a sequence of controlled vector spaces, and we obtain that $\pi^{\tau_n}_{F_n} D\phi_n$ converges to $D\phi$ uniformly in $L^2_{\ell-2}$ on $[-T, T]$. By the proof of Lemma 2.7.1, the limit $\tilde{\gamma}$ is a solution of the Seiberg–Witten equations. The proof from this point follows along the same lines as Theorem 2.3.3. ∎

3.5 The Seiberg–Witten invariant

In this section we repackage the construction of $\mathcal{SWF}_{[n]}(Y, \mathfrak{s})$ to take account of the choices made in the construction.

Definition 3.5.1. A 3-*manifold spectral system* (abbreviated as just a *spectral system*) for a family $\mathcal{F}$ of metrized spin^c 3-manifolds, with fiber $(Y, \mathfrak{s})$, is a tuple

$$\mathfrak{S} = \big(\mathbf{P}, \mathbf{Q}, \mathbf{W}_P, \mathbf{W}_Q, \{\eta^P_n\}_n, \{\eta^Q_n\}, \{\eta^{W_P}_n\}_n, \{\eta^{W_Q}_n\}_n\big), \tag{3.5.1}$$

where $\mathbf{P} = \{P_n\}_n$ (for $n \geq 0$) is a sequence of good (increasing) spectral sections of the Dirac operator $-D$; similarly, $\mathbf{Q} = \{Q_n\}_n$ is a sequence of good increasing spectral sections of D parameterized by $\operatorname{Pic}(\mathcal{F})$. The $\mathbf{W}_P = \{W_{P,n}\}_n$ are good spectral sections of the operator $-{*d}$; similarly, $\mathbf{W}_Q = \{W_{Q,n}\}_n$ are good spectral sections of ${*d}$. We require $W_{P,0}$ to be the sum of all negative eigenspaces of ${*d}$, as we may, since the nullspace of ${*d}$, acting on the bundle $L^2_k(\operatorname{im} d^*)$, is trivial, and similarly $W_{Q,0}$ will be the sum of positive eigenspaces. The η_n are exactly as in Theorem 3.1.1.

We have not established that there exist good sequences of spectral sections for ${*d}$ for all families $\mathcal{F}$. However, they exist in many situations, as for example when the family $\mathcal{F}$ is obtained as a mapping torus of a self-diffeomorphism preserving the fiber metric. In this case, $\mathcal{F}$ is a family over S^1 and the eigenvalues of ${*d}$ are constant functions on S^1. More generally, if there is a neighborhood U of b for each $b \in L$ such that $\mathcal{F}$ has a local trivialization $\mathcal{F}|_U \cong U \times Y$ preserving the fiber metric, then the eigenvalues of $*d$ are constants. So we have a good sequence of spectral sections of $*d$.

Definition 3.5.2. The *unparameterized Seiberg–Witten Floer spectrum*

$$\mathbf{SWF}^u(\mathcal{F}, \mathfrak{S}, k_+, k_-)$$

of a family $\mathcal{F}$ as in Definition 3.5.1 associated to a spectral system $\mathfrak{S}$, and $k_\pm$ half-integers with $k_\pm > 5$ and $|k_+ - k_-| \leq 1/2$, is the (partially defined) equivariant spectrum, whose sequence of spaces is defined as follows.

Let $\mathfrak{S}$ be a spectral system with components as named in (3.5.1). Let

$$D_n = (\dim(P_n - P_0), \dim(Q_n - Q_0), \dim(W_{P,n} - W_{P,0}), \dim(W_{Q,n} - W_{Q,0})),$$

whose components we denote D_n^ℓ for $\ell = 1, \ldots, 4$. Recall (cf. Appendix A.3) that we must assign, for a certain collection of representations, a space to each representation, together with structure maps. The spaces in the Seiberg–Witten Floer spectrum are most naturally defined at those representations $\mathbb{C}^{D_n^2} \oplus \mathbb{R}^{D_n^4}$; in order to define the spectra at other levels, we extrapolate from the definitions at these levels; see also Remark 3.5.13.

Let $\mathbb{N}_0$ be the set of nonnegative integers. For $(i_1, i_2) \in \mathbb{N}_0^2$ sufficiently large, let $A(i_1, i_2) = (A(i_1, i_2)_1, A(i_1, i_2)_2)$ denote the largest pair (D_n^2, D_n^4) among pairs (D_j^2, D_j^4) for which $(D_j^2, D_j^4) \leq (i_1, i_2)$. We can write

$$A(i_1, i_2) = (D^2_{n(i_1,i_2)}, D^4_{n(i_1,i_2)})$$

for some $n(i_1, i_2) \in \mathbb{N}_0$. Set $\mathbf{SWF}^u_{i_1,i_2}(\mathcal{F}, \mathfrak{S}, k_+, k_-)$ to be

$$\Sigma^{\mathbb{C}^{i_1 - A(i_1,i_2)_1} \oplus \mathbb{R}^{i_2 - A(i_1,i_2)_2}} \mathcal{SWF}^u_{[n(i_1,i_2)]}(\mathcal{F}, \mathfrak{S}, k_+, k_-).$$

Here, $\mathcal{SWF}^u_{[n(i_1,i_2)]}(\mathcal{F}, \mathfrak{S}, k_+, k_-)$ is the (unparameterized) Conley index with respect to the flow $\varphi_{n(i_1,i_2),k_+,k_-}$. If (i_1, i_2) is not sufficiently large, let $\mathbf{SWF}^u_{i_1,i_2}(\mathcal{F}, \mathfrak{S}, k_+, k_-)$ be a point. Define the transition map

$$\sigma_{(i,j),(i+1,j)} \colon \Sigma^{\mathbb{C}} \mathbf{SWF}^u_{i,j} \to \mathbf{SWF}^u_{i+1,j},$$

where $i + 1 \neq D_n^2$ for any n, as the identity (with the $\mathbb{C}$ factor contributing to the leftmost factor of $\Sigma^{\mathbb{C}^{i_1 - A(i_1,i_2)_1}}$), and similarly for transitions in the real coordinate. If $i + 1 = D_n^2$ for some n, we use the $(\eta_n)_*$ as defined in Theorem 3.1.1. Note that the $(\eta_n)_*$ are only well defined up to homotopy; we choose representatives in the homotopy class.

In the event that the family has a self-conjugate spin^c structure, and so that the spectral section $\mathfrak{S}$ is preserved by j, we use $\mathbb{H}$ instead of $\mathbb{C}$ above, as appropriate, so that $\mathbf{SWF}^u$ is indexed on the Pin(2)-universe described in Appendix A.1. To be more specific, we write $\mathbf{SWF}^{u,\mathrm{Pin}(2)}(\mathcal{F},\mathfrak{S})$ for the Pin(2)-spectrum invariant. In particular, $\mathbf{SWF}^{u,\mathrm{Pin}(2)}_{i,j}$, viewed as an S^1-space, is identified with $\mathbf{SWF}^u_{2i,j}$.

We will often suppress some arguments of $\mathbf{SWF}^u$ from the notation where they are clear from context.

At the point-set level, there is a choice of index pairs (at each level (i_1,i_2)) involved in Definition 3.5.2. However, the space $\mathcal{SWF}^u_{[n]}(\mathcal{F},\mathfrak{S},k_+,k_-)$ is well defined up to canonical homotopy, since the Conley index forms a connected simple system, Theorem A.2.3.

Remark 3.5.3. We would be able to repeat Definition 3.5.2 in the parameterized setting, replacing the spectrum $\mathbf{SWF}^u$ with a parameterized spectrum $\mathbf{SWF}$, except that it is not known that the parameterized Conley index forms a connected simple system in $\mathcal{K}_{G,B}$, the category considered in Appendix A.

The spaces $\mathbf{SWF}^u_{(i_1,i_2)}(\mathcal{F})$ for (i_1,i_2) not a pair (D^2_n, D^4_n), for some n, seem to have rather an awkward definition, because they do not naturally represent the Conley index of some fixed flow. However, they may be viewed as the Conley indices of a split flow on $\underline{V}\times_{\mathrm{Pic}(\mathcal{F})}\mathcal{SWF}_{[n]}(\mathcal{F})$, for $V=\mathbb{C}^{i_1-D^2_n}\oplus\mathbb{R}^{i_2-D^4_n}$ a vector space equipped with a linear (repelling) flow.

More generally, associated to a spectral system $\mathfrak{S}$, we define the virtual dimension of the vector bundle $F_n\oplus W_n$ as

$$D_n=(\dim(P_n-P_0),\dim(Q_n-Q_0),\dim(W_n^+),\dim(W_n^-)).$$

We write $\mathfrak{S}(\vec{i})$ for the vector bundle of virtual dimension $\vec{i}=(i_1,i_2,i_3,i_4)$. If the spectral section does not produce a vector bundle in that virtual dimension, we define

$$\mathfrak{S}(i_1,i_2,i_3,i_4)=\underline{V}\oplus F_n\oplus W_n,$$

where $F_n\oplus W_n$ is the largest vector bundle coming from $\mathfrak{S}$ with virtual dimension at most (i_1,i_2,i_3,i_4), and where we define $\underline{V}$ to be the trivial S^1 (or Pin(2), as appropriate) vector bundle with dimension $(i_1,i_2,i_3,i_4)-D_n$. When we need to distinguish between the contributions of $F_n\oplus W_n$ and $\underline{V}$ to $\mathfrak{S}(\vec{i})$, we call $F_n\oplus W_n$ the *geometric* bundle, and $\underline{V}$ the *virtual* bundle.

We can treat $\mathfrak{S}(i_1,i_2,i_3,i_4)$ as a vector bundle with a split flow, as discussed above; its unparameterized Conley index is (canonically, up to homotopy) homotopy equivalent to $\mathcal{SWF}^u_{(i_2,i_4)}(\mathcal{F},\mathfrak{S})$.

Let

$$\underline{V}(\vec{i},\vec{j})=\underline{\mathbb{C}}^{j_1-i_1}\oplus\underline{\mathbb{C}}^{j_2-i_2}\oplus\underline{\mathbb{R}}^{j_3-i_3}\oplus\underline{\mathbb{R}}^{j_4-i_4},$$

viewed as a vector bundle with linear flow, outward in the even factors, inward in the odd factors. Note that for any $\vec{j} \geq \vec{i}$ (that is, $j_1 \geq i_1, \ldots, j_4 \geq i_4$), there is a vector bundle morphism

$$\underline{V}(\vec{i}, \vec{j}) \oplus \mathfrak{S}(\vec{i}) \to \mathfrak{S}(\vec{j}), \tag{3.5.2}$$

as follows. Indeed, if $A(\vec{i}) = A(\vec{j})$, then (3.5.2) is defined by

$$\begin{aligned} \underline{V}(\vec{i}, \vec{j}) \oplus (\underline{V}(D_n, \vec{i}) \oplus F_n \oplus W_n) &= (\underline{V}(\vec{i}, \vec{j}) \oplus \underline{V}(D_n, \vec{i})) \oplus F_n \oplus W_n \\ &\to \underline{V}(D_n, \vec{j}) \oplus F_n \oplus W_n. \end{aligned}$$

If $\vec{j} = D_{n+1}$ and $\vec{i} = D_n$, the morphism (3.5.2) is just the structure map involved in the definition of a spectral system. For more general $\vec{j}, \vec{i}$, the morphism (3.5.2) is the composite coming from the sequence $\vec{i} \to D_{n_1} \to \cdots \to D_{n_k} = A(\vec{j}) \to \vec{j}$, where the rightmost factors of $\underline{V}(\vec{i}, \vec{j})$ are used first.

Similarly, we define $P(i_1) = \underline{\mathbb{C}}^{i_1 - D^1_{A(i_1)}} \oplus P_{A(i_1)}$, etc.

Definition 3.5.4. We call two spectral systems $\mathfrak{S}_1$ and $\mathfrak{S}_2$ for the same family $\mathcal{F}$ *equivalent* if there exists a collection of bundle isomorphisms,

$$\Phi_{P,i} \colon P^1(i) \to P^2(i),$$

and similarly for Q, W_P, W_Q, for all i sufficiently large, satisfying the following conditions. First, there exists some sufficiently large n, so that the $\Phi_{P,i}$ (respectively $\Phi_{Q,i}$ etc.), as i becomes large, must preserve the subbundles P^j_n for $j = 1, 2$ (similarly for Q^j_n etc.). (Indeed, for $\vec{i}$ sufficiently large, P^1_n (respectively Q^1_n etc.) will be contained in the geometric bundles of $P^2(i)$ (respectively $Q^2(i)$ etc.).)

Second, the Φ_i must be compatible with the structure maps of $\mathfrak{S}_1, \mathfrak{S}_2$ in that the following square commutes (as well as its analogs):

$$\begin{tikzcd} \underline{V} \oplus P^1(i) \arrow[r, "\mathrm{id} \oplus \Phi_{P,i}"] \arrow[d, "\eta"'] & \underline{V} \oplus P^2(i) \arrow[d, "\eta"] \\ P^1(j) \arrow[r, "\Phi_{P,j}"'] & P^2(j). \end{tikzcd}$$

We do not require the isomorphisms Φ_i (etc.) to preserve all of the P^j_n as n varies.

Note that a morphism of spectral systems as in Definition 3.5.4 also induces maps

$$\Phi_{\vec{i}} \colon \mathfrak{S}_1(\vec{i}) \to \mathfrak{S}_2(\vec{i})$$

for $\vec{i}$ sufficiently large, which preserve the subbundles $F^1_n \oplus W^1_n$ (which lie in $\mathfrak{S}_2(\vec{i})$ for $\vec{i}$ sufficiently large naturally), for some fixed large n, for $\vec{i}$ sufficiently large. There is also a commutative square:

$$
\begin{tikzcd}
\underline{V} \oplus \mathfrak{S}_1(\vec{i}) \arrow[r, "\mathrm{id} \oplus \Phi_{\vec{i}}"] \arrow[d, "\eta"'] & \underline{V} \oplus \mathfrak{S}_2(\vec{i}) \arrow[d, "\eta"] \\
\mathfrak{S}_1(\vec{j}) \arrow[r, "\Phi_{\vec{j}}"'] & \mathfrak{S}_2(\vec{j}).
\end{tikzcd}
$$

Proposition 3.5.5. *For $\mathcal{F}$ a family of* spin^c *3-manifolds, n sufficiently large and $\Phi\colon \mathfrak{S}_1 \to \mathfrak{S}_2$ an equivalence of spectral systems, there is a homotopy equivalence, well defined up to homotopy,*

$$\Phi^u_{n,*} : \mathcal{SWF}^u_{[n]}(\mathcal{F}, \mathfrak{S}_1) \to \mathcal{SWF}^u_{[n]}(\mathcal{F}, \mathfrak{S}_2).$$

In fact, there is a fiberwise-deforming homotopy equivalence,

$$\Phi_{n,*}\colon \mathcal{SWF}_{[n]}(\mathcal{F}, \mathfrak{S}_1) \to \mathcal{SWF}_{[n]}(\mathcal{F}, \mathfrak{S}_2),$$

so that $\Phi^u_{n,*} = \nu_! \Phi_{n,*}$. Here, ν is the map $\mathrm{Pic}(\mathcal{F}) \to *$ *sending* $\mathrm{Pic}(\mathcal{F})$ *to a point, and $\nu_!$ is defined as in Appendix A. (Note that $\Phi_{n,*}$ is not claimed to be well defined.) Analogous statements hold for* Pin(2)*-equivariant spectral sections.*

Proof. We consider the pullback of the flow φ_2 on $\mathfrak{S}_2(\vec{i})$ by the morphism (for some large $\vec{i}$)

$$\Phi_{\vec{i}}\colon \mathfrak{S}_1(\vec{i}) \to \mathfrak{S}_2(\vec{i}),$$

defining a flow on $\mathfrak{S}_1(\vec{i})$. Following the proof of Theorem 3.1.1, we see that there is a well-defined, up to homotopy, deformation of $\Phi^*_{\vec{i}}\varphi_2$ to φ_1. Deformation invariance of the Conley index gives a fiberwise-deforming homotopy equivalence

$$I(\varphi_1) \to I((\Phi_{\vec{i}})^*\varphi_2) \cong I(\varphi_2),$$

where the isomorphism is canonical (at the point-set level). Passing to the unparameterized Conley index, the morphism

$$I^u(\varphi_1) \to I^u((\Phi_{\vec{i}})^*\varphi_2)$$

is canonical (up to homotopy). This gives the proposition. ■

We write $[\mathfrak{S}]$ for the equivalence class of a spectral system $\mathfrak{S}$.

Remark 3.5.6. As usual, if Conjecture A.2.4 holds, then $\Phi_{n,*}$ appearing in Proposition 3.5.5, is well defined.

Theorem 3.5.7. *The equivariant parameterized stable homotopy type of*

$$\Sigma_B^{\mathbb{C}^{-D_n^2} \oplus \mathbb{R}^{-D_n^4}} \mathcal{SWF}_{[n]}(\mathcal{F}, [\mathfrak{S}])$$

is independent of the choices in its construction. That is, it is independent of

(1) *the choice of* k_+, k_-,

(2) *the element* $n \gg 0$,

(3) *a choice of spectral system* $\mathfrak{S}$ *representing the equivalence class* $[\mathfrak{S}]$,

(4) *the family of metrics on* $\mathcal{F}$.

Here, $\Sigma_B^{\mathbb{C}^{-D_n^2} \oplus \mathbb{R}^{-D_n^4}}$ *stands for the desuspension by* $\mathbb{C}^{D_n^2} \oplus \mathbb{R}^{D_n^4}$ *in the category* $PSW_{S^1,B}$. *See Appendix A.1.*

If the spin^c *structure is self-conjugate, a similar statement holds for*

$$\Sigma_B^{\mathbb{H}^{-D_n^2} \oplus \tilde{\mathbb{R}}^{-D_n^4}} \mathcal{SWF}_{[n]}(\mathcal{F}, [\mathfrak{S}]).$$

Proof. Proposition 3.5.5 addresses changes in the spectral section. Proposition 3.4.1 addresses varying of $k_\pm$. The choice of n was handled in Theorem 3.1.1, and the metric was addressed in Theorem 3.3.1. ∎

Definition 3.5.8. The *Seiberg–Witten Floer parameterized homotopy type*

$$\mathcal{SWF}(\mathcal{F}, [\mathfrak{S}])$$

is defined as the class of

$$\Sigma_B^{\mathbb{C}^{-D_n^2} \oplus \mathbb{R}^{-D_n^4}} \mathcal{SWF}_{[n]}(\mathcal{F}, [\mathfrak{S}]),$$

for any n.

When the spin^c structure is self-conjugate, the Pin(2)-*Seiberg–Witten Floer parameterized homotopy type* $\mathcal{SWF}^{\mathrm{Pin}(2)}(\mathcal{F}, [\mathfrak{S}])$ is defined as the class of

$$\Sigma_B^{\mathbb{H}^{-D_n^2} \oplus \tilde{\mathbb{R}}^{-D_n^4}} \mathcal{SWF}_{[n]}(\mathcal{F}, [\mathfrak{S}]).$$

Recall from Appendix A.3 that a *weak* morphism of spectra is a (collection of) maps that is only defined in sufficiently high degrees (this is also the case for ordinary morphisms in Adams' [2] category of spectra).

Theorem 3.5.9. *For* $\mathcal{F}$ *a family of* spin^c 3*-manifolds, and* $\Phi\colon \mathfrak{S}_1 \to \mathfrak{S}_2$ *an equivalence of spectral systems, there is a weak morphism which is a homotopy equivalence (see Appendix A.3), well defined up to homotopy:*

$$\Phi_*\colon \mathbf{SWF}^u(\mathcal{F}, \mathfrak{S}_1) \to \mathbf{SWF}^u(\mathcal{F}, \mathfrak{S}_2).$$

That is, the collection of spectra

$$\mathbf{SWF}^u(\mathcal{F},[\mathfrak{S}]) = \{\mathbf{SWF}^u(\mathcal{F},\mathfrak{S})\}_{\mathfrak{S}}$$

forms a connected simple system in spectra, if $\mathcal{F}$ admits a spectral system.

Proof. First, independence of $\mathbf{SWF}^u(\mathcal{F},[\mathfrak{S}])$ from the choice of Sobolev norms was handled in Proposition 3.4.1. Moreover, variation of metric, for a particular level $\mathcal{SWF}^u_{[n]}(\mathcal{F},[\mathfrak{S}])$, was handled in Theorem 3.3.1. We then need only show that an equivalence of spectral systems induces a well-defined, up to homotopy, morphism

$$\mathbf{SWF}^u(\mathcal{F},\mathfrak{S}_1) \to \mathbf{SWF}^u(\mathcal{F},\mathfrak{S}_2).$$

For this, we use Proposition 3.5.5 to define the maps levelwise, and we need only show that the following square homotopy commutes (the squares involving other vector bundles $\mathfrak{S}(i_1,i_2,i_3,i_4)$ are straightforward):

$$\begin{tikzcd}
\Sigma^{V_n}\mathcal{SWF}^u_{[n]}(\mathcal{F},\mathfrak{S}_1) \arrow[r, "\mathrm{id}\wedge\Phi_{n,*}"] \arrow[d, "\eta_{n,*}"'] & \Sigma^{V_n}\mathcal{SWF}^u_{[n]}(\mathcal{F},\mathfrak{S}_2) \arrow[d, "\eta_{n,*}"] \\
\mathcal{SWF}^u_{[n+1]}(\mathcal{F},\mathfrak{S}_1) \arrow[r, "\Phi_{n+1,*}"'] & \mathcal{SWF}^u_{[n+1]}(\mathcal{F},\mathfrak{S}_2).
\end{tikzcd}$$

Here, $V_n = \mathbb{C}^{D^2_{n+1}-D^2_n} \oplus \mathbb{R}^{D^4_{n+1}-D^4_n}$. This is a consequence of the two composites involved being Conley-index continuation maps associated to deformations of the flow. Observe that the composite deformations are related to each other by a deformation of deformations. By [47, Section 6.3], the square homotopy commutes (the necessary adjustments of Salamon's argument for equivariance are straightforward). ∎

As usual, subject to Conjecture A.2.4, Theorem 3.5.9 would hold in the parameterized case.

Moreover, it is easy to determine when two spectral systems are equivalent, as follows.

Lemma 3.5.10. *The set of spectral systems for a family $\mathcal{F}$ of* spin^c 3*-manifolds up to equivalence, if nonempty, is affine equivalent to* $K(\mathrm{Pic}(\mathcal{F})) \times K(\mathrm{Pic}(\mathcal{F}))$*, where the difference of systems $\mathfrak{S}_1$, $\mathfrak{S}_2$ is sent to* $([P^1_0 - P^2_0],[Q^1_0 - Q^2_0])$.

Proof. By its construction, an equivalence of spectral systems is determined by its value $(\Phi_{P,i},\Phi_{Q,i},\Phi_{W_P,i},\Phi_{W_Q,i})$ for any sufficiently large i. In the positive spectral section part of the spinor coordinate, to construct an equivalence $\mathfrak{S}_1 \to \mathfrak{S}_2$ it is sufficient (and necessary) to construct an isomorphism $P^1(i) - P^1_n \to P^2(i) - P^1_n$ for some i large, relative to a fixed (large) n. By definition, $P^1(i) - P_n^1$ is canonically some number of copies of $\underline{\mathbb{C}}$, and so such an isomorphism exists if and only if

$$[P^2(i) - P_n^1] = [\underline{\mathbb{C}}^{\dim(P^1(i)-P_n^1)}].$$

This condition is satisfied exactly when $[P_0^1 - P_0^2] = 0 \in K(\mathrm{Pic}(\mathcal{F}))$, as needed.

The 1-form coordinate is handled similarly, but the bundles $W_n^{\pm}$ there are always trivial. ∎

In particular, we note that there is a canonical choice, subject to a choice of Q_0, and up to adding trivial bundles, of a spectral section P_0, by requiring $P_0 - Q_0$ trivializable. We call these *normal* spectral sections; the set of equivalence classes of such is affine equivalent to $K(\mathrm{Pic}(Y))$, as above.

Definition 3.5.11. An (S^1-equivariant) *Floer framing* is an equivalence class of normal spectral sections. A Pin(2)-equivariant Floer framing is a (Pin(2))-equivalence class of normal spectral sections. Here, a Pin(2)-equivalence of (Pin(2)-equivariant) spectral sections is a collection of isomorphisms as in Definition 3.5.4 that are Pin(2)-equivariant.

There are various extensions of Lemma 3.5.10. Let us state a Pin(2)- equivariant version of the lemma.

Lemma 3.5.12. *The set of* Pin(2)*-spectral systems for a family* $\mathcal{F}$ *of* spin^c 3*-manifolds up to equivalence, if nonempty, is affine equivalent to*

$$KQ(\mathrm{Pic}(\mathcal{F})) \times KQ(\mathrm{Pic}(\mathcal{F})),$$

where the difference of systems $\mathfrak{S}_1$*,* $\mathfrak{S}_2$ *is sent to* $([P_0^1 - P_0^2], [Q_0^1 - Q_0^2])$*. Here,* KQ *is the quaternionic* K*-theory defined in* [19, 33].

Remark 3.5.13. We can define the spectrum $\mathbf{SWF}^u_{i_1,i_2}$ in a little different way. Fix a sufficiently large integer n and put

$$\mathbf{SWF}^u_{i_1,i_2} = \Sigma^{\mathbb{C}^{i_1-D_n^2} \oplus \mathbb{R}^{i_2-D_n^4}} \mathcal{SWF}^u_{[n]}$$

for $(i_1, i_2) \in \mathbb{N}_0^2$ with $i_1, i_2 \geq n$. The transition maps

$$\sigma_{(i_1,i_2),(i_1+1,i_2)} \colon \Sigma^{\mathbb{C}}\mathbf{SWF}^u_{i_1,i_2} \to \mathbf{SWF}^u_{i_1+1,i_2},$$
$$\sigma_{(i_1,i_2),(i_1,i_2+1)} \colon \Sigma^{\mathbb{R}}\mathbf{SWF}^u_{i_1,i_2} \to \mathbf{SWF}^u_{i_1,i_2+1}$$

are defined to be the identities. This spectrum is homotopy equivalent to the previous one.

In the previous definition of $\mathbf{SWF}^u$, we introduced $A(i_1, i_2)$, which allows us to avoid choosing a large integer n. This makes the definition of $\mathbf{SWF}^u$ more natural.

In the construction of $\mathcal{SWF}_{[n]}(\mathcal{F}, \mathfrak{S})$, we have a frame of the orthogonal complement of Q_n in Q_{n+1}. Using the frame, we have

$$\mathcal{SWF}_{[n+1]}(\mathcal{F}, \mathfrak{S}) \cong \Sigma_B^{\mathbb{C}^{k_{Q,n}} \oplus \mathbb{R}^{k_{W,-,n}}} \mathcal{SWF}_{[n]}(\mathcal{F}, \mathfrak{S}).$$

More generally, we can choose spectral sections Q_n such that the orthogonal complement of Q_n in Q_{n+1} does not necessarily have a frame. In this case, we have

$$\mathcal{SWF}_{[n+1]}(\mathcal{F}, \mathfrak{S}) \cong \Sigma_B^{(Q_{n+1}/Q_n) \oplus \mathbb{R}^{k_{W,-,n}}} \mathcal{SWF}_{[n]}(\mathcal{F}, \mathfrak{S}),$$

where Q_{n+1}/Q_n may not be trivialized. See Theorem 3.1.1. We can still define the Seiberg–Witten Floer stable homotopy type in a suitable stable homotopy category. The category is defined by taking R, W to be finite-dimensional, virtual G-vector bundles over B in Definition A.1.9, so that we can take desuspensions by nontrivial vector bundles. The Seiberg–Witten Floer stable homotopy type is defined to be the class of

$$\Sigma_B^{-(Q_n/Q_0) \oplus \mathbb{R}^{-D_n^4}} \mathcal{SWF}_{[n]}(\mathcal{F}, \mathfrak{S})$$

in the category, where n is a fixed large integer.

3.6 Elementary properties of $\mathcal{SWF}(Y, \mathfrak{s})$

Here we collect a few results about $\mathcal{SWF}(Y, \mathfrak{s})$ that follow almost directly from the definitions. We work only for a single $(Y, \mathfrak{s})$, but similar results hold in families.

Proposition 3.6.1. *The total space of $\mathcal{SWF}^u_{[n]}(Y, \mathfrak{s})$ has the homotopy type of a finite S^1-CW complex; respectively, the total space of $\mathcal{SWF}^{u,\mathrm{Pin}(2)}_{[n]}(Y, \mathfrak{s})$, when defined, is a finite* Pin(2)*-CW complex. As a consequence, for $G = S^1$ or* Pin(2)*, the Seiberg–Witten Floer spectrum $\mathbf{SWF}^{u,G}(Y, \mathfrak{s}, \mathfrak{S})$ is a finite G-CW spectrum.*

Proof. For this, we need to consider perturbations of the Seiberg–Witten equations. Recall the notion of cylinder functions from [28, Chapter 11]. As in [24, Definition 2.1], given a sequence of $\{C_j\}_{j=1}^{\infty}$ of positive real numbers and cylinder functions $\{\hat{f}_j\}_{j=1}^{\infty}$, let $\mathcal{P}$ be the Banach space

$$\mathcal{P} = \left\{ \sum_{j=1}^{\infty} \eta_j \hat{f}_j \, : \, \eta_j \in \mathbb{R}, \; \sum_{j=1}^{\infty} C_j |\eta_j| < \infty \right\}$$

with norm defined by $\| \sum_{j=1}^{\infty} \eta_j \hat{f}_j \| = \sum_{j=1}^{\infty} |\eta_j| C_j$. The elements of $\mathcal{P}$ are called *extended cylinder functions*.

For f an extended cylinder function, let grad $f = \mathfrak{q}$ be the L^2-gradient over $L^2_k(\mathbb{S}) \times \mathcal{H}^1(Y) \times L^2_k(\operatorname{im} d^*)$ of f. We write $(\mathfrak{q}_V, \mathfrak{q}_H, \mathfrak{q}_W)$ for the vertical, horizontal and 1-form components of $\mathfrak{q}$. Define the perturbed Seiberg–Witten equations by the downward gradient flow of $\mathcal{L} + f$, explicitly:

$$\begin{aligned} \frac{d\phi}{dt} &= -D_a\phi(t) - c_1(\gamma(t)) - \mathfrak{q}_V, \\ \frac{da}{dt} &= -X_H(\phi) - \mathfrak{q}_H, \\ \frac{d\omega}{dt} &= - * d\omega - c_2(\gamma(t)) - \mathfrak{q}_W. \end{aligned} \tag{3.6.1}$$

We may perform finite-dimensional approximation with the perturbed Seiberg–Witten equations in place of (2.3.2) (with the same spectral sections as for the unperturbed equations). It is straightforward but tedious to check that the proof of Theorem 2.3.3 holds also for (3.6.1), for k-extended cylinder functions f, where $k \geq \max\{k_+, k_-\} + \frac{1}{2}$. The key points are [24, Proposition 2.2] and [32, Lemma 4.10].

Moreover, for a family of perturbations, the analog of Theorem 2.3.3 continues to hold, by a similar argument. In particular, it is a consequence that $\mathcal{SWF}^u_{[n]}(Y, \mathfrak{s})$ is well defined up to canonical equivariant homotopy, independent of perturbation.

Finally, the space of perturbations $\mathcal{P}$ attains transversality for the Seiberg–Witten equations, in the sense that for a generic perturbation from $\mathcal{P}$, there are finitely many (all nondegenerate) stationary points for the perturbed formal gradient flow.

In particular, using the attractor–repeller sequence for the Conley index, together with the fact that the Conley index for a single nondegenerate critical point is a sphere, we observe that the Conley index $I^u(\varphi_{n,k_+,k_-})$ for n large is a finite G-CW complex. ∎

Proposition 3.6.2. *For $(Y, \mathfrak{s})$ a spin^c, oriented closed 3-manifold, and $\mathfrak{S}$ a spectral system, we have*

$$\mathcal{SWF}^u(Y, \mathfrak{s}, \mathfrak{S})^\vee \simeq \mathcal{SWF}^u(-Y, \mathfrak{s}, \mathfrak{S}^\vee),$$

where the spectral system $\mathfrak{S}^\vee$ is obtained by reversing the roles of P_n and Q_n in $\mathfrak{S}$.

Proof. This follows from the Spanier–Whitehead duality for the Conley index, Theorem A.2.8. ∎

Note that it would be desirable in Proposition 3.6.2 to have a similar result in the parameterized setting; the analog of Theorem A.2.8 in the parameterized setting has not been established, but would suffice.

Using the latter parts of Theorem 3.1.1, we have the following corollary.

Corollary 3.6.3. *The homotopy type of $\mathcal{SWF}_{[n]}(Y, \mathfrak{s}, \mathfrak{S})$ is independent of the spectral sections P_n for n large. That is, instead of $\mathcal{SWF}_{[n]}(Y, \mathfrak{s}, \mathfrak{S})$ depending on a* *choice in a set affine equivalent to* $K(\mathrm{Pic}(Y)) \times K(\mathrm{Pic}(Y))$, $\mathcal{SWF}_{[n]}(Y, \mathfrak{s}, \mathfrak{S})$ *is determined by a (relative) class in* $K(\mathrm{Pic}(Y))$.

Further,

$$\mathcal{SWF}_{[n]}(Y, \mathfrak{s}, \mathfrak{S}_1) \simeq \Sigma_B^{\mathfrak{S}_1 - \mathfrak{S}_2} \mathcal{SWF}_{[n]}(Y, \mathfrak{s}, \mathfrak{S}_2),$$

where $\mathfrak{S}_1 - \mathfrak{S}_2$ *is the bundle defined by Lemma* 3.5.10, *and where suspension is defined as in Remark* A.1.8.

We can now prove some of the results from the introduction.

Proof of Theorem 1.1.1. By [30], the vanishing of the triple-cup product on $H^1(Y; \mathbb{Z})$ implies that the family index of the Dirac operator on Y is trivial. Using this, fix a Floer framing $\mathfrak{P}$. In that case, Theorems 3.5.7 and 3.5.9 imply that $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P})$ and $\mathbf{SWF}(Y, \mathfrak{s}, \mathfrak{P})$ are well defined.

Proposition 3.6.1 gives the claim about finite CW structures.

Finally, when $b_1(Y) = 0$, the relationship with $SWF(Y, \mathfrak{s})$ is immediate from the definition of $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P})$, since the collection of linear subspaces used in the construction of $SWF(Y, \mathfrak{s})$ defines a spectral system as in Definition 3.5.1. ∎

Proof of Theorem 1.3.2. The argument is completely parallel to the proof of Theorem 1.1.1. ∎

Finally, we address the claims in the introduction about complex oriented cohomology theories. We start by reviewing the definition of an E-orientation of a vector bundle, where E is a multiplicative cohomology theory (see [3] for a discussion of orientability[1]). Indeed, let $V \to X$ be a topological vector bundle of rank m. Then an E-orientation is a class

$$u \in \widetilde{E}^m(\mathrm{Th}(V)),$$

so that, for all $x \in X$ and $i_x \colon S^m \to V$, the map associated to inclusion of a fiber over x, $i_x^* u$ is a unit in $\widetilde{E}^m(S^m) = \widetilde{E}^0(S^0)$ (the latter equality being the suspension isomorphism of the cohomology theory E).

Recall that a cohomology theory E is *complex oriented* if it is oriented on all complex vector bundles. There is a universal such cohomology theory, complex cobordism MU, in the sense that for any complex-oriented cohomology theory E, there is a map of ring spectra $MU \to E$ inducing the orientation on E.

The utility of a complex-oriented cohomology theory E for studying the stable homotopy type $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{S}_1)$ is as follows. By Theorem 3.1.1, we have, by changing the spectral system $\mathfrak{S}_1$ to $\mathfrak{S}_2$, that there is an (S^1-equivariant) parameterized equivalence

$$\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{S}_1) \to \Sigma^{\mathfrak{S}_1 - \mathfrak{S}_2} \mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{S}_2). \tag{3.6.2}$$

[1]nLab also has a nice discussion, which our presentation follows.

In Chapter 6, after having considered the 4-dimensional invariant, we will introduce a number $n(Y, \mathfrak{s}, g, P_0)$ associated to a spectral section P_0 of the Dirac operator over Y, and a metric g on $(Y, \mathfrak{s})$. By its construction, $n(Y, \mathfrak{s}, g, P_0) = n(Y, \mathfrak{s}, g, [\mathfrak{S}])$ is an invariant of a spectral system up to equivalence $[\mathfrak{S}]$, and its main property is that it changes appropriately to counteract the shift in (3.6.2). That is,

$$n(Y, \mathfrak{s}, g, [\mathfrak{S}_1]) - n(Y, \mathfrak{s}, g, [\mathfrak{S}_2]) = \dim[\mathfrak{S}_1 - \mathfrak{S}_2],$$

as follows immediately from (6.2.1).

For E an S^1-equivariant cohomology theory, let

$$FE^*(Y, \mathfrak{s}, \mathfrak{S}_1) = \widetilde{E}^{*-2n(Y,\mathfrak{s},g,\mathfrak{S}_1)}(\nu_! \mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{S}_1)).$$

We call $FE^*(Y, \mathfrak{s}, \mathfrak{S}_1)$ the *Floer E-cohomology* of the tuple $(Y, \mathfrak{s}, \mathfrak{S}_1)$.

More generally, we can also consider the notion of an *equivariant complex orientation*. This is more complicated to state; we follow [12] for the definition of equivariant complex orientability. That is, let A be an abelian compact Lie group, and fix a complete complex A-universe $\mathcal{U}$ (see Appendix A). A multiplicative equivariant cohomology theory $E_A^*(\cdot)$ is called *complex stable* if there are suspension isomorphisms:

$$\sigma_V \colon \widetilde{E}_A^n(X) \to \widetilde{E}_A^{n+\dim V}((V^+) \wedge X)$$

for all complex (finite-dimensional) A-representations V in $\mathcal{U}$. The natural transitivity condition on the σ_V is required, and the map σ_V is required to be given by multiplication by an element of $\widetilde{E}^{\dim V}(V^+)$ (necessarily a generator). A *complex orientation* of a complex stable theory E_A is a cohomology class $x(\varepsilon) \in E_A^*(\mathbb{C}P(\mathcal{U}, \mathbb{C}P(\varepsilon)))$ that restricts to a generator of

$$E_A^*(\mathbb{C}P(\alpha \oplus \varepsilon), \mathbb{C}P(\varepsilon)) \cong \widetilde{E}_A^*(S^{\alpha^{-1}}),$$

for all 1-dimensional representations α.

Building on the equivalence (3.6.2), we have the following claim.

Theorem 3.6.4. *Let E be an equivariant complex-oriented (nonparameterized) homology theory. Then, for any two spectral systems $\mathfrak{S}_1$, $\mathfrak{S}_2$, there is a canonical isomorphism*

$$\widetilde{E}^*(\nu_! \mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{S}_1)) \to \widetilde{E}^*(\nu_! \Sigma^{\mathfrak{S}_2 - \mathfrak{S}_1} \mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{S}_2)).$$

In particular, $FE^*(Y, \mathfrak{s}, \mathfrak{S}_1)$ is independent of $\mathfrak{S}_1$, and defines an invariant $FE^*(Y, \mathfrak{s})$.

Proof. The theorem is a consequence of the fact that, for an ex-space (X, r, s) over a base B, and a complex m-dimensional vector bundle V over B, with ν as usual the basepoint map $B \to *$,

$$\nu_! \Sigma_B^V X = \mathrm{Th}(r^* V). \tag{3.6.3}$$

This equality is a direct exercise in the definitions. In fact, if (X,r,s) is an S^1-ex-space, with base B on which S^1 acts trivially, the equality also holds at the level of S^1-spaces, where V is an S^1-equivariant vector bundle over B, inherited from its complex structure (so that the pullback r^*V is an S^1-equivariant vector bundle over the S^1-space X).

We have by (3.6.2),

$$\widetilde{E}^*(\nu_!\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S}_1)) = \widetilde{E}^*(\nu_!\Sigma^{\mathfrak{S}_1-\mathfrak{S}_2}\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S}_2)).$$

By (3.6.3),

$$\widetilde{E}^*(\nu_!\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S}_1)) = \widetilde{E}^*\big(\mathrm{Th}(r^*(\mathfrak{S}_1-\mathfrak{S}_2))\big),$$

where r is the restriction map of the ex-space $\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S}_2)$. However, the complex orientation on E induces an isomorphism,

$$\widetilde{E}^*\big(\mathrm{Th}(r^*(\mathfrak{S}_1-\mathfrak{S}_2))\big) \to \widetilde{E}^{*-2\dim(\mathfrak{S}_1-\mathfrak{S}_2)}(\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S}_2)),$$

which is exactly what we needed (the last isomorphism above, in the equivariant case, follows from the construction of Thom classes in [12, Theorem 6.3]).

The last claim of the theorem is then a consequence of the definition of FE^*. ∎

The most important equivariant complex orientable cohomology theory for us will be equivariant complex cobordism MU_G, defined by tom Dieck [50] for a compact Lie group G. It turns out, if G is abelian, that MU_G is the universal G-equivariant complex oriented cohomology theory, in the sense that any equivariant complex oriented cohomology theory E_G accepts a unique ring map of ring spectra $MU_G \to E_G$ so that the orientation on E_G is the image of the canonical orientation on MU_G. See [12].

We define $FMU^*(Y,\mathfrak{s})$ and $FMU^*_{S^1}(Y,\mathfrak{s})$ by

$$FMU^*(Y,\mathfrak{s}) = \widetilde{MU}^{*-2n(Y,\mathfrak{s},g,\mathfrak{S})}(\nu_!\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S})),$$
$$FMU^*_{S^1}(Y,\mathfrak{s}) = \widetilde{MU}^{*-2n(Y,\mathfrak{s},g,\mathfrak{S})}_{S^1}(\nu_!\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S})),$$

for some spectral sections $\mathfrak{S}$. By Theorem 3.6.4 and the complex orientation on MU and MU_{S^1}, these are well defined independent of a choice of $\mathfrak{S}$, and this proves Theorem 1.2.1.

For a spin structure $\mathfrak{s}$, we have the Pin(2)-equivariant Seiberg–Witten Floer stable homotopy type $\mathcal{SWF}^{\mathrm{Pin}(2)}(Y,\mathfrak{s},\mathfrak{S})$. To define Pin(2)-equivariant cohomology theory $FMU^*_{\mathrm{Pin}(2)}(Y,\mathfrak{s})$, we need to show that

$$\widetilde{MU}^{*-2n(Y,\mathfrak{s},\mathfrak{S})}_{\mathrm{Pin}(2)}(\nu_!\mathcal{SWF}^{\mathrm{Pin}(2)}(Y,\mathfrak{s},\mathfrak{S}))$$

is independent of the choice of $\mathfrak{S}$, which requires an orientation on $\widetilde{MU}^*_{\mathrm{Pin}(2)}$. But we cannot apply the argument in [12] to $\widetilde{MU}^*_{\mathrm{Pin}(2)}$ since Pin(2) is not abelian. We do not discuss orientations on $\widetilde{MU}^*_{\mathrm{Pin}(2)}$ in this memoir.

Chapter 4

Computation

In this chapter we provide a sample of calculations of the Seiberg–Witten Floer homotopy type.

4.1 Seiberg–Witten Floer homotopy type in reducible case

We will need the following lemma.

Lemma 4.1.1. *Let* $\varphi\colon M \times \mathbb{R} \to M$ *be a smooth flow on a smooth manifold* M *and* N *be a compact submanifold (with corners) of* M *with* $\dim M = \dim N$. *Assume that the following conditions are satisfied:*

(1) $\partial N = L_+ \cup L_-$, *where* L_+, L_- *are compact submanifolds (with corners) of* ∂N *with* $L_+ \cap L_- = \partial L_+ = \partial L_-$.

(2) *For* $x \in \operatorname{int}(L_+)$, *there is* $\varepsilon > 0$ *such that* $\varphi(x,t) \in \operatorname{int}(N)$ *for* $t \in (0,\varepsilon)$.

(3) *For* $x \in L_-$, *there is* $\varepsilon > 0$ *such that* $\varphi(x,t) \notin N$ *for* $t \in (0,\varepsilon)$.

Then N *is an isolating neighborhood and* (N, L_-) *is an index pair of* $\operatorname{inv}(N)$. *(See* [14] *for a similar statement.)*

Proof. By conditions (2) and (3), we have $\operatorname{inv}(N) \subset \operatorname{int}(N)$. It is easy to see that L_- is an exit set from the three conditions. Also, condition (3) implies that L_- is positively invariant in N. ■

Fix a spin^c 3-manifold $(Y, \mathfrak{s})$, along with a spectral system $\mathfrak{S}$, which we will usually suppress from the notation. Let $k_+, k_- > 5$ be half-integers with $|k_+ - k_-| \leq \frac{1}{2}$, $k = \min\{k_+, k_-\}$ and

$$\varphi_n = \varphi_{n,k_+,k_-}\colon (F_n \oplus W_n) \times \mathbb{R} \to F_n \oplus W_n$$

be the flow induced by the Seiberg–Witten equations.

Fix $R \gg 0$. Put

$$A_n(R) := (B_{k_+}(F_n^+; R) \times_B B_{k_-}(F_n^-; R)) \times_B (B_{k_+}(W_n^+; R) \times_B B_{k_-}(W_n^-; R)).$$

Let $I_n \to B = \operatorname{Pic}(Y)$ be the parameterized Conley index of $\operatorname{inv}(A_n(R), \varphi_n)$.

Theorem 4.1.2. *Assume that the following conditions are satisfied:*

(1) $\ker(D\colon \mathcal{E}_\infty \to \mathcal{E}_\infty) = 0$.

(2) *All solutions to the Seiberg–Witten equations* (2.3.4) *with finite energy are reducible.*

Let $\mathfrak{S}$ *be a spectral system such that* $P_0 = \mathcal{E}_0(D)^0_{-\infty}$. *Then for all* $n \gg 0$ *we have*

$$I_n \cong S_B^{F_n^- \oplus W_n^-},$$

as an S^1*-equivariant space, with the obvious projection to* B. *Hence the Seiberg–Witten Floer parameterized homotopy type is given by*

$$\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}]) \cong \Sigma_B^{\mathbb{C}^{-D_n^2} \oplus \mathbb{R}^{-D_n^4}} I_n \cong S_B^0$$

in $PSW_{S^1,B}$. *Here,* $D_n^2 = \operatorname{rank} F_n$, $D_n^4 = \operatorname{rank} W_n^-$ *and* $PSW_{S^1,B}$ *is the category defined in Definition* A.1.9.

If the spin^c *structure is self-conjugate, the* Pin(2)*-Seiberg–Witten Floer parameterized homotopy type is given by*

$$\mathcal{SWF}^{\mathrm{Pin}(2)}(Y, \mathfrak{s}, [\mathfrak{S}]) \cong S_B^0$$

in $PSW_{\mathrm{Pin}(2),B}$.

To prove this, we need the following.

Proposition 4.1.3. *Assume that all solutions to* (2.3.4) *with finite energy are reducible. For any* $\varepsilon > 0$, *there is* n_0 *such that for* $n > n_0$ *we have*

$$\operatorname{inv}(A_n(R)) \subset A_n(\varepsilon).$$

Proof. Put

$$\delta_n := \max\{\|\phi^+\|_{k_+} : (\phi, \omega) \in \operatorname{inv}(A_n(R))\}.$$

Let

$$\gamma_n = (\phi_n, \omega_n) \colon \mathbb{R} \to A_n(R)$$

be approximate Seiberg–Witten trajectories with

$$\|\phi_n^+(0)\|_{k_+} = \delta_n.$$

Then we have

$$\frac{d}{dt}\bigg|_{t=0} \|\phi_n^+(t)\|^2_{k_+} = 0.$$

As we have seen before, after passing to a subsequence, γ_n converges to a Seiberg–Witten trajectory γ with finite energy. By assumption, γ is reducible and we can write $\gamma = (0, \omega)$. As in Lemma 2.7.2, we can show that there is a constant $C > 0$ such that $\|\phi_n^+(0)\|_{k_+ + \frac{1}{2}} < C$ for all n. By the Rellich lemma, $\phi_n^+(0)$ converges to 0 in L^2_k. Therefore $\delta_n \to 0$.

Similarly,

$$\max\{\|\phi^-\|_{k_-} : (\phi, \omega) \in \mathrm{inv}(A_n(R))\},$$
$$\max\{\|\omega^+\|_{k_+} : (\phi, \omega) \in \mathrm{inv}(A_n(R))\},$$
$$\max\{\|\omega^-\|_{k_-} : (\phi, \omega) \in \mathrm{inv}(A_n(R))\}$$

go to 0 as $n \to 0$. ■

Proof of Theorem 4.1.2. Fix a small positive number ε with $\varepsilon^2 \ll \varepsilon$ and choose $n \gg 0$. By the proposition,

$$\mathrm{inv}(A_n(R)) \subset A_n(\varepsilon).$$

Put

$$\begin{aligned} L_{n,-}(\varepsilon) &= (B_{k_+}(F_n^+;\varepsilon) \times_B S_{k_-}(F_n^-;\varepsilon)) \times_B (B_{k_+}(W_n^+;\varepsilon) \times_B B_{k_-}(W_n^-;\varepsilon)) \\ &\quad \bigcup (B_{k_+}(F_n^+;\varepsilon) \times_B B_{k_-}(F_n^-;\varepsilon)) \times_B (B_{k_+}(W_n^+;\varepsilon) \times_B S_{k_-}(W_n^-;\varepsilon)), \\ L_{n,+}(\varepsilon) &= (S_{k_+}(F_n^+;\varepsilon) \times_B B_{k_-}(F_n^-;\varepsilon)) \times_B (B_{k_+}(W_n^+;\varepsilon) \times_B B_{k_-}(W_n^-;\varepsilon)) \\ &\quad \bigcup (B_{k_+}(F_n^+;\varepsilon) \times_B B_{k_-}(F_n^-;\varepsilon)) \times_B (S_{k_+}(W_n^+;\varepsilon) \times_B B_{k_-}(W_n^-;\varepsilon)). \end{aligned}$$

Then we have

$$\begin{aligned} \partial A_n(\varepsilon) &= L_{n,-}(\varepsilon) \cup L_{n,+}(\varepsilon), \\ L_{n,-}(\varepsilon) \cap L_{n,+}(\varepsilon) &= \partial L_{n,-}(\varepsilon) = \partial L_{n,+}(\varepsilon). \end{aligned}$$

We will show that the pair $(A_n(\varepsilon), L_{n,-}(\varepsilon))$ is an index pair. It is enough to check that $A_n(\varepsilon)$, $L_{n,-}(\varepsilon)$, $L_{n,+}(\varepsilon)$ satisfy conditions (2), (3) in Lemma 4.1.1. We consider the case when $k_+ \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.

Take an approximate Seiberg–Witten trajectory

$$\gamma = (\phi, \omega) \colon (-\delta, \delta) \to F_n \oplus W_n$$

for a small positive number δ.

Assume that

$$\|\phi^+(0)\|_{k_+} = \varepsilon.$$

We have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big|_{t=0} \|\phi^+(t)\|_{k_+} &= \frac{1}{2}\frac{d}{dt}\Big|_{t=0} \langle |D|^{k_+ + \frac{1}{2}} \pi^+ \phi(t), |D|^{k_+ - \frac{1}{2}} \pi^+ \phi(t) \rangle_0 \\ &= \langle (\nabla_{X_H} |D|^{k_+ + \frac{1}{2}}) \phi^+(0), |D|^{k_+ - \frac{1}{2}} \phi^+(0) \rangle_0 \\ &\quad + \langle |D|^{k_+ + \frac{1}{2}} \phi^+(0), (\nabla_{X_H} |D|^{k_+ - \frac{1}{2}}) \phi^+(0) \rangle_0 \\ &\quad + \langle (\nabla_{X_H} \pi^+) \phi(0), \phi^+(0) \rangle_{k_+} + \left\langle \frac{d\phi}{dt}(0), \phi^+(0) \right\rangle_{k_+}. \end{aligned}$$

Note that

$$\|X_H(\phi)\| = \|q(\phi)_{\mathcal{H}}\| \le C\varepsilon^2.$$

Hence we have

$$\begin{aligned}
\left|\langle(\nabla_{X_H}|D|^{k_+ + \frac{1}{2}})\phi^+(0), |D|^{k_+ - \frac{1}{2}}\phi^+(0)\rangle_0\right| &\le C\varepsilon^4, \\
\left|\langle|D|^{k_+ + \frac{1}{2}}\phi^+(0), (\nabla_{X_H}|D|^{k_+ - \frac{1}{2}})\phi^+(0)\rangle_0\right| &\le C\varepsilon^4, \\
\left|\langle(\nabla_{X_H}\pi^+)\phi(0), \phi^+(0)\rangle_{k_+}\right| &\le C\varepsilon^4,
\end{aligned}$$

by Proposition 2.5.5 and Lemma 2.5.8. Recall that $\pi^+ = 1 - \pi_{P_0}$, where π_{P_0} is the L^2-projection onto P_0. We have

$$\begin{aligned}
\left\langle \frac{d\phi}{dt}(0), \phi^+(0)\right\rangle_{k_+} &= -\langle(\nabla_{X_H}\pi_{F_n})\phi(0), \phi^+(0)\rangle_{k_+} - \langle\pi_{F_n}D\phi(0), \phi^+(0)\rangle_{k_+} \\
&\quad - \langle\pi_{F_n}c_1(\gamma(0)), \phi^+(0)\rangle_{k_+}
\end{aligned}$$

and

$$\begin{aligned}
&\langle(\nabla_{X_H}\pi_{F_n})\phi(0), \phi^+(0)\rangle_{k_+} = 0, \\
&\langle\pi_{F_n}D\phi(0), \phi^+(0)\rangle_{k_+} = \langle D\phi(0), \phi^+(0)\rangle_{k_+} \ge C\varepsilon^2, \\
&|\langle\pi_{F_n}c_1(\gamma(0)), \phi^+(0)\rangle_{k_+}| \le C\varepsilon^3.
\end{aligned}$$

Here we have used Lemma 2.5.11 for the first equality. Therefore

$$\left.\frac{d}{dt}\right|_{t=0}\|\phi^+(t)\|_{k_+}^2 \le -C\varepsilon^2 + C\varepsilon^3 < 0.$$

Assume that

$$\|\phi^-(0)\|_{k_-} = \varepsilon.$$

A similar calculation shows that

$$\left.\frac{d}{dt}\right|_{t=0}\|\phi^-(t)\|_{k_-}^2 > 0.$$

Similarly, if $\|\omega^+(0)\|_{k_+} = \varepsilon$ then $\frac{d}{dt}\big|_{t=0}\|\omega^+(t)\|_{k_+}^2 < 0$, and if $\|\omega^-(0)\|_{k_-} = \varepsilon$ then $\frac{d}{dt}\big|_{t=0}\|\omega^-(t)\|_{k_-}^2 > 0$. From these, it is easy to see that conditions (2), (3) in Lemma 4.1.1 are satisfied and we can apply Lemma 4.1.1 to conclude that the pair $(A_n(\varepsilon), L_n(\varepsilon))$ is an index pair.

Therefore we have

$$I_n = A_n(\varepsilon) \cup_{p_B} L_{n,-}(\varepsilon) \cong S_B^{F_n^- \oplus W_n^-}. \qquad \blacksquare$$

4.2 Examples

Example 4.2.1. Suppose that Y has a positive scalar curvature metric. Then the conditions of Theorem 4.1.2 are satisfied.

Example 4.2.2. Let Y be a nontrivial flat torus bundle over S^1 which is not the Hantzsche–Wendt manifold. Then Y has a flat metric and $b_1(Y) = 1$. Take a torsion spin^c structure $\mathfrak{s}$ of Y. All solutions to the unperturbed Seiberg–Witten equations on Y are reducible solutions $(A, 0)$ with $F_A = 0$. Also, all finite energy solutions to the unperturbed Seiberg–Witten equations on $Y \times \mathbb{R}$ are the reducible solutions $(\pi_Y^* A, 0)$, where A are the flat spin^c connections on Y and $\pi_Y : Y \times \mathbb{R} \to Y$ is the projection. Hence condition (2) of Theorem 4.1.2 is satisfied.

By [28, Lemma 37.4.1], if $\mathfrak{s}$ is not the torsion spin^c structure corresponding to the 2-plane field tangent to the fibers, condition (1) of Theorem 4.1.2 is satisfied.

We consider the sphere bundle of a complex line bundle over a surface Σ. We will make use of results from [42, 44] and [24, Section 8].

Let Σ be a closed, oriented surface of genus g and $p : N_d \to \Sigma$ be the complex line bundle on Σ of degree d. We will consider the sphere bundle $Y = S(N_d)$. We have

$$H^2(Y; \mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus (\mathbb{Z}/d\mathbb{Z}).$$

The direct summand $\mathbb{Z}/d\mathbb{Z}$ corresponds to the image

$$\mathrm{Pic}^t(\Sigma)/\mathbb{Z}[N_d] \xrightarrow{p^*} \mathrm{Pic}^t(Y) \xrightarrow{c_1} H^2(Y; \mathbb{Z}),$$

where $\mathrm{Pic}^t(\Sigma)$ is the set of isomorphism classes of topological complex line bundles on Σ.

Fix a torsion spin^c structure $\mathfrak{s}$. We consider a metric

$$g_{Y,r} = (r\eta)^{\otimes 2} \oplus g_\Sigma$$

on Y for $r > 0$. Here, $i\eta \in i\Omega^1(Y)$ is a constant-curvature connection 1-form of $S(N_d)$. Following [42, 44], we take the connection ∇^0 on TY which is trivial in the fiber direction and is equal to the pullback of the Levi-Civita connection on Σ on $\ker \eta$. For $a \in \mathcal{H}^1(Y)$, let $D_{r,a}$ be the Dirac operator induced by ∇^0. We have

$$D_{r,a} = D_a + \delta_r,$$

where $\delta_r = \frac{1}{2} r d$. See [42, Section 5.1] and [44, Section 2.1]. The family $\{D_{r,a}\}_{a \in \mathcal{H}^1(Y)}$ induces an operator

$$D_r : \mathcal{E}_\infty \to \mathcal{E}_\infty.$$

We consider the perturbed Seiberg–Witten equations for $\gamma = (\phi, \omega)\colon \mathbb{R} \to \mathcal{E}_\infty \times \operatorname{im} d^*$:

$$\begin{aligned} \Big(\frac{d\phi}{dt}(t)\Big)_H &= -D_r\phi(t) - c_1(\gamma(t)), \\ \Big(\frac{d\phi}{dt}(t)\Big)_V &= -X_H(\phi(t)), \\ \frac{d\omega}{dt}(t) &= -*d\omega(t) - c_2(\gamma(t)). \end{aligned} \tag{4.2.1}$$

These equations are the gradient flow equation of the perturbed Chern–Simons–Dirac functional

$$CSD_r(\phi, \omega) = CSD(\phi, \omega) + \delta_r \|\phi\|_{L^2}^2.$$

The term $\delta_r \|\phi\|_{L^2}^2$ is a tame perturbation. See [28, p. 171]. We can apply Theorem 2.3.3 to the perturbed Seiberg–Witten equations (4.2.1).

The following is a direct consequence of [42, Corollary 5.17 and Theorem 5.19]. See also [44, Section 3.2] and [24, Proposition 8.1, Section 8.2].

Proposition 4.2.3. *Let $\mathfrak{s}_0$ be the* spin^c *structure of Y with spinor bundle $\mathbb{S} = p^*K_\Sigma^{-1} \oplus \underline{\mathbb{C}}$. Denote by L_q the flat complex line bundle on Y with $c_1 \equiv q \bmod d$ in* Tor $H^2(Y;\mathbb{Z})$. *Put $\mathfrak{s}_q := \mathfrak{s}_0 \otimes L_q$. Assume that $0 < g < d$. Then for $q \in \{g, g+1, \dots, d-1\}$, all critical points of the functional CSD_r associated with $\mathfrak{s}_q$ are reducible and nondegenerate.*

Note that this proposition implies that $\ker D_r = 0$ and hence we have a natural spectral section P_0 of D_r:

$$P_0 = (\mathcal{E}_0(D_r))_{-\infty}^0.$$

The following proposition is proved in [24, proof of Theorem 7.5].

Proposition 4.2.4. *Under the same assumption as Proposition 4.2.3, any gradient trajectory of CSD_r (that is, a solution to (4.2.1)) with finite energy is reducible.*

We can apply the proof of Theorem 4.1.2 to the perturbed Seiberg–Witten equations (4.2.1) to show the following.

Theorem 4.2.5. *Take $q \in \{g, g+1, \dots, d-1\}$. Let $\mathfrak{S}$ be a spectral system with $P_0 = \mathcal{E}_0(D_r)_{-\infty}^0$. In the above notation, for r small, we have*

$$I_n \cong S_B^{F_n^- \oplus W_n^-}.$$

Therefore we have

$$\mathcal{SWF}(Y, \mathfrak{s}_q, [\mathfrak{S}]) \cong S_B^0$$

in $PSW_{S^1,B}$. If $\mathfrak{s}$ is self-conjugate,

$$\mathcal{SWF}^{\mathrm{Pin}(2)}(Y, \mathfrak{s}_q, [\mathfrak{S}]) \cong S_B^0$$

in $PSW_{\mathrm{Pin}(2),B}$.

Dai and the authors [17] computed the Seiberg–Witten Floer stable homotopy type for almost rational plumbed 3-manifolds which have $b_1 = 0$. The computation is based on surgery exact triangles in [48]. If we establish a surgery exact triangle for the Seiberg–Witten Floer stable homotopy type $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{S})$ defined in this memoir, it would be possible to compute for more 3-manifolds with $b_1 > 0$.

Chapter 5

Finite-dimensional approximation on 4-manifolds

5.1 Construction of the relative Bauer–Furuta invariant

Let $(X, \mathfrak{t})$ be a compact spin^c 4-manifold with boundary Y. Take a Riemannian metric $\hat{g}$ of X such that a neighborhood of Y in X is isometric to $Y \times (-1, 0]$. We assume that the restriction $\mathfrak{s}$ of $\mathfrak{t}$ to Y is a torsion spin^c structure. Put

$$\begin{aligned}\mathcal{E}^{\pm}_{X,k} &:= \mathcal{H}^1(X) \times_{H^1(X;\mathbb{Z})} L^2_k(\Gamma(\mathbb{S}^{\pm})),\\ \mathcal{W}_{X,k} &:= B_X \times L^2_k(\Omega^1_{CC}(X)).\end{aligned}$$

Here, $B_X = \mathrm{Pic}(X)$ and $\mathbb{S}^{\pm}$ are the spinor bundles on X and $\Omega^1_{CC}(X)$ is the space of 1-forms on X in double Coulomb gauge. See [23] for the double Coulomb gauge condition. Note that $\mathcal{E}^{\pm}_{X,k}$, $\mathcal{W}_{X,k}$ are Hilbert bundles over B_X. We have the Dirac operator

$$D_X \colon \mathcal{E}^+_{X,k} \to \mathcal{E}^-_{X,k-1}$$

on X, and as before, we can define the fiberwise norm $\|\cdot\|_k$ on $\mathcal{E}^{\pm}_{X,k}$ for each non-negative number k. Also we put

$$\begin{aligned}\mathcal{E}_{Y,k} &:= \mathcal{H}^1(Y) \times_{H^1(Y;\mathbb{Z})} L^2_k(\mathbb{S}),\\ \mathcal{W}_{Y,k} &:= B_Y \times L^2_k(\operatorname{im} d^*) \subset B_Y \times L^2_k(\Omega^1(Y)).\end{aligned}$$

Here, $P_Y = \mathrm{Pic}(Y)$.

Proposition 5.1.1. *For $k, l \geq 0$, there are constants $R_{X,k}, R_{Y,l} > 0$ such that for any solution $x \in \mathcal{E}^+_{X,2} \oplus \mathcal{W}_{X,2}$ to the Seiberg–Witten equations on X and any Seiberg–Witten trajectory $\gamma \colon \mathbb{R}_{\geq 0} \to \mathcal{E}_{Y,2} \oplus \mathcal{W}_{Y,2}$ with finite energy and with*

$$r_Y(x) = \gamma(0),$$

we have

$$\|x\|_k \leq R_{X,k}, \quad \|\gamma(t)\|_l \leq R_{Y,l}$$

for all $t \in \mathbb{R}_{\geq 0}$. Here, r_Y stands for the restriction to the boundary Y.

See [23, Section 4] for this proposition.

Let D_Y be the family of Dirac operators on Y parameterized by B_Y. Assume that $\operatorname{ind} D_Y = 0$ in $K^1(B_Y)$. Choose a spectral system $\mathfrak{S}$. As usual, put

$$F_n = P_n \cap Q_n, \quad W_n = W_{P,n} \cap W_{Q,n}.$$

Then F_n, W_n are subbundles of $\mathcal{E}_{Y,0}$, $\mathcal{W}_{Y,0}$ with finite rank.

From now on, we assume that k is a half-integer and $k > 5$ so that we can use the results in Chapters 2 and 3. We consider the map

$$\begin{aligned} SW_{X,n}\colon \mathcal{E}^+_{X,k} \oplus \mathcal{W}_{X,k} \\ \to \big(\mathcal{E}^-_{X,k-1} \times L^2_{k-1}(\Omega^+(X))\big) \times ((P_n \oplus W_{P,n}) \cap L^2_{k-\frac{1}{2}}) \end{aligned} \tag{5.1.1}$$

defined by

$$SW_{X,n}(\hat{\phi}, \widehat{\omega}) = (D_X\hat{\phi} + \rho(\widehat{\omega})\hat{\phi}, F^+_{\hat{A}} - q(\hat{\phi}), \pi_{P_n} r_Y \hat{\phi}, \pi_{W_{P,n}} r_Y \widehat{\omega}).$$

Here, π_{P_n}, $\pi_{W_{P,n}}$ are the L^2-projection, where we have also written P_n for the total space of the spectral section P_n. We will take subbundles U_n, U'_n of $\mathcal{E}^+_{X,k}$, $\mathcal{E}^-_{X,k-1}$ with finite rank as follows. The operator

$$(D_X, \pi_{P_0} r_Y)\colon \mathcal{E}^+_{X,k} \to \mathcal{E}^-_{X,k-1} \oplus r^*_Y(P_0 \cap L^2_{k-\frac{1}{2}})$$

is Fredholm. (See [40], [28, Section 17.2] and Section 2.1.) Hence there is a fiberwise linear operator

$$\mathfrak{p}\colon \underline{\mathbb{C}}^m \to \mathcal{E}^-_{X,k-1} \oplus r^*_Y(P_0 \cap L^2_{k-\frac{1}{2}})$$

such that

$$(D_X, \pi_{P_0} r_Y) + \mathfrak{p} : \mathcal{E}^+_{X,k} \oplus \underline{\mathbb{C}}^m \to \mathcal{E}^-_{X,k-1} \oplus r^*_Y(P_0 \cap L^2_{k-\frac{1}{2}}) \tag{5.1.2}$$

is surjective. Here, $\underline{\mathbb{C}}^m = B_X \times \mathbb{C}^m$ is the trivial bundle over B_X.

Lemma 5.1.2. *For any n and any subbundle U' in $\mathcal{E}^-_{X,k-1}$, $U' \oplus r^*_Y F_n$ and the image of*

$$(D_X, \pi_{P_n} r_Y) + \mathfrak{p}\colon \mathcal{E}^+_{X,k} \oplus \underline{\mathbb{C}}^m \to \mathcal{E}^-_{X,k-1} \oplus r^*_Y(P_n \cap L^2_{k-\frac{1}{2}})$$

are transverse in $\mathcal{E}^-_{X,k-1} \oplus r^*_Y(P_n \cap L^2_{k-\frac{1}{2}})$.

Proof. Take any element (x', y) from $\mathcal{E}^-_{X,k-1} \oplus r^*_Y(P_n \cap L^2_{k-\frac{1}{2}})$. There is $(x, v) \in \mathcal{E}^+_{X,k} \oplus \underline{\mathbb{C}}^m$ such that

$$((D_X, \pi_{P_0} r_Y) + \mathfrak{p})(x, v) = (x', \pi_{P_0}(y)).$$

Note that

$$P_n \cap (P_0)^\perp = F^+_n.$$

We can write

$$(D_X, \pi_{P_n} r_Y + \mathfrak{p})(x, v) = \big((D_X, (\pi_{P_0} + \pi_{F^+_n}) r_Y) + \mathfrak{p}\big)(x, v) = (x', \pi_{P_0}(y) + z),$$

where $z = \pi_{F_n^+}(r_Y x) \in F_n^+ \subset F_n$. Hence

$$\begin{aligned}(x', y) &= (x', \pi_{P_0}(y) + z) + (0, \pi_{F_n^+}(y) - z)\\ &\in \operatorname{im}((D_X, \pi_{P_n} r_Y) + \mathfrak{p}) + F_n.\end{aligned}$$

■

Take a sequence of finite-dimensional subbundles U_n' of $\mathcal{E}^-_{X,k-1}$ such that $\pi_{U_n'} \to \mathrm{id}_{\mathcal{E}^-_{X,k-1}}$ strongly as $n \to \infty$ and put

$$U_n := ((D_X, \pi_{P_n} r_Y) + \mathfrak{p})^{-1}(U_n' \oplus r_Y^* F_n). \tag{5.1.3}$$

By Lemma 5.1.2, U_n are subbundles of $\mathcal{E}^+_{X,k} \oplus \underline{\mathbb{C}}^m$. Note that

$$[U_n] - [U_n' \oplus r_Y^* F_n] - [\underline{\mathbb{C}}^m] = [\operatorname{ind}(D_X, P_n)] \in K(B_X).$$

Here, the right-hand side is the index bundle defined in [40, Section 6].

Choose finite-dimensional subbundles

$$V_n' = B_X \times V_{n,0}'$$

of $B_X \times L^2_{k-1}(\Omega^+(X))$ with $\pi_{V_n'} \to \mathrm{id}_{B_X \times L^2_{k-1}(\Omega^+(X))}$ strongly as $n \to \infty$ and put

$$V_n := (d^+, \pi_{W_{P,n}} r_Y)^{-1}(V_n' \oplus W_n) \subset \mathcal{W}_{X,k}.$$

We consider the maps

$$\begin{aligned}SW_{X,n,\mathfrak{p}} &:= (D_X, d^+) + \mathfrak{p} + \pi_{U_n' \oplus V_n'} c_X \colon U_n \oplus V_n \to U_n' \oplus V_n',\\ \widetilde{SW}_{X,n,\mathfrak{p}} &:= (SW_{X,n,\mathfrak{p}}, \pi_{P_n} r_Y, \pi_{W_{P,n}} r_Y, \mathrm{id}_{\underline{\mathbb{C}}^m}) \colon\\ &\qquad U_n \oplus V_n \to U_n' \oplus V_n' \oplus r_Y^*(F_n \oplus W_n) \oplus \underline{\mathbb{C}}^m,\end{aligned} \tag{5.1.4}$$

where

$$c_X(\hat{\phi}, \hat{\omega}) = (\rho(\hat{\omega})\hat{\phi}, F^+_{\hat{A}_0} + q(\hat{\phi}))$$

for a fixed connection $\hat{A}_0$ on X. Fix positive numbers R, R' with $0 \ll R' \ll R$. Put

$$A_n := (B_{k-\frac{1}{2}}(F_n^+; R) \times_{B_Y} B_k(F_n^-; R)) \times_{B_Y} (B_{k-\frac{1}{2}}(W_n^+; R) \times_{B_Y} B_k(W_n^-; R)).$$

Here, $B_{k-\frac{1}{2}}(F_n^+; R)$ is the ball in F_n^+ of radius R with respect to $L^2_{k-\frac{1}{2}}$, and similarly for $B_k(F_n^-; R)$, $B_{k-\frac{1}{2}}(W_n^+; R)$, $B_k(W_n^-; R)$. Note that we take different norms $L^2_{k-\frac{1}{2}}$ and L^2_k for F_n^+, W_n^+ and F_n^-, W_n^-. By Theorem 2.3.3, for $n \gg 0$, A_n is an isolating neighborhood of the flow $\varphi_{n,k-\frac{1}{2},k}$, for suitable k. For $\varepsilon > 0$, we define compact subsets $K_{n,1}(\varepsilon)$, $K_{n,2}(\varepsilon)$ of A_n by

$$\begin{aligned}K_{n,1}(\varepsilon) := \big\{y \in A_n : \exists(\hat{\phi}, v, \hat{\omega}) &\in B_k(U_n \oplus V_n; R'),\ (\hat{\phi}, v) \in U_n \subset \mathcal{E}^+_{X,k} \oplus \underline{\mathbb{C}}^m,\\ &\hat{\omega} \in V_n,\ \|(SW_{X,n,\mathfrak{p}}, \mathrm{id}_{\mathbb{C}^m})(\hat{\phi}, v, \hat{\omega})\|_{k-1} \le \varepsilon,\\ &y = \pi_{P_n \oplus W^{\mu_n}_{-\infty}} r_Y(\hat{\phi}, \hat{\omega})\big\},\end{aligned}$$

and

$$
\begin{aligned}
K_{n,2}(\varepsilon) := \big\{y \in A_n : \exists(\hat{\phi}, v, \hat{\omega}) &\in \partial B_k(U_n \oplus V_n; R'), \\
&\|(SW_{X,n,\mathfrak{p}}, \mathrm{id}_{\mathbb{C}^m})(\hat{\phi}, v, \hat{\omega})\|_{k-1} \le \varepsilon, \\
&y = \pi_{P_n \oplus W_{-\infty}^{\mu_n}} r_Y(\hat{\phi}, \hat{\omega})\big\} \\
\cup(\partial A_n \cap K_{n,1}(\varepsilon)).&
\end{aligned}
$$

Here,

$$\|(SW_{X,n,\mathfrak{p}}, \mathrm{id}_{\mathbb{C}^m})(\hat{\phi}, v, \hat{\omega})\|_{k-1} = \|SW_{X,n,\mathfrak{p}}(\hat{\phi}, \hat{\omega})\|_{k-1} + \|v\|.$$

We will show that we can find a regular index pair containing $(K_{1,n}(\varepsilon), K_{2,n}(\varepsilon))$. See Appendix A.2 for the definition of a regular index pair.

Proposition 5.1.3. *There is an $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, for n large, we can find a regular index pair (N_n, L_n) of* $\mathrm{inv}(A_n; \varphi_{n,k-\frac{1}{2},k})$ *with*

$$K_{n,1}(\varepsilon) \subset N_n \subset A_n, \quad K_{n,2}(\varepsilon) \subset L_n.$$

Proof. We write φ_n for $\varphi_{n,k-\frac{1}{2},k}$. We denote by $A_n^{[0,\infty)}$ the set

$$\big\{y \in A_n : \forall t \in [0, \infty),\ \varphi_n(y, t) \in A_n\big\}.$$

By [35, Theorem 4], it is sufficient to prove the following for n large and ε small:

(i) if $y \in K_{n,1}(\varepsilon) \cap A_n^{[0,\infty)}$ then we have $\varphi_n(y, t) \notin \partial A_n$ for all $t \in [0, \infty)$,

(ii) $K_{n,2}(\varepsilon) \cap A_n^{[0,\infty)} = \emptyset$.

Furthermore, any index pair as constructed by [35, Theorem 4] may be thickened to give a regular index pair still satisfying the conditions of the proposition. See [47, Remark 5.4].

Note that for $y \in K_{n,1}(\varepsilon)$ we have

$$\|y^+\|_{k-\frac{1}{2}} < R \tag{5.1.5}$$

for all n since the restriction $L^2_k(X) \to L^2_{k-\frac{1}{2}}(Y)$ is bounded and $R' \ll R$.

First, we will prove that (i) holds for n large and ε small. Assume that this is not true. Then there is a sequence $\varepsilon_n \to 0$ such that after passing to a subsequence, we have $y_n \in A_n$, $(\hat{\phi}_n, v_n, \hat{\omega}_n) \in B_k(U_n \oplus V_n; R')$, $t_n \in [0, \infty)$ with

$$
\begin{aligned}
&y_n = \pi_{P_n \oplus W_{P,n}} r_Y(\hat{\phi}_n, \hat{\omega}_n), \\
&\|SW_{X,n,\mathfrak{p}}(\hat{\phi}_n, \omega_n)\|^2_{k-1} + \|v_n\|^2 \le \varepsilon_n^2, \\
&\varphi_n(y_n, [0, \infty)) \subset A_n, \\
&\varphi_n(y_n, t_n) \in \partial A_n.
\end{aligned}
$$

Note that $v_n \to 0$. Let

$$\gamma_n = (\phi_n, \omega_n) \colon [0, \infty) \to F_n \oplus W_n$$

be the approximate Seiberg–Witten trajectory defined by

$$\gamma_n(t) = \varphi_n(y_n, t).$$

After passing to a subsequence, one of the following holds for all n:

(a) $\phi_n^+(t_n) \in S_{k-\frac{1}{2}}(F_n^+; R)$,
(b) $\phi_n^-(t_n) \in S_k(F_n^-; R)$,
(c) $\omega_n^+(t_n) \in S_{k-\frac{1}{2}}(W_n^+; R)$,
(d) $\omega_n^-(t_n) \in S_k(W_n^-; R)$.

Note that in cases (a) and (c), we have $t_n > 0$ because of (5.1.5).

As in the proof of Theorem 2.3.3, we can show that there is a Seiberg–Witten trajectory

$$\gamma = (\phi, \omega) \colon [0, \infty) \to \mathcal{E}_{Y,k-\frac{3}{2},k-1} \oplus \mathcal{W}_{Y,k-\frac{3}{2},k-1}$$

such that after passing to a subsequence, γ_n converges to γ uniformly in $L^2_{k-\frac{3}{2}}$ on each compact set in $[0, \infty)$. Also, after passing to a subsequence, $(\hat{\phi}_n, \widehat{\omega}_n)$ converges to a solution $(\hat{\phi}, \widehat{\omega})$ to the Seiberg–Witten equations on X uniformly in L^2_{k-1} on each compact set in the interior of X. We have

$$r_Y(\hat{\phi}, \widehat{\omega}) = \gamma(0).$$

Assume that case (a) happens for all n. As mentioned, $t_n > 0$. Hence we have

$$\frac{d}{dt}\bigg|_{t=t_n} \|\phi_n^+(t)\|^2_{k-\frac{1}{2}} = 0.$$

As in Lemma 2.7.2, we can show that there is $C > 0$ such that $\|\phi_n^+(t_n)\|_k < C$ for all n. After passing to a subsequence, $t_n \to t_\infty \in \mathbb{R}_{\geq 0}$ or $t_n \to \infty$. First assume that $t_n \to t_\infty$. By the Rellich lemma, $\phi_n^+(t_n)$ converges in $L^2_{k-\frac{1}{2}}$ strongly. This implies that

$$\|\phi^+(t_\infty)\|_{k-\frac{1}{2}} = R,$$

which contradicts Proposition 5.1.1.

Next we consider the case $t_n \to \infty$. Let

$$\underline{\gamma}_n = (\underline{\phi}_n, \underline{\omega}_n) \colon [-t_n, \infty) \to F_n \oplus W_n$$

be the approximate Seiberg–Witten trajectory defined by

$$\underline{\gamma}_n(t) := \varphi_n(y_n, t + t_n).$$

As before, we can show that there is a Seiberg–Witten trajectory

$$\underline{\gamma}\colon \mathbb{R} \to \mathcal{E}_{Y,k-\frac{3}{2},k-1} \oplus \mathcal{W}_{Y,k-\frac{3}{2},k-1}$$

such that after passing to a subsequence, $\underline{\gamma}_n$ converges to $\underline{\gamma}$ uniformly in $L^2_{k-\frac{3}{2}}$ on each compact set in $\mathbb{R}$. As before we can show that the sequence $\|\underline{\phi}_n^+(0)\|_k$ is bounded and hence $\underline{\phi}_n^+(0)$ converges to $\underline{\phi}^+(0)$ in $L^2_{k-\frac{1}{2}}$ strongly. Therefore $\|\underline{\phi}^+(0)\|_{k-\frac{1}{2}} = R$, which contradicts Proposition 2.3.2. Thus (a) cannot happen.

Let us consider the case when (b) holds for all n. We have

$$\frac{d}{dt}\bigg|_{t=t_n} \|\phi_n^-(t)\|_k^2 \le 0.$$

As in the proof of Lemma 2.7.3,

$$\begin{aligned} 0 &\ge \frac{d}{dt}\bigg|_{t=t_n} \|\phi_n^-(t)\|_k^2 \\ &\ge -\langle D'\phi_n^-(t_n), \phi_n^-(t_n)\rangle_k - CR^2\|\phi_n^-(t_n)\|_{k+\frac{1}{2}} - CR^2 \\ &= \|\phi_n^-(t_n)\|_{k+\frac{1}{2}}^2 - CR^2\|\phi_n^-(t_n)\|_{k+\frac{1}{2}} - CR^2. \end{aligned}$$

This implies that the sequence $\|\phi_n^-(t_n)\|_{k+\frac{1}{2}}$ is bounded and there is a subsequence such that $\phi_n^-(t_n)$ converges in L^2_k strongly. We have a contradiction as before.

In the case when (c) or (d) holds for all n, we have a contradiction similarly. We have proved that (i) holds for n large and ε small.

Next we will prove that (ii) holds for n large and ε small. If this is not true, there is a sequence $\varepsilon_n \to 0$ such that after passing to a subsequence, one of the following cases holds for all n:

(a) We have $(\hat{\phi}_n, v_n, \hat{\omega}_n) \in \partial B_k(U_n \oplus V_n; R')$, $y_n \in A_n^{[0,\infty)}$ with

$$\|SW_{X,n,\mathfrak{p}}(\hat{\phi}_n, \hat{\omega}_n)\|_{k-1} + \|v_n\| \le \varepsilon_n, \quad y_n = \pi_{P_n \oplus W_{P,n}} r_Y(\hat{\phi}_n, \hat{\omega}_n).$$

(b) We have $(\hat{\phi}_n, v_n, \hat{\omega}_n) \in B_k(U_n \oplus V_n; R')$, $y_n \in \partial A_n \cap A_n^{[0,\infty)}$ with

$$\|SW_{X,n,\mathfrak{p}}(\hat{\phi}_n, \hat{\omega}_n)\|_{k-1} + \|v_n\| \le \varepsilon_n, \quad y_n = \pi_{P_n \oplus W_{P,n}} r_Y(\hat{\phi}_n, \hat{\omega}_n).$$

First we consider the case (a). Let

$$\gamma_n = (\phi_n, \omega_n)\colon [0,\infty) \to F_n \oplus W_n$$

be the approximate Seiberg–Witten trajectory defined by

$$\gamma_n(t) = \varphi_n(y_n, t).$$

As before, there is a Seiberg–Witten trajectory

$$\gamma = (\phi, \omega): [0, \infty) \to \mathcal{E}_{Y,k-\frac{3}{2},k-1} \oplus \mathcal{W}_{Y,k-\frac{3}{2},k-1}$$

such that after passing to a subsequence, γ_n converges to γ uniformly in $L^2_{k-\frac{3}{2}}$ on each compact set in $[0, \infty)$. Also, there is a solution $(\hat{\phi}, \widehat{\omega})$ to the Seiberg–Witten equations on X such that after passing to a subsequence, $(\hat{\phi}_n, \widehat{\omega}_n)$ converges to $(\hat{\phi}, \widehat{\omega})$ in L^2_{k-1} on each compact set in the interior of X. We have

$$r_Y(\hat{\phi}, \widehat{\omega}) = (\phi(0), \omega(0)).$$

Since $y_n \in A_n$, we have

$$\|y_n^-\|_k = \|(\phi_n^-(0), \omega_n^-(0))\|_k \le R.$$

Hence, after passing to subsequence, $(\phi_n^-(0), \omega_n^-(0))$ converges to $(\phi^-(0), \omega^-(0))$ in $L^2_{k-\frac{1}{2}}(Y)$ strongly. By the standard elliptic estimate, we have

$$\begin{aligned} &\|\hat{\phi}_n - \hat{\phi}\|_{L^2_k(X)} \\ &\quad \le C\big(\|\hat{\phi}_n - \hat{\phi}\|_{L^2(X)} + \|D_X(\hat{\phi}_n - \hat{\phi})\|_{L^2_{k-1}(X)} + \|\phi_n^-(0) - \phi^-(0)\|_{L^2_{k-\frac{1}{2}}(Y)}\big). \end{aligned}$$

From the condition that

$$\|SW_{X,n,\mathfrak{p}}(\hat{\phi}_n, \widehat{\omega}_n)\|_{k-1} + \|v_n\| \le \varepsilon_n,$$

we have

$$\|D_X(\hat{\phi}_n - \hat{\phi})\|_{k-1} \le C(\|c_X(\hat{\phi}_n, \widehat{\omega}_n) - c_X(\hat{\phi}, \widehat{\omega})\|_{k-1} + \varepsilon_n).$$

Since $c_X(\hat{\phi}_n, \widehat{\omega}_n)$ converges to $c_X(\hat{\phi}, \widehat{\omega})$ in L^2_{k-1} strongly, $\hat{\phi}_n$ converges to $\hat{\phi}$ in L^2_k strongly.

Similarly, $\widehat{\omega}_n$ converges to $\widehat{\omega}$ in L^2_k strongly. Hence,

$$\|(\hat{\phi}, \widehat{\omega})\|_k = R'.$$

This contradicts Proposition 5.1.1, so case (a) cannot happen.

Next we consider case (b). Let

$$y_n = (\phi_n, \omega_n).$$

After passing to a subsequence, $\phi_n^- \in S_k(F_n^-; R)$ for all n, or $\omega_n^- \in S_k(W_n^-; R)$ for all n. Note that the cases $\phi_n^+ \in S_{k-\frac{1}{2}}(F_n^+; R)$, $\omega_n^+ \in S_{k-\frac{1}{2}}(W_n^+; R)$ do not happen because of (5.1.5).

We consider the case $\phi_n^- \in S_k(F_n^-; R)$. Put

$$\gamma_n(t) = (\phi_n(t), \omega_n(t)) = \varphi_n(y_n, t)$$

for $t \geq 0$. As in the proof of Lemma 2.7.3,

$$\begin{aligned} 0 &\geq \frac{d}{dt}\bigg|_{t=0} \|\phi_n^-(t)\|_k^2 \\ &\geq \|\phi_n^-\|_{k+\frac{1}{2}}^2 - CR^2\|\phi_n^-\|_{k+\frac{1}{2}} - CR^2. \end{aligned}$$

Therefore the sequence $\|\phi_n^-\|_{k+\frac{1}{2}}$ is bounded. By the Rellich lemma, ϕ_n^- converges to ϕ^- in L_k^2 strongly and hence

$$\|\phi^-\|_k = R,$$

which contradicts Proposition 5.1.1. Similarly, if $\omega_n^- \in S_k(W_n^-; R)$ for all n, we obtain a contradiction. We have proved that (ii) holds for n large and ε small. ■

Remark 5.1.4. To get (5.1.5), we used the $L_{k-\frac{1}{2}}^2$-norm on the positive component. On the other hand, in the case (ii)-(a), we used the condition that $\|\phi_n^-(0)\|_k$ is bounded (rather than $\|\phi_n^-(0)\|_{k-\frac{1}{2}}$) to have that $\phi_n^-(0)$ converges to $\phi^-(0)$ in $L_{k-\frac{1}{2}}^2$. This is why we used the L_k^2-norm on the negative component to define $K_{n,1}(\varepsilon)$, $K_{n,2}(\varepsilon)$.

In the case where $b_1(Y) = 0$, we can use the $L_{k-\frac{1}{2}}^2$-norm on both of the positive and negative component. See the proofs of [35, Proposition 6] and [23, Lemma 4.4]. In those proofs, to get the $L_{k-\frac{1}{2}}^2$-convergence of $\phi_n^-(0)$, the following identity was used:

$$e^D\phi_n^-(1) - \phi_n^-(0) = \int_0^1 \frac{d}{dt}(e^{tD}\pi^-\phi_n(t))\,dt. \tag{5.1.6}$$

In the case where $b_1(Y) > 0$, we have

$$\begin{aligned} \frac{d}{dt}(e^{tD}\pi^-\phi_n(t)) &= e^{tD}(D + \nabla_{X_H} D)\pi^-\phi_n(t) + e^{tD}(\nabla_{X_H}\pi^-)\phi_n(t) \\ &\quad - e^{tD}\pi^-\{(\pi_n D + \nabla_{X_H}\pi_{F_n})\phi_n(t) + q(\phi_n(t))\}. \end{aligned}$$

Since $(\nabla_{X_H}\pi_{F_n})\phi_n(t)$ does not converge in $L_{k-\frac{1}{2}}^2$, we cannot deduce that $\phi_n^-(0)$ converges in $L_{k-\frac{1}{2}}^2$ from (5.1.6).

For n large and ε small, let (N_n, L_n) be a regular index pair of $\mathrm{inv}(\varphi_n, A_n)$ with

$$K_{1,n}(\varepsilon) \subset N_n, \quad K_{2,n}(\varepsilon) \subset L_n.$$

Put

$$S_{B_X}^{U_n \oplus V_n} := \bigcup_{a \in B_X} B((U_n \oplus V_n)_a; R)/S((U_n \oplus V_n)_a; R),$$
$$S_{B_X}^{U_n' \oplus V_n' \oplus \underline{\mathbb{C}}^m} := \bigcup_{a \in B_X} B((U_n' \oplus V_n' \oplus \underline{\mathbb{C}}^m)_a; \varepsilon)/S((U_n' \oplus V_n' \oplus \underline{\mathbb{C}}^m)_a; \varepsilon),$$

which are sphere bundles over B_X, and let I_n be the Conley index:

$$I_n := N_n \cup_{p_{B_Y}|_{L_n}} B_Y.$$

Here, $p_{B_Y} \colon N_n \to B_Y$ is the projection. We obtain a map

$$\mathcal{BF}_{[n]}(X, \mathfrak{t}) \colon S_{B_X}^{U_n \oplus V_n} \to S_{B_X}^{U_n' \oplus V_n' \oplus \underline{\mathbb{C}}^m} \wedge_{B_X} r_Y^* I_n \tag{5.1.7}$$

defined by

$$\mathcal{BF}_{[n]}(X, \mathfrak{t})([\hat{\phi}, v, \hat{\omega}])$$
$$= \begin{cases} [SW_{X,n,\mathfrak{p}}(\hat{\phi}, v, \hat{\omega}), v] \wedge [\pi_{P_n \oplus W_{P,n}} r_Y(\hat{\phi}, \hat{\omega})] & \text{if (5.1.8) holds,} \\ *_a & \text{otherwise.} \end{cases}$$

Here, $a = p_{B_X}(\hat{\phi}, \hat{\omega})$, $*_a$ denotes the base point of the sphere $S^{(U_n' \oplus V_n' \oplus \underline{\mathbb{C}}^m)_a}$ and we have the following condition:

$$\begin{aligned} \|SW_{X,n,\mathfrak{p}}(\hat{\phi}, v, \hat{\omega})\|_{k-1}^2 + \|v\|^2 \le \varepsilon, \\ \pi_{P_n \oplus W_{P,n}} r_Y(\hat{\phi}, \hat{\omega}) \in K_{n,1}(\varepsilon). \end{aligned} \tag{5.1.8}$$

We refer to the map $\mathcal{BF}_n(X, \mathfrak{t})$ as the (relative, nth) *pre-Bauer–Furuta invariant* of $(X, \mathfrak{t})$, to emphasize that it is not yet an invariant of the construction (rather, its stable homotopy equivalence class will turn out to be an invariant).

An alternative version of this relative Bauer–Furuta invariant is obtained instead by considering the map of B_Y spaces:

$$\mathcal{BF}_{[n]}(X, \mathfrak{t}) \colon S_{B_X}^{U_n \oplus V_n} \to S_{B_X}^{U_n' \oplus V_n' \oplus \underline{\mathbb{C}}^m} \wedge_{B_Y} N_n/_{B_Y} L_n,$$

where $S_{B_X}^{U_n \oplus V_n}$ is a B_Y space using r_Y, and where $N_n/_B L_n$ is the fiberwise quotient.

5.2 Well-definedness of the relative Bauer–Furuta invariant

We next consider how the construction of the relative Bauer–Furuta invariant in (5.1.7) depends on the choices involved. This is very similar to Chapter 3, so we will abbreviate many of the arguments.

First, we address the perturbation $\mathfrak{p}$.

Lemma 5.2.1. *Let $\mathfrak{p}_1$ be a perturbation for which (5.1.2) is surjective. Let $\mathfrak{q}$ be a linear operator $\mathbb{C}^{m_2} \to \mathcal{E}^-_{X,k-1} \oplus r^*_Y(P_0 \cap L^2_{k-\frac{1}{2}})$. Let $U_n(\mathfrak{p})$, respectively $U_n(\mathfrak{p}+\mathfrak{q})$ be the bundles defined as in (5.1.3) with respect to the perturbations $\mathfrak{p}$, respectively $\mathfrak{p}+\mathfrak{q}$. Let $\mathcal{BF}_{[n],\mathfrak{p}}(X,\mathfrak{t})$, respectively $\mathcal{BF}_{[n],\mathfrak{p}+\mathfrak{q}}(X,\mathfrak{t})$, be the maps defined in (5.1.7) with respect to the perturbations $\mathfrak{p}$ and $\mathfrak{p}+\mathfrak{q}$. Then there is the following commutative diagram:*

$$\begin{tikzcd}
\Sigma^{\mathbb{C}^{m_2}} S_{B_X}^{U_n(\mathfrak{p})\oplus V_n} \arrow[r, "\Sigma^{\mathbb{C}^{m_2}} \mathcal{BF}_{[n],\mathfrak{p}}"] \arrow[d] & S_{B_X}^{U_n'\oplus V_n'\oplus \mathbb{C}^m \oplus \mathbb{C}^{m_2}} \wedge_{B_Y} I_n \arrow[d] \\
S_{B_X}^{U_n(\mathfrak{p}+\mathfrak{q})\oplus V_n} \arrow[r, "\mathcal{BF}_{[n],\mathfrak{p}+\mathfrak{q}}"'] & S_{B_X}^{U_n'\oplus V_n'\oplus \mathbb{C}^m \oplus \mathbb{C}^{m_2}} \wedge_{B_Y} I_n.
\end{tikzcd}$$

Moreover, a choice of map $L\colon \mathbb{C}^{m_2} \to \mathcal{E}^+_{X,k} \oplus \mathbb{C}^m$ so that $((D_X, \pi_{P_0} r_Y) + \mathfrak{p}) \circ L = \mathfrak{q}$ determines the vertical arrows in the diagram.

Proof. Such a choice of L as at the end of the statement exists for any such $\mathfrak{p}$, $\mathfrak{q}$, by surjectivity of (5.1.2). We show how to define maps as in the commutative diagram in terms of such L. Of course, if $\mathfrak{q} = 0$, this is obvious, with $L = 0$.

More generally, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{E}^+_{X,k} \oplus \mathbb{C}^m \oplus \mathbb{C}^{m_2} \arrow[r] \arrow[d, "\tilde{L}"'] & \mathcal{E}^-_{X,k-1} \oplus r^*_Y(P_0 \cap L^2_{k-\frac{1}{2}}) \arrow[d, "\mathrm{id}"] \\
\mathcal{E}^+_{X,k} \oplus \mathbb{C}^m \oplus \mathbb{C}^{m_2} \arrow[r] & \mathcal{E}^-_{X,k-1} \oplus r^*_Y(P_0 \cap L^2_{k-\frac{1}{2}}),
\end{tikzcd} \tag{5.2.1}$$

where $\tilde{L}$ is the identity on $\mathcal{E}^+_{X,k} \oplus \mathbb{C}^m$, and $L \oplus \mathrm{id}_{\mathbb{C}^{m_2}}$ on $\mathbb{C}^{m_2}$. The horizontal arrows are $(D_X, \pi_{P_0} r_Y) \oplus \mathfrak{p} \oplus 0$ and $(D_X, \pi_{P_0} r_Y) \oplus \mathfrak{p} \oplus \mathfrak{q}$, respectively.

Comparing with the definition of the Seiberg–Witten map (5.1.1), we see that there is a commutative diagram analogous to (5.2.1), but with the maps $\widetilde{SW}_{X,n,\mathfrak{p}}$ (and similarly for $\mathfrak{q}$) from (5.1.4) along the horizontal arrows.

The definition of $\mathcal{BF}_{[n]}(X,\mathfrak{t})$ then gives the commutative diagram in the lemma statement. ∎

As in Chapter 3, the proof of well-definedness is related to the definition of a families invariant. Let $\mathcal{F}$ be a family of (metrized, spin^c) 4-manifolds with boundary, over a base B, with fiber $(X,\mathfrak{t})$, and let $\mathcal{G}$ be the boundary family (naturally over the base B), where we write $\partial(X,\mathfrak{t}) = (Y,\mathfrak{s})$. See Section 3.2 for family of spin^c manifolds. Assume that we have fixed a sequence of good spectral sections P_n, Q_n on the boundary family.

Assume also that we have fixed a sequence of good spectral sections $W_{P,n}$, $W_{Q,n}$ of $*d$ of the boundary family, and assume $W_{P,0}$ is the orthogonal complement of $W_{Q,0}$.

As at the beginning of the section, we now have bundles $\mathcal{E}^{\pm}_{\mathcal{F},k}$ and $\mathcal{W}_{\mathcal{F},k}$, where the fibers over $b \in B$ (with associated 4-manifold $(X, \mathfrak{t})$) are

$$\begin{aligned}\mathcal{E}^{\pm}_{\mathcal{F},k,b} &:= \mathcal{H}^1(\mathcal{F}_b) \times_{H^1(X;\mathbb{Z})} L^2_k(\Gamma(\mathbb{S}^{\pm}_b)),\\ \mathcal{W}_{\mathcal{F},k,b} &:= \mathrm{Pic}(\mathcal{F}_b) \times L^2_k(\Omega^1_{CC}(\mathcal{F}_b)).\end{aligned}$$

Furthermore, the space of sections $L^2_{k-1}(\Omega^+(\mathcal{F}))$ now defines a bundle over B as well, with fiber $L^2_{k-1}(\Omega^+(\mathcal{F}_b))$, the L^2_{k-1}-self-dual 2-forms on the fiber.

The 4-dimensional Seiberg–Witten equations (5.1.1) now define a fiberwise map:

$$SW_{\mathcal{F},n} : \mathcal{E}^+_{\mathcal{F},k} \oplus \mathcal{W}_{\mathcal{F},k} \to \left(\mathcal{E}^-_{\mathcal{F},k-1} \oplus L^2_{k-1}(\Omega^+(\mathcal{F}))\right) \oplus r^*_{\mathcal{G}}(P_n \oplus W_{P,n}).$$

Define U_n as in (5.1.3), and V_n similarly. Exactly as before, define A_n; note that A_n is now a fiber bundle over the total space of the fibration $\mathrm{Pic}(\mathcal{F}) \to B$, a fiber of this latter fibration is $\mathrm{Pic}(\mathcal{F}_b)$. Define subspaces (themselves spaces over the total space of $\mathrm{Pic}(\mathcal{G}) \to B$) $K_{n,1}(\varepsilon)$ and $K_{n,2}(\varepsilon)$ with fibers $K_{n,1,b}(\varepsilon)$ and $K_{n,2,b}(\varepsilon)$ according to

$$\begin{aligned}K_{n,1,b}(\varepsilon) := \big\{y \in A_n : \exists(\hat{\phi}, v, \hat{\omega}) \in B_k(U_n \oplus V_n; R'),\ &(\hat{\phi}, v) \in U_n \subset \mathcal{E}^+_{X,k} \oplus \underline{\mathbb{C}}^m,\\ &\hat{\omega} \in V_n,\ \|(SW_{X,n,\mathfrak{p},b}, \mathrm{id}_{\mathbb{C}^m})(\hat{\phi}, v, \hat{\omega})\|_{k-1} \leq \varepsilon,\\ &y = \pi_{P_n \oplus W_{P,n}} r_{\mathcal{G}_b}(\hat{\phi}, \hat{\omega})\big\}\end{aligned}$$

and

$$\begin{aligned}K_{n,2,b}(\varepsilon) := \big\{y \in A_n : \exists&(\hat{\phi}, v, \hat{\omega}) \in \partial B_k(U_n \oplus V_n; R'),\\ &\|(SW_{X,n,\mathfrak{p},b}, \mathrm{id}_{\underline{\mathbb{C}}^m})(\hat{x}, v, \hat{\omega})\|_{k-1} \leq \varepsilon,\\ &y = \pi_{P_n \oplus W_{P,n}} r_{\mathcal{G}_b}(\hat{\phi}, \hat{\omega})\big\}\\ \cup(\partial A_n &\cap K_{n,1,b}(\varepsilon)).\end{aligned}$$

The proof of Proposition 5.1.3 is only changed in this setting according to the procedure in Chapter 3. In particular, the following proposition also relies on a families version of [35, Theorem 4]; the proof thereof is only notationally different from that appearing in [35]. A families version of Proposition 5.1.1 is also used; its proof is a modification of that in [23, Section 4]. We obtain the following proposition.

Proposition 5.2.2. *There is an $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, for n large, we can find a regular fiberwise index pair (N_n, L_n) of $\mathrm{inv}(A_n; \varphi_{n,k,k-\frac{1}{2}})$ with*

$$K_{n,1}(\varepsilon) \subset N_n \subset A_n, \quad K_{n,2}(\varepsilon) \subset L_n.$$

Put

$$S^{U_n \oplus V_n}_{\mathrm{Pic}(\mathscr{F})} := \bigcup_{a \in \mathrm{Pic}(\mathscr{F})} B((U_n \oplus V_n)_a; R)/S((U_n \oplus V_n)_a; R),$$

$$S^{U'_n \oplus V'_n \oplus \underline{\mathbb{C}}^m}_{\mathrm{Pic}(\mathscr{F})} := \bigcup_{a \in \mathrm{Pic}(\mathscr{F})} B((U'_n \oplus V'_n \oplus \underline{\mathbb{C}}^m)_a; \varepsilon)/S((U'_n \oplus V'_n \oplus \underline{\mathbb{C}}^m)_a; \varepsilon).$$

Let

$$I_n(\mathscr{G}) = N_n \cup_{p_{\mathrm{Pic}(\mathscr{G})|_{L_n}}} \mathrm{Pic}(\mathscr{G}),$$

where $p_{\mathrm{Pic}(\mathscr{G})}$ is the projection to $\mathrm{Pic}(\mathscr{G})$ of $F \times W$.

We obtain a fiber-preserving map over $\mathrm{Pic}(\mathscr{G})$:

$$\mathcal{BF}_{[n]}(\mathscr{F}) \colon S^{U_n \oplus V_n}_{\mathrm{Pic}(\mathscr{G})} \to S^{U'_n \oplus V'_n \oplus \underline{\mathbb{C}}^m}_{\mathrm{Pic}(\mathscr{G})} \wedge_{\mathrm{Pic}(\mathscr{G})} I_n(\mathscr{G}).$$

Here, $S^{U_n \oplus V_n}_{\mathrm{Pic}(\mathscr{G})}$ and $S^{U'_n \oplus V'_n \oplus \underline{\mathbb{C}}^m}_{\mathrm{Pic}(\mathscr{G})}$ are spaces over $\mathrm{Pic}(\mathscr{G})$ by pushing forward $S^{U_n \oplus V_n}_{\mathrm{Pic}(\mathscr{F})}$ and $S^{U'_n \oplus V'_n \oplus \underline{\mathbb{C}}^m}_{\mathrm{Pic}(\mathscr{F})}$ along the restriction map $\mathrm{Pic}(\mathscr{F}) \to \mathrm{Pic}(\mathscr{G})$ (see Appendix A.1).

In particular, we obtain that the homotopy class of the map $\mathcal{BF}_{[n]}(X, \mathfrak{t})$ in (5.1.7) is independent of the metric on X used in its construction. To be more precise, we have the following lemma.

Lemma 5.2.3. *Let $(X, \mathfrak{t})$ be a compact* spin^c 4*-manifold with boundary (admitting a Floer framing) $(Y, \mathfrak{s})$. Let g_t for $t \in [0, 1]$ be a path of metrics on X, along with a path of perturbations $\mathfrak{p}_t$ with surjectivity in* (5.1.2) *for all t. There exist good spectral sections $P_{n,t}$, $Q_{n,t}$, $W_{P,n,t}$, $W_{Q,n,t}$ on the boundary Y, say, forming a spectral system $\mathfrak{S}$. Let $I_n = \mathcal{SWF}_{[n]}(Y, \mathfrak{s}, \mathfrak{S})$ denote the family Seiberg–Witten invariant of the boundary. Let p denote the projection $p \colon B_Y \times I \to B_Y$, where $I = [0, 1]$. Then there exists a map*

$$\mathcal{BF}_{[n],I}(X, \mathfrak{t}) \colon S^{U_n \oplus V_n}_{B_X \times I} \to S^{U'_n \oplus V'_n \oplus \underline{\mathbb{C}}^m}_{B_X \times I} \wedge_{B_Y \times I} p^* I_n.$$

The map $\mathcal{BF}_{[n],I}(X, \mathfrak{t})$ is a map respecting the projection on each side to $B_Y \times I$.

In particular, for a fixed trivialization *of the families $U_{n,t}$, $V_{n,t}$, $U'_{n,t}$, $V'_{n,t}$ and I_n over I_+, together with a path of perturbations $\mathfrak{p}_t$, there is an (equivariant) homotopy equivalence from $\mathcal{BF}_{[n],0,\mathfrak{p}_0}$ and $\mathcal{BF}_{[n],1,\mathfrak{p}_1}$ which is well defined up to (equivariant) homotopy.*

Proof. The existence of the spectral sections follows from Chapter 2. Otherwise the lemma is a restatement of the definition of the families relative Bauer–Furuta invariant. There is no issue in choosing a good spectral section for $*d$ of the boundary family in this situation, since on $[0, 1]$, each $*d$ may be written as a (small) compact perturbation of $*_g d$, where g is some fixed metric. ∎

Further, the homotopy class of $\mathcal{BF}_{[n]}(X,\mathfrak{t})$ does not depend on the Sobolev norm used in its construction. The proof of the following lemma is analogous to the work in Section 3.4, and is left to the reader. We state the result for the unparameterized case; the parameterized case is not substantially different.

Lemma 5.2.4. *Let $(X,\mathfrak{t})$ be a compact* spin^c 4*-manifold with boundary (admitting a Floer framing) $(Y,\mathfrak{s})$. Let U_n' be a sequence of finite-dimensional subbundles of $\mathcal{E}^-_{X,k}$ for $k>11/2$, and $V_n'=B_X\times V_{n,0}'$ be a sequence of finite-dimensional subbundles of $B_X\times L^2_k(\Omega^+(X))$, where $V_{n,0}'\subset L^2_k(\Omega^+(X))$, with $\pi_{U_n'}\to \mathrm{id}_{\mathcal{E}^-_{X,k}}$ and $\pi_{V_n'}\to \mathrm{id}_{B_X\times L^2_k(\Omega^+(X))}$ strongly. Let $\mathcal{BF}_{[n],k+1}(X)$ and $\mathcal{BF}_{[n],k}(X)$ be the pre-Bauer–Furuta invariants defined with respect to the L^2_{k+1} and L^2_k-norms respectively. Write I for the interval* $[0,1]$. *Then there is a family of maps over the interval,*

$$\mathcal{BF}_{[n],I}(X,\mathfrak{t})\colon S^{U_n\oplus V_n}_{B_X\times I}\to S^{U_n'\oplus V_n'\oplus\underline{\mathbb{C}}^m}_{B_X\times I}\wedge_{B_Y\times I}\mathcal{SWF}_{[n]}(Y)_I,$$

where $\mathcal{SWF}_{[n]}(Y)_I$ is the parameterized Conley index coming from the I-family of flows used in the proof of Proposition 3.4.1. In particular, for the given homotopy equivalence in Proposition 3.4.1, the maps $\mathcal{BF}_{[n],k}(X,\mathfrak{t})$ and $\mathcal{BF}_{[n],k+1}(X,\mathfrak{t})$ are homotopic by a homotopy well defined up to homotopy.

We next consider the effect of stabilization on $\mathcal{BF}_{[n]}$. There are two separate stabilizations: increasing U_n', V_n', or increasing P_n, Q_n, $W_n^{\pm}$. Fix trivializations of $U_{n+1}'/U_n'=\underline{\mathbb{C}}^{c_n}$ and $V_{n+1}'/V_n'=\underline{\mathbb{R}}^{d_n}$. Recall the definition of a *spectral system* from Definition 3.5.1. By construction, U_{n+1} is naturally identified with $U_n\oplus\underline{\mathbb{C}}^{k_{Q_n}+c_n}$ for $k_{P,n}$, $k_{Q,n}$ as in Theorem 3.1.1, using the isomorphism $\eta\colon P_{n+1}\to P_n\oplus\underline{\mathbb{C}}^{k_{P,n}}$, and similarly for $k_{Q,n}$. Analogously, V_{n+1} is identified with $V_n\oplus\underline{\mathbb{R}}^{k_{W,-,n}+d_n}$. Let $\varphi_{n+1,t}$ denote the family of flows as in Theorem 3.1.1, with n chosen large enough. Recall that there is an induced homotopy equivalence

$$\Sigma_{B_Y}^{\underline{\mathbb{C}}^{k_{Q,n}}\oplus\underline{\mathbb{R}}^{k_{W,-,n}}}\mathcal{SWF}_{[n]}(Y)\to\mathcal{SWF}_{[n+1]}(Y)$$

as in Theorem 3.1.1.

Stabilization of the Bauer–Furuta invariant is as follows. Let $c_n'=c_n+k_{Q,n}$ and $d_n'=d_n+k_{W,-,n}$.

Proposition 5.2.5. *For appropriate choices of index pairs, there is a homotopy-commuting square of parameterized spaces, defined by Conley index continuation maps:*

$$\begin{tikzcd}
S^{\underline{\mathbb{C}}^{c_n'}\oplus\underline{\mathbb{R}}^{d_n'}}_{B_X}\wedge_{B_X}S^{T_n}_{B_X} \arrow[r] \arrow[d, "\mathrm{id}\wedge_{B_X}\mathcal{BF}_{[n]}"'] & S^{T_{n+1}}_{B_X} \arrow[d, "\mathcal{BF}_{[n+1]}"] \\
S^{\underline{\mathbb{C}}^{c_n'}\oplus\underline{\mathbb{R}}^{d_n'}\oplus T_n'\oplus\underline{\mathbb{C}}^m}_{B_X}\mathcal{SWF}_{[n]}(Y) \arrow[r] & S^{T_{n+1}'\oplus\underline{\mathbb{C}}^m}_{B_X}\wedge_{B_Y}\mathcal{SWF}_{[n+1]}(Y),
\end{tikzcd}\tag{5.2.2}$$

$T_n = U_n \oplus V_n$, $T'_n = U'_n \oplus V'_n$. In particular, (5.2.2) is a homotopy-commuting square of (unparameterized) connected simple systems.

Proof. The proof is similar to the proof of Theorem 3.1.1, and we will only roughly sketch the details. Indeed, the bottom arrow of (5.2.2) is exactly the map defined in that theorem.

Recall that we have fixed identifications $U_{n+1}/U_n = \underline{\mathbb{C}}^{c_n + k_{Q,n}}$ To obtain that (5.2.2) homotopy-commutes, we deform $\widetilde{SW}_{X,n+1,\mathfrak{p}} = \widetilde{SW}_{X,n+1,\mathfrak{p},0}$ by a family $\widetilde{SW}_{X,n+1,\mathfrak{p},t}$, by removing (linearly in t) the nonlinear terms in $SW_{X,n,\mathfrak{p}}$ on the U_{n+1}/U_n and V_{n+1}/V_n-factors to a map $\widetilde{SW}_{X,n+1,\mathfrak{p},1}$ which is the sum of maps

$$H\colon U_{n+1}/U_n \oplus V_{n+1}/V_n \to U'_{n+1}/U'_n \oplus V'_{n+1}/V'_n \oplus \underline{\mathbb{C}}^{k_{Q,n}} \oplus \underline{\mathbb{R}}^{k_{W,-,n}}$$

and

$$\widetilde{SW}_{X,n,\mathfrak{p}}\colon U_n \oplus V_n \to U'_n \oplus V'_n \oplus r_Y^*(F_n \oplus W_n) \oplus \underline{\mathbb{C}}^m.$$

Here, H is some linear isomorphism (from the linearization of $SW_{X,n}$).

We define A_n as before, and require that A_n is an isolating neighborhood of the flow $\varphi_{n+1,t}$ for all $t \in [0,1]$.

We then define

$$\begin{aligned} K_{n,1}(\varepsilon) := \big\{(y,t) \in A_n \times [0,1] : \; & \exists(\hat{\phi}, v, \hat{\omega}) \in B_k(U_n \oplus V_n; R'), \\ & \|(SW_{X,n,\mathfrak{p},t}, \mathrm{id}_{\mathbb{C}^m})(\hat{\phi}, v, \hat{\omega})\|_{k-1} \le \varepsilon, \\ & y = \pi_{P_n \oplus W_{P,n}} r_Y(\hat{\phi}, \hat{\omega})\big\} \end{aligned}$$

and

$$\begin{aligned} K_{n,2}(\varepsilon) := \big\{(y,t) \in A_n \times [0,1] : \; & \exists(\hat{\phi}, v, \hat{\omega}) \in \partial B_k(U_n \oplus V_n; R'), \\ & \|(SW_{X,n,\mathfrak{p},t}, \mathrm{id}_{\underline{\mathbb{C}}^m})(\hat{x}, v, \hat{\omega})\|_{k-1} \le \varepsilon, \\ & y = \pi_{P_n \oplus W_{P,n}} r_Y(\hat{\phi}, \hat{\omega})\big\} \\ \cup \big(((\partial A_n) \times [0,1]) \cap K_{n,1}(\varepsilon)\big). & \end{aligned}$$

One then establishes the analog of Proposition 5.1.3 for the family of flows $\varphi_{n+1,t}$.

Writing $I = [0,1]$, there results a map

$$\begin{aligned} \mathcal{BF}_{[n+1],I}(X,\mathfrak{t})\colon & S_{B_X \times I}^{\underline{\mathbb{C}}^{c_n + k_{Q,n}} \oplus \underline{\mathbb{R}}^{d_n + k_{W,-,n}}} \wedge_{B_Y \times I} S_{B_X \times I}^{U_n \oplus V_n} \\ & \to S_{B_X \times I}^{U'_{n+1} \oplus V'_{n+1} \oplus \underline{\mathbb{C}}^m} \wedge_{B_Y \times I} \mathcal{SWF}_{[n+1]}(Y). \end{aligned}$$

At $t = 1$ this is the composite from first going down in (5.2.2), while for $t = 0$, this restricts to $\mathcal{BF}_{[n+1]}$. The homotopy commutativity of (5.2.2) follows.

The claim on the well-definedness of the maps in (5.2.2) follows from Theorem A.2.3. ∎

Proposition 5.2.6. *The map $\mathcal{BF}_{[n]}$ is independent of the choice of regular index pair (N_n, L_n) with $K_{n,1}(\varepsilon) \subset N_n$, $K_{n,2}(\varepsilon) \subset L_n$ for n large and ε small, up to isomorphisms in $PSW_{S^1,B}$.*

Proof. We will follow the argument in [23, Appendix]. Take another regular index pair (N'_n, L'_n) with $K_{1,n}(\varepsilon) \subset N'_n$, $K_{2,n}(\varepsilon) \subset L'_n$ for n large and ε small. Let I'_n denote the parameterized Conley index associated to (N'_n, L'_n).

First we consider the case when $(N_n, L_n) \subset (N'_n, L'_n)$. The map

$$\iota_n \colon I_n \to I'_n$$

induced by the inclusion is an isomorphism in $PSW_{S^1,B}$ by [43, Theorem 6.2] and the following diagram is commutative:

$$\begin{tikzcd}
 & S_{B_X}^{U'_n \oplus V'_n \oplus \underline{\mathbb{C}}^m} \wedge_{B_Y} I_n \arrow[dd, "\mathrm{id} \wedge \iota_n"] \\
S_{B_X}^{U_n \oplus V_n} \arrow[ur, "\mathcal{BF}_{[n]}"] \arrow[dr, "\mathcal{BF}'_{[n]}"'] & \\
 & S_{B_X}^{U'_n \oplus V'_n \oplus \underline{\mathbb{C}}^m} \wedge_{B_Y} I'_n.
\end{tikzcd}$$

Next we consider the general case. As shown in [23, p. 1653], we have index pairs $(\widetilde{N}_n, \tilde{L}_n)$, $(N_{n,1}, L_{n,1})$, $(N'_{n,1}, L'_n)$ such that

$$(N_n, L_n) \subset (N_{n,1}, L_{n,1}), \quad (N'_n, L'_n) \subset (N'_{n,1}, L'_{n,1}),$$
$$(K_{n,1}(\varepsilon), K_{2,n}(\varepsilon)) \subset (\widetilde{N}_n, \tilde{L}_n) \subset (N_{n,1}, L_{n,1}) \cap (N'_{n,1}, L'_{n,1}).$$

We can assume that $(\widetilde{N}_n, \tilde{L}_n)$, $(N_{n,1}, L_{n,1})$, $(N'_{n,1}, L'_{n,1})$ are all regular by thickening the exits slightly ([47, Remark 5.4]). The statement follows from the commutative diagram

$$\begin{tikzcd}
(N_{n,1}, L_{n,1}) & & (N'_{n,1}, L'_{n,1}) \\
 & (\widetilde{N}_n, \tilde{L}_n) \arrow[ul, hook] \arrow[ur, hook] & \\
(N_n, L_n) \arrow[uu, hook] & & (N'_n, L'_n). \arrow[uu, hook]
\end{tikzcd}$$

■

Recall that we have defined the virtual bundle $\mathrm{ind}(D_X, P)$ following equation (5.1.3). For a normal spectral system $\mathfrak{P}$ whose nth section is P_n, we write $\mathrm{ind}(D_X, \mathfrak{P})$, since $\mathrm{ind}(D_X, P_n)$ and $\mathrm{ind}(D_X, P_{n+1})$ are canonically identified for all n. For $V =$

$V_1 \ominus V_2$ a virtual vector bundle over a base B, we define an element S_B^V of the stable-homotopy category PSW_B (see Definition A.1.9) by $(S_B^{V_1}, -V_2)$, where $S_B^{V_1}$ is the sphere bundle associated to V_1; the stable-homotopy type of this space does not depend on a choice of universe.

For V a vector bundle over B, let Th_B^V denote the Thom space of V; we will abuse notation and also write Th_B^V for the suspension spectrum of Th_B^V. Write $\ker(D_X, \mathfrak{P})$ for the kernel of the map in (5.1.2), which depends on the perturbation $\mathfrak{p}$.

For topological spaces W, Z, a *map class* from W to Z will refer to a homotopy class $W \to Z$, up to self-homotopy-equivalence of W, Z. We can now prove Theorem 1.3.1 from the introduction, which we restate as follows.

Corollary 5.2.7. *Fix a Floer framing $\mathfrak{P}$ on Y. There is a well-defined (parameterized, equivariant, stable) map class*

$$\mathcal{BF}(X, \mathfrak{t})\colon S_{\mathrm{Pic}(X)}^{\mathrm{ind}(D_X, \mathfrak{P})} \to \mathcal{SWF}(Y, \mathfrak{P}).$$

For a choice of perturbation $\mathfrak{p}$ as in (5.1.2)*, there is a well-defined (equivariant, unparameterized) weak map of spectra:*

$$\mathbf{BF}_{\mathfrak{p}}(X, \mathfrak{t})\colon \mathrm{Th}_{\mathrm{Pic}(X)}^{\ker(D_X, \mathfrak{P})} \to \Sigma^{\mathbb{C}^m} \mathbf{SWF}^u(Y, \mathfrak{P}).$$

Moreover, if $\mathfrak{p}_0$ and $\mathfrak{p}_1$ are related by a family $\mathfrak{p}_t$ of perturbations satisfying (5.1.2)*, $\mathbf{BF}_{\mathfrak{p}_0}$ is homotopic to $\mathbf{BF}_{\mathfrak{p}_1}$.*

Proof. The class $\mathcal{BF}_{\mathfrak{p}}$ is well defined by Proposition 5.2.5. Independence (as a map class) from $\mathfrak{p}$ follows from Lemma 5.2.1.

The unparameterized case follows from Proposition 5.2.5, and an argument for families as before. ∎

Analogous results hold for the Pin(2)-equivariant versions, mutatis mutandis.

Chapter 6

Frøyshov-type invariants

In this chapter we will generalize the Frøyshov-type invariants [20, 37] defined for rational homology 3-spheres to 3-manifolds with $b_1 > 0$, making use of the Seiberg–Witten Floer stable homotopy type constructed in this memoir. As applications, we will prove restrictions on the intersection forms of smooth 4-manifolds with boundary.

It may be of interest to compare the material of this section with work of Levine–Ruberman, where similar invariants are defined in the Heegaard Floer setting [31]; also see [9] for further work in the Heegaard Floer setting.

6.1 Equivariant cohomology

We will recall a basic fact about the S^1-equivariant Borel cohomology. For a pointed S^1-CW complex W, we let $\widetilde{H}^*_{S^1}(W;\mathbb{R})$ be the reduced S^1-equivariant Borel cohomology:

$$\widetilde{H}^*_{S^1}(W;\mathbb{R}) = \widetilde{H}^*(W \wedge_{S^1} ES^1_+;\mathbb{R}),$$

where ES^1_+ is a union of ES^1 and a disjoint base point. Note that $\widetilde{H}^*_{S^1}(S^0;\mathbb{R})$ is isomorphic to $\mathbb{R}[T]$ and that $\widetilde{H}^*_{S^1}(W;\mathbb{R})$ is an $\mathbb{R}[T]$-module. We have the following (see [16, Proposition 1.18.2] and [38, Proposition 2.2]).

Proposition 6.1.1. *Let V be an S^1-representation space and $\mathcal{V}$ be the vector bundle*

$$\mathcal{V} = (W \times ES^1) \times_{S^1} V \to W \times_{S^1} ES^1$$

over $W \times_{S^1} ES^1$. The Thom isomorphism for $\mathcal{V}$ induces an $\mathbb{R}[T]$-module isomorphism

$$\widetilde{H}^{*+\dim_{\mathbb{R}} V}_{S^1}(\Sigma^V W;\mathbb{R}) \cong \widetilde{H}^*_{S^1}(W;\mathbb{R}).$$

6.2 Frøyshov-type invariant

Let B be a compact CW-complex and choose a base point $b_0 \in B$. We view B as an S^1-CW-complex, with the trivial action of S^1. The following definition is an S^1-ex-space version of [38, Definition 2.7].

Definition 6.2.1. Let $\mathbf{U} = (W, r, s)$ be a well-pointed S^1-ex-space over B such that W is S^1-homotopy equivalent to an S^1-CW complex. We say that $\mathbf{U}$ is of SWF type at level t if there is an equivalence, as ex-spaces, from $W^{S^1} \to S_B^{\mathbb{R}^t}$, and so that the S^1-action on $W \setminus W^{S^1}$ is free.

Note that in the situation above, W^{S^1} inherits the structure of an ex-space, as a subspace of W, naturally. Spaces of SWF type are meant to be the class of spaces that are produced by the Seiberg–Witten Floer homotopy-type construction. Indeed, note that in the case that B is a point, spaces of SWF type over B are exactly spaces of SWF type as in [38]. For us, B will always be a Picard torus.

Moreover, for $\mathbf{U} = \mathcal{SWF}(Y)$ for some 3-manifold Y admitting a spectral section (with torsion spin^c structure and spectral section suppressed from the notation), more is true, in that the fixed point set W^{S^1} is actually fiber-preserving homotopy-equivalent, relative to $s(B)$, to $S_B^{\mathbb{R}^t}$, although for the definition of the Frøyshov invariant, this is not strictly needed.

Definition 6.2.2. Let $\mathbf{U} = (W, r, s)$ be a well-pointed S^1-ex-space of SWF type at level t over B. We denote by $\mathcal{I}_\Lambda(\mathbf{U})$ the submodule in $\widetilde{H}^*(B_+;\mathbb{R}) \otimes \mathbb{R}[\![T]\!]$, viewed as a module over the formal power series ring $\mathbb{R}[\![T]\!]$, generated by the image of the homomorphism induced by the inclusion $\iota\colon W^{S^1} \hookrightarrow W$:

$$\begin{aligned}\widetilde{H}^{*+t}_{S^1}(W/s(B);\mathbb{R}) &\xrightarrow{\iota^*} \widetilde{H}^{*+t}_{S^1}(W^{S^1}/s(B);\mathbb{R}) \cong \widetilde{H}^{*+t}_{S^1}(S^{\mathbb{R}^t} \wedge B_+;\mathbb{R}) \\ &= H^*(B;\mathbb{R}) \otimes \mathbb{R}[T] \hookrightarrow H^*(B;\mathbb{R}) \otimes \mathbb{R}[\![T]\!].\end{aligned}$$

We obtain a more specific invariant by considering only $H^0(B;\mathbb{R})$, in the case that B is connected; we impose this condition on B from now on. Let $\mathcal{I}(\mathbf{U})$ denote the ideal in $\mathbb{R}[\![T]\!]$ which is the image of

$$\begin{aligned}&\widetilde{H}^{*+t}_{S^1}(W/s(B);\mathbb{R}) \xrightarrow{\iota^*} \widetilde{H}^{*+t}_{S^1}(W^{S^1}/s(B);\mathbb{R}) \\ &\qquad \cong \widetilde{H}^{*+t}_{S^1}(S^{\mathbb{R}^t} \wedge B_+;\mathbb{R}) \to \widetilde{H}^{*+t}_{S^1}(S^{\mathbb{R}^t};\mathbb{R}) = \mathbb{R}[T] \hookrightarrow \mathbb{R}[\![T]\!]\end{aligned}$$

obtained using the inclusion of a fiber $S^{\mathbb{R}^t} \to S^{\mathbb{R}^t} \wedge B_+$.

Then there is a nonnegative integer h such that $\mathcal{I}(\mathbf{U}) = (T^h)$. Here, (T^h) is the ideal generated by T^h. We denote this integer by $h(\mathbf{U})$.

The invariant $h(\mathbf{U})$ defined above is most similar to d_{bot} as in [31], while $\mathcal{I}_\Lambda(\mathbf{U})$ is, roughly, in line with the collection of their "intermediate invariants".

Remark 6.2.3. We also note that the cohomology group $\widetilde{H}^*_{S^1}(W/s(B);\mathbb{R})$ admits an action by $H^*(B)$ as follows. Using the projection map $r\colon W \to B$, we have an algebra morphism $r^*\colon H^*(B;\mathbb{R}) \to H^*(W;\mathbb{R})$. The Mayer–Vietoris sequence for (B, W) splits because of the map $s\colon B \to W$, and we obtain

$$H^*(W;\mathbb{R}) = H^*(W/s(B);\mathbb{R}) \oplus H^*(B;\mathbb{R}),$$

and in fact this splitting is at the level of $H^*(B;\mathbb{R})$-modules, so that the cohomology group $H^*(W/s(B);\mathbb{R})$ inherits an $H^*(B;\mathbb{R})$-action. This is not strictly necessary in the definition of invariants from $\mathcal{I}_\Lambda(\mathbf{U})$ above, but is indicative of the structure of $\mathcal{I}_\Lambda(\mathbf{U})$.

From Proposition 6.1.1, we can see the following.

Lemma 6.2.4. *Let $\mathbf{U}=(W,r,s)$ be a well-pointed S^1-ex-space of SWF type over B. If V is a real vector space, we have*

$$h(\Sigma_B^V\mathbf{U})=h(\mathbf{U}).$$

If V is a complex vector space, we have

$$h(\Sigma_B^V\mathbf{U})=h(\mathbf{U})+\dim_{\mathbb{C}}V.$$

Proposition 6.2.5. *Let $\mathbf{U}_0=(W_0,r_0,s_0)$, $\mathbf{U}_1=(W_1,r_1,s_1)$ be well-pointed S^1-ex-spaces of SWF type at level t over B_0 and B_1, and assume we are given a map $\rho\colon B_0\to B_1$. Let $\rho_!\mathbf{U}_0$ denote the pushforward of $\mathbf{U}_0$, as an ex-space over B_1. Assume that there is a fiberwise-deforming S^1-map*

$$f\colon \rho_!\mathbf{U}_0\to\mathbf{U}_1$$

such that the restriction to

$$f^{S^1}\colon \rho_!W_0^{S^1}\to W_1^{S^1},$$

as a fiberwise-deforming morphism over B_1, is homotopy equivalent to

$$\mathrm{id}\wedge\rho\colon (\mathbb{R}^t)^+\times B_0\cup_{B_0}B_1\to(\mathbb{R}^t)^+\times B_1.$$

Then

$$h(\mathbf{U}_0)\le h(\mathbf{U}_1).$$

As a special case, if B_0 is a point, the hypothesis is that the map f, restricted to fixed point sets, $f^{S^1}\colon W_0^{S^1}\to W_1^{S^1}/s(W_1)$, be homotopic to the inclusion of a fiber.

Proof of Proposition 6.2.5. We have the following diagram:

$$\begin{tikzcd}
\widetilde{H}^{*+t}(W_0/s(B_0);\mathbb{R}) \arrow[d] & & \widetilde{H}^{*+t}(W_1/s(B_1);\mathbb{R}) \arrow[ll, "f^*"'] \arrow[d] \\
\widetilde{H}^{*+t}((\mathbb{R}^t)^+\times B_0/s(B_0);\mathbb{R}) \arrow[d] & & \widetilde{H}^{*+t}((\mathbb{R}^t)^+\times B_1/s(B_1);\mathbb{R}) \arrow[ll, "\rho^*"'] \arrow[d] \\
\widetilde{H}^{*+t}((\mathbb{R}^t)^+;\mathbb{R})=\mathbb{R}[T] \arrow[dr] & & \widetilde{H}^{*+t}((\mathbb{R}^t)^+;\mathbb{R})=\mathbb{R}[T] \arrow[ll, "f^*"', "\cong"] \arrow[dl] \\
& \mathbb{R}[\![T]\!]. &
\end{tikzcd}$$

From this diagram, we obtain

$$(T^{h(\mathbf{U}_0)}) \supset (T^{h(\mathbf{U}_1)}),$$

which implies that $h(\mathbf{U}_0) \leq h(\mathbf{U}_1)$. ∎

Definition 6.2.6. For $m, n \in \mathbb{Z}$ and S^1-ex-space $\mathbf{U}$ of SWF type over B, we define

$$h(\Sigma_B^{\mathbb{R}^m \oplus \mathbb{C}^n} \mathbf{U}) = h(\mathbf{U}) + n.$$

Note that this definition is compatible with Lemma 6.2.4.

Definition 6.2.7. For $m_0, n_0, m_0, n_1 \in \mathbb{Z}$ and S^1-ex-spaces $\mathbf{U}_0$, $\mathbf{U}_1$ of SWF type over B, we say that $\Sigma_B^{\mathbb{R}^{m_0} \oplus \mathbb{C}^{n_0}} \mathbf{U}_0$ and $\Sigma_B^{\mathbb{R}^{m_1} \oplus \mathbb{C}^{n_1}} \mathbf{U}_1$ are locally equivalent if there is $N \in \mathbb{Z}_{\geq 0}$ with $N + m_0, N + n_0, N + m_1, N + n_1 \geq 0$ and fiberwise-deforming maps

$$\begin{aligned} f &\colon \Sigma_B^{\mathbb{R}^{N+m_0} \oplus \mathbb{C}^{N+n_0}} \mathbf{U}_0 \to \Sigma_B^{\mathbb{R}^{N+m_1} \oplus \mathbb{C}^{N+n_1}} \mathbf{U}_1, \\ g &\colon \Sigma_B^{\mathbb{R}^{N+m_1} \oplus \mathbb{C}^{N+n_1}} \mathbf{U}_1 \to \Sigma_B^{\mathbb{R}^{N+m_0} \oplus \mathbb{C}^{N+n_0}} \mathbf{U}_0 \end{aligned}$$

such that the restrictions

$$\begin{aligned} f^{S^1} &\colon \Sigma_B^{\mathbb{R}^{N+m_0}} (\mathbf{U}_0)^{S^1} \to \Sigma_B^{\mathbb{R}^{N+m_1}} (\mathbf{U}_1)^{S^1}, \\ g^{S^1} &\colon \Sigma_B^{\mathbb{R}^{N+m_1}} (\mathbf{U}_1)^{S^1} \to \Sigma_B^{\mathbb{R}^{N+m_0}} (\mathbf{U}_0)^{S^1} \end{aligned}$$

are homotopy equivalent to

$$\mathrm{Id} \colon B \times (\mathbb{R}^t) \to B \times (\mathbb{R}^t)^+$$

as fiberwise-deforming morphisms over B.

It is easy to see that the local equivalence is an equivalence relation.

Corollary 6.2.8. *If $\Sigma_B^{\mathbb{R}^{m_0} \oplus \mathbb{C}^{n_0}} \mathbf{U}_0$ and $\Sigma_B^{\mathbb{R}^{m_1} \oplus \mathbb{C}^{n_1}} \mathbf{U}_1$ are locally equivalent,*

$$h(\Sigma_B^{\mathbb{R}^{m_0} \oplus \mathbb{C}^{n_0}} \mathbf{U}_0) = h(\Sigma_B^{\mathbb{R}^{m_1} \oplus \mathbb{C}^{n_1}} \mathbf{U}_1).$$

Proof. This is a direct consequence of Proposition 6.2.5. ∎

Let Y be a closed 3-manifold, g be a Riemannian metric, $\mathfrak{s}$ be a torsion spin^c structure on Y. Let B_Y be the Picard torus $\mathrm{Pic}(Y)$ of Y. Assume that $\operatorname{ind} D_Y = 0$ in $K^1(B_Y)$. We take a spectral system

$$\mathfrak{S} = (\mathbf{P}, \mathbf{Q}, \mathbf{W}_P, \mathbf{W}_Q, \{\eta_n^P\}_n, \{\eta_n^Q\}, \{\eta_n^{W_P}\}_n, \{\eta_n^{W_Q}\}_n)$$

for Y. See Definition 3.5.1. Put

$$F_n = P_n \cap Q_n, \quad W_n = W_{P,n} \cap W_{Q,n}$$

as before. Take half-integers k_+, k_- with $k_+, k_- > 5$ and with $|k_+ - k_-| \leq \frac{1}{2}$. We have the approximate Seiberg–Witten flow

$$\varphi_n = \varphi_{n,k_+,k_-} \colon (F_n \oplus W_n) \times \mathbb{R} \to F_n \oplus W_n.$$

Put

$$A_n = (B_{k_+}(F_n^+; R) \times_{B_Y} B_{k_-}(F_n^-; R)) \times_{B_Y} (B_{k_+}(W_n^+; R) \times_{B_Y} B_{k_-}(W_n^-; R))$$

for $R \gg 0$. Recall that A_n is an isolating neighborhood for $n \gg 0$ (Theorem 2.3.3).

Lemma 6.2.9. *Let* $\mathbf{U}_n = (I_n, r_n, s_n)$ *be the* S^1*-equivariant Conley index for the isolated invariant set* $\operatorname{inv}(A_n, \varphi_n)$ *for* $n \gg 0$*. Then* $\mathbf{U}_n$ *is of SWF type at level* $\operatorname{rank}_{\mathbb{R}} W_n^-$.

Proof. We first note that I_n is of the homotopy type of an S^1-CW complex by Proposition 3.6.1. The S^1-fixed point set $(I_n, r_n, s_n)^{S^1}$ is the Conley index for

$$\operatorname{inv}(\varphi_n|_{W_n}, B_{k_+}(W_n^+; R) \times_{B_Y} B_{k_-}(W_n^-; R)).$$

Note that if $\phi = 0$, the quadratic terms $c_1(\gamma)$, $c_2(\gamma)$, $X_H(\phi)$ are all zero. See (2.3.3). Hence the restriction of the flow φ_n to W_n is the flow induced by the linear map $-{*}d|_{W_n}$. In particular, the flow $\varphi_n|_{W_n}$ preserves each fiber of the trivial bundle $W_n = B_Y \times L_k^2(\operatorname{im} d^*)_{\lambda_n}^{\mu_n}$ over B_Y. Hence there is an equivalence, as ex-spaces, $(I_n)^{S^1} \cong S_B^{W_n^-}$. (In fact, more is true: there is a fiber-preserving homotopy equivalence $(I_n)^{S^1} \cong S_B^{W_n^-}$.) ∎

Let $\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}])$ be the Seiberg–Witten Floer parameterized homotopy type (Definition 3.5.8).

Recall that $\eta_n^P, \eta_n^Q, \eta_n^{W_P}, \eta_n^{W_Q}$ are isomorphisms

$$\begin{aligned}
P_{n+1} &\xrightarrow{\cong} P_n \oplus \mathbb{C}^{k_{P,n}}, \\
Q_{n+1} &\xrightarrow{\cong} Q_n \oplus \mathbb{C}^{k_{Q,n}}, \\
W_{n+1}^P &\xrightarrow{\cong} W_n^+ \oplus \mathbb{R}^{k_{W,+,n}}, \\
W_{n+1}^Q &\xrightarrow{\cong} W_n^- \oplus \mathbb{R}^{k_{W,-,n}}.
\end{aligned}$$

These induce an S^1-equivariant homotopy equivalence

$$I(\varphi_{n+1}) \cong \Sigma_B^{\mathbb{C}^{k_{Q,n}} \oplus \mathbb{R}^{k_{W,-,n}}} I(\varphi_n)$$

for $n \gg 0$, whose restriction to the S^1-fixed point set is a fiber-preserving homotopy equivalence. See Theorem 3.1.1. This implies that the number

$$h(\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}])) = h(I(\varphi_n)) - D_n^2$$

is independent of the choice of $n \gg 0$ by Lemma 6.2.4 and Corollary 6.2.8. Here, $D_n^2 = \dim(Q_n - Q_0)$.

Also, it follows from Proposition 3.4.1 that $h(\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}]))$ is independent of $k_{\pm}$. Hence $h(\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}]))$ is well defined.

We will introduce another number. We can take a spin^c 4-manifold $(X, \mathfrak{t})$ with boundary $(Y, \mathfrak{s})$. Since $c_1(\mathfrak{t})|_Y$ is torsion in $H^2(Y; \mathbb{Z})$, there is a positive integer m such that

$$mc_1(\mathfrak{t}) \in H^2(X, Y; \mathbb{Z}).$$

Put

$$c_1(\mathfrak{t})^2 := \frac{1}{m}\langle (mc_1(\mathfrak{t})) \cup c_1(\mathfrak{t}), [X] \rangle \in \mathbb{Q},$$

where $\langle \cdot, \cdot \rangle$ is the pairing

$$H^4(X, Y; \mathbb{Z}) \otimes H_4(X; \mathbb{Z}) \to \mathbb{Z}.$$

We define

$$\begin{aligned} n(Y, g, \mathfrak{s}, P_0) &:= \dim \operatorname{ind}(D_X, P_0) - \frac{c_1(\mathfrak{t})^2 - \sigma(X)}{8} \in \mathbb{Q} \\ &= \frac{1}{2}\eta_{D,P_0} - \frac{1}{8}\eta_{Y,\mathrm{sign}}. \end{aligned} \tag{6.2.1}$$

Here, D_X is the Dirac operator on X, $\operatorname{ind}(D, P_0)$ is the index defined in Proposition 2.1.3 and η_{D,P_0}, $\eta_{Y,\mathrm{sign}}$ are the η-invariants of the Dirac operator and signature operator. We have used the index formula [5, 40]. See also [35, Section 6].

Definition 6.2.10. We define $h(Y, \mathfrak{s}) \in \mathbb{Q}$ by

$$h(Y, \mathfrak{s}) := h(\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}])) - n(Y, g, \mathfrak{s}, P_0).$$

A priori, the expression in Definition 6.2.10 may depend on both the metric and the spectral system. However, for two spectral systems $\mathfrak{S}_0, \mathfrak{S}_1$ with $\dim \operatorname{ind}(D_X, P_0^0) = \dim \operatorname{ind}(D_X, P_0^1)$, we see that the h-invariants agree, since $\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}_0])$ differs from $\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}_1])$ by suspension by a virtual complex vector bundle of formal dimension zero. In order to see this, we first note that S^1-equivariant Borel cohomology is an S^1-equivariant complex orientable cohomology theory by [12], so that for an S^1-equivariant complex vector bundle V over B and an S^1-ex-space (X, r, s) over B, there is a canonical isomorphism

$$\widetilde{H}_{S^1}^{*+2\operatorname{rank}_{\mathbb{C}} V}(\nu_! \Sigma_B^V X) \cong \widetilde{H}_{S^1}^{*+2\operatorname{rank}_{\mathbb{C}} V}(\mathrm{Th}(r^*V)) \cong \widetilde{H}_{S^1}^{*}(X).$$

Here, $\nu\colon B \to *$ and we have used (3.6.3). This implies that

$$h(\Sigma_B^V X) = h(X) + 2\,\mathrm{rank}_{\mathbb{C}}\, V.$$

It follows in particular that

$$h(\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}_0])) = h(\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}_1])).$$

Changes in the metric and changes in $\dim \mathrm{ind}(D_X, P_0)$ are treated in a similar way, so we only address the latter. Indeed, if we replace $\mathfrak{S}_0$ with a spectral system $\mathfrak{S}_1$ so that the K-theory class is

$$[\mathfrak{S}_1 - \mathfrak{S}_0] = \underline{\mathbb{C}} \in K(B_Y),$$

then

$$h(\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}_1])) = h(\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}_0])) - 1,$$

but $n(Y, g, \mathfrak{s}, P_0^1) = n(Y, g, \mathfrak{s}, P_0^0) - 1$, as needed.

Finally, in the case that $b_1(Y) = 0$, this agrees (by definition) with the δ-invariant defined in [38].

In particular, it is natural to consider the parameterized equivariant homotopy type of the formal desuspension:

$$\Sigma_{B_Y}^{-n(Y,g,\mathfrak{s},P_0)\mathbb{C}} \mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}]),$$

which one can think of as a desuspension so that the grading of a reducible element of $\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}])$ has been specified. We note that $n(S^1 \times S^2, g, \mathfrak{s}, P_0) = 0$, where g is the product metric on $S^1 \times S^2$, $\mathfrak{s}$ is the torsion spin^c structure and P_0 is the standard spectral section (since the Dirac operator has trivial kernel for each flat connection, this is specified). That is, with our conventions, the grading of each reducible in

$$\mathrm{Pic}(S^1 \times S^2) \simeq \mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}])$$

is zero. This differs from the convention in Heegaard–Floer homology, for which each reducible should be $-\frac{1}{2}$-graded, as in [45].

We will prove a generalization of [20, Theorem 4].

Theorem 6.2.11. *Let Y_0 be a rational homology 3-sphere and Y_1 be a closed, oriented 3-manifold such that the triple-cup product*

$$\Lambda^3 H^1(Y_1; \mathbb{Z}) \to \mathbb{Z},$$
$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \mapsto \langle \alpha_1 \cup \alpha_2 \cup \alpha_3, [Y_1] \rangle$$

is zero. Let $(X, \mathfrak{t})$ be a compact, spin^c negative semidefinite 4-manifold with boundary $-Y_0 \coprod Y_1$ such that $c_1(\mathfrak{t})|_{\partial X}$ is torsion. Then we have

$$\frac{c_1(\mathfrak{t})^2 + b_2^-(X)}{8} + h(Y_0, \mathfrak{t}|_{Y_0}) \leq h(Y_1, \mathfrak{t}|_{Y_1}).$$

Proof. Since the triple-cup product is zero, we have $\operatorname{ind} D_{Y_1} = 0$ in $K^1(B_{Y_1})$ by the index formula. (See [30, Proposition 6].) Note that the map $\mathcal{BF}_{[n]}(X, \mathfrak{t})$ constructed in Chapter 5 is a fiber-preserving map. We consider the restriction of $\mathcal{BF}_{[n]}(X, \mathfrak{t})$ to the fiber over a point $[0] \in B_X$. The restriction $\mathcal{BF}_{[n]}(X, \mathfrak{t})$ to the fiber and the duality map

$$I_n(Y_0) \wedge I_n(-Y_0) \to S^{F_n(Y_0) \oplus W_n(Y_0)},$$

defined in [36, Section 2.5], induce an S^1-map

$$f_n \colon \Sigma^{\mathbb{R}^{m_0} \oplus \mathbb{C}^{n_0 + a}} I_n(Y_0) \to \Sigma^{\mathbb{R}^{m_1} \oplus \mathbb{C}^{n_1}} (I_n(Y_1)/s_n(B_{Y_1}))$$

for $n \gg 0$, where

$$\begin{aligned} m_0 - m_1 &= \operatorname{rank}_{\mathbb{R}} W_n(Y_1)^- - \dim_{\mathbb{R}} W_n(Y_0)^-, \\ n_0 - n_1 &= \operatorname{rank}_{\mathbb{C}} F_n(Y_1)^- - \dim_{\mathbb{C}} F_n(Y_0)^-, \\ a &= \dim \operatorname{ind} D_{X, P_0} \\ &= \frac{c_1(\mathfrak{t})^2 + b_2^-(X)}{8} + n(Y_1, g|_{Y_1}, \mathfrak{t}|_{Y_1}, P_0) - n(Y_0, g|_{Y_0}, \mathfrak{t}|_{Y_0}). \end{aligned}$$

The restriction of f_n to the S^1-fixed point set $\Sigma^{\mathbb{R}^{m_0}} (I_n(Y_0))^{S^1}$ is induced by the operator

$$D' = (d^+, \pi^0_{-\infty} r_{-Y_0}, \pi^0_{-\infty} r_{Y_1}) \colon \Omega^1_{CC}(X) \to \Omega^+(X) \oplus (\mathcal{W}_{-Y_0})^0_{-\infty} \oplus (\mathcal{W}_{Y_1})^0_{-\infty}.$$

The operator D' is an isomorphism. Therefore the restriction

$$f_n^{S^1} : \Sigma^{\mathbb{R}^{m_0}} (I_n(Y_0))^{S^1} \to \Sigma^{\mathbb{R}^{m_1}} (I_n(Y_1))^{S^1}_{[0]}$$

is a homotopy equivalence. Here, $[0] \in B_{Y_1}$ is the restriction of $[0] \in B_X$ to Y and $(I_n(Y_1))^{S^1}_{[0]}$ is the fiber over $[0]$.

By Lemma 6.2.4 and Proposition 6.2.5, we have

$$\frac{c_1(\mathfrak{t})^2 + b_2^-(X)}{8} + h(Y_0, \mathfrak{t}|_{Y_0}) \leq h(Y_1, \mathfrak{t}|_{Y_1}). \qquad \blacksquare$$

Remark 6.2.12. There is an apparent discrepancy with the statement of [31, Theorem 4.7]. We note that in the translation between these statements, we expect $h(Y, \mathfrak{s})$ to correspond to $\frac{d_{\mathrm{bot}}(Y, \mathfrak{s})}{2} + \frac{b_1(Y)}{4}$, due to the difference in the grading conventions on the reducible; with this observation, the statements are consistent.

Remark 6.2.13. In order to generalize Theorem 6.2.11 to the case $b_1(Y_0) > 0$, we need to establish the duality for the Seiberg–Witten Floer parameterized homotopy types $\mathcal{SWF}(Y_0, \mathfrak{t}|_{Y_0}, [\mathfrak{S}])$ and $\mathcal{SWF}(-Y_0, \mathfrak{t}|_{Y_0}, [\mathfrak{S}_0^\vee])$ to get the parameterized Bauer–Furuta map

$$\mathcal{SWF}(Y_0, \mathfrak{t}|_{Y_0}, [\mathfrak{S}_0]) \to \mathcal{SWF}(Y_1, \mathfrak{t}|_{Y_1}, [\mathfrak{S}_1]).$$

We do not discuss it in this memoir. See Proposition 3.6.2.

Corollary 6.2.14. *Let Y be a closed, connected, oriented 3-manifold such that the triple-cup product is zero. Let $(X, \mathfrak{t})$ be a compact, negative semidefinite,* spin^c *4-manifold with $\partial X = Y$ such that $c_1(\mathfrak{t})|_Y$ is torsion. Then we have*

$$\frac{c_1(\mathfrak{t})^2 + b_2^-(X)}{8} \leq h(Y, \mathfrak{t}|_Y).$$

Proof. Removing a small ball from X, we get a compact spin^c 4-manifold X' with boundary $S^3 \coprod Y$. Applying Theorem 6.2.11 to X', we get the inequality. ∎

Example 6.2.15. Let T^2 be a torus $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$. Put

$$Y := \mathbb{R} \times T^2/(x, \theta_1, \theta_2) \sim (x + 1, -\theta_1, -\theta_2).$$

Then Y is a flat T^2 bundle over S^1, which has a flat metric and $b_1(Y) = 1$. We have

$$H^2(Y; \mathbb{Z}) \cong H_1(Y; \mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}).$$

There are four spin^c structures $\mathfrak{s}_0, \dots, \mathfrak{s}_3$. Let $\mathfrak{s}_0$ be the spin^c structure corresponding to the 2-plane field tangent to the fibers. As stated in Example 4.2.2, for $j = 1, 2, 3$, $(Y, \mathfrak{s}_j)$ satisfies the conditions of Theorem 4.1.2. We have

$$\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}]) \cong S^0_{B_Y}.$$

Here, $\mathfrak{S}$ is a spectral system with $P_0 = \mathcal{E}_0(D)^0_{-\infty}$. As stated in [24, p. 2112],

$$n(Y, \mathfrak{s}_j, g, P_0) = 0$$

for $j = 1, 2, 3$. Therefore we obtain

$$h(Y, \mathfrak{s}_j) = h(\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}])) - n(Y, \mathfrak{s}_j, g, P_0) = 0.$$

Example 6.2.16. Let Σ be a closed, oriented surface with $g(\Sigma) > 0$ and Y be the sphere bundle of the complex line bundle over Σ of degree d. Suppose that $0 < g < d$, where $g := g(\Sigma)$. Let $\mathfrak{s}_q$ be the spin^c structure in Proposition 4.2.3. For $q \in \{g, g+1, \dots, d-1\}$, we have

$$\mathcal{SWF}(Y, \mathfrak{s}_q, [\mathfrak{S}]) \cong S^0_B$$

by Theorem 4.2.5. Here, $\mathfrak{S}$ is a spectral system with $P_0 = \mathcal{E}_0(D_r)^0_{-\infty}$. The value of $n(Y, g_r, \mathfrak{s}_q, P_0)$ was computed in [24, Section 8.2] and we have

$$n(Y, g_r, \mathfrak{s}_q, P_0) = -\frac{d-1}{8} - \frac{(g-1-q)(d+g-1-q)}{2d}. \tag{6.2.2}$$

(Note that the definition of $n(Y, g, \mathfrak{s}_q, P_0)$ of this memoir is -1 times that of [24].) Hence

$$\begin{aligned} h(Y, \mathfrak{s}_q, g) &= h(\mathcal{SWF}(Y, \mathfrak{s}_q, [\mathfrak{S}])) - n(Y, g, \mathfrak{s}_q, P_0) \\ &= \frac{d-1}{8} + \frac{(g-1-q)(d+g-1-q)}{2d}. \end{aligned}$$

6.3 K-theoretic Frøyshov invariant

In analogy to the previous section on the (homological) Frøyshov invariant, we now generalize the invariant $\kappa(Y)$ constructed in [37]. For details on Pin(2)-equivariant complex K-theory, we refer to [37].

Let $\widetilde{\mathbb{R}}$ be the nontrivial real representation of $\mathrm{Pin}(2) = S^1 \coprod jS^1$. Let B be a compact, connected Pin(2)-CW complex with a Pin(2)-fixed marked (though we do not consider B itself to be an object in the category of pointed spaces) point $b_0 \in B^{\mathrm{Pin}(2)}$, such that the S^1-action on B is trivial and the action of j is an involution.

Definition 6.3.1. Let $\mathbf{U} = (W, r, s)$ be a well-pointed Pin(2)-ex-space over B such that W is Pin(2)-homotopy equivalent to a Pin(2)-CW complex. We say that $\mathbf{U}$ is of SWF type at level t if there is an ex-space Pin(2)-homotopy equivalence from W^{S^1} to $S_B^{\widetilde{\mathbb{R}}^t}$ and if the Pin(2)-action on $W \setminus W^{S^1}$ is free.

As before, in fact for us there is the stronger condition that there is a fiber-preserving (equivariant) homotopy equivalence $W^{S^1} \to S_B^{\widetilde{\mathbb{R}}^t}$.

Let $R(\mathrm{Pin}(2))$ be the representation ring of Pin(2). That is,

$$R(\mathrm{Pin}(2)) \cong \mathbb{Z}[z, w]/(w^2 - 2w, zw - 2w),$$

where

$$w = 1 - [\widetilde{\mathbb{C}}], \quad z = 2 - [\mathbb{H}].$$

We will generalize [37, Definition3] to Pin(2)-ex-spaces.

Definition 6.3.2. Let $\mathbf{U} = (W, r, s)$ be a well-pointed Pin(2)-ex-space of SWF type at level $2t$ over B so that W is Pin(2)-homotopy equivalent to a Pin(2)-CW complex. We denote by $\mathcal{I}_\Lambda(\mathbf{U})$ the submodule in $K_{\mathbb{Z}/2}(B)$, viewed as a module over $R(\mathrm{Pin}(2))$, generated by the image of the homomorphism induced by the inclusion $\iota\colon W^{S^1} \hookrightarrow W$:

$$\begin{aligned}\widetilde{K}_{\mathrm{Pin}(2)}(W/s(B)) \xrightarrow{\iota^*} \widetilde{K}_{\mathrm{Pin}(2)}(W^{S^1}/s(B)) \cong \widetilde{K}_{\mathrm{Pin}(2)}(S^{\widetilde{\mathbb{C}}^t} \wedge B_+) \\ = K_{\mathbb{Z}/2}(B).\end{aligned}$$

We obtain a more specific invariant by considering only a single fiber. Let $\mathcal{I}(\mathbf{U})$ denote the ideal in $R(\mathrm{Pin}(2))$ which is the image of

$$\begin{aligned}&\widetilde{K}_{\mathrm{Pin}(2)}(W/s(B)) \xrightarrow{\iota^*} \widetilde{K}_{\mathrm{Pin}(2)}(W^{S^1}/s(B)) \\ &\qquad \cong \widetilde{K}_{\mathrm{Pin}(2)}(S^{\widetilde{\mathbb{C}}^t} \wedge B_+) \to \widetilde{K}_{\mathrm{Pin}(2)}(S^{\widetilde{\mathbb{C}}^t}; \mathbb{R}) = R(\mathrm{Pin}(2))\end{aligned}$$

obtained using the inclusion of a fiber $S^{\mathbb{R}^t} \to S^{\mathbb{R}^t} \wedge B_+$, over the marked point $b_0 \in B^{\mathrm{Pin}(2)}$. In particular, the invariant $k(\mathbf{U})$ depends on a choice of the point $b_0 \in B$, which does not appear in the notation.

We define $k(\mathbf{U}) \in \mathbb{Z}_{\geq 0}$ by

$$k(\mathbf{U}) = \min\{k \in \mathbb{Z}_{\geq 0} : \exists x \in \mathcal{I}(\mathbf{U}),\ wx = 2^k w\}.$$

If $\mathcal{I}(\mathbf{U})$ is of the form (z^k) for some nonnegative integer k, we say that $\mathbf{U}$ is $K_{\mathrm{Pin}(2)}$-split.

Lemma 6.3.3.

$$k(\Sigma_B^{\tilde{\mathbb{C}}}\mathbf{U}) = k(\mathbf{U}), \quad k(\Sigma_B^{\mathbb{H}}\mathbf{U}) = k(\mathbf{U}) + 1.$$

Proof. Since

$$(\Sigma_B^{\tilde{\mathbb{C}}} W)/s(B) = \Sigma^{\tilde{\mathbb{C}}}[W/s(B)],$$
$$(\Sigma_B^{\mathbb{H}} W)/s(B) = \Sigma^{\mathbb{H}}[W/s(B)],$$

we can apply [37, Lemma 3.4]. ∎

Proposition 6.3.4. *Let* $\mathbf{U}_0 = (W_0, r_0, s_0)$, $\mathbf{U}_1 = (W_1, r_1, s_1)$ *be* Pin(2)*-ex-spaces of SWF type at level* $2t_0$, $2t_1$ *over* B_0 *and* B_1, *and assume we are given an inclusion* $\rho\colon B_0 \to B_1$. *Let* $\rho_!\mathbf{U}_0$ *denote the pushforward of* $\mathbf{U}_0$, *as an ex-space over* B_1. *Assume that there is a fiberwise-deforming* S^1*-map*

$$f\colon \rho_!\mathbf{U}_0 \to \mathbf{U}_1$$

such that the restriction to

$$f^{S^1}\colon \rho_! W_0^{S^1} \to W_1^{S^1},$$

as a fiberwise-deforming morphism over B_1, *is homotopy equivalent to*

$$\ell \cup \rho\colon ((\tilde{\mathbb{C}}^{t_0})^+ \times B_0) \cup_{B_0} B_1 \to (\tilde{\mathbb{C}}^{t_1})^+ \times B_1,$$

where ℓ *is the map on one-point compactifications induced by a map of representations* $\tilde{\mathbb{C}}^{t_0} \to \tilde{\mathbb{C}}^{t_1}$, *which is an inclusion if* $t_0 \leq t_1$. *Say that* ρ *sends the marked point* $b_0 \in B_0$ *to* $b_1 \in B_1$*:*

(1) *If* $t_0 \leq t_1$, *we have*

$$k(\mathbf{U}_0) + t_0 \leq k(\mathbf{U}_1) + t_1.$$

(2) *If* $t_0 < t_1$ *and* $\mathbf{U}_0$ *is* $K_{\mathrm{Pin}(2)}$*-split, we have*

$$k(\mathbf{U}_0) + t_0 + 1 \leq k(\mathbf{U}_1) + t_1.$$

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
\widetilde{K}_{\mathrm{Pin}(2)}(W_0/s_0(B_0)) \arrow[d, "\iota_0^*"'] & & \widetilde{K}_{\mathrm{Pin}(2)}(W_1/s_1(B_1)) \arrow[ll, "f^*"'] \arrow[d, "\iota_1^*"] \\
\widetilde{K}_{\mathrm{Pin}(2)}(((\widetilde{\mathbb{C}}^{t_0})^+ \times B_0) \cup_{B_0} B_1/s(B_1)) \arrow[d, "\iota^*"'] & & \widetilde{K}_{\mathrm{Pin}(2)}((\widetilde{\mathbb{C}}^{t_1})^+ \times B_1/s(B_1)) \arrow[ll, "(\ell \cup \rho)^*"'] \arrow[d, "\iota^*"] \\
\widetilde{K}_{\mathrm{Pin}(2)}((\widetilde{\mathbb{C}}^{t_0})^+) \arrow[d, "\cdot w^{t_0}"'] & & \widetilde{K}_{\mathrm{Pin}(2)}((\widetilde{\mathbb{C}}^{t_1})^+) \arrow[ll, "\ell^*"'] \arrow[d, "\cdot w^{t_1}"] \\
\widetilde{K}_{\mathrm{Pin}(2)}(S^0) & & \widetilde{K}_{\mathrm{Pin}(2)}(S^0). \arrow[ll, "\mathrm{id}"]
\end{tikzcd}$$

Here we have used ι to denote various inclusions. Note that f^* in the first row is well defined, because $s_0(B_0) \subset s_0(B_1)$, using the definition of the pushforward $\rho_! \mathbf{U}_0$ (this does not require that ρ be an inclusion). In fact, more is true, in that $\rho_! W_0/s_0(B_1)$ is exactly $W_0/s_0(B_0)$.

We can apply the arguments in the proofs of [37, Lemmas 3.10 and 3.11] so that the result follows. ∎

Definition 6.3.5. For $m, n \in \mathbb{Z}$ and Pin(2)-ex-space $\mathbf{U}$ of SWF type at even level, we define

$$k(\Sigma_B^{\widetilde{\mathbb{R}}^{2m} \oplus \mathbb{H}^n} \mathbf{U}) = k(\mathbf{U}) + n.$$

Note that this definition is compatible with Lemma 6.3.3.

Definition 6.3.6. For $m_0, n_0, m_1, n_1 \in \mathbb{Z}$ and Pin(2)-ex-spaces $\mathbf{U}_0$, $\mathbf{U}_1$ of SWF type at even level over B, we say that $\Sigma_B^{\widetilde{\mathbb{R}}^{2m_0} \oplus \mathbb{H}^{n_0}} \mathbf{U}_0$ and $\Sigma_B^{\widetilde{\mathbb{R}}^{2m_1} \oplus \mathbb{H}^{n_1}} \mathbf{U}_1$ are locally equivalent if there are $N \in \mathbb{Z}$ with $N + m_0, N + n_0, N + m_1, N + n_1 \geq 0$ and Pin(2)-fiberwise deforming maps

$$f \colon \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_0)} \oplus \mathbb{H}^{N+n_0}} \mathbf{U}_0 \to \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_1)} \oplus \mathbb{H}^{N+n_1}} \mathbf{U}_1,$$
$$g \colon \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_1)} \oplus \mathbb{H}^{N+n_1}} \mathbf{U}_1 \to \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_0)} \oplus \mathbb{H}^{N+n_0}} \mathbf{U}_0,$$

such that the restrictions

$$f^{S^1} \colon \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_0)}} \mathbf{U}_0^{S^1} \to \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_1)}} \mathbf{U}_1^{S^1}, \quad g^{S^1} \colon \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_1)}} \mathbf{U}_1^{S^1} \to \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_0)}} \mathbf{U}_0^{S^1}$$

are homotopy equivalent to

$$\mathrm{Id} \colon B \times (\mathbb{R}^t)^+ \to B \times (\mathbb{R}^t)^+$$

as Pin(2)-fiberwise-deforming morphisms.

Corollary 6.3.7. *If* $\Sigma_B^{\widetilde{\mathbb{R}}^{2m_0}\oplus\mathbb{H}^{n_0}}\mathbf{U}_0$ *and* $\Sigma_B^{\widetilde{\mathbb{R}}^{2m_1}\oplus\mathbb{H}^{n_1}}\mathbf{U}_1$ *are locally equivalent, we have*

$$k(\Sigma_B^{\widetilde{\mathbb{R}}^{2m_0}\oplus\mathbb{H}^{n_0}}\mathbf{U}_0) = k(\Sigma_B^{\widetilde{\mathbb{R}}^{2m_1}\oplus\mathbb{H}^{n_1}}\mathbf{U}_1).$$

Proof. This is a direct consequence of Proposition 6.3.4. ∎

Let $\mathfrak{s}$ be a spin structure (not just a self-conjugate spin^c structure, although we will also write $\mathfrak{s}$ for the induced self-conjugate spin^c structure) of Y. Then the Seiberg–Witten equations (2.3.4) and the finite-dimensional approximations (2.3.10) have Pin(2)-symmetry. Let B_Y be the Picard torus of Y, which is homeomorphic to the torus $\widetilde{\mathbb{R}}^{b_1(Y)}/\mathbb{Z}^{b_1(Y)}$, where we have chosen coordinates so that $0 \in \widetilde{\mathbb{R}}^{b_1(Y)}$ corresponds to the selected spin structure on Y. We choose $[0] \in B_Y$ as base point. Assume that $\operatorname{ind} D_Y = 0$ in $KQ^1(B_Y)$. By Theorem 2.4.8, we can choose a Pin(2)-spectral system

$$\mathfrak{S} = (\mathbf{P}, \mathbf{Q}, \mathbf{W}_P, \mathbf{W}_Q, \{\eta_n^P\}_n, \{\eta_n^Q\}, \{\eta_n^{W_P}\}_n, \{\eta_n^{W_Q}\}_n)$$

for Y. Put

$$F_n = P_n \cap Q_n, \quad W_n = W_{P,n} \cap W_{Q,n}.$$

We have the Pin(2)-equivariant Conley index (I_n, r_n, s_n) for the isolated invariant set $\operatorname{inv}(A_n, \varphi_{k_+,k_-,n})$ for $n \gg 0$.

Lemma 6.3.8. *The* Pin(2)*-equivariant Conley index* (I_n, r_n, s_n) *is of SWF type at level* $\operatorname{rank}_{\mathbb{R}} W_n^-$ *for* $n \gg 0$.

Proof. The proof is similar to that of Lemma 6.2.9 and omitted. ∎

Let $\mathcal{SWF}^{\mathrm{Pin}(2)}(Y, \mathfrak{s}, [\mathfrak{S}])$ be the Pin(2)-Seiberg–Witten Floer parameterized homotopy type. As before, the local equivalence class of $\mathcal{SWF}^{\mathrm{Pin}(2)}(Y, \mathfrak{s}, [\mathfrak{S}])$ is independent of $k_\pm$, n. See [49] for the study of the local equivalence class of the Pin(2)-Seiberg–Witten Floer homotopy type in the case $b_1(Y) = 0$. We may assume that $\dim_{\mathbb{R}} W_n^-$ are even for all n. Then we have the well-defined number

$$k(\mathcal{SWF}^{\mathrm{Pin}(2)}(Y, \mathfrak{s}, [\mathfrak{S}])) \in \mathbb{Z}.$$

Definition 6.3.9. Fix $(Y, \mathfrak{s})$ as above. We define $\kappa(Y, \mathfrak{s}) \in \mathbb{Q} \cup \{-\infty\}$ by

$$\kappa(Y, \mathfrak{s}) := \inf_{g,\mathfrak{S}} 2\Big(k(\mathcal{SWF}^{\mathrm{Pin}(2)}(Y, \mathfrak{s}, [\mathfrak{S}])) - \frac{1}{2}n(Y, g, \mathfrak{s}, P_0)\Big).$$

We say that $(Y, \mathfrak{s})$ is Floer $K_{\mathrm{Pin}(2)}$-split if (I_n, r_n, s_n) is $K_{\mathrm{Pin}(2)}$-split for n large, where (I_n, r_n, s_n) realizes equality in the definition of $\kappa(Y, \mathfrak{s})$.

Note that this invariant indeed depends a priori on $\mathfrak{s}$ as a spin structure, in what we have chosen as the marked point in B_Y that is used in the definition of κ.

Unlike the case for homology, we have not shown that the invariant

$$k(\mathcal{SWF}^{\mathrm{Pin}(2)}(Y,\mathfrak{s},[\mathfrak{S}]))$$

is invariant under changes of spectral section that lie in $\widetilde{KQ}(B)$ (essentially since we do not have access to a notion of Pin(2)-complex orientable cohomology theories). We expect that the quantity appearing in the inf is, in fact, independent of $[\mathfrak{S}]$, however.

We do not know whether a self-conjugate spin^c structure may have different κ-invariants associated to different underlying spin structures. The invariant $\kappa(Y,\mathfrak{s})$, for Y a rational homology 3-sphere, agrees with Manolescu's definition [37], by construction.

Corollary 6.3.10. *The reduction* mod2 *of the κ invariant satisfies*

$$\mu(Y,\mathfrak{s}) = \kappa(Y,\mathfrak{s}) \bmod 2,$$

where $\mu(Y,\mathfrak{s})$ is the Rokhlin invariant of $(Y,\mathfrak{s})$.

Proof. Indeed, $n(Y,g,\mathfrak{s},P_0)$ mod 2 is the Rokhlin invariant of $(Y,\mathfrak{s})$ by its construction. The corollary then follows from the definition of κ and the fact that k is an integer. ∎

Corollary 6.3.10 indicates that $\kappa(Y,\mathfrak{s})$ may depend on $\mathfrak{s}$, as a spin structure. Note that if $(Y,\mathfrak{s})$ admits a Pin(2)-equivariant spectral section, for a self-conjugate spin^c structure $\mathfrak{s}$, then $\mu(Y,-)$ is constant on all spin structures underlying $\mathfrak{s}$; by Lin's result [33], this condition, coupled with the triple-cup product vanishing, characterizes 3-manifolds which admit a Pin(2)-equivariant spectral section. However, if the Pin(2)-equivariant K-theory could be extended to 3-manifolds without a Pin(2)-spectral section, so that Corollary 6.3.10 held, it would of course also imply that $\kappa(Y,\mathfrak{s})$ depends on the spin structure and not just the spin^c structure.

Using our invariant $\kappa(Y,\mathfrak{s})$, we can prove a $\frac{10}{8}$-type inequality for smooth 4-manifolds with boundary, which generalizes the results of [21] and [37].

Theorem 6.3.11. *Let $(Y_0,\mathfrak{s}_0)$ be a spin, rational homology 3-sphere and $(Y_1,\mathfrak{s}_1)$ be a closed, spin 3-manifold such that the index* ind D_{Y_1} *is zero in $KQ^1(B_{Y_1})$.*

(1) *Let $(X,\mathfrak{t})$ be a compact, smooth, spin, negative semidefinite 4-manifold with boundary $-(Y_0,\mathfrak{s}_0)\coprod(Y_1,\mathfrak{s}_1)$. Then we have*

$$\frac{1}{8}b_2^-(X) + \kappa(Y_0,\mathfrak{s}_0) \le \kappa(Y_1,\mathfrak{s}_1).$$

(2) *Let $(X,\mathfrak{t})$ be a compact, smooth, spin 4-manifold with boundary $-(Y_0,\mathfrak{s}_0)\coprod(Y_1,\mathfrak{s}_1)$. Then we have*

$$-\frac{\sigma(X)}{8} + \kappa(Y_0,\mathfrak{s}_0) - 1 \le b^+(X) + \kappa(Y_1,\mathfrak{s}_1).$$

Moreover, if Y_0 is Floer $K_{\mathrm{Pin}(2)}$-split and $b^+(X) > 0$, we have

$$-\frac{\sigma(X)}{8} + \kappa(Y_0, \mathfrak{s}_0) + 1 \le b^+(X) + \kappa(Y_1, \mathfrak{s}_1).$$

Proof. Let $[0] \in B_X = \mathrm{Pic}(X)$ be the element corresponding to the flat spin connection. Recall that $\mathcal{BF}_{[n]}$ is a fiber-preserving map. The restriction $\mathcal{BF}_{[n]}(X, \mathfrak{t})$ to the fiber over $[0]$ and the duality map

$$I_n(Y_0) \wedge I_n(-Y_0) \to S^{F_n(Y_0) \oplus W_n(Y_0)}$$

defined in [36, Section 2.5], give a Pin(2)-map

$$f_n \colon \Sigma^{\tilde{\mathbb{R}}^{m_0} \oplus \mathbb{H}^{n_0}} I_n(Y_0) \to \Sigma^{\tilde{\mathbb{R}}^{m_1} \oplus \mathbb{H}^{n_1}} (I_n(Y_1)/s_n(B_{Y_1}))$$

such that

$$f_n((\Sigma^{\tilde{\mathbb{R}}^{m_0} \oplus \mathbb{H}^{n_0}} I_n(Y_0))^{S^1}) \subset (\Sigma^{\tilde{\mathbb{R}}^{m_1} \oplus \mathbb{H}^{n_1}} I_n(Y_1)_{[0]})^{S^1},$$
$$f_n((\Sigma^{\tilde{\mathbb{R}}^{m_0} \oplus \mathbb{H}^{n_0}} I_n(Y_0))^{\mathrm{Pin}(2)}) \subset (\Sigma^{\tilde{\mathbb{R}}^{m_1} \oplus \mathbb{H}^{n_1}} I_n(Y_1)_{[0]})^{\mathrm{Pin}(2)}.$$

Here, $[0] \in \mathrm{Pic}(Y_1)$ is the element corresponding to the flat spin connection, and

$$\begin{aligned} m_0 - m_1 &= \mathrm{rank}_{\mathbb{R}}\, W_n(Y_1)^- - \dim_{\mathbb{R}} W_n(Y_0)^- - b^+(X), \\ n_0 - n_1 &= \mathrm{rank}_{\mathbb{H}}\, F_n(Y_1)^- - \dim_{\mathbb{H}} F_n(Y_0)^- \\ &\quad + \frac{1}{2} n(Y_1, g|_{Y_1}, \mathfrak{t}|_{Y_1}, P_0) - \frac{1}{2} n(Y_0, g|_{Y_0}, \mathfrak{t}|_{Y_0}) - \frac{\sigma(X)}{16}. \end{aligned}$$

The restriction of f_n to $(\Sigma^{\tilde{\mathbb{R}}^{m_0} \oplus \mathbb{H}^{n_0}} I_n(Y_0))^{S^1}$ is induced by the operator

$$(d^+, \pi^0_{-\infty} r_{-Y_0}, \pi^0_{-\infty} r_{Y_1}) \colon \Omega^1_{CC}(X) \to \Omega^+(X) \oplus (\mathcal{W}_{-Y_0})^0_{-\infty} \oplus (\mathcal{W}_{Y_1,[0]})^0_{-\infty}$$

and is a homotopy equivalence

$$(\Sigma^{\tilde{\mathbb{R}}^{m_0} \oplus \mathbb{H}^{n_0}} I_n(Y_0))^{\mathrm{Pin}(2)} \to (\Sigma^{\tilde{\mathbb{R}}^{m_1} \oplus \mathbb{H}^{n_1}} I_n(Y_1)_{[0]})^{\mathrm{Pin}(2)};$$

indeed, both of these are just S^0 consisting of 0 and the base point. Moreover, if $b^+(X) = 0$, the restriction of f_n to $(\Sigma^{\tilde{\mathbb{R}}^{m_0} \oplus \mathbb{H}^{n_0}} I_n(Y_0))^{S^1}$ is a Pin(2)-homotopy equivalence

$$\Sigma^{\tilde{\mathbb{R}}^{m_0}} I_n(Y_0)^{S^1} \to \Sigma^{\tilde{\mathbb{R}}^{m_1}} I_n(Y_1)^{S^1}_{[0]}.$$

We may assume that m_0, m_1 are even and we can use Proposition 6.3.4 (1) to get the first statement.

If $b^+(X)$ is even, $\Sigma^{\tilde{\mathbb{R}}^{m_0} \oplus \mathbb{H}^{n_0}} I_n(Y_0)$ and $\Sigma^{\tilde{\mathbb{R}}^{m_1} \oplus \mathbb{H}^{n_1}} I_n(Y_1)$ are of SWF type at even levels and we can apply Proposition 6.3.4 (1), (2) to f_n to obtain the second statement. If $b^+(X)$ is odd, we take a connected sum $X \# S^2 \times S^2$, and then we can apply Proposition 6.3.4. In this second part, we take advantage of the fact that $\kappa(Y, \mathfrak{s})$ mod 2 agrees with the Rokhlin invariant, as is used in [37, Proof of Theorem 1.4]. ∎

Corollary 6.3.12. *Let* $(X, \mathfrak{t})$ *be a compact spin* 4*-manifold with boundary* Y*. Assume that the index bundle* ind D_Y *is zero in* $KQ^1(B_Y)$*. Then we have*

$$-\frac{\sigma(X)}{8} - 1 \leq b^+(X) + \kappa(Y, \mathfrak{t}|_Y).$$

Moreover, if $b^+(X) > 0$ *we have*

$$-\frac{\sigma(X)}{8} + 1 \leq b^+(X) + \kappa(Y, \mathfrak{t}|_Y).$$

Proof. Removing a small disk from X, we get a bordism X' with boundary $S^3 \coprod Y$. Since $\kappa(S^3) = 0$ and S^3 is Floer $K_{\mathrm{Pin}(2)}$-split, applying Theorem 6.3.11 to X', we obtain the inequalities. ■

Since the spin bordism group Ω_3^{spin} is zero, we obtain the following.

Corollary 6.3.13. $\kappa(Y, \mathfrak{s}) > -\infty$.

Example 6.3.14. Let $\mathfrak{s}$ be a spin structure on $S^1 \times S^2$. Since $S^1 \times S^2$ has a positive scalar curvature metric g, the conditions of Theorem 4.1.2 are satisfied. Hence $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{S}) \cong S^0_{B_Y}$. Here, $\mathfrak{S}$ is a spectral system with $P_0 = \mathcal{E}_0(D)^0_{-\infty}$. Also we have $n(S^1 \times S^2, g, \mathfrak{s}, P_0) = 0$, because there is an orientation-reversing diffeomorphism of $S^1 \times S^2$. So we obtain

$$\kappa(S^1 \times S^2, \mathfrak{s}) \leq 0.$$

Note that $\mathfrak{s}$ extends to a spin structure $\mathfrak{t}$ on $S^1 \times D^3$. Applying Theorem 6.3.12 to $(S^1 \times D^3)\#(S^2 \times S^2)$, we get $\kappa(S^1 \times S^2, \mathfrak{s}) \geq 0$. Hence

$$\kappa(S^1 \times S^2, \mathfrak{s}) = 0.$$

If X is a compact, oriented, spin 4-manifold with boundary $S^1 \times S^2$ and with $b^+(X) > 0$, we have

$$-\frac{\sigma(X)}{8} + 1 \leq b^+(X)$$

by Corollary 6.3.12. This inequality can be also obtained from the $\frac{10}{8}$-inequality [21] for the closed 4-manifold $X \cup (S^1 \times D^3)$ and the additivity of the signature.

Example 6.3.15. Let Y be the flat 3-manifold and $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{s}_3$ be the spin^c structures in Example 6.2.15. As in Example 6.2.15, for any underlying spin structure, we have

$$\kappa(Y, \mathfrak{s}_j) \leq 0$$

for $j = 1, 2, 3$.

Example 6.3.16. Let $p\colon Y \to \Sigma$ be the sphere bundle of the complex line bundle N_d on a closed, oriented surface Σ of degree d. Assume that d is even and that $0 < g(\Sigma) < \frac{d}{2} + 1$. Using a connection on N_d, we have an identification

$$TN_d = p^*T\Sigma \oplus p^*N_d.$$

Let $s\colon Y \to p^*N_d|_Y$ be the tautological section. Then we have

$$TY = p^*T\Sigma \oplus i\mathbb{R}s. \tag{6.3.1}$$

Choose spin structures of Σ and N_d. This is equivalent to choosing complex line bundles $K_\Sigma^{\frac{1}{2}}$, $N_d^{\frac{1}{2}}$ and isomorphisms $K_\Sigma^{\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}} \cong K_\Sigma$, $N_d^{\frac{1}{2}} \otimes N_d^{\frac{1}{2}} \cong N_d$. Also we consider the natural spin structure of the trivial bundle $i\mathbb{R}s$. The spin structures of Σ, $i\mathbb{R}s$ and (6.3.1) induce a spin structure $\mathfrak{s}'$ on Y. Note that $p^*(N_d^{\frac{1}{2}} \otimes N_d^{\frac{1}{2}}) \cong p^*N_d = \underline{\mathbb{C}}$ and hence the structure group of $p^*N_d^{\frac{1}{2}}$ is $\{\pm 1\}$. Put $\mathfrak{s} := \mathfrak{s}' \otimes p^*N_d^{\frac{1}{2}}$. Then $\mathfrak{s}$ is a spin structure of Y with spinor bundle $\mathbb{S} = p^*((K_\Sigma^{-\frac{1}{2}} \oplus K_\Sigma^{\frac{1}{2}}) \otimes N_d^{\frac{1}{2}})$. The spin^c structure induced by $\mathfrak{s}$ is $\mathfrak{s}_{g-1+\frac{d}{2}}$ of Proposition 4.2.3. Since $g \leq g - 1 + \frac{d}{2} < d$, we can apply Theorem 4.2.5 and we get

$$\mathcal{SWF}^{\mathrm{Pin}(2)}(Y, \mathfrak{s}, [\mathfrak{S}]) \cong S_B^0.$$

Here, $\mathfrak{S}$ is as in Theorem 4.2.5. Taking q to be $g - 1 + \frac{d}{2}$ in (6.2.2), we have

$$n(Y, \mathfrak{s}, g_r, P_0) = \frac{1}{8}.$$

Thus we obtain

$$\kappa(Y, \mathfrak{s}) \leq -\frac{1}{8}.$$

Appendix A

The Conley index and parameterized stable homotopy

In this appendix we define the category in which the Seiberg–Witten stable homotopy type lives, and variations thereon, as well as some background on the Conley index. Let G be a compact Lie group for this section. In Section A.1 we define parameterized homotopy categories we will be interested in. In Section A.2 we give basic definitions for the Conley index. In Section A.3 we give a definition of spectra suitable for the construction. The main point is Theorem A.2.1, which states that the parameterized homotopy class of the (parameterized) Conley index is well defined as a parameterized equivariant homotopy class in $\mathcal{K}_{G,Z}$.

A.1 The unstable parameterized homotopy category

This section is intended both to introduce some notation and to point out that the notions introduced in [43] are compatible with parameterized, equivariant homotopy theory, as considered in [16, 39].[1] In the first part, we follow the discussion of Costenoble–Waner [16, Chapter II] and Mrozek–Reineck–Srzednicki [43, Section 3]. In particular, we will occasionally use the notation of model categories, but the reader unfamiliar with this language may safely ignore these aspects. The main points are Lemma A.1.4, which lets us translate properties from the language of [43] to that of [39], and Proposition A.1.6, which is used in describing the change of the Conley index of approximate Seiberg–Witten flows upon changing the finite-dimensional approximation.

Definition A.1.1. Fix a compactly generated space Z with a continuous G-action. A triple $\mathbf{U} = (U, r, s)$ consisting of a G-space U and G-equivariant continuous maps $r: U \to Z$ and $s: Z \to U$ such that $r \circ s = \mathrm{id}_Z$ is called an (equivariant) *ex-space* over Z.[2] Let $\mathcal{K}_{G,Z}$ be the category of ex-spaces, where morphisms $(U, r, s) \to (U', r', s')$ are given by maps $f: U \to U'$ so that $r' f = r$ and $fs = s'$.

In comparison to the ordinary homotopy category, passing to the parameterized homotopy category results in many more maps (for a highbrow definition of the parameterized homotopy category, refer to Remark A.1.3).

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[1] Establishing that [43] and [16, 39] are compatible is, in fact, straightforward. However, at the time that [43] appeared, the May–Sigurdsson parameterized homotopy category had not yet appeared.

[2] In [43], ex-spaces are called *fiberwise-deforming spaces*.

Definition A.1.2. A *fiberwise-deforming map* $f: \mathbf{U} \to \mathbf{U}'$ is an equivariant continuous map $f: (U, s(Z)) \to (U', s'(Z))$ so that $r' \circ f$ is (equivariantly) homotopic to r, relative to $s(Z)$. We say that fiberwise-pointed spaces $\mathbf{U}$ and $\mathbf{U}'$ are *fiberwise-deforming homotopy equivalent* if there exist continuous G-equivariant maps $f: \mathbf{U} \to \mathbf{U}'$, $g: \mathbf{U}' \to \mathbf{U}$ so that

$$\begin{aligned} f \circ s &= s', & g \circ s' &= s, \\ r' \circ f &\simeq r \operatorname{rel} s(Z), & r \circ g &\simeq r' \operatorname{rel} s'(Z), \\ g \circ f &\simeq \mathrm{id}_U \operatorname{rel} s(Z), & f \circ g &\simeq \mathrm{id}_{U'} \operatorname{rel} s'(Z). \end{aligned}$$

We write $[\mathbf{U}]$ for the fiberwise homotopy type of $\mathbf{U}$. We will call a fiberwise-deforming map, along with the choice of a homotopy h between $r' \circ f$ and r, a *lax map*, following [16].

We can also consider homotopies of fiberwise-deforming maps. A *homotopy* of fiberwise-deforming maps will mean a collection of fiberwise-deforming maps $F_t: \mathbf{U} \to \mathbf{U}'$, so that $F: U \times I \to U'$ is continuous. Homotopy of lax maps is similar, but requiring that the homotopy involved in the definition of a lax map is compatible, as we will define below.

Remark A.1.3. There is a model structure (what May–Sigurdsson call the *q-model structure*) on $\mathcal{K}_{G,Z}$ given by declaring a map in $\mathcal{K}_{G,Z}$ to be a weak equivalence, fibration, or cofibration, if it is such after forgetting the base Z, but May–Sigurdsson point out technical difficulties with this model structure. They define a variant, the *qf-model structure* on $\mathcal{K}_{G,Z}$, whose weak equivalences are those of the q-model structure, but with a smaller class of cofibrations. Let $\operatorname{Ho} \mathcal{K}_{G,Z}$ denote the homotopy category of the qf-model structure; we call this the *parameterized homotopy category* and write $[X, Y]_{G,Z}$ for the morphism sets of $\operatorname{Ho} \mathcal{K}_{G,Z}$ – these turn out to be the same as the lax maps X to Y up to homotopy, as in [16, Section 2.1].

Let ΛZ denote the set of *Moore paths* of Z:

$$\Lambda Z = \left\{ (\lambda, \ell) \in Z^{[0,\infty]} \times [0, \infty) : \lambda(r) = \lambda(\ell) \text{ for } r \geq \ell \right\}.$$

Recall that Moore paths have a strictly associative composition:

$$(\lambda\mu)(t) = \begin{cases} \lambda(t) & \text{if } t \leq \ell_\lambda, \\ \mu(t - \ell_\lambda) & \text{if } t \geq \ell_\lambda. \end{cases}$$

Given $r: X \to Z$, the *Moore path fibration* $LX = L(X, r)$ is defined by

$$LX = X \times_Z \Lambda Z,$$

and there is an inherited projection map $Lr: LX \to Z$ by $Lr((x, \lambda)) = \lambda(\infty)$, as well as an inherited section map $Ls: Z \to LX$ given by $Ls(b) = (s(b), b)$, the path with length zero at $s(b)$. Finally, there is a natural inclusion $\iota\colon X \to LX$, which is a weak-equivalence on total spaces, and hence a weak equivalence in the qf-model structure.

Note that a lax map $X \to Y$ is equivalent to the data of a genuine map $X \to LY$ in $\mathcal{K}_{G,Z}$ (using that Y and LY are weakly equivalent, and basic properties of model categories). In particular, any lax map defines an element of $[X, Y]_{G,Z}$, which may or may not be represented by a map $X \to Y$ in $\mathcal{K}_{G,Z}$. The following lemma is then immediate from the definitions.

Lemma A.1.4. *Fiberwise-deforming homotopy-equivalent spaces are weakly equivalent in $\mathcal{K}_{G,Z}$.*

A *homotopy* between lax maps $f_0\colon X \to Y$ and $f_1\colon X \to Y$ is a lax map $X \wedge_Z [0,1]_+ \to Y$ so that $f|_{X \wedge i} = f_i$ for $i = 0, 1$. By [16, Section 2.1] the homotopy classes of lax maps are in agreement with $[X, Y]_{G,Z}$.

We will encounter collections of fiberwise-deforming spaces related by suspensions. We have the following definition.

Definition A.1.5 ([43, Section 3.10]). Let $\mathbf{U} = (U, r, s)$ and $\mathbf{U}' = (U', r', s')$ be ex-spaces over Z, Z', where U, Z are G-spaces and U', Z' are G'-spaces, for G, G' compact Lie groups. Define an equivalence relation $\sim_\wedge$ on $U \times U'$ by $(u, u') \sim_\wedge (v, v')$ if $(u, u') = (v, v')$ or $u = v \in s(Z)$, $r'(u') = r'(v')$ or $r(u) = r(v)$, $u' = v' \in s'(Z')$. Define the *fiberwise smash product* by

$$U \wedge U' := U \times U' / \sim_\wedge .$$

We call an ex-space $\mathbf{U}$ *well pointed* if the inclusion $s(Z) \to U$ is a cofibration in the category of G-spaces. That is, we require that $s(Z) \subset U$ admits a G-equivariant Strøm structure (for a definition see [43, Section 3]). We record the following result from [43] (the proof in the equivariant case is identical to that for the nonequivariant case).

Proposition A.1.6 ([43, Proposition 3.10]). *Assume that* $\mathbf{U}$, $\mathbf{U}'$, $\mathbf{V}$, $\mathbf{V}'$ *are fiberwise well-pointed spaces, with* $[\mathbf{U}] = [\mathbf{U}']$ *and* $[\mathbf{V}] = [\mathbf{V}']$. *Then* $[\mathbf{U} \wedge \mathbf{V}] = [\mathbf{U}' \wedge \mathbf{V}']$.

There is also a pushforward for ex-spaces defined in [39]. Fix an ex-object $\mathbf{U}$ given by $Z \to^s U \to^r Z$ and a map $f\colon Z \to Y$. Define $f_!\mathbf{U} = (f_!U, t, q)$ by the retract diagram

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathbf{U} & \longrightarrow & f_!U \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y, \end{array}$$

where the top square is a pushout, and the bottom is defined by the universal property of pushouts, along with the requirement that $q \circ t = \mathrm{id}$.

Proposition A.1.7 ([39, Proposition 7.3.4]). *Say that* $\mathbf{U}$ *and* $\mathbf{U}'$ *are weakly equivalent* G*-ex-spaces. Then* $f_!\mathbf{U} \simeq f_!\mathbf{U}'$.

Note the simple example that for $\mathbf{U}$ a sectioned spherical fibration over Z, and $f\colon Z \to *$ the collapse, $f_!\mathbf{U}$ is the Thom complex.

For W a real G-vector space and $\mathbf{U} \in \mathcal{K}_{G,Z}$, we define $\Sigma^W \mathbf{U} = \mathbf{U} \wedge W^+$, where W^+ is considered as a parameterized space over a point (we consider $\mathbf{U} \wedge W^+$ as a G-fiberwise deforming space by pulling back along the diagonal map $G \to G \times G$). By Proposition A.1.7, this is well defined on the level of homotopy categories.

Remark A.1.8. For two ex-spaces $\mathbf{U}, \mathbf{U}'$, there is a fiberwise product $\mathbf{U} \times_Z \mathbf{U}'$, which is naturally an ex-space (whose structure maps are inherited from the universal properties of pullbacks), and similarly we obtain a fiberwise smash product $\mathbf{U} \wedge_Z \mathbf{U}'$. That is, we have a functor $\wedge_Z\colon \mathrm{Ho}\,\mathcal{K}_{G,Z} \times \mathrm{Ho}\,\mathcal{K}_{G,Z} \to \mathrm{Ho}\,\mathcal{K}_{G,Z}$. By [39, Proposition 7.3.1], $\wedge_Z$ descends to homotopy categories. The main implication of this from our perspective is that it is legitimate to suspend Conley indices by nontrivial sphere bundles over the base Z.

Definition A.1.9. Fix B a finite G-CW complex. The G-equivariant *parameterized Spanier–Whitehead category* PSW_B is defined as follows. The objects are pairs $(\mathbf{U}, R)$, also denoted by $\Sigma_B^R \mathbf{U}$, for $\mathbf{U}$ an element of $\mathcal{K}_{G,Z}$ (with total space U a finite G-CW complex) and R a virtual real finite-dimensional G-vector space (in a fixed universe). Morphisms are given by

$$\hom((\mathbf{U}, R), (\mathbf{U}', R')) = \operatorname{colim}_W [\Sigma^{W+R}\mathbf{U}, \Sigma^{W+R'}\mathbf{U}']_{G,B},$$

where the colimit is over sufficiently large W. A *stable homotopy equivalence* in $PSW_{G,B}$ will be a stable map that admits some representative which is a weak equivalence. We write $(\mathbf{U}, R) \simeq_{PSW} (\mathbf{U}', R')$ to denote stable homotopy equivalence, omitting the subscript if clear from the context. A *parameterized* G*-equivariant stable homotopy type* is an equivalence class of objects in $PSW_{G,B}$ up to stable homotopy equivalence.

In Definition A.1.9, the colimit may be taken over any sequence of representations which is cofinal in the universe. In particular, in the case of S^1 and $\mathrm{Pin}(2)$-spaces, we will fix the following definitions.

Let $\mathcal{U}_{S^1} = \underline{\mathbb{C}}^{\oplus\infty} \oplus \underline{\mathbb{R}}^{\oplus\infty}$, where $\mathbb{C}$ is the standard representation of $U(1)$, and $\mathbb{R}$ is the trivial representation. Let $\mathcal{U}_{\mathrm{Pin}(2)} = \underline{\mathbb{H}}^{\oplus\infty} \oplus \underline{\widetilde{\mathbb{R}}}^{\oplus\infty}$, where $\mathbb{H}$ is the quaternion representation of $\mathrm{Pin}(2)$ and $\widetilde{\mathbb{R}}$ is the sign representation. There is a full subcategory $\mathfrak{C}_{S^1}$ of $PSW_{S^1,B}$ obtained by considering only those spaces $(\mathbf{U}, R)$ with $R = \underline{\mathbb{C}}^{\oplus n} \oplus \underline{\mathbb{R}}^{\oplus m}$, with $m, n \in \mathbb{Z}$; we use the shorthand $(\mathbf{U}, -2n, -m)$ to denote $(\mathbf{U}, R)$ in $\mathfrak{C}_{S^1}$.

Note that every element of $PSW_{S^1,B}$ on $\mathcal{U}_{S^1}$ is stable homotopy equivalent to an element of $\mathfrak{C}_{S^1}$. Similarly, we write $\mathfrak{C}_{\mathrm{Pin}(2)}$ for the subcategory whose objects are tuples $(\mathbf{U}, R)$ in $PSW_{\mathrm{Pin}(2),B}$ with

$$R = \underline{\mathbb{H}}^{\oplus n} \oplus \underline{\widetilde{\mathbb{R}}}^{\oplus m}.$$

We write $(\mathbf{U}, -4n, -m)$ for the resulting element (so that the notation is consistent with the forgetful functor from Pin(2)-spaces to S^1-spaces).

We note that PSW_*, the parameterized Spanier–Whitehead category over a point, is exactly the ordinary Spanier–Whitehead category. The next lemma follows from the definitions.

Lemma A.1.10. *Let $f\colon B \to *$. There is an induced functor $f_!\colon PSW_B \to PSW_*$ defined by $f_!(\mathbf{U}, R) = (f_!\mathbf{U}, R)$ so that $(\mathbf{U}, R) \simeq_{PSW_B} (\mathbf{U}', R')$ implies $f_!(\mathbf{U}, R) \simeq_{PSW_*} f_!(\mathbf{U}', R')$.*

We have the following corollary.

Corollary A.1.11. *Let $f\colon B \to *$. Then stable-homotopy equivalence classes in PSW_B give well-defined stable-homotopy classes in PSW_*.*

Finally, we remark that May–Sigurdsson [39, Chapters 20–22] define many parameterized homology theories, suitably generalizing the usual definition of a (usual) homology theory, and giving convenient invariants from objects of PSW_*.

A.2 The parameterized Conley index

In this subsection we review the *parameterized Conley index* from [43] (see also Bartsch [7]); we note that we work in considerably less generality than they present. We start by giving the basic definitions in Conley index theory, following [35, Section 5]. Note that the authors of [43] work nonequivariantly; the proofs in the equivariant case are similar.

Let M be a finite-dimensional manifold and φ a flow on M; for a subset $N \subset M$, we define the following sets:

$$\begin{aligned} N^+ &= \{x \in N : \forall t > 0,\ \varphi_t(x) \in N\}, \\ N^- &= \{x \in N : \forall t < 0,\ \varphi_t(x) \in N\}, \\ \operatorname{inv} N &= N^+ \cap N^-. \end{aligned}$$

A compact subset $S \subset M$ is called an *isolated invariant set* if there exists a compact neighborhood $S \subset N$ so that $S = \operatorname{inv}(N) \subset \operatorname{int}(N)$. Such a set N is called an *isolating neighborhood* of S.

A pair (N, L) of compact subsets $L \subset N \subset M$ is an *index pair* for S if the following hold:

(1) $\mathrm{inv}(N \setminus L) = S \subset \mathrm{int}(N \setminus L)$.

(2) L is an exit set for N, that is, for any $x \in N$ and $t > 0$ so that $\varphi_t(x) \notin N$, there exists $\tau \in [0, t)$ with $\varphi_\tau(x) \in L$.

(3) L is *positively invariant* in N. That is, for $x \in L$ and $t > 0$, if $\varphi_{[0,t]}(x) \subset N$, then $\varphi_{[0,t]}(x) \subset L$.

For an index pair $P = (P_1, P_2)$ of an isolated invariant set S, we define $\tau_P \colon P_1 \to [0, \infty]$ by

$$\tau_P(x) = \begin{cases} \sup\{t \geq 0 : \varphi_{[0,t]}(x) \subset P_1 \setminus P_2\} & \text{if } x \in P_1 \setminus P_2, \\ 0 & \text{if } x \in P_2. \end{cases}$$

We say that an index pair P is *regular* if τ_P is continuous.

For Z a Hausdorff space, $\omega \colon M \to Z$ a continuous map, and a regular index pair $P = (P_1, P_2)$, define the *parameterized Conley index* $I_\omega(P)$ as $P_1 \cup_{\omega|_{P_2}} Z$, namely,

$$I_\omega(P) = (Z \times 0) \cup (P_1 \times 1)/\sim,$$

where $(x, 1) \sim (\omega(x), 0)$ for all $x \in P_2 \times 1$.

The space $I_\omega(P)$ is naturally an ex-space, with embedding $s_P \colon Z \to I_\omega(P)$ given by $z \to [z, 0]$, and projection $r_P \colon I_\omega(P) \to Z$ given by $r_P([x, 1]) = \omega(x)$, $r_P([z, 0]) = z$. By construction, $r_P \circ s_P = \mathrm{id}_Z$.

For $Z = *$, we sometimes write $I^u(P)$ for $I_\omega(P)$, to specify the "unparameterized" Conley index.

Theorem A.2.1 ([43, Theorem 2.1]). *If P and Q are two regular index pairs for an isolated invariant set S, then $(I_\omega(P), r_P, s_P)$ and $(I_\omega(Q), r_Q, s_Q)$ have the same equivariant homotopy type over Z, and are both fiberwise well pointed.*

Proof. In [43] it is proved that the two indices have the same fiberwise-deforming type; Lemma A.1.4 then implies the statement. The well-pointedness is [43, Proposition 6.1]. ∎

Definition A.2.2 ([13], [47, Definition 2.6]). A *connected simple system* is a collection I_0 of pointed spaces along with a collection of I_h of homotopy classes of maps among them, so that

(1) for each pair $X, X' \in I_0$, there is a unique class $[f] \in I_h$ from $X \to X'$;

(2) for $f, f' \in I_h$ with $f \colon X \to X'$ and $f' \colon X' \to X''$, the composite $f' \circ f$ is in I_h;

(3) for each $X \in I_0$, the morphism $f \colon X \to X$ is $[\mathrm{id}]$.

Of course, the notion of a connected simple system has an obvious generalization in any category with an associated homotopy category.

Theorem A.2.3 ([47]). *Fix notation as in Theorem A.2.1. The* unparameterized Conley indices $I^u(P) = I_\omega(P)/Z$*, ranging over regular index pairs for* S*, form a* connected simple system.

We conjecture that in fact the parameterized Conley indices also have this property.

Conjecture A.2.4. Fix notation as in Theorem A.2.1. Then the parameterized Conley indices $(I_\omega(P), r_P, s_P)$, running over all regular index pairs for the isolated invariant set S, form a connected simple system.

In Chapter 3 we encounter the parameterized Conley indices for product flows. We have the following theorem.

Theorem A.2.5 ([43, Theorem 2.4]). *Let* S*,* S' *be isolated invariant sets for* φ*,* φ'*. Then*

$$I_{\omega\times\omega'}(S \times S', \varphi \times \varphi') \simeq I_\omega(S, \varphi) \wedge I_{\omega'}(S', \varphi').$$

Moreover, the usual deformation invariance of the Conley index continues for the parameterized Conley index.

Theorem A.2.6 ([43, Theorem 2.5], [47, Corollary 6.8]). *If* N *is an isolating neighborhood with respect to flows* φ^λ *continuously depending on* $\lambda \in [0,1]$*, with a continuous family of isolated invariant sets* S^λ *inside of* N*, then the fiberwise-deforming homotopy type of* $I_\omega(S^\lambda, \varphi^\lambda)$ *is independent of* λ*.*

In the case of the unparameterized Conley index, for each $\lambda_1, \lambda_2 \in [0,1]$*, there is a well-defined, up to homotopy, map of connected simple systems:*

$$F(\lambda_1, \lambda_2)\colon I^u(S^{\lambda_1}, \varphi^{\lambda_1}) \to I^u(S^{\lambda_2}, \varphi^{\lambda_2}).$$

Furthermore, for all $\lambda_1, \lambda_2, \lambda_3 \in [0,1]$*,*

$$F(\lambda_2, \lambda_3) \circ F(\lambda_1, \lambda_2) \sim F(\lambda_1, \lambda_3),$$
$$F(\lambda_1, \lambda_1) \sim \mathrm{id}.$$

Lemma A.2.7. *Fix a flow* φ *on a manifold* X*, along with a map* $p\colon X \to B$*, and write* $\pi\colon B \to *$ *as the map collapsing* B *to a point. Then the pushforward of the parameterized Conley index* $I(\varphi)$*, namely* $\pi_! I(\varphi)$*, is the ordinary Conley index* $I^u(\varphi)$*.*

Proof. This is immediate from the definitions. ∎

We also note the behavior under time reversal.

Theorem A.2.8 ([15, Theorem 3.5], [38, Proposition 3.8]). *Let* M *be a stably parallelized* G*-manifold for a compact Lie group* G*. For* φ *a flow on* M*, the (unparameterized) Conley index of an isolated invariant set* S *with respect to the time-reversed flow* $-\varphi$*, denoted* $I^u(S, -\varphi)$*, is equivariantly Spanier–Whitehead dual to* $I^u(S, \varphi)$*.*

A.3 Spectra

For G a compact Lie group, we define a G-*universe* $\mathcal{U}$ to be a countably infinite-dimensional orthogonal representation of G.

Definition A.3.1. Let $\mathcal{U}$ be a universe with a direct sum decomposition $\mathcal{U} = \bigoplus_{i=1}^{n} V_i^{\infty}$, for finite-dimensional G-representations V_i. A sequential G-spectrum X on $\mathcal{U}$ is a collection $X(V)$ of spaces, indexed on the subspaces of $\mathcal{U}$ of the form $V = \bigoplus_{i=1}^{n} V_i^{k_i}$ for some $k_i \geq 0$, along with transition maps, whenever $W \subset V$,

$$\sigma_{V-W}\colon \Sigma^{V-W} X(W) \to X(V),$$

where $V - W$ is the orthogonal complement of W in V. For $V = W$, the transition map is required to be the identity, and the maps σ are required to be transitive in the usual way. The space $X(V)$ is sometimes referred to as the Vth *level* of the spectrum.

If σ_{V-W} is a homotopy equivalence for V, W sufficiently large, we say that X is a G-*suspension spectrum*.

We will only work with suspension spectra in this memoir.

A *morphism* of spectra $X \to Y$ will be a collection of morphisms

$$\phi_V\colon X(V) \to Y(V)$$

compatible with the transition maps.

We will also consider a generalization of morphisms, as follows.

Definition A.3.2. A *weak* morphism of spectra $\phi\colon X \to Y$ is a collection of morphisms

$$\phi_V\colon X(V) \to Y(V)$$

for V sufficiently large, so that the diagram

$$\begin{tikzcd} \Sigma^{W-V} X(V) \arrow[r, "\Sigma^{W-V}\phi_V"] \arrow[d] & \Sigma^{W-V} Y(V) \arrow[d] \\ X(W) \arrow[r, "\phi_W"'] & Y(W) \end{tikzcd}$$

homotopy commutes for W sufficiently large. Weak morphisms ϕ_0, ϕ_1 are said to be *homotopic* if there exists a weak morphism $\phi_{[0,1]}\colon X \wedge [0,1]_+ \to Y$ restricting to ϕ_j at $X \wedge \{j\}$ for $j = 0, 1$.

We will also need the notion of a *connected simple system of spectra*. Indeed, instead of using the direct generalization for spaces, the Seiberg–Witten Floer spectrum, as currently defined, requires that we work with weak morphisms instead, as follows.

Definition A.3.3. A *connected simple system of G-spectra* is a collection I_0 of G-spectra, along with a collection I_h of weak homotopy classes of maps between them, so that the analogs of (1)–(3) of Definition A.2.2 are satisfied.

Remark A.3.4. In Section 3.5 we could have used nonsequential G-spectra, but we have no need for the added generality in the memoir, and it slightly complicates the notation.

Remark A.3.5. If higher naturality is established for the Conley index, then it would be possible to replace *weak* morphisms in the definition of **SWF**, and Definition A.3.3 could be replaced with ordinary morphisms of spectra.

Afterword: Finite-dimensional approximation in other settings

Outside of Seiberg–Witten theory, we expect that the notion of parameterized finite-dimensional approximation may be applicable in some cases in symplectic topology. The methods of this memoir rely, roughly speaking, on a few special features of the Seiberg–Witten equations, relative to other Floer-type problems:

(1) The configuration space is naturally a bundle over a compact, finite-dimensional manifold.

(2) Bubbling phenomena do not occur.

(3) With respect to the bundle structure in (1), the Seiberg–Witten equations are "close to linear" on the fibers.

(4) There is a relatively good understanding of the spectrum of the Dirac operator.

Perhaps the item most likely to elicit worry more generally is (1). However, we note that it is classical that for any compact subset K of a Hilbert manifold, there is an open sub-Hilbert-manifold B containing K which is diffeomorphic to the total space of a Hilbert bundle over a compact finite-dimensional manifold.

Lemma 1. *Let M be a separable Hilbert manifold and $K \subset M$ a compact subset. Then there exists some open $U \supset K$ diffeomorphic to $V \times H$, where H is a separable Hilbert space, and V is a finite-dimensional smooth manifold.*

Proof. By compactness, choose a good open cover $\mathcal{C}'$ of K, with finite subcover $\mathcal{C} = \{U_i\}_i$, which is once again good, with $U = \bigcup_i U_i$. The nerve $N(\mathcal{C})$ is then homotopy equivalent to U. Moreover, $N(\mathcal{C})$ may be embedded in some finite-dimensional Euclidean space and has a regular neighborhood which is a smooth manifold V, with $N(\mathcal{C}) \simeq V$. By [10,41], separable infinite-dimensional homotopy-equivalent Hilbert manifolds are diffeomorphic. Then $V \times H$ is diffeomorphic to U, as needed. ∎

In particular, (1) holds locally around the moduli space (of gradient flows of the Chern–Simons functional, symplectic action, etc.) in many situations of interest (there is the technical point that a version of Lemma 1 which respected L^2_k-norms for multiple values of k would be more appropriate, but we have not attempted it). Although it is not at all clear how to perform finite-dimensional approximation in the presence of bubbling, nonetheless items (2) and (4) also hold in various geometric situations. The problem then amounts to establishing appropriate versions of (3) in specific situations; this appears challenging except when the configuration space is very special.

We finally note that the finite-dimensional approximation process of this memoir can also be applied locally. In particular, it can be applied in the neighborhood of a broken trajectory. Here, the base space is some smooth trajectory very close to the broken trajectory, so that there is a neighborhood containing the broken trajectory, and on which (1)–(4) hold. Finite-dimensional approximation then produces a sequence of flows, whose finite-energy integral curves converge to solutions of the Seiberg–Witten equations. Assuming nondegeneracy, one may be able to assemble these locally constructed approximating submanifolds into the data of a flow category as in [11]. The hoped-for result of this process would be replacing the need to give a smooth structure to the corners for the moduli spaces of the Seiberg–Witten equations themselves, with the problem of putting a smooth structure on the trajectory spaces of a finite-dimensional approximation. The main obstruction to this approach is likely the need to establish that the approximating submanifolds constructed this way are suitably independent of the choices involved in their construction, which may be difficult.

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Seiberg–Witten Floer Spectra for $b_1 > 0$

The Seiberg–Witten Floer spectrum is a stable homotopy refinement of the monopole Floer homology of Kronheimer and Mrowka. The Seiberg–Witten Floer spectrum was defined by Manolescu for closed, spin^c 3-manifolds with $b_1 = 0$ in an S^1-equivariant stable homotopy category and has been producing interesting topological applications. Lidman and Manolescu showed that the S^1-equivariant homology of the spectrum is isomorphic to the monopole Floer homology.

For closed spin^c 3-manifolds Y with $b_1(Y) > 0$, there are analytic and homotopy-theoretic difficulties in defining the Seiberg–Witten Floer spectrum. In this memoir, we address the difficulties and construct the Seiberg–Witten Floer spectrum for Y, provided that the first Chern class of the spin^c structure is torsion and that the triple-cup product on $H^1(Y;\mathbb{Z})$ vanishes. We conjecture that its S^1-equivariant homology is isomorphic to the monopole Floer homology.

For a 4-dimensional spin^c cobordism X between Y_0 and Y_1, we define the Bauer–Furuta map on these new spectra of Y_0 and Y_1, which is conjecturally a refinement of the relative Seiberg–Witten invariant of X. As an application, for a compact spin 4-manifold X with boundary Y, we prove a $\frac{10}{8}$-type inequality for X which is written in terms of the intersection form of X and an invariant $\kappa(Y)$ of Y.

In addition, we compute the Seiberg–Witten Floer spectrum for some 3-manifolds.

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