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Reciprocity Laws for (φ_L, Γ_L)-Modules over Lubin–Tate Extensions

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Abstract

In the Lubin–Tate setting, we study pairings for analytic (φ_L, Γ_L)-modules and prove an abstract reciprocity law which then implies a relation between the analogue of Perrin-Riou's big exponential map as developed by Berger and Fourquaux and a p-adic regulator map whose construction relies on the theory of Kisin–Ren modules generalizing the concept of Wach modules to the Lubin–Tate situation.

Contents

Chapter 1

Introduction

Classically explicit reciprocity laws or formulae usually mean an explicit computation of Hilbert symbols or (local) cup products using, e.g., differential forms, (Coleman) power series, etc., and a bunch of manifestations of this idea exists in the literature due to Artin–Hasse, Iwasawa, Wiles, Kolyvagin, Vostokov, Brückner, Coleman, Sen, de Shalit, Fesenko, Bloch–Kato, Benois, ... In the same spirit, Perrin-Riou's reciprocity law gives an explicit calculation of the Iwasawa cohomology pairing in terms of big exponential and regulator maps for crystalline representations of $G_{\mathbb{Q}_p}$; more precisely, the latter maps are adjoint to each other when also involving the crystalline duality pairing after base change to the distribution algebra corresponding to the cyclotomic situation. We refer the interested reader to the survey [92] concerning the historical development of explicit reciprocity laws.

The motivation for this article is the question of what happens if one replaces the cyclotomic $\mathbb{Z}_p$-extension by a Lubin–Tate extension L_∞ over some finite extension L over $\mathbb{Q}_p$ with Galois group $\Gamma_L = \mathrm{Gal}(L_\infty/L)$ and Lubin–Tate character $\chi_{\mathrm{LT}} : G_L \to o_L^\times$ which all arise from a Lubin–Tate formal group attached to a prime $\pi_L \in o_L$, the additive group of the ring of integers o_L of L; by q we denote the cardinality of the residue field $o_L/o_L\pi_L$. This situation is considerably more complicated for various reasons. First of all, for $L \neq \mathbb{Q}_p$, the group Γ_L requires more than one topological generator and in order to achieve again a sort of one-dimensional theory one has to work in the "L-analytic case". Secondly, the quotient $\frac{q}{\pi_L}$ related to the two meanings of p in the cyclotomic case as uniformizer and cardinality of the residue field is no longer a unit and appears directly in formulae. We try to extend the above sketched cyclotomic picture to the Lubin–Tate case at least for L-analytic crystalline representations of the absolute Galois group G_L of L. As pointed out in [80] already, the character $\tau := \chi_{\mathrm{cyc}} \cdot \chi_{\mathrm{LT}}^{-1}$ plays a crucial role, again related to $\frac{q}{\pi_L}$.

To this aim we study (φ_L, Γ_L)-modules over different Robba rings with coefficients in suitable complete intermediate fields $L \subseteq K \subseteq \mathbb{C}_p$. The starting point is the theory of Schneider and Teitelbaum. The p-adic functional analysis of the additive group of p-adic integers $\mathbb{Z}_p$ is based on the fact that the continuous p-adic characters of $\mathbb{Z}_p$ are parametrized by the points of the p-adic open unit disk. The generalization of this basic feature to the additive group o_L was constructed in [74]: a rigid analytic group variety $\mathfrak{X} = \mathfrak{X}_L$ over L whose points parametrize the locally analytic characters of o_L. The Fourier isomorphism of [74] identifies the ring of holomorphic functions $\mathcal{O}_L(\mathfrak{X})$ with the locally analytic distribution algebra $D(o_L, L)$ of o_L. A connection to the Lubin–Tate setting is also established in loc. cit. Under the assumption that the period Ω of the dual of the fixed Lubin–Tate group belongs to K an isomorphism

$\kappa : \mathbf{B}_K \cong \mathfrak{X}_K$ of rigid analytic varieties over K, called the Lubin–Tate isomorphism, is constructed. Here $\mathbf{B}$ denotes the rigid analytic open unit disk and the index K indicates base change to K. In other words, $\mathfrak{X}_L$ is a form of the open unit disk, which is non-trivial if $L \neq \mathbb{Q}_p$.

The multiplicative group $o_L^\times$ acts by multiplication on o_L and hence acts naturally on $\mathfrak{X}$ and $\mathcal{O}_L(\mathfrak{X})$. In [8], the ring $\mathcal{O}_L(\mathfrak{X})$ with its $o_L^\times$-action was enlarged by a geometric construction to the Robba ring $\mathcal{R}_L(\mathfrak{X})$ with $o_L^\times$-action. Some details of this will be recalled in Section 4.1.1. This made it possible to introduce the notion of a (φ_L, Γ_L)-module over $\mathcal{R}_L(\mathfrak{X})$. The main result of [8] relates the category of these new (φ_L, Γ_L)-modules to the Lubin–Tate (φ_L, Γ_L)-modules considered in [44, 73] and therefore to Galois representations. In this paper, we pursue the further systematic development of the theory of (φ_L, Γ_L)-modules over $\mathcal{R}_L(\mathfrak{X})$ in two directions.

First of all, the construction of the variety $\mathfrak{X}_L$ and its corresponding rings is not specific to the additive group o_L. In Sections 4.1.2 and 4.3.4, this will be worked out first for the multiplicative group $o_L^\times$ and then for the Galois group Γ_L (which, by the Lubin–Tate character χ_{LT}, is isomorphic to $o_L^\times$). The corresponding Robba rings will be denoted by $\mathcal{R}_L(\mathfrak{X}^\times)$ and $\mathcal{R}_L(\Gamma_L)$, respectively. We will study the question to which extent the $\Gamma_L \cong o_L^\times$-action on $\mathcal{R}_L(\mathfrak{X})$, resp., on any (φ_L, Γ_L)-module over $\mathcal{R}_L(\mathfrak{X})$, can be extended to an action of the full ring $\mathcal{R}_L(\Gamma_L)$. A first indication of how this works is given by the Mellin transformation in Lemma 4.1.6. The open and closed inclusion of locally analytic manifolds $o_L^\times \hookrightarrow o_L$ induces an isomorphism between the multiplicative distribution algebra $D(o_L^\times, L)$ and the kernel of the ψ-operator on the additive distribution algebra $D(o_L, L)$. This fairly simple observation has a vast conceptual generalization to the following new structural result in Theorem 4.3.23.

Theorem 1 (Theorem 4.3.23). *For any complete intermediate field $L \subseteq K \subseteq \mathbb{C}_p$ and any L-analytic (φ_L, Γ_L)-module M over $\mathcal{R}_K(\mathfrak{X})$, we have the following:*

- *The Γ_L-action on M extends functorially and continuously to an $\mathcal{R}_K(\Gamma_L)$-module structure on $M^{\psi_L = 0}$.*

- *Any basis of M over $\mathcal{R}_K(\mathfrak{X})$ can be transformed by a very simple explicit recipe into a basis of $M^{\psi_L = 0}$ over $\mathcal{R}_K(\Gamma_L)$ of the same cardinality.*

The proof will be given in Sections 4.3.5–4.3.8. For $\mathbf{B}$ instead of $\mathfrak{X}$ an analogous statement holds if K contains Ω; technically, this is the case we prove first (see Theorem 4.3.21) which then, after involving the Lubin–Tate isomorphism, descends to the above theorem.

Secondly, we will systematically develop the self-duality properties of the Robba rings under consideration in Section 4.2.3. This is based on the Serre duality for the coherent cohomology of rigid analytic spaces as developed in [11, 20, 90]. This is recalled and complemented in Sections 4.2.1 and 4.2.2. In order to apply this to Robba rings these will be related to the cohomology with compact support of the

underlying character varieties. This results in residue pairings

$$\Omega^1_{\mathcal{R}_K(-)} \times \mathcal{R}_K(-) \to K$$

for the differentials $\Omega^1_{\mathcal{R}_K(-)}$. The main new results are Corollary 4.2.7 and Proposition 4.2.12. The latter proposition then has to be evaluated for our two cases of main interest $\mathfrak{X}$ and $\mathfrak{X}^{\times}$ in Section 4.2.3. This leads to two duality pairings,

$$\langle\, ,\, \rangle_{\mathfrak{X}} : \mathcal{R}_L(\mathfrak{X}) \times \mathcal{R}_L(\mathfrak{X}) \to L \quad \text{and} \quad \langle\, ,\, \rangle_{\mathfrak{X}^{\times}} : \mathcal{R}_L(\mathfrak{X}^{\times}) \times \mathcal{R}_L(\mathfrak{X}^{\times}) \to L.$$

The relation between these two pairings will be investigated in Section 4.5.3 and will be the base of our main results in Section 4.5. The right-hand pairing induces topological isomorphisms,

$$\operatorname{Hom}_{K,\mathrm{cont}}(\mathcal{R}_K(\Gamma_L), K) \cong \mathcal{R}_K(\Gamma_L)$$

and

$$\operatorname{Hom}_{K,\mathrm{cont}}(\mathcal{R}_K(\Gamma_L)/D(\Gamma_L, K), K) \cong D(\Gamma_L, K).$$

For an L-analytic (φ_L, Γ_L)-module M over $\mathcal{R} := \mathcal{R}_K(\mathfrak{Y})$ with $\mathfrak{Y}$ equal to either $\mathfrak{X}$ or $\mathbf{B}$, we finally use these isomorphisms to define on the one hand the two Iwasawa pairings

$$\{\, ,\, \}^0_{M,\mathrm{Iw}} : \breve{M}^{\psi_L=0} \times M^{\psi_L=0} \to \mathcal{R}_K(\Gamma_L)$$

and

$$\{\, ,\, \}_{M,\mathrm{Iw}} : \breve{M}^{\psi_L=\frac{q}{\pi_L}} \times M^{\psi_L=1} \to D(\Gamma_L, K),$$

where $\breve{M} := \operatorname{Hom}_{\mathcal{R}}(M, \Omega^1_{\mathcal{R}})$. They are linked by the commutative diagram

$$
\begin{array}{ccccc}
\{\, ,\, \}_{M,\mathrm{Iw}} : \breve{M}^{\psi_L=\frac{q}{\pi_L}} & \times & M^{\psi_L=1} & \longrightarrow & D(\Gamma_L, K) \\
{\scriptstyle \varphi_L-1}\downarrow & & {\scriptstyle \frac{\pi_L}{q}\varphi_L-1}\downarrow & & \downarrow \\
\{\, ,\, \}^0_{M,\mathrm{Iw}} : \breve{M}^{\psi_L=0} & \times & M^{\psi_L=0} & \longrightarrow & \mathcal{R}_K(\Gamma_L).
\end{array}
$$

Now assume that M arises as $D^{\dagger}_{\mathrm{rig}}(W)$ under Berger's equivalence of categories, if $\mathfrak{Y} = \mathbf{B}$, and as $D^{\dagger}_{\mathrm{rig}}(W)_{\mathfrak{X}}$ under the equivalence from [8], if $\mathfrak{Y} = \mathfrak{X}$ (see Theorem 4.5.28), from an L-analytic, crystalline representation W of G_L, whence $\breve{M} \cong D^{\dagger}_{\mathrm{rig}}(W^*(\chi_{\mathrm{LT}}))$ and $\breve{M} \cong D^{\dagger}_{\mathrm{rig}}(W^*(\chi_{\mathrm{LT}}))_{\mathfrak{X}}$, respectively. Then, on the other hand, we obtain the pairing

$$[\, ,\,]_{D_{\mathrm{cris},L}(W)} : \mathcal{R}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(W^*(\chi_{\mathrm{LT}})\big) \times \mathcal{R}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}(W) \to \mathcal{R}_K(\Gamma_L)$$

by base extension of the usual crystalline duality pairing, if $\mathfrak{Y} = \mathbf{B}$ assuming $\Omega \in K$, see (4.94). The work of Kisin–Ren and Berger–Schneider–Xie, respectively, provides comparison isomorphisms

$$\operatorname{comp}_M : M\Big[\frac{1}{t_{\mathfrak{Y}}}\Big] \cong \mathfrak{R}\Big[\frac{1}{t_{\mathfrak{Y}}}\Big] \otimes_L D_{\mathrm{cris},L}(W)$$

and

$$\mathrm{comp}_{\breve{M}} : \breve{M}\left[\frac{1}{t_{\mathfrak{Y}}}\right] \cong \mathfrak{R}\left[\frac{1}{t_{\mathfrak{Y}}}\right] \otimes_L D_{\mathrm{cris},L}\left(W^*(\chi_{\mathrm{LT}})\right).$$

Here $t_{\mathbf{B}} := t_{\mathrm{LT}} := \log_{\mathrm{LT}}(Z) \in \mathcal{R}$ denotes the Lubin–Tate period which stems from the Lubin–Tate logarithm while $t_{\mathfrak{x}} = \log_{\mathfrak{x}}$ as defined before Remark 4.2.9. The Lubin–Tate character χ_{LT} induces isomorphism $\Gamma_L \xrightarrow{\cong} o_L^\times$ as well as $\mathrm{Lie}(\Gamma_L) \xrightarrow{\cong} L$, and we let $\nabla \in \mathrm{Lie}(\Gamma_L)$ be the preimage of 1. Then the abstract reciprocity law we prove is the following statement.

Theorem 2 (Theorem 4.5.32). *For all $x \in \breve{M}^{\psi_L=0}$ and $y \in M^{\psi_L=0}$, for which the crystalline pairing is defined via the comparison isomorphism, it holds that*

$$\frac{q-1}{q}\{\nabla x, y\}_{M,\mathrm{Iw}}^0 = [x, y]_{D_{\mathrm{cris},L}(W)}$$

if $\mathfrak{Y} = \mathfrak{X}$, while the analogous statement for $\mathfrak{Y} = \mathbf{B}$ holds upon assuming $\Omega \in K$.

As explained in more detail at the beginning of Section 4.5, the proof of this abstract reciprocity law is mainly based on the insight, how the residue maps of $\mathfrak{X}$ and $\mathfrak{X}^\times$ and hence their associated pairings $\langle\,,\,\rangle_{\mathfrak{x}}$ and $\langle\,,\,\rangle_{\mathfrak{x}^\times}$ are related to each other by Theorem 4.5.12 in Section 4.5.3.

As an application for $\mathfrak{Y} = \mathbf{B}$ we show in Chapter 5 the adjointness of big exponential and regulator maps. Recall that already Fourquaux [32], who initiated the investigation of Perrin-Riou's approach for Lubin–Tate extensions in his thesis in 2005, had achieved a generalization of Colmez's construction of the *Perrin-Riou logarithm*. Moreover, Berger and Fourquaux [7] have constructed for V an L-analytic representation of G_L and an integer $h \geq 1$ such that

- $\mathrm{Fil}^{-h} D_{\mathrm{cris},L}(V) = D_{\mathrm{cris},L}(V)$ and

- $D_{\mathrm{cris},L}(V)^{\varphi_L = \pi_L^{-h}} = 0$,

a *big exponential map* à la Perrin-Riou

$$\Omega_{V,h} : \left(\mathcal{O}_K(\mathbf{B})\right)^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}(V) \to D_{\mathrm{rig}}^\dagger(V)^{\psi_L=\frac{q}{\pi_L}},$$

which up to comparison isomorphism is for $h = 1$ given by $f = (1 - \varphi_L)x \mapsto \nabla x$ and which interpolates Bloch–Kato exponential maps $\exp_{L,V(\chi_{\mathrm{LT}}^r)}$.

On the other hand, based on an extension of the work of Kisin and Ren [44] in the first section, we construct for a lattice $T \subseteq V$, such that $V(\tau^{-1})$ is L-analytic and crystalline and such that V does not have any quotient isomorphic to $L(\tau)$, a *regulator map* à la Loeffler and Zerbes [52]

$$\mathcal{L}_V^0 : H_{\mathrm{Iw}}^1(L_\infty/L, T) \cong D_{\mathrm{LT}}\left(T(\tau^{-1})\right)^{\psi_L=1} \to \left(\mathcal{O}_K(\mathbf{B})\right)^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\left(V(\tau^{-1})\right)$$

as applying the operator

$$1 - \frac{\pi_L}{q}\varphi_L$$

up to comparison isomorphism. Then we derive from the abstract version above with $W = V(\tau^{-1})$ the following reciprocity formula.

Theorem 3 (Theorem 5.2.1). *Assume that $V^*(1)$ is L-analytic. We suppose further that* $\mathrm{Fil}^{-1} D_{\mathrm{cris},L}(V^*(1)) = D_{\mathrm{cris},L}(V^*(1))$ *and additionally*

$$D_{\mathrm{cris},L}\big(V^*(1)\big)^{\varphi_L = \pi_L^{-1}} = D_{\mathrm{cris},L}\big(V^*(1)\big)^{\varphi_L = 1} = 0.$$

Then the following diagram commutes:

$$
\begin{array}{ccccc}
D_{\mathrm{rig}}^\dagger\big(V^*(1)\big)^{\psi_L = \frac{q}{\pi_L}} & \times & D\big(V(\tau^{-1})\big)^{\psi_L = 1} & \xrightarrow{\ \{\,,\}_{\mathrm{Iw}}\ } & D(\Gamma_L, \mathbb{C}_p) \\
\Big\uparrow{\scriptstyle \Omega_{V^*(1),1}} & & {\scriptstyle \mathscr{L}_V^0}\Big\downarrow & & \Big\| \\
\big(\mathcal{O}_K(\mathbf{B})\big)^{\psi_L = 0} \otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big) \times \big(\mathcal{O}_K(\mathbf{B})\big)^{\psi_L = 0} \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) & & & \xrightarrow{[\,,]} & D(\Gamma_L, \mathbb{C}_p).
\end{array}
$$

While the crystalline pairing satisfies an interpolation property (Proposition 5.2.22) for trivial reasons, the statement that the second Iwasawa pairing interpolates Tate's cup product pairing is more subtle (Corollary 5.2.20). Eventually, the interpolation property of Berger and Fourquaux for $\Omega_{V,h}$ combined with the adjointness of the latter with $\mathscr{L}_V^0$ implies an interpolation formula for the regulator map, which interpolates dual Bloch–Kato exponential maps, see Theorem 5.2.26.

As alluded to at the beginning of this introduction, we expect that the new regulator map $\mathscr{L}_V^0$ will play the same role as the one of Loeffler and Zerbes; i.e., it should be related to ε-isomorphisms and p-adic L-functions. With regard to the first topic, this regulator map should be compared to the one defined in [92, §6.6], which is used in [55] to generalize Nakamura's explicit reciprocity law from the cyclotomic to the Lubin–Tate setting. Moreover, we hope that our regulator map can be used to also generalize [51] to the Lubin–Tate situation.

With respect to the second topic we explain at the end of Section 5.1.1 how in the context of an CM-elliptic curve with supersingular reduction at p the attached p-adic L-function is the image of the Euler system of elliptic units under this regulator map. While this is merely a reinterpretation of the work [74, §5], Manji is working out in his PhD thesis [56] another manifestation of this principle in the context of an ordinary automorphic representation defined over the unitary group $\mathrm{GU}(2, 1)$, see also [92, Ex. 7.2] for slightly more details.

In the remaining part of this introduction, we briefly outline the content of the various sections.

After fixing some general notation in Chapter 2, we dedicate Chapter 3 to the generalization of Wach modules to the Lubin–Tate situation. First of all, we recall in Section 3.1 the work [44] of Kisin and Ren, who already accomplished most of this task. Then we check a missing compatibility of their constructions in Lemma 3.1.5, i.e., the commutativity of diagram (3.2), and add Corollary 3.1.15 to the picture to

identify how to recover $D_{\mathrm{cris},L}(V)$ from the Kisin–Ren module attached to an L-analytic presentation V. Moreover, we compare in Proposition 3.1.10 their approach to the construction in [5, Prop. II.1.1], in particular, applying the concept of *positive* representations. After calculating the determinant of the crystalline comparison isomorphism in Section 3.2, we generalize in Section 3.3 for non-negative Hodge–Tate weights Appendix A of [4] to the Lubin–Tate situation culminating in the statement in Lemma 3.3.6 that under certain conditions the part of the (φ_L, Γ_L)-module fixed under the ψ-operator is contained in the associated Kisin–Ren module.

The main section of our article is Chapter 4 concerning the theory of (φ_L, Γ_L)-modules over different types of Robba rings as explained above. In particular, following the ideas of [42] in the cyclotomic setting, we deduce from the residue pairing in Section 4.5.1 for an L-analytic (φ_L, Γ_L)-module M and its dual $\widetilde{M}$ the corresponding Iwasawa pairing in Section 4.5.4. In Section 4.5.5, we recall as Theorem 4.5.28 the equivalences of categories between L-analytic Galois representation and L-analytic, étale (φ_L, Γ_L)-modules over the Robba rings $\mathcal{R}_L(\mathbf{B})$ and $\mathcal{R}_L(\mathcal{X})$, respectively. The study of the compatibility of the Iwasawa pairings under the Kisin–Ren comparison isomorphisms (4.91) and also using in the proof of Lemma 4.5.26 the consequence (4.86) from Theorem 4.5.12 leads to our main result, the abstract reciprocity law in Theorem 4.5.32.

In Chapter 5, we apply first the results from Chapter 3 concerning the Wach modules à la Kisin–Ren to construct under certain technical assumptions the regulator map in Section 5.1. As a reality check and for further motivation, we study how previous work [74, §5] by the first named author and Teitelbaum fits into this picture. In Section 5.2, we recall Berger's and Fourquaux's construction of their big exponential map. From this and the definition of our regulator map the adjointness in Theorem 5.2.1 of the big exponential map and the regulator map is an immediate consequence of the abstract reciprocity law in Theorem 4.5.32. After fixing some notation of homological algebra in Section 5.2.1 and recalling the self-duality of Koszul complexes in Section 5.2.2, we study in Section 5.2.3 continuous and analytic cohomology in order to consider various generalized Herr complexes of (φ_L, Γ_L)-modules. The version using Koszul complexes being the most explicit one is then used to prove Proposition 5.2.18 and Corollary 5.2.20, i.e., the statement that Iwasawa pairing interpolates the local cup product. Based on this compatibility of the dualities, we shall derive in Section 5.2.4 the interpolation property for the regulator map from that for the big exponential map, see Theorem 5.2.26.

In Appendix A, we first show how to construct a quasi-isomorphism between the two Herr complexes built of continuous cocycle and an appropriate Koszul complex, respectively. In order to compare the cup product pairings between (φ_L, Γ_L)-modules over different coefficient rings, we have to generalize some technical lemmata from [19, 42] to our setting, especially concerning the cokernel of the operator $\psi - 1$. In particular, the derived finite dimensionality of some cohomology group h^2 is needed

for the interchange of taking continuous duals with forming cohomology h^1 in the sense of Remark 5.2.6.

Finally, Appendix B recalls the quasi-isomorphism (B.5) between the complex defining Iwasawa cohomology in the sense of Nekovar and the complex given by the operator $\psi - 1$ by [45] and calculates its descent, which is needed for the interpolation property of the Iwasawa pairing.

Chapter 2

Notation

Let $\mathbb{Q}_p \subseteq L \subset \mathbb{C}_p$ be a field of finite degree d over $\mathbb{Q}_p$, o_L the ring of integers of L, $\pi_L \in o_L$ a fixed prime element, $k_L = o_L/\pi_L o_L$ the residue field, $q := |k_L|$, e the absolute ramification index of L and $G_L := \mathrm{Gal}(\overline{L}/L)$. We always use the absolute value $|\ |$ on $\mathbb{C}_p$ which is normalized by $|\pi_L| = q^{-1}$. We *warn* the reader, though, that we will repeatedly use the references [8, 33, 49, 68, 69, 74, 76] in which the absolute value is normalized differently from this paper by $|p| = p^{-1}$. Our absolute value is the dth power of the one in these references. The transcription of certain formulae to our convention will usually be done silently.

Let further $\mathrm{Rep}_L(G_L)$ denote the category of finite-dimensional L-vector spaces equipped with a continuous linear G_L-action.

We fix a Lubin–Tate formal o_L-module $\mathrm{LT} = \mathrm{LT}_{\pi_L}$ over o_L corresponding to the prime element π_L. We always identify LT with the open unit disk around zero, which gives us a global coordinate Z on LT. The o_L-action then is given by formal power series $[a](Z) \in o_L[\![Z]\!]$. For simplicity the formal group law will be denoted by $+_{\mathrm{LT}}$.

The power series $\frac{\partial(X +_{\mathrm{LT}} Y)}{\partial Y}|_{(X,Y)=(Z,0)}$ is a unit in $o_L[\![Z]\!]$ and we let $g_{\mathrm{LT}}(Z)$ denote its inverse. Then $g_{\mathrm{LT}}(Z)dZ$ is, up to scalars, the unique invariant differential form on LT (cf. [37, §5.8]). We also let

$$\log_{\mathrm{LT}}(Z) = Z + \cdots \tag{2.1}$$

denote the unique formal power series in $L[\![Z]\!]$ whose formal derivative is g_{LT}. This $\log_{\mathrm{LT}}$ is the logarithm of LT (cf. [47, §8.6]). In particular, $g_{\mathrm{LT}}dZ = d\log_{\mathrm{LT}}$. The invariant derivation ∂_{inv} corresponding to the form $d\log_{\mathrm{LT}}$ is determined by

$$f'dZ = df = \partial_{\mathrm{inv}}(f)d\log_{\mathrm{LT}} = \partial_{\mathrm{inv}}(f)g_{\mathrm{LT}}dZ$$

and hence is given by

$$\partial_{\mathrm{inv}}(f) = g_{\mathrm{LT}}^{-1} f'. \tag{2.2}$$

For any $a \in o_L$ we have

$$\log_{\mathrm{LT}}\big([a](Z)\big) = a \cdot \log_{\mathrm{LT}} \quad \text{and hence} \quad ag_{\mathrm{LT}}(Z) = g_{\mathrm{LT}}\big([a](Z)\big) \cdot [a]'(Z) \tag{2.3}$$

(cf. [47, 8.6 Lemma 2]).

Let T_π be the Tate module of LT. Then T_π is a free o_L-module of rank one, say with generator $\eta = (\eta_n)$, and the action of G_L on T_π is given by a continuous character $\chi_{\mathrm{LT}} : G_L \to o_L^\times$. Let T_π' denote the Tate module of the p-divisible group Cartier dual to LT with period Ω (depending on the choice of a generator of T_π'),

which again is a free o_L-module of rank one. The Galois action on $T'_\pi \cong T^*_\pi(1)$ is given by the continuous character $\tau := \chi_{\mathrm{cyc}} \cdot \chi_{\mathrm{LT}}^{-1}$, where χ_{cyc} is the cyclotomic character.

For $n \geq 0$ we let L_n/L denote the extension (in $\mathbb{C}_p$) generated by the π_L^n-torsion points of LT, and we put $L_\infty := \bigcup_n L_n$. The extension L_∞/L is Galois. We let $\Gamma_L := \mathrm{Gal}(L_\infty/L)$ and $H_L := \mathrm{Gal}(\overline{L}/L_\infty)$. The Lubin–Tate character χ_{LT} induces an isomorphism $\Gamma_L \xrightarrow{\cong} o_L^\times$.

Henceforth, we use the same notation as in [80]. In particular, the ring endomorphisms induced by sending Z to $[\pi_L](Z)$ are called φ_L where applicable, e.g., for the ring $\mathcal{A}_L$ defined to be the π_L-adic completion of $o_L[\![Z]\!][Z^{-1}]$ or $\mathcal{B}_L := \mathcal{A}_L[\pi_L^{-1}]$ which denotes the field of fractions of $\mathcal{A}_L$. Recall that we also have introduced the unique additive endomorphism ψ_L of $\mathcal{B}_L$ (and then $\mathcal{A}_L$) which satisfies

$$\varphi_L \circ \psi_L = \pi_L^{-1} \cdot \mathrm{trace}_{\mathcal{B}_L/\varphi_L(\mathcal{B}_L)}.$$

Moreover, projection formula

$$\psi_L\big(\varphi_L(f_1)f_2\big) = f_1 \psi_L(f_2) \quad \text{for any } f_i \in \mathcal{B}_L$$

and the formula

$$\psi_L \circ \varphi_L = \tfrac{q}{\pi_L} \cdot \mathrm{id}$$

hold. An étale (φ_L, Γ_L)-module M comes with a Frobenius operator φ_M semilinear with respect to φ_L and an induced operator denoted by ψ_M.

Let $\widetilde{\mathbf{E}}^+ := \varprojlim o_{\mathbb{C}_p}/p o_{\mathbb{C}_p}$ with the transition maps being given by the Frobenius $\varphi(a) = a^p$. We may also identify $\widetilde{\mathbf{E}}^+$ with $\varprojlim o_{\mathbb{C}_p}/\pi_L o_{\mathbb{C}_p}$ with the transition maps being given by the q-Frobenius $\varphi_q(a) = a^q$. Recall that $\widetilde{\mathbf{E}}^+$ is a complete valuation ring with residue field $\overline{\mathbb{F}_p}$ and its field of fractions $\widetilde{\mathbf{E}} = \varprojlim \mathbb{C}_p$ being algebraically closed of characteristic p. Let $\mathfrak{m}_{\widetilde{\mathbf{E}}}$ denote the maximal ideal in $\widetilde{\mathbf{E}}^+$.

The q-Frobenius φ_q first extends by functoriality to the rings of the Witt vectors $W(\widetilde{\mathbf{E}}^+) \subseteq W(\widetilde{\mathbf{E}})$ and then o_L-linearly to

$$W(\widetilde{\mathbf{E}}^+)_L := W(\widetilde{\mathbf{E}}^+) \otimes_{o_{L_0}} o_L \subseteq W(\widetilde{\mathbf{E}})_L := W(\widetilde{\mathbf{E}}) \otimes_{o_{L_0}} o_L,$$

where L_0 is the maximal unramified subextension of L. The Galois group G_L obviously acts on $\widetilde{\mathbf{E}}$ and $W(\widetilde{\mathbf{E}})_L$ by automorphisms commuting with φ_q. This G_L-action is continuous for the weak topology on $W(\widetilde{\mathbf{E}})_L$ (cf. [73, Lem. 1.5.3]).

Sometimes we omit the index q, L, or M from the Frobenius operator, but we always write φ_p when dealing with the p-Frobenius.

The evaluation of the global coordinate Z of LT at π_L-power torsion points induces a map (not a homomorphism of abelian groups) $\iota : T_\pi \to \widetilde{\mathbf{E}}^+$. Namely, if $t = (z_n)_{n \geq 1} \in T_\pi$ with $[\pi_L](z_{n+1}) = z_n$ and $[\pi_L](z_1) = 0$, then $z_{n+1}^q \equiv z_n \bmod \pi_L$ and hence $\iota(t) := (z_n \bmod \pi_L)_n \in \widetilde{\mathbf{E}}^+$. As before, we fix an o_L-generator η of T_π

and put $\omega := \iota(\eta)$. Then there exists a (unique) lift $\omega_{\mathrm{LT}} \in W(\widetilde{\mathbf{E}}^+)_L$ of ω satisfying the following (cf. [80, Lem. 4.1]):

(i)   if $\eta' = a\eta$ with $a \in o_L^\times$ denotes another generator of T_π, then $\omega'_{\mathrm{LT}} = [a](\omega_{\mathrm{LT}})$ is the corresponding lift;

(ii) $\varphi_q(\omega_{\mathrm{LT}}) = [\pi_L](\omega_{\mathrm{LT}})$;

(iii) $\sigma(\omega_{\mathrm{LT}}) = [\chi_{\mathrm{LT}}(\sigma)](\omega_{\mathrm{LT}})$ for any $\sigma \in G_L$.

Chapter 3

Wach modules à la Kisin–Ren

3.1 Wach modules

In this section, we recall the theory of Wach modules à la Kisin–Ren [44] (with the simplifying assumption that – in their notation – $K = L$, $m = 1$, etc.).

By sending Z to $\omega_{\mathrm{LT}} \in W(\widetilde{\mathbf{E}}^+)_L$ we obtain a G_L-equivariant, Frobenius compatible embedding of rings

$$o_L[\![Z]\!] \to W(\widetilde{\mathbf{E}}^+)_L,$$

the image of which we call $\mathbf{A}_L^+$; it is a subring of $\mathbf{A}_L$ (the image of $\mathcal{A}_L$ in $W(\widetilde{\mathbf{E}})_L$). The latter ring is a complete discrete valuation ring with prime element π_L and residue field of the image $\mathbf{E}_L$ of $k_L(\!(Z)\!) \hookrightarrow \widetilde{\mathbf{E}}$ sending Z to $\omega := \omega_{\mathrm{LT}} \bmod \pi_L$. We form the maximal integral unramified extension (= strict henselization) $\mathbf{A}_L^{nr}$ of $\mathbf{A}_L$ inside $W(\widetilde{\mathbf{E}})_L$. Its p-adic completion $\mathbf{A}$ still is contained in $W(\widetilde{\mathbf{E}})_L$. Note that $\mathbf{A}$ is a complete discrete valuation ring with prime element π_L and residue field of the separable algebraic closure $\mathbf{E}_L^{\mathrm{sep}}$ of $\mathbf{E}_L$ in $\widetilde{\mathbf{E}}$. By the functoriality properties of strict henselizations, the q-Frobenius φ_q preserves $\mathbf{A}$. According to [44, Lem. 1.4], the G_L-action on $W(\widetilde{\mathbf{E}})_L$ respects $\mathbf{A}$ and induces an isomorphism

$$H_L = \ker(\chi_{\mathrm{LT}}) \xrightarrow{\cong} \mathrm{Aut}^{\mathrm{cont}}(\mathbf{A}/\mathbf{A}_L).$$

We set $\mathbf{A}^+ := \mathbf{A} \cap W(\widetilde{\mathbf{E}}^+)_L$.

Set $Q := \frac{[\pi_L](\omega_{\mathrm{LT}})}{\omega_{\mathrm{LT}}} \in \mathbf{A}_L^+$, which satisfies by definition $\varphi_L(\omega_{\mathrm{LT}}) = Q \cdot \omega_{\mathrm{LT}}$.

Following [44] we write $\mathcal{O} = \mathcal{O}_L(\mathbf{B})$ for the ring of rigid analytic functions on the open unit disk $\mathbf{B}$ over L, or equivalently the ring of power series in Z over L converging in $\mathbf{B}$. Via sending ω_{LT} to Z we view $\mathbf{A}_L^+$ as a subring of $\mathcal{O}$. We denote by $\mathrm{Mod}_{\mathcal{O}}^{\varphi_L, \Gamma_L, \mathrm{an}}$ the category consisting of finitely generated free $\mathcal{O}$-modules $\mathcal{M}$ together with the following data:

(i)   an isomorphism $1 \otimes \varphi_{\mathcal{M}} : (\varphi_L^* \mathcal{M})[\frac{1}{Q}] \cong \mathcal{M}[\frac{1}{Q}]$,[1]

(ii)   a semilinear Γ_L-action on $\mathcal{M}$, commuting with $\varphi_{\mathcal{M}}$ and such that the induced action on $D(\mathcal{M}) := \mathcal{M}/\omega_{\mathrm{LT}}\mathcal{M}$ is trivial.

We note that since $\mathcal{M}/\omega_{\mathrm{LT}}\mathcal{M} = \mathcal{M}[\frac{1}{Q}]/\omega_{\mathrm{LT}}\mathcal{M}[\frac{1}{Q}]$, the map $\varphi_{\mathcal{M}}$ induces an L-linear endomorphism of $D(\mathcal{M})$, which we denote by $\varphi_{D(\mathcal{M})}$. As a consequence of (i), in fact, it is an automorphism.

---

[1]By $\varphi_L^* \mathcal{M}$ we understand the module $\mathcal{O} \otimes_{\mathcal{O}, \varphi_L} \mathcal{M}$.

The Γ_L-action on $\mathcal{M}$ is differentiable (cf. [8, Lem. 3.4.13]),[2] and the corresponding derived action of $\mathrm{Lie}(\Gamma_L)$ is L-bilinear (cf. [8, Rem. 3.4.15]).[3]

Similarly, we denote by $\mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L, \Gamma_L, \mathrm{an}}$ the category consisting of finitely generated free $\mathbf{A}_L^+$-modules N together with the following data:

(i)   an isomorphism $1 \otimes \varphi_N : (\varphi_L^* N)[\frac{1}{Q}] \cong N[\frac{1}{Q}]$,[4]

(ii)   a semilinear Γ_L-action on N, commuting with φ_N and such that the induced action on $N/\omega_{\mathrm{LT}} N$ is trivial.

The map φ_N induces an L-linear automorphism of $D(N) := N[\frac{1}{p}]/\omega_{\mathrm{LT}} N[\frac{1}{p}]$ which we denote by $\varphi_{D(N)}$.

Obviously, we have the base extension functor

$$\mathcal{O} \otimes_{\mathbf{A}_L^+} - : \mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L, \Gamma_L, \mathrm{an}} \to \mathrm{Mod}_{\mathcal{O}}^{\varphi_L, \Gamma_L, \mathrm{an}}.$$

It satisfies

$$D(\mathcal{O} \otimes_{\mathbf{A}_L^+} N) = D(N). \tag{3.1}$$

We write $\mathrm{Mod}_{\mathcal{O}}^{\varphi_L, \Gamma_L, 0}$ for the full subcategory of $\mathrm{Mod}_{\mathcal{O}}^{\varphi_L, \Gamma_L, \mathrm{an}}$ consisting of all $\mathcal{M}$ such that $\mathcal{R} \otimes_{\mathcal{O}} \mathcal{M}$ is pure of slope 0. Here $\mathcal{R}$ denotes the Robba ring, which will be recalled in detail in Section 4.3.1.

By $\mathrm{Mod}_L^{F, \varphi_q}$ we denote the category of finite-dimensional L-vector spaces D equipped with an L-linear automorphism $\varphi_q : D \xrightarrow{\cong} D$ and a decreasing, separated, and exhaustive filtration, indexed by $\mathbb{Z}$, by L-subspaces. In $\mathrm{Mod}_L^{F, \varphi_q}$ we have the full subcategory $\mathrm{Mod}_L^{F, \varphi_q, \mathrm{wa}}$ of weakly admissible objects (cf. [44, (2.3.2)]). For D in $\mathrm{Mod}_L^{F, \varphi_q, \mathrm{wa}}$ let

$$V_L(D) := (B_{\mathrm{cris}, L} \otimes_L D)^{\varphi_q = 1} \cap \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_L D),$$

where, as usual, $B_{\mathrm{cris}, L} := B_{\mathrm{cris}} \otimes_{L_0} L$. In order to formulate the crystalline comparison theorem in this context, we also consider the category $\mathrm{Mod}_{L_0 \otimes_{\mathbb{Q}_p} L}^{F, \varphi}$ of finitely generated free $L_0 \otimes_{\mathbb{Q}_p} L$-modules $\mathfrak{D}$ equipped with a $(\varphi_p \otimes \mathrm{id})$-linear automorphism $\varphi : \mathfrak{D} \xrightarrow{\cong} \mathfrak{D}$ as well as a decreasing, separated, and exhaustive filtration on $\mathfrak{D}_L := \mathfrak{D} \otimes_{L_0} L$, indexed by $\mathbb{Z}$, by $L \otimes_{\mathbb{Q}_p} L$-submodules. For $\mathfrak{D}$ in $\mathrm{Mod}_{L_0 \otimes_{\mathbb{Q}_p} L}^{F, \varphi}$

---

[2]Note that the statements in loc. cit. are all over the character variety; but by the introduction to §3.4 they are also valid over the open unit ball – with even easier proofs.

[3]In [44], being L-analytic is an extra condition in the definition of $\mathrm{Mod}_{\mathcal{O}}^{\varphi_L, \Gamma_L, \mathrm{an}}$, which by this remark is automatically satisfied, whence the corresponding categories with and without the superscript "an" in [44] coincide!

[4]By $\varphi_L^* N$ we understand the module $\mathbf{A}_L^+ \otimes_{\mathbf{A}_L^+, \varphi_L} N$, and formally φ_N is a map from N to $N[\frac{1}{Q}]$.

we define, as usual,

$$V(\mathfrak{D}) := (B_{\mathrm{cris}} \otimes_{L_0} \mathfrak{D})^{\varphi=1} \cap \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_L \mathfrak{D}_L).$$

Let $\mathrm{Rep}_{o_L,f}(G_L)$ denote the category of finitely generated free o_L-modules with a continuous linear G_L-action and $\mathrm{Rep}_{o_L,f}^{\mathrm{cris,an}}(G_L)$ the full subcategory of those T which are free over o_L and such that the representation $V := L \otimes_{o_L} T$ is crystalline and *analytic*, i.e., satisfying that, if $D_{\mathrm{dR}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_L}$, the filtration on $D_{\mathrm{dR}}(V)_{\mathfrak{m}}$ is trivial for each maximal ideal $\mathfrak{m}$ of $L \otimes_{\mathbb{Q}_p} L$ which does not correspond to the identity $\mathrm{id} : L \to L$. Correspondingly, we let $\mathrm{Rep}_L^{\mathrm{cris}}(G_L)$, resp., $\mathrm{Rep}_L^{\mathrm{cris,an}}(G_L)$, denote the category of continuous G_L-representations in finite-dimensional L-vector spaces which are crystalline, resp., crystalline and analytic. The base extension functor $L \otimes_{o_L} -$ induces an equivalence of categories

$$\mathrm{Rep}_{o_L,f}^{\mathrm{cris,an}}(G_L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\simeq} \mathrm{Rep}_L^{\mathrm{cris,an}}(G_L).$$

Here applying $\otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ to a $\mathbb{Z}_p$-linear category means applying this functor to the Hom-modules. For V in $\mathrm{Rep}_L^{\mathrm{cris,an}}(G_L)$ we set $D_{\mathrm{cris},L}(V) := (B_{\mathrm{cris},L} \otimes_L V)^{G_L} = (B_{\mathrm{cris}} \otimes_{L_0} V)^{G_L}$ and $D_{\mathrm{cris}}(V) := (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_L}$. The usual crystalline comparison theorem says that D_{cris} and V are equivalences of categories between $\mathrm{Rep}_L^{\mathrm{cris}}(G_L)$ and the subcategory of weakly admissible objects in $\mathrm{Mod}_{L_0 \otimes_{\mathbb{Q}_p} L}^{F,\varphi}$.

Lemma 3.1.1 ([78, Lem. 5.3] and subsequent discussion, or [44, Cor. 3.3.1]). *There is a fully faithful $\otimes$-functor*

$$\sim \,:\, \mathrm{Mod}_L^{F,\varphi_q} \to \mathrm{Mod}_{L_0 \otimes_{\mathbb{Q}_p} L}^{F,\varphi}$$
$$D \mapsto \widetilde{D} := L_0 \otimes_{\mathbb{Q}_p} D,$$

whose essential image consists of all analytic *objects, i.e., those for which the filtration on the non-identity components is trivial. A quasi-inverse functor from the essential image is given by sending $\mathfrak{D}$ to the base extension $L \otimes_{L_0 \otimes_{\mathbb{Q}_p} L} \mathfrak{D}$ for the multiplication map $L_0 \otimes_{\mathbb{Q}_p} L \to L$.*

Lemma 3.1.1 implies that

$$D_{\mathrm{cris},L}(V)^{\sim} \cong D_{\mathrm{cris}}(V) \quad \text{for any } V \text{ in } \mathrm{Rep}_L^{\mathrm{cris,an}}(G_L).$$

We denote by $\mathfrak{M}^{\mathrm{et}}(\mathbf{A}_L)$ the category of étale (φ_q, Γ_L)-modules over $\mathbf{A}_L$ (cf. [80, Def. 3.7]) and by $\mathfrak{M}_f^{\mathrm{et}}(\mathbf{A}_L)$ the full subcategory consisting of those objects, which are finitely generated free as $\mathbf{A}_L$-module. For M in $\mathfrak{M}_f^{\mathrm{et}}(\mathbf{A}_L)$, resp., for T in $\mathrm{Rep}_{o_L,f}(G_L)$, we put $V(M) := (\mathbf{A} \otimes_{\mathbf{A}_L} M)^{\varphi_q \otimes \varphi_M = 1}$, resp., $D_{\mathrm{LT}}(T) := (\mathbf{A} \otimes_{o_L} T)^{\ker(\chi_{\mathrm{LT}})}$.

Having defined all of the relevant categories (and most of the functors) we now contemplate the following diagram of functors:

$$
\begin{array}{c}
\mathfrak{M}_f^{\mathrm{et}}(\mathbf{A}_L) \xrightarrow[\substack{\simeq \\ \overleftarrow{D_{\mathrm{LT}}}}]{V} \mathrm{Rep}_{o_L,f}(G_L) \\
\end{array}
$$

$$(3.2)$$

The arrows without decoration are the obvious natural ones. The following pairs of functors are quasi-inverse $\otimes$-equivalences of $\otimes$-categories:

- (D_{LT}, V) by [44, Thm. 1.6];
- $(D_{\mathrm{cris},L}, V_L)$ by combining the crystalline comparison theorem (cf. [31, Rem. 3.6.7]) and Lemma 3.1.1;
- $(D, \mathcal{M})$ by [44, Prop. 2.2.6] (or [8, Thm. 3.4.16]) and [44, Cor. 2.4.4], to which we also refer for the definition of the functor $\mathcal{M}$.

In particular, all functors in the above diagram are $\otimes$-functors. The second arrow in the left column, resp., the left arrow in the second horizontal row, is an equivalence of categories by [44, Cor. 2.4.2], resp., by [44, Cor. 3.3.8]. The lower square and the upper quadrangle are commutative for trivial reasons.

Remark 3.1.2. We list a few additional properties of these functors.

(i)   For any M in $\mathfrak{M}_f^{\mathrm{et}}(\mathbf{A}_L)$ the inclusion $V(M) \subseteq \mathbf{A} \otimes_{\mathbf{A}_L} M$ extends to an isomorphism

$$
\mathbf{A} \otimes_{o_L} V(M) \xrightarrow{\cong} \mathbf{A} \otimes_{\mathbf{A}_L} M, \tag{3.3}
$$

which is compatible with the φ_q- and Γ_L-actions on both sides.

(ii) The functors D_{LT}, V, and $V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} -)$ respect exact sequences (of abelian groups).

(iii)   (Cf. [8, Prop. 3.4.14].) For any object $\mathcal{M}$ in $\mathrm{Mod}_{\mathcal{O}}^{\varphi_L, \Gamma_L, \mathrm{an}}$ the projection map $\mathcal{M}[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}] \to D(\mathcal{M})$ restricts to an isomorphism $\mathcal{M}[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}]^{\Gamma_L} \xrightarrow{\cong} D(\mathcal{M})$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{M}[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}]^{\Gamma_L} & \xrightarrow{\cong} & D(\mathcal{M}) \\
\varphi_{\mathcal{M}} \downarrow & & \downarrow \varphi_{D(\mathcal{M})} \\
\mathcal{M}[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}]^{\Gamma_L} & \xrightarrow{\cong} & D(\mathcal{M})
\end{array}
$$

is commutative; moreover, we have

$$
\mathcal{M}\left[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\right] = \mathcal{O}\left[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\right] \otimes_L \mathcal{M}\left[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\right]^{\Gamma_L} \cong \mathcal{O}\left[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\right] \otimes_L D(\mathcal{M}).
$$

Now we recall that A_{cris} is the p-adic completion of a divided power envelope of $W(\widetilde{\mathbf{E}}^+)$ (cf. [62, §1.4.1]) and let $A_{\mathrm{cris},L} := A_{\mathrm{cris}} \otimes_{o_{L_0}} o_L$. The inclusion $W(\widetilde{\mathbf{E}}^+) \subseteq A_{\mathrm{cris}}$ induces an embedding $\mathbf{A}_L^+ \subseteq W(\widetilde{\mathbf{E}}^+)_L \subseteq A_{\mathrm{cris},L}$.

We observe that $t_{\mathrm{LT}} = \log_{\mathrm{LT}}(\omega_{\mathrm{LT}})$ belongs to $B_{\mathrm{cris},L}^{\times}$. Indeed, by [22, §III.2] we know that $\varphi_p(B_{\max}) \subseteq B_{\mathrm{cris}} \subseteq B_{\max}$, whence we obtain

$$
\varphi_q(B_{\max} \otimes_{L_0} L) \subseteq B_{\mathrm{cris},L} \subseteq B_{\max} \otimes_{L_0} L,
$$

where the definition of $B_{\max}$ can be found in loc. cit. By [24, Prop. 9.10, Lem. 9.17, §9.7], t_{LT} and ω_{LT} are invertible in $B_{\max,L} \subseteq B_{\max} \otimes_{L_0} L$ (this reference assumes that the power series $[\pi_L](Z)$ is a polynomial. But, by some additional convergence considerations, the results can be seen to hold in general (cf. [73, §2.1] for more details)). Hence, by the above inclusions and using that $\varphi_q(t_{\mathrm{LT}}) = \pi_L t_{\mathrm{LT}}$, we see that t_{LT} is a unit in $B_{\mathrm{cris},L}$. In particular, we have an inclusion $A_{\mathrm{cris},L}[\frac{1}{\pi_L}, \frac{1}{t_{\mathrm{LT}}}] \subseteq B_{\mathrm{cris},L}$. Moreover, since $\varphi_q(\omega_{\mathrm{LT}}) = Q\omega_{\mathrm{LT}}$ is invertible in $\varphi_q(B_{\max} \otimes_{L_0} L)$, the elements ω_{LT} and Q are units in $B_{\mathrm{cris},L}$ as well. In particular, we have an inclusion

$$
\mathbf{A}^+\left[\frac{1}{\omega_{\mathrm{LT}}}\right] \subseteq B_{\mathrm{cris},L}. \tag{3.4}
$$

Next we shall recall in Lemma 3.1.4 below that the above inclusion $\mathbf{A}_L^+ \subseteq A_{\mathrm{cris},L}$ extends to a (continuous) ring homomorphism

$$
\mathcal{O} \to A_{\mathrm{cris},L}\left[\frac{1}{\pi_L}\right] \subseteq B_{\mathrm{cris},L}. \tag{3.5}
$$

For $\alpha \in \widetilde{\mathbf{E}}^+ \cong \mathrm{proj}\lim_n o_{\mathbb{C}_p}$ (cf. [73, Lem. 1.4.5]) we denote by $\alpha^{(0)}$ as usual its zero-component.

Lemma 3.1.3. *The following diagram of o_{L_0}-modules is commutative:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & W(\widetilde{\mathbf{E}}^+) & \xrightarrow{\Theta} & o_{\mathbb{C}_p} & \longrightarrow & 0 \\
& & \downarrow & & u \downarrow & & \| & & \\
0 & \longrightarrow & \ker(\Theta_L) & \longrightarrow & W(\widetilde{\mathbf{E}}^+)_L & \xrightarrow{\Theta_L} & o_{\mathbb{C}_p} & \longrightarrow & 0,
\end{array} \tag{3.6}
$$

where

$$J := \ker(\Theta),$$

$$\Theta : \sum_{n \geq 0} [\alpha_n] p^n \mapsto \sum_{n \geq 0} \alpha_n^{(0)} p^n, \quad \text{and similarly}$$

$$\Theta_L : \sum_{n \geq 0} [\alpha_n] \pi_L^n \mapsto \sum_{n \geq 0} \alpha_n^{(0)} \pi_L^n,$$

while u denotes the canonical map as defined in [30, Lem. 1.2.3], *which sends Teichmüller lifts* $[\alpha]$ *with respect to* $W(\widetilde{\mathbf{E}}^+)$ *to the Teichmüller lift* $[\alpha]$ *with respect to* $W(\widetilde{\mathbf{E}}^+)_L$.

Proof. First of all, we recall from [73, Lem. 1.6.1] that Θ and Θ_L are continuous and show that also u is continuous, each time with respect to the weak topology. Let V_p and V_{π_L} denote the Verschiebung on $W(\widetilde{\mathbf{E}}^+)$ and $W(\widetilde{\mathbf{E}}^+)_L$, respectively.

A fundamental system of open neighborhoods in these rings consists of

$$U_{\mathfrak{a},m} := \big\{ (b_0, b_1, \ldots) \in W(\widetilde{\mathbf{E}}^+) \mid b_0, \ldots, b_{m-1} \in \mathfrak{a} \big\} = \sum_{i=0}^{m-1} V_p^i \big([\mathfrak{a}]\big) + p^m W(\widetilde{\mathbf{E}}^+)$$

and similarly $U_{\mathfrak{a},m}^L := \{ (b_0, b_1, \ldots) \in W(\widetilde{\mathbf{E}}^+)_L | b_0, \ldots, b_{m-1} \in \mathfrak{a} \}$ for open ideals $\mathfrak{a}$ of $\widetilde{\mathbf{E}}^+$ and $m \geq 0$; see §1.5 in loc. cit. By o_{L_0}-linearity, we see that $u(p^m W(\widetilde{\mathbf{E}}^+)) \subseteq p^m W(\widetilde{\mathbf{E}}^+)_L$. Using the relation

$$u(V_p x) = \frac{p}{\pi_L} V_{\pi_L} \big(u(F^{f-1} x) \big)$$

from [30, Lem. 1.2.3][5] one easily concludes that

$$u \big(V_p^i([b]) \big) = \big(\tfrac{p}{\pi_L} \big)^i V_{\pi_L}^i \big([b^{p^{i(f-1)}}] \big),$$

whence $u(U_{\mathfrak{a},m}) \subseteq U_{\mathfrak{a},m}^L$ and the continuity of u follows.

Since the commutativity is clear on Teichmüller lifts and on p by o_{L_0}-linearity, which generate a dense ideal, the result follows by continuity. ∎

The following lemma generalizes parts from [62, Prop. 1.5.2].

Lemma 3.1.4. *Sending $f = \sum_{n \geq 0} a_n Z^n$ to $f(\omega_{\mathrm{LT}})$ induces a continuous map*

$$\mathcal{O} \to A_{\mathrm{cris},L} \big[\tfrac{1}{\pi_L} \big],$$

where the source carries the Fréchet topology while the target is a topological o_{L_0}-module, of which the topology is uniquely determined by requiring that $A_{\mathrm{cris},L}$ is open; i.e., the system $p^m A_{\mathrm{cris},L}$ with $m \geq 0$ forms a basis of open neighborhoods of 0.

---

[5]Note the typos in loc. cit., where $u(V_p x) = \frac{p}{\pi_L} V_{\pi_L} (F^{f-1} u(x))$ is stated. Moreover, one has the relation $u(F^f x) = F u(x)$.

Proof. First of all, the relation $J^p \subseteq pA_{\text{cris}}$ from [62, §1.4.1, bottom of p. 96] (note that $J^p \subseteq W_p(R)$ regarding the notation in loc. cit. for the last object) implies easily by flat base change

$$J_L^p \subseteq pA_{\text{cris},L} \tag{3.7}$$

with $J_L := J \otimes_{o_{L_0}} o_L$. By [73, Lem. 2.1.12] we know that ω_{LT} belongs to $\ker(\Theta_L)$. Now we claim that there exists a natural number r' such that $\omega_{\text{LT}}^{r'}$ lies in $W_1 := J_L + pW(\widetilde{\mathbf{E}}^+)_L$, whence for $r := pr'$ we have $\omega_{\text{LT}}^r \in W_p$ with $W_m := W_1^m$ for all $m \geq 0$. To this aim, note that diagram (3.6) induces the following commutative diagram with exact lines:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & W_1 & \longrightarrow & W(\widetilde{\mathbf{E}}^+) \otimes_{o_{L_0}} o_L & \overset{\Theta}{\to} & (o_{\mathbb{C}_p} \otimes_{o_{L_0}} o_L)/p(o_{\mathbb{C}_p} \otimes_{o_{L_0}} o_L) & \to & 0 \\
& & \downarrow & & \cong \downarrow & & \mu \downarrow & & \\
0 & \to & \ker(\Theta_L) + pW(\widetilde{\mathbf{E}}^+)_L & \longrightarrow & W(\widetilde{\mathbf{E}}^+)_L & \overset{\Theta_L}{\longrightarrow} & o_{\mathbb{C}_p}/po_{\mathbb{C}_p} & \longrightarrow & 0,
\end{array}
$$

where the map μ is induced by sending $a \otimes b$ to ab and a reference for the middle vertical isomorphism is [73, Prop. 1.1.26]. By the snake lemma the cokernel of the left vertical map is isomorphic to

$$
\begin{aligned}
\ker(\mu) &\subseteq \ker\left((o_{\mathbb{C}_p} \otimes_{o_{L_0}} o_L)/p(o_{\mathbb{C}_p} \otimes_{o_{L_0}} o_L) \to \bar{k}\right) \\
&= \ker(o_{\mathbb{C}_p}/po_{\mathbb{C}_p} \otimes_k o_L/po_L \to o_{\mathbb{C}_p}/\mathfrak{m}_{\mathbb{C}_p} \otimes_k o_L/\pi_L o_L) \\
&= \mathfrak{m}_{\mathbb{C}_p} \otimes_k o_L/po_L + o_{\mathbb{C}_p}/po_{\mathbb{C}_p} \otimes_k \pi_L o_L/po_L
\end{aligned}
$$

and thus consists of nilpotent elements, whence the claim follows. Here $\mathfrak{m}_{\mathbb{C}_p}$ denotes the maximal ideal of $o_{\mathbb{C}_p}$.

Now let $f = \sum_{n \geq 0} a_n Z^n$ satisfy that $|a_n|\rho^n$ tends to zero for all $\rho < 1$. Writing $n = q_n r + r_n$ with $0 \leq r_n < r$, we have

$$a_n \omega_{\text{LT}}^n = a_n \omega_{\text{LT}}^{r_n} (\omega_{\text{LT}}^r)^{q_n} \in a_n W_{pq_n} \subseteq a_n p^{q_n} A_{\text{cris},L},$$

where the last inclusion follows from (3.7). But $|a_n p^{q_n}| \leq |a_n|_p p^{1-\frac{n}{r}}$ tends to 0 for $n \to \infty$. Thus, the series $\sum_{n \geq 0} a_n \omega_{\text{LT}}^n$ converges in $A_{\text{cris},L}[\frac{1}{\pi_L}]$.

Moreover, since one has $\sup|a_n p^{-1+\frac{n}{r}}| \leq p\|f\|_\rho$ for the usual norms $\| \; \|_\rho$ if $1 > \rho > p^{-\frac{1}{r}}$, we obtain for any m that

$$\{f \mid \|f\|_\rho < p^{-m-1}\} \subseteq \{f \in \mathcal{O} \mid f(\omega_{\text{LT}}) \in p^m A_{\text{cris},L}\},$$

whence the latter set, which is the preimage of $p^m A_{\text{cris},L}$, is open. This implies continuity. ∎

Lemma 3.1.5. *The big square in the middle of diagram* (3.2) *is a commutative square of $\otimes$-functors (up to a natural isomorphism of $\otimes$-functors).*

Proof. We have to establish a natural isomorphism

$$L \otimes_{o_L} V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N) \cong V_L\big(D(\mathcal{O} \otimes_{\mathbf{A}_L^+} N)\big) \quad \text{for any } N \text{ in } \mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L,\Gamma_L,\mathrm{an}}. \quad (3.8)$$

In fact, we shall prove the dual statement, i.e., using (3.1), that

$$\big(L \otimes_{o_L} V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N)\big)^* \cong V_L\big(D(N)\big)^*, \quad (3.9)$$

where $*$ indicates the L-dual. From the canonical isomorphisms

$$\mathrm{Hom}_{\mathbf{A}_L,\varphi_q}(M, \mathbf{A}) \cong \mathrm{Hom}_{\mathbf{A},\varphi_q}(\mathbf{A} \otimes_{\mathbf{A}_L} M, \mathbf{A})$$
$$\cong \mathrm{Hom}_{\mathbf{A},\varphi_q}\big(\mathbf{A} \otimes_{o_L} V(M), \mathbf{A}\big)$$
$$\cong \mathrm{Hom}_{o_L}\big(V(M), \mathbf{A}^{\varphi_q=1}\big)$$
$$\cong \mathrm{Hom}_{o_L}\big(V(M), o_L\big),$$

where we used (3.3) for the second isomorphism and write M for $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N$, we conclude that the left-hand side of (3.9) is canonically isomorphic to $\mathrm{Hom}_{\mathbf{A}_L,\varphi_q}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N, \mathbf{A}) \otimes_{o_L} L$. Let $\mathbf{A}^+ := \mathbf{A} \cap W(\tilde{\mathbf{E}}^+)_L$. On the one hand, by [44, Lem. 3.2.1], base extension induces an isomorphism

$$\mathrm{Hom}_{\mathbf{A}_L^+,\varphi_q}\big(N, \mathbf{A}^+\big[\tfrac{1}{\omega_{\mathrm{LT}}}\big]\big) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{A}_L,\varphi_q}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N, \mathbf{A}).$$

On the other hand, in [44, Prop. 3.2.3], Kisin and Ren construct a natural isomorphism

$$\mathrm{Hom}_{\mathbf{A}_L^+,\varphi_q}\big(N, \mathbf{A}^+\big[\tfrac{1}{\omega_{\mathrm{LT}}}\big]\big) \otimes_{o_L} L \xrightarrow{\cong} \mathrm{Hom}_{L,\varphi_q,\mathrm{Fil}}\big((N/\omega_{\mathrm{LT}}N)[\tfrac{1}{p}], B_{\mathrm{cris},L}\big).^6 \quad (3.10)$$

Therefore, the left-hand side of (3.9) becomes naturally isomorphic to

$$\mathrm{Hom}_{L,\varphi_q,\mathrm{Fil}}\big(D(N), B_{\mathrm{cris},L}\big) \cong V_L\big(D(N)^*\big), \quad (3.11)$$

where the last isomorphism is straightforward. Thus, the proof of (3.9) is reduced to the canonical identity

$$V_L\big(D(N)^*\big) \cong V_L\big(D(N)\big)^*. \quad (3.12)$$

This can be proved in the same way as in [31, Rem. 3.4.5 (iii), Rem. 3.6.7]: Since V_L is a rigid $\otimes$-functor, it preserves inner Hom-objects, in particular duals.

In order to see that (3.8) is compatible with tensor products, note that base change, taking L-duals or applying comparison isomorphisms, is $\otimes$-compatible. Thus the claim is reduced to the tensor compatibility of the isomorphism (3.10), the construction of which we therefore recall from [44]. It is induced by a natural map

$$\mathrm{Hom}_{\mathbf{A}_L^+}\big(N, \mathbf{A}^+\big[\tfrac{1}{\omega_{\mathrm{LT}}}\big]\big) \otimes_{o_L} L \to \mathrm{Hom}_L\big((N/\omega_{\mathrm{LT}}N)[\tfrac{1}{p}], B_{\mathrm{cris},L}\big)$$

---

[6] $\otimes_{o_L} L$ is missing in the reference!

which comes about as follows. Let $f \in \mathrm{Hom}_{\mathbf{A}_L^+}(N, \mathbf{A}^+[\frac{1}{\omega_{\mathrm{LT}}}])$. By composing f with the inclusion (3.4) we obtain $f_1 : N \to B_{\mathrm{cris},L}$. By base extension to $\mathcal{O}$ via (3.5) and then localization in Q the map f_1 gives rise to a map $f_2 : (\mathcal{O} \otimes_{\mathbf{A}_L^+} N)[\frac{1}{Q}] \to B_{\mathrm{cris},L}$. We precompose this one with the isomorphism $1 \otimes \varphi_N$ to obtain

$$f_3 : (\mathcal{O} \otimes_{\mathbf{A}_L^+, \varphi_L} N)\left[\tfrac{1}{Q}\right] \xrightarrow{\cong} (\mathcal{O} \otimes_{\mathbf{A}_L^+} N)\left[\tfrac{1}{Q}\right] \to B_{\mathrm{cris},L}.$$

Now we observe the inclusions

$$(\mathcal{O} \otimes_{\mathbf{A}_L^+, \varphi_L} N)\left[\tfrac{1}{Q}\right] \subseteq (\mathcal{O} \otimes_{\mathbf{A}_L^+, \varphi_L} N)\left[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\right] \supseteq (\mathcal{O} \otimes_{\mathbf{A}_L^+, \varphi_L} N)\left[\varphi_L\left(\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\right)\right]$$

$$= \mathcal{O} \otimes_{\mathcal{O}, \varphi_L} \left((\mathcal{O} \otimes_{\mathbf{A}_L^+} N)\left[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\right]\right).$$

They only differ by elements which are invertible in $B_{\mathrm{cris},L}$. Therefore giving the map f_3 is equivalent to giving a map $f_4 : \mathcal{O} \otimes_{\mathcal{O}, \varphi_L} ((\mathcal{O} \otimes_{\mathbf{A}_L^+} N)[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}]) \to B_{\mathrm{cris},L}$. Finally, we use Remark 3.1.2 (iii) which gives the map

$$\xi : (N/\omega_{\mathrm{LT}} N)\left[\tfrac{1}{p}\right] \xleftarrow[\mathrm{pr}]{\cong} \left((\mathcal{O} \otimes_{\mathbf{A}_L^+} N)\left[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\right]\right)^{\Gamma_L} \xrightarrow{\subseteq} (\mathcal{O} \otimes_{\mathbf{A}_L^+} N)\left[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\right].$$

By precomposing f_4 with $1 \otimes \xi$ we at last arrive at a map $f_5 : (N/\omega_{\mathrm{LT}} N)[\frac{1}{p}] \to B_{\mathrm{cris},L}$. From this description the compatibility with tensor products is easily checked. ∎

Suppose that N is in $\mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L, \Gamma_L, \mathrm{an}}$ and put $T := V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N)$ in $\mathrm{Rep}_{o_L, f}^{\mathrm{cris}, \mathrm{an}}(G_L)$. Then, by Remark 3.1.2 (iii) and Lemma 3.1.5, we have a natural isomorphism of $\otimes$-functors

$$\mathrm{comp} : \mathcal{O}\left[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\right] \otimes_{\mathbf{A}_L^+} N \xrightarrow{\cong} \mathcal{O}\left[\tfrac{\omega_{\mathrm{LT}}}{t}_{\mathrm{LT}}\right] \otimes_L D_{\mathrm{cris},L}(L \otimes_{o_L} T) \tag{3.13}$$

which is compatible with the diagonal φ's on both sides.

In the proof of [44, Cor. 3.3.8], it is shown that, for any T in $\mathrm{Rep}_{o_L, f}^{\mathrm{cris}, \mathrm{an}}(G_L)$, there exists an $\mathbf{A}_L^+$-submodule $\mathfrak{M} \subseteq D_{\mathrm{LT}}(T)$ which

(N1) lies in $\mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L, \Gamma_L, \mathrm{an}}$ with $\varphi_{\mathfrak{M}}$ and the Γ_L-action on $\mathfrak{M}$ being induced by the (φ_q, Γ_L)-structure of $D_{\mathrm{LT}}(T)$ and

(N2) satisfies $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathfrak{M} = D_{\mathrm{LT}}(T)$.

Note that property (N2) implies that $\mathfrak{M}$ is *p-saturated* in $D_{\mathrm{LT}}(T)$, i.e., $\mathfrak{M}[\frac{1}{p}] \cap D_{\mathrm{LT}}(T) = \mathfrak{M}$, since $\mathbf{A}_L^+$ is obviously p-saturated in $\mathbf{A}_L$.

We once and for all pick such an $N(T) := \mathfrak{M}$. This defines a functor

$$N : \mathrm{Rep}_{o_L, f}^{\mathrm{cris}, \mathrm{an}}(G_L) \to \mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L, \Gamma_L, \mathrm{an}}$$

which is quasi-inverse to the undecorated horizontal arrow in the above big diagram (3.2). Note that N is in a unique way a $\otimes$-functor by [65, §I.4.4.2.1].

Remark 3.1.6. For T in $\mathrm{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L)$ and $N := N(T)$ in $\mathrm{Mod}^{\varphi_L,\Gamma_L,\mathrm{an}}_{\mathbf{A}^+_L}$ we have the following:

(i) If $V := L \otimes_{o_L} T$ is a positive (the Hodge–Tate weights are non-positive, i.e., $gr^j D_{\mathrm{dR}}(V) \neq 0$ implies that $j \geq 0$) analytic crystalline representation, then N is stable under φ_N.

(ii) If the Hodge–Tate weights of $L \otimes_{o_L} T$ are all ≥ 0, then we have

$$N \subseteq \mathbf{A}^+_L \cdot \varphi_N(N),$$

where the latter means the $\mathbf{A}^+_L$-span generated by $\varphi_N(N)$.

Proof. The corresponding assertions for $\mathcal{M} := \mathcal{O} \otimes_{\mathbf{A}^+_L} N$ are contained in [8, Cor. 3.4.9]. Let $n_1, \ldots, n_d$ be an $\mathbf{A}^+_L$-basis of N.

For (i) we have to show that $\varphi_N(n_j) \in N$ for any $1 \leq j \leq d$. Writing $\varphi_N(n_j) = \sum_{i=1}^d f_{ij} n_i$ we know from the definition of the category $\mathrm{Mod}^{\varphi_L,\Gamma_L,\mathrm{an}}_{\mathbf{A}^+_L}$ that $f_{ij} \in \mathbf{A}^+_L[\frac{1}{Q}]$ and from the above observation that $f_{ij} \in \mathcal{O}$. This reduces us to showing the inclusion $o_L[\![Z]\!][\frac{1}{Q}] \cap \mathcal{O} \subseteq o_L[\![Z]\!]$. Suppose therefore that $Q^r h = f$ for some $r \geq 1$, $h \in \mathcal{O}$, and $f \in o_L[\![Z]\!]$. The finitely many zeros of $Q \in o_L[\![Z]\!]$, which are the non-zero π_L-torsion points of the Lubin–Tate formal group, all lie in the open unit disk. By Weierstrass preparation it follows that Q must divide f already in $o_L[\![Z]\!]$. Hence, $h \in o_L[\![Z]\!]$.

For (ii) we have to show that $n_j = \sum_{i=1}^d f_{ij}\varphi_N(n_i)$, for any $1 \leq j \leq d$, with $f_{ij} \in \mathbf{A}^+_L$. For the same reasons as in the proof of (1) we have

$$n_j = \sum_{i=1}^d f'_{ij}\varphi_N(n_i) = \sum_{i=1}^d f''_{ij}\varphi_N(n_i)$$

with $f'_{ij} \in \mathbf{A}^+_L[\frac{1}{Q}]$ and $f''_{ij} \in \mathcal{O}$. Then $\sum_{i=1}^d (f'_{ij} - f''_{ij})\varphi_N(n_i) = 0$. But, again by the definition of the category $\mathrm{Mod}^{\varphi_L,\Gamma_L,\mathrm{an}}_{\mathbf{A}^+_L}$, the $\varphi_N(n_i)$ are linearly independent over $\mathbf{A}^+_L[\frac{1}{Q}]$ and hence over $\mathcal{O}[\frac{1}{Q}]$. It follows that $f'_{ij} = f''_{ij} \in \mathbf{A}^+_L$. ∎

First, we further investigate any T in $\mathrm{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L)$ whose *Hodge–Tate weights are all ≤ 0*, i.e., which is positive. For this purpose we need the ring $\mathbf{A}^+ = \mathbf{A} \cap W(\widetilde{\mathbf{E}}^+)_L$. One has the following general fact.

Lemma 3.1.7. *Let F be any non-archimedean valued field which contains $o_L/\pi_L o_L$, and let o_F denote its ring of integers; we have the following:*

(i) *Let $\alpha \in W(F)_L$ be any element; if the $W(o_F)_L$-submodule of $W(F)_L$ generated by $\{\varphi_q^i(\alpha)\}_{i \geq 0}$ is finitely generated, then $\alpha \in W(o_F)_L$.*

(ii) *Let X be a finitely generated free o_L-module, and let M be a finitely generated $W(o_F)_L$-submodule of $W(F)_L \otimes_{o_L} X$; if M is $\varphi_q \otimes$ id-invariant, then $M \subseteq W(o_F)_L \otimes_{o_L} X$.*

Proof. (i) This is a simple explicit calculation as given, for example, in the proof of [23, Lem. III.5]. (ii) This is a straightforward consequence of (i). ∎

Proposition 3.1.8. *For positive T in $\mathrm{Rep}_{o_L,f}^{\mathrm{cris,an}}(G_L)$ we have*

$$N(T) \subseteq D_{\mathrm{LT}}^+(T) := (\mathbf{A}^+ \otimes_{o_L} T)^{\ker(\chi_{\mathrm{LT}})},$$

and $N(T)$ is p-saturated in $D_{\mathrm{LT}}^+(T)$.

Proof. By Remark 3.1.6 (i) the $\mathbf{A}_L^+$-submodule $N(T)$ of $W(\widetilde{\mathbf{E}})_L \otimes_{o_L} T$ is $\varphi_q \otimes \mathrm{id}$-invariant (and finitely generated). Hence, we may apply Lemma 3.1.7 (ii) to $M := W(\widetilde{\mathbf{E}}^+)_L \cdot N(T)$ and obtain that

$$N(T) \subseteq \left(W(\widetilde{\mathbf{E}}^+)_L \otimes_{o_L} T\right) \cap (\mathbf{A} \otimes_{o_L} T)^{\ker(\chi_{\mathrm{LT}})} = D_{\mathrm{LT}}^+(T).$$

Since $N(T)$ is even p-saturated in $D_{\mathrm{LT}}(T)$, the same holds with respect to the smaller $D_{\mathrm{LT}}^+(T)$. ∎

Corollary 3.1.9. *For positive T in $\mathrm{Rep}_{o_L,f}^{\mathrm{cris,an}}(G_L)$ the $\mathbf{A}_L^+$-module $D_{\mathrm{LT}}^+(T)$ is free of the same rank as $N(T)$.*

Proof. By the argument in the proof of [23, Lem. III.3] the $\mathbf{A}_L^+$-module $D_{\mathrm{LT}}^+(T)$ is always free of a rank less than or equal to the rank of $N(T)$. The equality of the ranks in the positive case then is a consequence of Proposition 3.1.8. ∎

Next we relate $N(T)$ to the construction in [5, Prop. II.1.1].

Proposition 3.1.10. *Suppose that T in $\mathrm{Rep}_{o_L,f}^{\mathrm{cris,an}}(G_L)$ is positive. For $N := N(T)$ we then have the following:*

(i) N *is the unique $\mathbf{A}_L^+$-submodule of $D_{\mathrm{LT}}(T)$ which satisfies (N1) and (N2).*

(ii) N *is also the unique $\mathbf{A}_L^+$-submodule of $D_{\mathrm{LT}}^+(T)$ which satisfies the following:*

    (a) N *is free of rank equal to the rank of $D_{\mathrm{LT}}^+(T)$;*

    (b) N *is Γ_L-invariant;*

    (c) *the induced Γ_L-action on $N/\omega_{\mathrm{LT}}N$ is trivial;*

    (d) $\omega_{\mathrm{LT}}^r D_{\mathrm{LT}}^+(T) \subseteq N$ *for some $r \geq 0$.*

Proof. Let $P = P(\mathbf{A}_L^+)$ denote the set of height one prime ideals of $\mathbf{A}_L^+$. It contains the prime ideal $\mathfrak{p}_0 := (\omega_{\mathrm{LT}})$.

Step 1. We show the existence of a unique $\mathbf{A}_L^+$-submodule N' of $D_{\mathrm{LT}}^+(T)$ which satisfies (a)–(d), and we show that this N' is φ_q-invariant.

Existence. We begin by observing that the $\mathbf{A}_L^+$-submodule $N := N(T)$ of $D_{\mathrm{LT}}^+(T)$ has the properties (a), (b), and (c), but possibly not (d). In particular, the quotient

$D_{\mathrm{LT}}^+(T)/N$ is an $\mathbf{A}_L^+$-torsion module. Hence, the localizations $N_{\mathfrak{p}} = D_{\mathrm{LT}}^+(T)_{\mathfrak{p}}$ coincide for all but finitely many $\mathfrak{p} \in P$. By [16, VII.4.3 Thm. 3] there exists a unique intermediate $\mathbf{A}_L^+$-module $N \subseteq N' \subseteq D_{\mathrm{LT}}^+(T)$ which is finitely generated and reflexive and such that $N'_{\mathfrak{p}_0} = N_{\mathfrak{p}_0}$ and $N'_{\mathfrak{p}} = D_{\mathrm{LT}}^+(T)_{\mathfrak{p}}$ for any $\mathfrak{p} \in P \setminus \{\mathfrak{p}_0\}$. Since $\mathbf{A}_L^+$ is a two-dimensional regular local ring, the finitely generated reflexive module N' is actually free and then, of course, must have the same rank as N and $D_{\mathrm{LT}}^+(T)$. We also have $N' = \bigcap_{\mathfrak{p}} N'_{\mathfrak{p}} = N_{\mathfrak{p}_0} \cap \bigcap_{\mathfrak{p} \neq \mathfrak{p}_0} D_{\mathrm{LT}}^+(T)_{\mathfrak{p}}$. Since $\mathfrak{p}_0$ is preserved by $\varphi_{D_{\mathrm{LT}}^+(T)}$ and Γ_L, it follows that N' is $\varphi_{D_{\mathrm{LT}}^+(T)}$- and Γ_L-invariant. Next the identities

$$L \otimes_{o_L} N/\omega_{\mathrm{LT}}N = N_{\mathfrak{p}_0}/\omega_{\mathrm{LT}}N_{\mathfrak{p}_0} = N'_{\mathfrak{p}_0}/\omega_{\mathrm{LT}}N'_{\mathfrak{p}_0} = L \otimes_{o_L} N'/\omega_{\mathrm{LT}}N' \supseteq N'/\omega_{\mathrm{LT}}N'$$

show that the induced Γ_L-action on $N'/\omega_{\mathrm{LT}}N'$ is trivial. By using [16, VII.4.4 Thm. 5] we obtain, for some $m_1, \ldots, m_d \geq 0$, a homomorphism of $\mathbf{A}_L^+$-modules $D_{\mathrm{LT}}^+(T)/N' \to \bigoplus_{i=1}^d \mathbf{A}_L^+/\mathfrak{p}_0^{m_i}\mathbf{A}_L^+$ whose kernel is finite. Any finite $\mathbf{A}_L^+$-module is annihilated by a power of the maximal ideal in $\mathbf{A}_L^+$. We see that $D_{\mathrm{LT}}^+(T)/N'$ is annihilated by a power of $\mathfrak{p}_0$, which proves (d).

Uniqueness. Observing that

$$\gamma(\omega_{\mathrm{LT}}) = \big[\chi_{\mathrm{LT}}(\gamma)\big](\omega_{\mathrm{LT}}) \quad \text{for any } \gamma \in \Gamma_L$$

(cf. [73, Lem. 2.1.15]), this is exactly the same computation as in the uniqueness part of the proof of [5, Prop. II.1.1].

Step 2. We show that N' is p-saturated in $D_{\mathrm{LT}}^+(T)$. By construction we have $(N')_{(\pi_L)} = D_{\mathrm{LT}}^+(T)_{(\pi_L)}$. This implies that the p-torsion in the quotient $D_{\mathrm{LT}}^+(T)/N'$ is finite. On the other hand, both modules, N' and $D_{\mathrm{LT}}^+(T)$, are free of the same rank. Hence, the finitely generated $\mathbf{A}_L^+$-module $D_{\mathrm{LT}}^+(T)/N'$ has projective dimension ≤ 1 and therefore has no non-zero finite submodule (cf. [59, Prop. 5.5.3 (iv)]).

Step 3. We show that $N' = N$. Since both, N and N', are p-saturated in $D_{\mathrm{LT}}^+(T)$, it suffices to show that the free $\mathbf{B}_L^+$-modules $N(V) := N[\frac{1}{p}]$ and $N'(V) := N'[\frac{1}{p}]$ over the principal ideal domain $\mathbf{B}_L^+ := \mathbf{A}_L^+[\frac{1}{p}]$ coincide. As they are both Γ_L-invariant, so is the annihilator ideal

$$I := \mathrm{ann}_{\mathbf{B}_L^+}\big(N'(V)/N(V)\big).$$

Hence, by a standard argument as in [5, Lem. 1.3.2], the ideal I is generated by an element f of the form $\omega_{\mathrm{LT}}^{\alpha_0} \prod_{n \geq 1}^s \varphi_L^{n-1}(Q)^{\alpha_n}$ with certain $\alpha_n \geq 0$, $0 \leq n \leq s$, for some (minimal) $s \geq 0$. Since $N(V)_{(\omega_{\mathrm{LT}})} = N'(V)_{(\omega_{\mathrm{LT}})}$ by the construction of N', it follows that $\alpha_0 = 0$. Assuming that $M := N'(V)/N(V) \neq 0$, we conclude that $s \geq 1$ (with $\alpha_s \geq 1$), i.e., that, with $\mathfrak{p}_n := (\varphi_L^{n-1}(Q))$, we have $M_{\mathfrak{p}_s} \neq 0$ while $M_{\mathfrak{p}_{s+1}} = 0$. We claim that $(\varphi_L^* M)_{\mathfrak{p}_{s+1}} \neq 0$. First, note that we have an exact sequence

$$0 \to (\mathbf{B}_L^+)^d \xrightarrow{A} (\mathbf{B}_L^+)^d \to M \to 0,$$

with f dividing $\det(A) \in \mathbf{B}_L^+ \setminus (\mathbf{B}_L^+)_{\mathfrak{p}_s}^\times$, which induces an exact sequence

$$0 \to (\mathbf{B}_L^+)^d \xrightarrow{\varphi_L(A)} (\mathbf{B}_L^+)^d \to \varphi_L^* M \to 0.$$

Since $\varphi_L(f) = \prod_{n \geq 2}^{s+1} \varphi_L^{n-1}(Q)^{\alpha_{n-1}}$ divides $\det(\varphi_L(A))$, we conclude that $\det(\varphi_L(A))$ belongs to $\mathfrak{p}_{s+1}$ which implies the claim.

Now consider the following diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \left(\varphi_L^* N(V)\right)\left[\frac{1}{Q}\right] & \xrightarrow{\;\cong\;} & N(V)\left[\frac{1}{Q}\right] & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \left(\varphi_L^* N'(V)\right)\left[\frac{1}{Q}\right] & \longrightarrow & N'(V)\left[\frac{1}{Q}\right] & \longrightarrow & C & \longrightarrow & 0.
\end{array}
$$

The upper isomorphism comes from the definition of the category $\mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L, \Gamma_L, \mathrm{an}}$ in which N lies. The map

$$\left(\varphi_L^* N'(V)\right)\left[\tfrac{1}{Q}\right] \to N'(V)\left[\tfrac{1}{Q}\right]$$

is injective since $\varphi_L^* N' \to N'$ is the restriction of the isomorphism

$$\varphi_L^* D_{\mathrm{LT}}(T) \xrightarrow{\;\cong\;} D_{\mathrm{LT}}(T).$$

By the snake lemma and as $Q \notin \mathfrak{p}_{s+1}$ we obtain an injection

$$0 \neq (\varphi_L^* M)_{\mathfrak{p}_{s+1}} \hookrightarrow M_{\mathfrak{p}_{s+1}} = 0,$$

which is a contradiction. Thus, $M = 0$ as had to be shown. $\blacksquare$

Remark 3.1.11. The following are two examples:

(i)   We have $N(o_L(\chi_{\mathrm{LT}}^{-1})) = \omega_{\mathrm{LT}} \mathbf{A}_L^+ \otimes_{o_L} o_L \eta^{\otimes -1}$ and $N(o_L) = \mathbf{A}_L^+$ (recall that η denotes our fixed generator of T_π).

(ii)   Let $o_L(\chi) = o_L t_0$ with $\chi : G_L \to o_L^\times$ unramified. Then there exists an $a \in W(\bar{k}_L)_L^\times$ with $\sigma a = \chi^{-1}(\sigma) a$ for all $\sigma \in G_L$ by Remark 3.2.4;[7] in particular,

$$N\big(o_L(\chi)\big) = D_{\mathrm{LT}}^+\big(o_L(\chi)\big) = \mathbf{A}_L^+ n_0 \quad \text{for } n_0 = a \otimes t_0,$$

where Γ_L fixes n_0 and $\varphi_{N(o_L(\chi))}(n_0) = c n_0$ with $c := \frac{\varphi_L(a)}{a} \in o_L^\times$.

Proof. Each case belongs to a positive representation T: in all cases the right-hand side of the equality satisfies the properties characterizing $N(T)$ in Proposition 3.1.10 (ii) (cf. [73, Lem. 2.1.15]). $\blacksquare$

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[7]Since π_L has trivial G_L-action, the period a there can be normalized such that it becomes a unit in $W(\bar{k}_L)_L$.

Lemma 3.1.12. *For any $T \in \mathrm{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L)$ we have the following:*

- $N(T)$ *is the unique $\mathbf{A}_L^+$-submodule of $D_{\mathrm{LT}}(T)$ which satisfies* (N1) *and* (N2);

- $N(T(\chi_{\mathrm{LT}}^{-r})) \cong \omega_{\mathrm{LT}}^r N(T) \otimes_{o_L} o_L \eta^{\otimes -r}.$

Proof. First, we choose $r \geq 0$ such that $T(\chi_{\mathrm{LT}}^{-r})$ is positive. Sending N to

$$\omega_{\mathrm{LT}}^r N(T) \otimes_{o_L} o_L \eta^{\otimes -r} \subseteq D_{\mathrm{LT}}(T) \otimes_{o_L} o_L \eta^{\otimes -r}$$

viewed in $D_{\mathrm{LT}}(T) \otimes_{o_L} o_L \eta^{\otimes -r} \cong D_{\mathrm{LT}}(T(\chi_{\mathrm{LT}}^{-r}))$ sets up a bijection between the $\mathbf{A}_L^+$-submodules of $D_{\mathrm{LT}}(T)$ and $D_{\mathrm{LT}}(T(\chi_{\mathrm{LT}}^{-r}))$, respectively. One checks that N satisfies (N1) and (N2) if and only if its image does. Hence, (i) and (ii) (for such r) are a consequence of Proposition 3.1.10 (i). That (ii) holds in general follows from the obvious transitivity property of the above bijections. ∎

Proposition 3.1.13. *Let T be in $\mathrm{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L)$ of o_L-rank d and such that $V = L \otimes_{o_L} T$ is positive with Hodge–Tate weights $-r = -r_d \leq \cdots \leq -r_1 \leq 0$. Taking* (3.13) *as an identification, we then have*

$$\left(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\right)^r \mathcal{O} \otimes_{\mathbf{A}_L^+} N(T) \subseteq \mathcal{O} \otimes_L D_{\mathrm{cris},L}(L \otimes_{o_L} T) \subseteq \mathcal{O} \otimes_{\mathbf{A}_L^+} N(T) \qquad (3.14)$$

with elementary divisors

$$\left[\mathcal{O} \otimes_{\mathbf{A}_L^+} N(T) : \mathcal{O} \otimes_L D_{\mathrm{cris},L}(L \otimes_{o_L} T)\right] = \left[\left(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\right)^{r_1} : \cdots : \left(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\right)^{r_d}\right].$$

Proof. We abbreviate $D := D_{\mathrm{cris},L}(V)$. By the definition of the functor $\mathcal{M}$ in [44] we have

$$\mathcal{O} \otimes_L D \subseteq \mathcal{M}(D) \subseteq \left(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\right)^{-r} \mathcal{O} \otimes_L D. \qquad (3.15)$$

On the other hand, the commutativity of the big diagram before Remark 3.1.2 says that $\mathcal{M}(D) \cong \mathcal{O} \otimes_{\mathbf{A}_L^+} N(T)$. This implies the inclusions (3.14).

Concerning the second part of the assertion, we first of all note that although $\mathcal{O}$ is only a Bézout domain, it does satisfy the elementary divisor theorem (cf. [75, proof of Prop. 4.4]). We may equivalently determine the elementary divisors of the $\mathcal{O}$-module $\mathcal{M}(D)/(\mathcal{O} \otimes_L D)$. The countable set $\mathbb{S}$ of zeros of the function $\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}} \in \mathcal{O}$ coincides with the set of non-zero torsion points of our Lubin–Tate formal group, each occurring with multiplicity one. The first part of the assertion implies that the $\mathcal{O}$-module $\mathcal{M}(D)/(\mathcal{O} \otimes_L D)$ is supported on $\mathbb{S}$. Let $\mathcal{M}_z(D)$, resp., $\mathcal{O}_z$, denote the stalk in $z \in \mathbb{S}$ of the coherent sheaf on the open unit disk $\mathbf{B}$ defined by $\mathcal{M}(D)$, resp., $\mathcal{O}$. The argument in the proof of [8, Prop. 1.1.10] then shows that we have

$$\mathcal{M}(D)/(\mathcal{O} \otimes_L D) = \prod_{z \in \mathbb{S}} \mathcal{M}_z(D)/(\mathcal{O}_z \otimes_L D).$$

The ring $\mathcal{O}_z$ is a discrete valuation ring with maximal ideal $\mathfrak{m}_z$ generated by $\frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}$. We consider on its field of fractions $\mathrm{Fr}(\mathcal{O}_z)$ the $\mathfrak{m}_z$-adic filtration and then on $\mathrm{Fr}(\mathcal{O}_z) \otimes_L D$ the tensor product filtration. By [43, Lem. 1.2.1 (2)] (or [8, Lem. 3.4.4]) we have

$$\mathcal{M}_z(D) \cong \mathrm{Fil}^0(\mathrm{Fr}(\mathcal{O}_z) \otimes_L D) \quad \text{for any } z \in \mathbb{S},$$

and this isomorphism preserves $\mathcal{O}_z \otimes_L D$. At this point we let $0 \leq s_1 < \cdots < s_m < r$ denote the jumps of the filtration $\mathrm{Fil}^\bullet D$, i.e., the r_j but without repetition. We write

$$D = D_1 \oplus \cdots \oplus D_m \quad \text{such that} \quad \mathrm{Fil}^{s_i} D = D_i \oplus \cdots \oplus D_m.$$

For the following computation let, for notational simplicity, R denote any L-algebra which is a discrete valuation ring with maximal ideal $\mathfrak{m}$. We compute

$$
\begin{aligned}
\mathrm{Fil}^0\big(\mathrm{Fr}(R) \otimes_L D\big) &= \sum_{j \in \mathbb{Z}} \mathfrak{m}^{-j} \otimes_L \mathrm{Fil}^j D = \sum_{j=0}^{r} \mathfrak{m}^{-j} \otimes_L \mathrm{Fil}^j D \\
&= \sum_{i=1}^{m} \mathfrak{m}^{-s_i} \otimes_L \mathrm{Fil}^{s_i} D = \sum_{i=1}^{m} \sum_{j=i}^{m} \mathfrak{m}^{-s_i} \otimes_L D_j \\
&= \sum_{j=1}^{m} \Big(\sum_{i=1}^{j} \mathfrak{m}^{-s_i} \Big) \otimes_L D_j = \sum_{j=1}^{m} \mathfrak{m}^{-s_j} \otimes_L D_j.
\end{aligned}
$$

Hence, we obtain

$$\mathrm{Fil}^0\big(\mathrm{Fr}(R) \otimes_L D\big)/(R \otimes_L D) = \bigoplus_{j=1}^{m} \mathfrak{m}^{-s_j}/R \otimes_L D_j \cong \bigoplus_{j=1}^{m} R/\mathfrak{m}^{s_j} \otimes_L D_j.$$

By combining all of the above, we finally arrive at

$$
\begin{aligned}
\mathcal{M}(D)/(\mathcal{O} \otimes_L D) &= \prod_{z \in \mathbb{S}} \mathcal{M}_z(D)/(\mathcal{O}_z \otimes_L D) \\
&\cong \prod_{z \in \mathbb{S}} \mathrm{Fil}^0\big(\mathrm{Fr}(\mathcal{O}_z) \otimes_L D\big)/(\mathcal{O}_z \otimes_L D) \\
&\cong \prod_{z \in \mathbb{S}} \Big(\bigoplus_{j=1}^{m} \mathcal{O}_z/\mathfrak{m}_z^{s_j} \otimes_L D_j \Big) \\
&= \bigoplus_{j=1}^{m} \Big(\prod_{z \in \mathbb{S}} \big(\mathcal{O}_z/\big(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\big)^{s_j} \mathcal{O}_z \otimes_L D_j \big) \Big) \\
&= \bigoplus_{j=1}^{m} \Big(\prod_{z \in \mathbb{S}} \mathcal{O}_z/\big(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\big)^{s_j} \mathcal{O}_z \Big) \otimes_L D_j \\
&= \bigoplus_{j=1}^{m} \mathcal{O}/\big(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\big)^{s_j} \mathcal{O} \otimes_L D_j. \qquad \blacksquare
\end{aligned}
$$

For a first application of this result we recall the comparison isomorphism

$$
\begin{array}{c}
N(V) \\
\cap\big\downarrow \\
N(V)\big[\tfrac{1}{Q}\big] \xrightarrow{\ \subseteq\ } \mathcal{O}\big[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\big] \otimes_{\mathbf{A}_L^+} N(V) \xrightarrow[\cong]{\ \mathrm{comp}\ } \mathcal{O}\big[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\big] \otimes_L D_{\mathrm{cris},L}(V)
\end{array}
\tag{3.16}
$$

for any T in $\mathrm{Rep}^{\mathrm{cris},\mathrm{an}}_{o_L,f}(G_L)$, $V := L \otimes_{o_L} T$, and $N(V) := N(T)[\tfrac{1}{\pi_L}]$. The left horizontal inclusion comes from the fact that $\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}$ is a multiple of Q in $\mathcal{O}$. In particular, we have the commutative diagram

$$
\begin{array}{ccc}
 & N(V) \xrightarrow{\ \mathrm{comp}\ } & \mathcal{O}\big[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\big] \otimes_L D_{\mathrm{cris},L}(V) \\
{\scriptstyle\varphi_{N(V)}}\swarrow \quad \big\downarrow{\scriptstyle\varphi_{N(V)}} & & \big\downarrow{\scriptstyle\varphi_L\otimes\varphi_{\mathrm{cris}}} \\
N^{(\varphi)}(V) \xrightarrow{\ \subseteq\ } N(V)\big[\tfrac{1}{Q}\big] \xrightarrow{\ \mathrm{comp}\ } & \mathcal{O}\big[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\big] \otimes_L D_{\mathrm{cris},L}(V),
\end{array}
$$

where we let $N^{(\varphi)}(V)$ be the $\mathbf{A}_L^+$-submodule of $N(V)[\tfrac{1}{Q}]$ generated by the image of $N(V)$ under $\varphi_{N(V)}$ and where φ_{cris} denotes the q-Frobenius on $D_{\mathrm{cris},L}(V)$. We note that since Q is invertible in $\mathbf{A}_L$, $N^{(\varphi)}(V)$ can also be viewed as the $\mathbf{A}_L^+$-submodule of $D_{\mathrm{LT}}(V) = \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N(V)$ generated by the image of $N(V)$ under $\varphi_{D_{\mathrm{LT}}(V)}$. From this one easily deduces (use the projection formula for the ψ-operator) that the map $\psi_{D_{\mathrm{LT}}(V)}$ on $D_{\mathrm{LT}}(V)$ restricts to an operator

$$
\psi_{N(V)} : N^{(\varphi)}(V) \to N(V).
$$

Corollary 3.1.14. *Assume that the Hodge–Tate weights of V are all in $[0,r]$. Then we have*

$$
\begin{aligned}
\mathrm{comp}\big(N(V)\big) &\subseteq \mathcal{O} \otimes_L D_{\mathrm{cris},L}(V), \\
\mathrm{comp}\big(N^{(\varphi)}(V)\big) &\subseteq \mathcal{O} \otimes_L D_{\mathrm{cris},L}(V), \\
\mathrm{comp}\big(N^{(\varphi)}(V)^{\psi_{N(V)}=0}\big) &\subseteq \mathcal{O}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}(V).
\end{aligned}
\tag{3.17}
$$

Proof. Apply Proposition 3.1.13 to $T(\chi_{\mathrm{LT}}^{-r})$, and then divide the resulting (left) inclusion in (3.14) by t_{LT}^r and tensor with $o_L(\chi_{\mathrm{LT}}^r)$. This gives the first inclusion by Lemma 3.1.12 upon noting that

$$
t_{\mathrm{LT}}^r D_{\mathrm{cris},L}(L \otimes_{o_L} T) \otimes_L L\eta^{\otimes -r} = D_{\mathrm{cris},L}\big(L \otimes_{o_L} T(\chi_{\mathrm{LT}}^{-r})\big).
$$

The second inclusion easily derives from the first by using that the map comp is compatible with the φ's.

For the third inclusion we consider any element $x = \sum_i f_i \varphi_{N(V)}(x_i) \in N^{(\varphi)}(V)$, with $f_i \in \mathbf{A}_L^+$ and $x_i \in N(V)$, such that $\psi_{N(V)}(x) = \sum_i \psi_L(f_i) x_i = 0$. We choose an L-basis $e_1,\dots,e_m$ of $D_{\mathrm{cris},L}(V)$ and write $\mathrm{comp}(x_i) = \sum_j f_{ij} \otimes e_j$ with $f_{ij} \in \mathcal{O}$. Then

$$
0 = \mathrm{comp}\big(\psi_{N(V)}(x)\big) = \sum_i \psi_L(f_i)\mathrm{comp}(x_i) = \sum_i \sum_j \psi_L(f_i) f_{ij} \otimes e_j
$$

and it follows that

$$\psi_L \Big(\sum_i f_i \varphi_L(f_{ij}) \Big) = \sum_i \psi_L(f_i) f_{ij} = 0,$$

i.e., that $\sum_i f_i \varphi_L(f_{ij}) \in \mathcal{O}^{\psi_L = 0}$. On the other hand, we compute

$$\text{comp}(x) = \sum_i f_i \varphi_{N(V)}(x_i) = \sum_i f_i (\varphi_L \otimes \varphi_{\text{cris}}) \big(\text{comp}(x_i) \big)$$

$$= \sum_i \sum_j f_i \big(\varphi_L(f_{ij}) \otimes \varphi_{\text{cris}}(e_j) \big)$$

$$= \sum_j \Big(\sum_i f_i \varphi_L(f_{ij}) \Big) \otimes \varphi_{\text{cris}}(e_j). \qquad \blacksquare$$

Corollary 3.1.15. *In the situation of Proposition 3.1.13, we have*

$$D_{\text{cris},L}(V) \cong \big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(T) \big)^{\Gamma_L}.$$

Proof. We set $\mathcal{M} := \mathcal{O} \otimes_{\mathbf{A}_L^+} N(T)$ and identify the modules $D(\mathcal{M})$ and $D_{\text{cris},L}(V)$ by using Lemma 3.1.5 and (3.14). The proof of [44, Prop. 2.2.6] combined with Remark 3.1.2 (iii) implies the commutativity of the following diagram:

$$D(\mathcal{M}) \xhookrightarrow[\text{incl.}]{} \mathcal{O}\big[\tfrac{\omega_{\text{LT}}}{t_{\text{LT}}}\big] \otimes_L D(\mathcal{M}) \xrightarrow[\cong]{\xi} \mathcal{M}\big[\tfrac{\omega_{\text{LT}}}{t_{\text{LT}}}\big]$$

in which the right vertical map is the canonical inclusion while the left vertical map stems from the definition of the functor $\mathcal{M}$ as in (3.15) (which also implies the commutativity of the left triangle). Taking Γ_L-invariants and using the fact that the upper line induces the isomorphism $D(\mathcal{M}) \cong \mathcal{M}[\tfrac{\omega_{\text{LT}}}{t_{\text{LT}}}]^{\Gamma_L}$ in Remark 3.1.2 (iii), the result follows. $\qquad \blacksquare$

Corollary 3.1.16. *In the situation of Proposition 3.1.13, we have $Q^r N(V) \subseteq N^{(\varphi)}(V)$.*

Proof. In the present situation, $\varphi_{N(V)} : N(V) \to N(V)$ is a semilinear endomorphism of $N(V)$ by Remark 3.1.6 (i). Then

$$\varphi_{\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)} = \varphi_L \otimes \varphi_{N(V)} : \mathcal{O} \otimes_{\mathbf{A}_L^+} N(V) \to \mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)$$

is an endomorphism as well. The corresponding linearized maps are

$$\varphi_{N(V)}^{\text{lin}} : \mathbf{A}_L^+ \otimes_{\mathbf{A}_L^+, \varphi_L} N(V) \xrightarrow{\cong} N^{(\varphi)}(V) \subseteq N(V)$$

$$f \otimes x \mapsto f \varphi_{N(V)}(x)$$

and

$$\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+}N(V)} = \mathrm{id}_{\mathcal{O}} \otimes \varphi^{\mathrm{lin}}_{N(V)} : \mathcal{O} \otimes_{\mathbf{A}_L^+,\varphi_L} N(V)$$
$$= \mathcal{O} \otimes_{\mathbf{A}_L^+} \left(\mathbf{A}_L^+ \otimes_{\mathbf{A}_L^+,\varphi_L} N(V)\right) \to \mathcal{O} \otimes_{\mathbf{A}_L^+} N(V).$$

Since $\mathcal{O}$ is flat over $\mathbf{A}_L^+[\frac{1}{\pi_L}]$, it follows that

$$\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V) / \mathrm{im}\left(\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+}N(V)}\right) = \mathcal{O} \otimes_{\mathbf{A}_L^+} \left(N(V)/N^{(\varphi)}(V)\right).$$

But $\mathcal{O}$ is even faithfully flat over $\mathbf{A}_L^+[\frac{1}{\pi_L}]$. Hence, the natural map

$$N(V)/N^{(\varphi)}(V) \to \mathcal{O} \otimes_{\mathbf{A}_L^+} N(V) / \mathrm{im}\left(\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+}N(V)}\right)$$

is injective. This reduces us to proving that

$$Q^r\left(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)\right) \subseteq \mathrm{im}\left(\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+}N(V)}\right).$$

As for any object in the category $\mathrm{Mod}_{\mathcal{O}}^{\varphi_L,\gamma_L,\mathrm{an}}$, we do have

$$Q^h\left(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)\right) \subseteq \mathrm{im}\left(\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+}N(V)}\right)$$

for some sufficiently big integer h. On the other hand, (3.14) says that

$$\left(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\right)^r \mathrm{comp}\left(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)\right) \subseteq \mathcal{O} \otimes_L D_{\mathrm{cris},L}(V) \subseteq \mathrm{comp}\left(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)\right).$$

Since φ_{cris} is bijective, we can sharpen the right-hand inclusion to

$$\mathcal{O} \otimes_L D_{\mathrm{cris},L}(V) \subseteq \mathrm{comp}\left(\mathrm{im}(\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+}N(V)})\right).$$

It follows that

$$\left(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\right)^r \left(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)\right) \subseteq \mathrm{im}\left(\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+}N(V)}\right).$$

Since the greatest common divisor of Q^h and $(\frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}})^r$ is $Q^{\min(h,r)}$, we finally obtain that

$$Q^r\left(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)\right) \subseteq \mathrm{im}\left(\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+}N(V)}\right). \qquad \blacksquare$$

Corollary 3.1.17. *In the situation of Proposition 3.1.13, we have, with regard to an* $\mathbf{A}_L^+$*-basis of* $N := N(T)$ *and with* $s := \sum_{i=1}^d r_i$*, that*

$$\det\left(\varphi_N : N(T) \to N(T)\right) = \det\left(\varphi_{N(V)} : N(V) \to N(V)\right) = Q^s$$

up to an element in $o_L^\times \cdot (\varphi_L - 1)((\mathbf{A}_L^+)^\times)$.

Proof. Note first that N is φ_N-stable by Remark 3.1.6 (i). Moreover, the determinant of φ_N acting on $N(V)$ equals the determinant of $\varphi_L \otimes \varphi_N$ acting on $\mathcal{O} \otimes_{\mathbf{A}_L^+} N(T)$ since we can take for both an $\mathbf{A}_L^+$-basis of $N(T)$. Since $\varphi_L(\frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}) = \frac{\pi_L}{Q} \frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}$, by Proposition 3.1.13, the latter determinant equals $(\frac{\pi_L}{Q})^{-s}$ multiplied by the determinant of $\varphi_L \otimes \mathrm{Frob}$ acting on $\mathcal{O} \otimes_L D_{\mathrm{cris},L}(V)$. The latter is equal to the determinant of Frob on $D_{\mathrm{cris},L}(V)$, which is π_L^s up to a unit in o_L since the filtered Frobenius module $D_{\mathrm{cris},L}(V)$ is weakly admissible. This shows the claim up to an element in $o_L^\times \cdot (\varphi_L - 1)(\mathcal{O}^\times)$. But $\mathcal{O}^\times = \pi_L^{\mathbb{Z}} \times (\mathbf{A}_L^+)^\times$ by [48, (4.8)]. Hence, $(\varphi_L - 1)(\mathcal{O}^\times) = (\varphi_L - 1)((\mathbf{A}_L^+)^\times)$. ∎

3.2 The determinant of the crystalline comparison isomorphism

Let T be any object in $\mathrm{Rep}_{o_L,f}^{\mathrm{cris,an}}(G_L)$ of o_L-rank d and such that $V = L \otimes_{o_L} T$ has Hodge–Tate weights $-r = -r_d \leq \cdots \leq -r_1$; we set $s := \sum_{i=1}^d r_i$, $N := N(T)$ and $\mathcal{M} = \mathcal{O} \otimes N$. Consider the integral lattice

$$\mathfrak{D} := \mathfrak{D}(T) \subseteq D_{\mathrm{cris},L}(V)$$

which is defined as the image of $N/\omega_{\mathrm{LT}}N \subseteq D(N)$ under the natural isomorphisms $D(N) \cong D(\mathcal{M}) \cong D_{\mathrm{cris},L}(V)$ arising from Lemma 3.1.5 and (3.1). Then with $N(-)$ also $\mathfrak{D}(-)$ is a $\otimes$-functor. The aim of this subsection is to prove the following result.

Proposition 3.2.1. *With regard to bases of T and $\mathfrak{D}$, the determinant of the crystalline comparison isomorphism*

$$B_{\mathrm{cris},L} \otimes_L V \cong B_{\mathrm{cris},L} \otimes_L D_{\mathrm{cris},L}(V)$$

belongs to $t_{\mathrm{LT}}^s W(\bar{k}_L)_L^\times$.

We write $\bigwedge V$ for the highest exterior power of V over L.

Remark 3.2.2. If V is L-analytic (Hodge–Tate, crystalline), then so is $\bigwedge V$.

Since $D_{\mathrm{cris},L}$ is a tensor functor, we are mainly reduced to consider characters $\rho : G_L \to L^\times$, for which we denote by V_ρ its representation space.

Remark 3.2.3. Let V_ρ be Hodge–Tate. We then have the following:

    (i)   The character ρ coincides on an open subgroup of the inertia group I_L of G_L with

$$\prod_{\sigma \in \Sigma_L} \sigma^{-1} \circ \chi_{\sigma L,\mathrm{LT}}^{n_\sigma},$$

        for some integers n_σ, where Σ_L denotes the set of embeddings of L into $\bar{L}$ and $\chi_{\sigma L,\mathrm{LT}}$ is the Lubin–Tate character for σL and $\sigma(\pi_L)$.

    (ii)   If, in addition, V_ρ is L-analytic, then ρ coincides on an open subgroup of the inertia group I_L with χ_{LT}^n for some integer n.

Proof. This follows from [81, III.A4 Prop. 4 as well as III.A5 Thm. 2 and its corollary]. ∎

Remark 3.2.4. Let ρ be a crystalline (hence, Hodge–Tate) and L-analytic character. We then have the following:

(i)   If ρ factorizes through $\mathrm{Gal}(L'/L)$ for some discretely valued Galois extension L' of L, then the determinant of the crystalline comparison isomorphism for V_ρ belongs to $(W(\bar{k}_L)_L[\frac{1}{p}])^\times$ (with respect to arbitrary bases of V and $D_{\mathrm{cris},L}(V)$).

(ii) If ρ has Hodge–Tate weight $-s$, then the determinant of the crystalline comparison isomorphism for V_ρ lies in $t_{\mathrm{LT}}^s(W(\bar{k}_L)_L[\frac{1}{p}])^\times$.

(iii) ρ is of the form $\chi_{\mathrm{LT}}^n \chi^{un}$ with an integer n and an unramified character χ^{un}.[8]

Proof. We shall write K_0 for the maximal absolutely unramified subextension of K, any algebraic extension of $\mathbb{Q}_p$. Taking $G_{L'}$-invariants of the comparison isomorphism shows that the latter is already defined over

$$B_{\mathrm{cris},L}^{G_{L'}} = (L \otimes_{L_0} B_{\mathrm{cris}})^{G_{L'}} = L \otimes_{L_0} (B_{\mathrm{cris}})^{G_{L'}} = L \otimes_{L_0} \widehat{L_0'} \subseteq W(\bar{k}_L)_L[\tfrac{1}{p}],$$

whence (i). Using Remark 3.2.3 (ii) and applying (i) to $\rho\chi_{\mathrm{LT}}^{-n}$ gives (ii). By the same argument it suffices to prove (iii) in the case of Hodge–Tate weight 0. Then its period lies in the completion of the maximal unramified extension of L by (i), whence the claim that ρ is unramified follows, as the inertia subgroup of G_L must act trivially. ∎

By Proposition 3.1.8 we have

$$N(T) \subseteq D_{\mathrm{LT}}^+(T) \subseteq \mathbf{A}^+ \otimes_{o_L} T$$

if T is positive. Using (N2) and the isomorphism

$$\mathbf{A} \otimes_{\mathbf{A}_L} D_{\mathrm{LT}}(T) \cong \mathbf{A} \otimes_{o_L} T,$$

we obtain a canonical injection

$$\mathbf{A}^+ \otimes_{\mathbf{A}_L^+} N(T) \hookrightarrow \mathbf{A}^+ \otimes_{o_L} T. \qquad (3.18)$$

Proposition 3.2.5. *If T is positive, then the determinant of (3.18) with respect to the bases of $N(T)$ and T is contained in $\omega_{\mathrm{LT}}^s(\mathbf{A}_L^+)^\times \cdot W(\bar{k}_L)_L^\times$.*

---

[8]Also the converse statement is true: Indeed, any unramified character is locally algebraic (by definition, see [81]), whence HT. By [81, A3 Prop. 3, A5 Thm. 2] the Hodge–Tate weights are all zero, whence ρ is L-analytic. It is crystalline as it is admissible; i.e., there is a period in $\mathbb{C}_p^\times$, which in this case needs to lie in the fixed field under inertia, which is contained in B_{cris}.

Proof. Let $M \in M_d(\mathbf{A}^+)$ be the matrix of a basis of $N(T)$ with respect to a basis of T and $P \in M_d(\mathbf{A}_L^+)$ the matrix of φ_L with respect to the same basis of $N(T)$. Then we have $\varphi_L(M) = MP$. By Corollary 3.1.17 we have $\det(P) = Q^s \varphi_L(f) f^{-1} u$ for some $f \in (\mathbf{A}_L^+)^\times$ and $u \in o_L^\times$. But $Q = \varphi_L(\omega_{\mathrm{LT}}) \omega_{\mathrm{LT}}^{-1}$. We deduce that

$$\varphi_L\big(\det(M)\big) = \varphi_L(\omega_{\mathrm{LT}}^s f)(\omega_{\mathrm{LT}}^s f)^{-1} u \det(M),$$

i.e., that

$$(\omega_{\mathrm{LT}}^s f a)^{-1} \det(M) \in \mathbf{A}^{\varphi_L = 1} = o_L$$

with $a \in W(\bar{k}_L)_L^\times$ such that $\varphi_L(a)/a = u$. It follows that

$$\det(M) \in \omega_{\mathrm{LT}}^s o_L (\mathbf{A}_L^+)^\times \cdot W(\bar{k}_L)_L^\times.$$

But we also have $\det(M) \in \mathbf{A}^\times$. Hence, we finally obtain[9]

$$\det(M) \in \omega_{\mathrm{LT}}^s o_L (\mathbf{A}_L^+)^\times \cdot W(\bar{k}_L)_L^\times \cap \mathbf{A}^\times = \omega_{\mathrm{LT}}^s (\mathbf{A}_L^+)^\times \cdot W(\bar{k}_L)_L^\times. \qquad \blacksquare$$

Remark 3.2.6. For $T = o_L(\chi)$ with unramified χ as in Remark 3.1.11 the map (3.18) maps the basis n_0 to $a \otimes t_0$.

Lemma 3.2.7. *If T is positive, then we have the following:*

   (i) $\mathcal{O} \otimes_{o_L} \mathfrak{D}(T) = \mathcal{O} \otimes_L D_{\mathrm{cris},L}(V) \subseteq \mathrm{comp}(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(T))$;

   (ii) *the determinant of the inclusion in* (i) *with respect to the bases of $\mathfrak{D}(T)$ and $N(T)$ belongs to $(\frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}})^s (\mathbf{A}_L^+)^\times$;*

  (iii) *for $T = o_L(\chi)$ with unramified χ as in Remark 3.1.11, $\mathrm{comp}(n_0) = \varphi_L(a) \otimes t_0 = ca \otimes t_0 \in D_{\mathrm{cris},L}(V)$ with $c = \frac{\varphi_L(a)}{a} \in o_L^\times$; in particular, the element $a \otimes t_0$ is a basis of $\mathfrak{D}(T)$.*

Proof. By construction the comparison isomorphism (3.13) is of the form

$$\mathrm{comp} = \mathrm{id}_{\mathcal{O}[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}]} \otimes_L \mathrm{comp}_0$$

with

$$\mathrm{comp}_0 : \big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}]\big)^{\Gamma_L} \overset{\cong}{\underset{\mathrm{pr}}{\to}} N/\omega_{\mathrm{LT}} N[\tfrac{1}{p}] = D(N) \overset{\cong}{\to} D_{\mathrm{cris},L}(V)$$

the right-hand arrow being the natural isomorphism from Lemma 3.1.5. For positive T we know in addition from the proof of Corollary 3.1.15 that

$$\big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N\big)^{\Gamma_L} = \big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}]\big)^{\Gamma_L}.$$

We deduce that

$$\mathrm{comp}\big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N\big) \supseteq \mathcal{O} \otimes_L \mathrm{comp}_0\big((\mathcal{O} \otimes_{\mathbf{A}_L^+} N)^{\Gamma_L}\big) = \mathcal{O} \otimes_L D_{\mathrm{cris},L}(V).$$

---

[9] The argument also shows that $\omega_{\mathrm{LT}}^s \mathbf{A}^+ \otimes_{o_L} T \subseteq \mathbf{A}^+ \otimes_{\mathbf{A}_L} N(T) \subseteq \mathbf{A}^+ \otimes_{o_L} T$. But the left inclusion does even hold with r instead of s (cf. [83] following [5] in the cyclotomic case).

By Proposition 3.1.13 we know that the determinant in (ii) is of the form $\left(\frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\right)^s f(\omega_{\mathrm{LT}})$ with $f(\omega_{\mathrm{LT}}) \in \mathcal{O}^\times$. On the other hand, if we base change the inclusion in (i) to $L = \mathcal{O}/\omega_{\mathrm{LT}}\mathcal{O}$, then we obtain the base change from o_L to L of the isomorphism $\mathcal{D} \cong N/\omega_{\mathrm{LT}}N$. By our choice of bases the determinant of the latter lies in $o_L^\times$. Since evaluation in zero maps $\left(\frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\right)^s f(\omega_{\mathrm{LT}})$ to $f(0)$, it follows that $f(0)$ belongs to $o_L^\times$ and hence (cf. [48, (4.8)]) that $f(\omega_{\mathrm{LT}})$ belongs to $(\mathbf{A}_L^+)^\times$.

Now we prove (iii): By the above description of comp_0 we have to show that the image $\bar{n}_0 \in D(N(T))$ of n_0 is mapped to $ca \otimes t_0$ under the natural isomorphism from Lemma 3.1.5. Since under the crystalline comparison isomorphisms these elements are sent to $a \otimes (a^{-1} \otimes \bar{n}_0) \in B_{\mathrm{cris},L} \otimes_L V_L(D(N))$ and $ca \otimes t_0 \in B_{\mathrm{cris},L} \otimes_{o_L} T$, respectively, it suffices to show that the map (3.8) sends $a^{-1} \otimes n_0 \in L \otimes_{o_L} V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N)$ (which corresponds to t_0 under the canonical isomorphism $T \cong V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N)$) to $(ca)^{-1} \otimes \bar{n}_0 \in V_L(D(N))$. Dualizing, this is equivalent to the claim that the map (3.9) sends the dual basis $\delta_{a^{-1} \otimes n_0} \in (L \otimes_{o_L} V(M))^*$ of $a^{-1} \otimes n_0$ to $\delta_{(ca)^{-1} \otimes \bar{n}_0} \in V_L(D(N))^*$. Note that the isomorphism

$$\left(L \otimes_{o_L} V(M)\right)^* \cong L \otimes_{o_L} \mathrm{Hom}_{\mathbf{A}_L, \varphi_q}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N, \mathbf{A})$$
$$\cong L \otimes_{o_L} \mathrm{Hom}_{\mathbf{A}_L^+, \varphi_q}\left(N, \mathbf{A}^+\left[\tfrac{1}{\omega_{\mathrm{LT}}}\right]\right)$$

sends $\delta_{a^{-1} \otimes n_0}$ to $a\delta_{n_0}$. Thus, it suffices to show that the map (3.10) sends $a\delta_{n_0}$ to $ca\delta_{\bar{n}_0}$ in $\mathrm{Hom}_{L, \varphi_q, \mathrm{Fil}}((N/\omega_{\mathrm{LT}}N)[\tfrac{1}{p}], B_{\mathrm{cris},L})$ since the latter corresponds under (3.11) to $ca \otimes \delta_{\bar{n}_0} \in V_L(D(N)^*)$ which in turn corresponds to $\delta_{(ca)^{-1} \otimes \bar{n}_0}$ under (3.12).

If $f = a\delta_{n_0}$, which is the map which sends n_0 to a, then – in the notation of the proof of Lemma 3.1.5 – f_1 and f_2 share this property, while f_3 (and hence f_4) sends $c^{-1}n_0$ to a because $\varphi_N(c^{-1}n_0) = c^{-1}\varphi_L(a)a^{-1}n_0 = n_0$. Then f_5 sends $c^{-1}\bar{n}_0$ to a because $\xi(c^{-1}\bar{n}_0) = c^{-1}n_0$. Altogether this means that $a\delta_{n_0}$ is mapped to $\varphi_L(a)\delta_{n_0} = ca\delta_{\bar{n}_0}$ as claimed. ∎

Proof of Proposition 3.2.1. The functor $D_{\mathrm{cris},L}(-)$ on crystalline Galois representations is a $\otimes$-functor and commutes with exterior powers, and the crystalline comparison isomorphism is compatible with tensor products and exterior powers. The analogous facts hold for the functor $N(-)$ and hence for the functor $\mathcal{D}(-)$ (by base change). The case of the functor $N(-)$ reduces, by using the properties (N1) and (N2) in Lemma 3.1.12 (i), to the case of the functor $D_{\mathrm{LT}}(-)$. Here the properties can easily be seen by the comparison isomorphism (3.3).

Upon replacing T by its highest exterior power we may and do assume that the o_L-module T has rank 1. In addition, by twisting T if necessary with a power of χ_{LT} we may and do assume that T is positive with $s = 0$, i.e., unramified Remark 3.2.4. In this case it is clear that – using the notation of Lemma 3.2.7 (iii) – the crystalline comparison isomorphism sends t_0 to $a \otimes t_0$. Since the latter is also a basis of $\mathcal{D}(T)$ by the same lemma, the proposition follows. ∎

3.3 Non-negative Hodge–Tate weights

Now assume that for T in $\mathrm{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L)$ the *Hodge–Tate weights are all* ≥ 0 and set $N := N(T)$. By [80, Rem. 3.2 (i)–(ii)] the map ψ_L preserves $\mathbf{A}^+_L$. It follows that $\psi_{D_{\mathrm{LT}}(T)}$ maps $\mathbf{A}^+_L \cdot \varphi_N(N)$ – and hence N by Remark 3.1.6 (i) (2) – into N. The following lemmata generalize those of [4, Appx. A].

Lemma 3.3.1. *For $m \geq 1$, there exists $Q_m \in o_L[\![Z]\!]$ such that*

$$\psi_L\left(\frac{1}{\omega_{\mathrm{LT}}^m}\right) = \frac{\pi_L^{m-1} + \omega_{\mathrm{LT}} Q_m(\omega_{\mathrm{LT}})}{\omega_{\mathrm{LT}}^m}.$$

Proof. According to the paragraph after Remark 2.1 combined with Remark 3.2 (ii) in [80], we have that

$$h(\omega_{\mathrm{LT}}) := \omega_{\mathrm{LT}}^m \psi_L\left(\frac{1}{\omega_{\mathrm{LT}}^m}\right) = \psi_L\left(\frac{[\pi_L]^m}{\omega_{\mathrm{LT}}^m}\right) \in \mathbf{A}^+_L.$$

Obviously, there exists $Q_m \in o_L[\![Z]\!]$ such that

$$h(\omega_{\mathrm{LT}}) - h(0) = \omega_{\mathrm{LT}} Q_m(\omega_{\mathrm{LT}}).$$

Thus, the claim follows from

$$
\begin{aligned}
h(0) &= \varphi_L\big(h(\omega_{\mathrm{LT}})\big)_{|\omega_{\mathrm{LT}}=0} \\
&= \varphi_L \circ \psi_L\left(\frac{[\pi_L]^m}{\omega_{\mathrm{LT}}^m}\right)_{|\omega_{\mathrm{LT}}=0} \\
&= \pi_L^{-1} \sum_{a \in \mathrm{LT}_1} \left(\frac{[\pi_L]^m(a +_{\mathrm{LT}} \omega_{\mathrm{LT}})}{(a +_{\mathrm{LT}} \omega_{\mathrm{LT}})^m}\right)_{|\omega_{\mathrm{LT}}=0} \\
&= \pi_L^{-1} \sum_{a \in \mathrm{LT}_1} \left(\frac{[\pi_L](\omega_{\mathrm{LT}})}{a +_{\mathrm{LT}} \omega_{\mathrm{LT}}}\right)^m_{|\omega_{\mathrm{LT}}=0} \\
&= \pi_L^{m-1}
\end{aligned}
$$

because $\left(\frac{[\pi_L](\omega_{\mathrm{LT}})}{a+_{\mathrm{LT}}\omega_{\mathrm{LT}}}\right)_{|\omega_{\mathrm{LT}}=0} = \pi_L$ for $a = 0$ and $= 0$ otherwise. ∎

Lemma 3.3.2. *We have*

$$\psi_{D_{\mathrm{LT}}(T)}\big(\pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-1} N(T)\big) \subseteq \pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-1} N(T)$$

and, for $k \geq 1$,

$$\psi_{D_{\mathrm{LT}}(T)}\big(\pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-(k+1)} N(T)\big) \subseteq \pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-k} N(T).$$

Proof. By Remark 3.1.6 (ii), we can write any $x \in N(T)$ in the form $x = \sum a_i \varphi_N(x_i)$ with $a_i \in \mathbf{A}_L^+$ and $x_i \in N(T)$. Therefore, $\psi_{D_{\mathrm{LT}}(T)}(\omega_{\mathrm{LT}}^{-(k+1)} x) = \sum \psi_L(\omega_{\mathrm{LT}}^{-(k+1)} a_i) x_i$ by the projection formula. Since ψ_L preserves $\mathbf{A}_L^+$ and is o_L-linear, we conclude by Lemma 3.3.1 that $\psi_L(\omega_{\mathrm{LT}}^{-(k+1)} a_i)$ belongs to $\pi_L \mathbf{A}_L + \omega_{\mathrm{LT}}^{-k} \mathbf{A}_L^+$, whenever $k \geq 1$, from which the second claim follows as $\psi_{D_{\mathrm{LT}}(T)}(\pi_L D_{\mathrm{LT}}(T)) \subseteq \pi_L D_{\mathrm{LT}}(T)$ by o_L-linearity of $\psi_{D_{\mathrm{LT}}(T)}$. For $k = 0$ finally, $\psi_L(\omega_{\mathrm{LT}}^{-1} a_i)$ belongs to $\omega_{\mathrm{LT}}^{-1} \mathbf{A}_L^+$, from which the first claim follows. ∎

Lemma 3.3.3. *If $k \geq 1$ and $x \in D_{\mathrm{LT}}(T)$ satisfies $\psi_{D_{\mathrm{LT}}(T)}(x) - x \in \pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-k} N(T)$, then x belongs to $\pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-k} N(T)$.*

Proof. Since $D_{\mathrm{LT}}(T)/\pi_L D_{\mathrm{LT}}(T)$ is a finitely generated (free) $k_L((\omega_{\mathrm{LT}}))$-module, there exists an integer $m \geq 0$ such that $x \in \pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-m} N(T)$; let l denote the smallest among them. Assume that $l > k$. Then Lemma 3.3.2 shows that

$$\psi_{D_{\mathrm{LT}}(T)}(x) \in \pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-(l-1)} N(T).$$

Hence, the element $\psi_{D_{\mathrm{LT}}(T)}(x) - x$ would belong to $\pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-l} N(T)$ but not to $(\pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-(l-1)} N(T))$, a contradiction to our assumption. It follows that $l \leq k$, and we are done. ∎

Lemma 3.3.4. *It holds that $D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1} \subseteq \omega_{\mathrm{LT}}^{-1} N(T)$, i.e.,*

$$D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1} = \left(\omega_{\mathrm{LT}}^{-1} N(T)\right)^{\psi_{D_{\mathrm{LT}}(T)}=1}.$$

Proof. By induction on $k \geq 1$ we will show that $D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1} \subseteq \pi_L^k D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-1} N(T)$, i.e., writing $x = \pi_L^k y_k + n_k \in D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1}$, the sequence n_k will π_L-adically converge in $\omega_{\mathrm{LT}}^{-1} N(T)$ with limit x.

In order to show the claim, assume $x \in D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1}$. As in the previous proof, there exists some minimal integer $m \geq 0$ such that $x \in \pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-m} N(T)$. Then $m = 1$ and we are done since otherwise Lemma 3.3.3 implies that m can be decreased by 1. This proves the claim for $k = 1$.

By our induction hypothesis we can write $x \in D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1}$ as $x = \pi_L^k y + n$ with $y \in D_{\mathrm{LT}}(T)$ and $n \in \omega_{\mathrm{LT}}^{-1} N(T)$. The equation $\psi_{D_{\mathrm{LT}}(T)}(x) = x$ implies that $\psi_{D_{\mathrm{LT}}(T)}(n) - n = \pi_L^k(\psi_{D_{\mathrm{LT}}(T)}(y) - y)$. In the proof of Lemma 3.3.2 we have seen that $\psi_{D_{\mathrm{LT}}(T)}(n) - n \in \omega_{\mathrm{LT}}^{-1} N(T)$. Note that $\pi_L^k D_{\mathrm{LT}}(T) \cap \omega_{\mathrm{LT}}^{-1} N(T) = \pi_L^k \omega_{\mathrm{LT}}^{-1} N(T)$ because $\mathbf{A}_L / \omega_{\mathrm{LT}}^{-1} \mathbf{A}_L^+$ has no π_L-torsion. Therefore, $\psi_{D_{\mathrm{LT}}(T)}(y) - y \in \omega_{\mathrm{LT}}^{-1} N(T)$, whence y, by Lemma 3.3.3, belongs to $\pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-1} N(T)$ so that we can write $x = \pi_L^k(\pi_L y' + n') + n = \pi_L^{k+1} y' + (\pi_L^k n' + n)$ as desired. ∎

Set $V := T \otimes_{o_L} L$.

Lemma 3.3.5. *If $D_{\mathrm{cris},L}(V)^{\varphi_q=1} \neq 0$, then V has the trivial representation L as quotient; i.e., the co-invariants V_{G_L} are non-trivial.*

Proof. Let $W = V^*$ be the L-dual of V. Then, by [80, (51)] we have

$$(V_{G_L})^* \cong H^0(L, W) \cong D_{\mathrm{cris},L}(W)^{\varphi_q=1} \cap (B_{\mathrm{dR}}^+ \otimes_L W)^{G_L} = D_{\mathrm{cris},L}(W)^{\varphi_q=1} \neq 0$$

because

$$(B_{\mathrm{dR}}^+ \otimes_L W)^{G_L} = (B_{\mathrm{dR}} \otimes_L W)^{G_L} \supseteq D_{\mathrm{cris},L}(W)$$

since the Hodge–Tate weights of W are ≤ 0.   ∎

Lemma 3.3.6. *If V does not have any quotient isomorphic to the trivial representation L, then $D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1} \subseteq N(T)$, i.e.,*

$$D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1} = N(T)^{\psi_{D_{\mathrm{LT}}(T)}=1}.$$

Proof. Because of Lemma 3.3.4, it suffices to show that $(\omega_{\mathrm{LT}}^{-1} N(T))^{\psi_{D_{\mathrm{LT}}(T)}=1} \subseteq N(T)$. Let $e_1, \ldots, e_d$ be a basis of $N := N(T)$ over $\mathbf{A}_L^+$. Then, by Remark 3.1.6 (ii) there exist $\beta_{ij} = \sum_{\ell \geq 0} \beta_{ij,\ell} \omega_{\mathrm{LT}}^\ell \in \mathbf{A}_L^+$ such that $e_i = \sum_{j=1}^d \beta_{ij} \varphi_N(e_j)$. Now assume that $\omega_{\mathrm{LT}}^{-1} n = \sum_{i=1}^d \alpha_i e_i = \sum_{i,j} \alpha_i \beta_{ij} \varphi_N(e_j)$ belongs to $(\omega_{\mathrm{LT}}^{-1} N)^{\psi_{D_{\mathrm{LT}}(T)}=1}$ with $\alpha_i = \sum_{\ell \geq -1} \alpha_{i,\ell} \omega_{\mathrm{LT}}^\ell \in \omega_{\mathrm{LT}}^{-1} \mathbf{A}_L^+$. By the projection formula this implies, for $1 \leq j \leq d$,

$$\alpha_j = \psi_L\left(\sum_{i=1}^d \alpha_i \beta_{ij} \right) \equiv \omega_{\mathrm{LT}}^{-1} \sum_{i=1}^d \alpha_{i,-1} \beta_{ij,0} \mod \mathbf{A}_L^+$$

because $\psi_L(\omega_{\mathrm{LT}}^{-1}) \equiv \omega_{\mathrm{LT}}^{-1} \mod \mathbf{A}_L^+$ by Lemma 3.3.1, whence

$$\varphi_L(\omega_{\mathrm{LT}})\varphi_L(\alpha_j) \equiv \sum_{i=1}^d \alpha_{i,-1} \beta_{ij,0} \mod \omega_{\mathrm{LT}} \mathbf{A}_L^+.$$

It follows from the definition of β_{ij} that

$$\begin{aligned}
\varphi_N(n) &= \sum_j \varphi_L(\omega_{\mathrm{LT}})\varphi_L(\alpha_j)\varphi_N(e_j) \\
&\equiv \sum_{j,i} \alpha_{i,-1} \beta_{ij,0} \varphi_N(e_j) \\
&\equiv \sum_i \alpha_{i,-1} e_i \\
&\equiv n \mod \omega_{\mathrm{LT}} N,
\end{aligned}$$

i.e., that $D_{\mathrm{cris},L}(V) \cong N/\omega_{\mathrm{LT}} N[\frac{1}{p}]$ (by (3.1) and Lemma 3.1.5) contains an eigenvector for φ_q with eigenvalue 1 if $\omega_{\mathrm{LT}}^{-1} n$ does not belong to N. Now the result follows from Lemma 3.3.5.   ∎

Chapter 4

(φ_L, Γ_L)-modules over the Robba ring

4.1 Robba rings of character varieties

Throughout our coefficient field K is a complete intermediate extension $L \subseteq K \subseteq \mathbb{C}_p$. For any reduced affinoid variety $\mathfrak{Y}$ over $\mathbb{Q}_p$ of L we let $|\ |_{\mathfrak{Y}}$ denote the supremum norm on the affinoid algebra $\mathcal{O}_K(\mathfrak{Y})$ of K-valued holomorphic functions on $\mathfrak{Y}$. It is submultiplicative and defines the intrinsic Banach topology of this algebra.

4.1.1 The additive character variety and its Robba ring

Let $\mathbf{B}_1$ denote the rigid $\mathbb{Q}_p$-analytic open disk of radius one around the point $1 \in \mathbb{Q}_p$. The rigid analytic group variety

$$\mathfrak{X}_0 := \mathbf{B}_1 \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p)$$

over $\mathbb{Q}_p$ (which non-canonically is a d-dimensional open unit polydisk) parametrizes the locally $\mathbb{Q}_p$-analytic characters of the additive group o_L: the point $z \otimes \beta$ is sent to the character $\chi_{z \otimes \beta}(a) := z^{\beta(a)}$. It is shown in [74, §2] that the rigid analytic group variety $\mathfrak{X}$ over L, which parametrizes the locally L-analytic characters of o_L, is the common zero set in $\mathfrak{X}_{0/L}$ of the functions

$$\sum_{j=1}^{d} z_j \otimes \beta_j \mapsto \sum_{j=1}^{d} \left(\beta_j(t_i) - t_i \cdot \beta_j(1)\right) \cdot \log(z_j)$$

for $1 \leq i \leq d$; here $t_1, \ldots, t_d$ is a $\mathbb{Z}_p$-basis of o_L and $\beta_1, \ldots, \beta_d$ is the corresponding dual basis. It is one dimensional, smooth, and connected. As a closed analytic subvariety of the Stein space $\mathfrak{X}_0$, the rigid variety $\mathfrak{X}$ is Stein as well.

For any $a \in o_L$ the map $b \mapsto ab$ on o_L is locally L-analytic. This induces an action of the multiplicative monoid $o_L \setminus \{0\}$ first on the $\mathbb{Z}_p$-module $\mathrm{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p)$ and then on the varieties $\mathfrak{X}_0$ and $\mathfrak{X}$. The latter actions further induce actions on the rings of K-valued holomorphic functions $\mathcal{O}_K(\mathfrak{X}_0) \twoheadrightarrow \mathcal{O}_K(\mathfrak{X})$, which we will denote by $(a, f) \mapsto a_*(f)$.

Furthermore, we have induced translation actions of $o_L \setminus \{0\}$ on the vector spaces $C^{\mathrm{an}}_{\mathbb{Q}_p}(o_L, K)$, resp., $C^{\mathrm{an}}(o_L, K)$, of K-valued locally $\mathbb{Q}_p$-analytic, resp., L-analytic, functions on o_L and then by duality on the spaces $D_{\mathbb{Q}_p}(o_L, K) \twoheadrightarrow D(o_L, K)$ of locally $\mathbb{Q}_p$-analytic and locally L-analytic distributions on o_L, respectively; they will

be denoted by $(a, \lambda) \mapsto a_*(\lambda)$. By [74, Thm. 2.3] we have the Fourier isomorphism

$$D(o_L, K) \xrightarrow{\cong} \mathcal{O}_K(\mathfrak{X})$$
$$\lambda \mapsto F_\lambda(\chi) = \lambda(\chi).$$

One easily checks that this isomorphism is $o_L \setminus \{0\}$-equivariant. In the following, we will denote the endomorphism $(\pi_L)_*$ in all situations also by φ_L. The Fourier isomorphism maps the Dirac distribution δ_a, for any $a \in o_L$, to the evaluation function $\mathrm{ev}_a(\chi) := \chi(a)$.

The ψ-operator and the Mellin transform

Lemma 4.1.1. *The endomorphism φ_L makes $\mathcal{O}_K(\mathfrak{X})$ into a free module over itself of rank equal to the cardinality of $o_L/\pi_L o_L$; a basis is given by the functions ev_a for a running over a fixed system of representatives for the cosets in $o_L/\pi_L o_L$.*

Proof. This is most easily seen by using the Fourier isomorphism which reduces the claim to the corresponding statement about the distribution algebra $D(o_L, K)$. But here the ring homomorphism φ_L visibly induces an isomorphism between $D(o_L, K)$ and the subalgebra $D(\pi_L o_L, K)$ of $D(o_L, K)$. Let $R \subseteq o_L$ denote a set of representatives for the cosets in $o_L/\pi_L o_L$. Then the Dirac distributions $\{\delta_a\}_{a \in R}$ form a basis of $D(o_L, K)$ as a $D(\pi_L o_L, K)$-module. ∎

Lemma 4.1.2. *The $o_L^\times$-action on $D(o_L, K) \cong \mathcal{O}_K(\mathfrak{X})$ extends naturally to a (jointly) continuous $D(o_L^\times, K)$-module structure.*

Proof. In a first step we consider the case $K = L$, so that K is spherically complete. By [75, Cor. 3.4] it suffices to show that $C^{\mathrm{an}}(G, K)$ as an $o_L^\times$-representation is locally analytic. This means we have to establish that, for any $f \in C^{\mathrm{an}}(G, K)$, the orbit map $a \mapsto a^*(f)$ on $o_L^\times$ is locally analytic. But this map is the image of the locally analytic function $(a, g) \mapsto f(ag)$ under the isomorphism $C^{\mathrm{an}}(o_L^\times \times G, K) = C^{\mathrm{an}}(o_L^\times, C^{\mathrm{an}}(G, K))$ in [77, Lem. A.1].

Now let K be general. All tensor products in the following are understood to be formed with the projective tensor product topology. By the universal property of the latter the jointly continuous bilinear map $D(o_L^\times, L) \times \mathcal{O}_L(\mathfrak{X}) \to \mathcal{O}_L(\mathfrak{X})$ extends uniquely to a continuous linear map $D(o_L^\times, L) \,\widehat{\otimes}_L\, \mathcal{O}_L(\mathfrak{X}) \to \mathcal{O}_L(\mathfrak{X})$. This further extends to the right-hand map in the sequence of continuous K-linear maps

$$\left(K \,\widehat{\otimes}_L\, D(o_L^\times, L)\right) \widehat{\otimes}_K \left(K \,\widehat{\otimes}_L\, \mathcal{O}_L(\mathfrak{X})\right) \to K \,\widehat{\otimes}_L\, \left(D(o_L^\times, L) \,\widehat{\otimes}_L\, \mathcal{O}_L(\mathfrak{X})\right)$$
$$\to K \,\widehat{\otimes}_L\, \mathcal{O}_L(\mathfrak{X}).$$

The left-hand map is the obvious canonical one. We refer to [61, §10.6] for the basics on scalar extensions of locally convex vector spaces. The same reasoning as in the

proof of [8, Prop. 2.5 (ii)] shows that $K \hat{\otimes}_L \mathcal{O}_L(\mathfrak{X}) = \mathcal{O}_K(\mathfrak{X})$. It remains to check that $K \hat{\otimes}_L D(o_L^\times, L) = D(o_L^\times, K)$ holds true as well. For any open subgroup $U \subseteq o_L^\times$ we have $D(o_L^\times, -) = \bigoplus_{a \in o_L^\times/U} \delta_a D(U, -)$. Hence, it suffices to check that $K \hat{\otimes}_L D(U, L) = D(U, K)$ for one appropriate U. But $o_L^\times$ contains such a subgroup U which is isomorphic to the additive group o_L so that $D(U, -) \cong D(o_L, -) \cong \mathcal{O}_-(\mathfrak{X})$. In this case we had established our claim already. ∎

The operator φ_L has a distinguished K-linear continuous left inverse ψ_L^D which is defined to be the dual of the map

$$C^{\mathrm{an}}(o_L, K) \to C^{\mathrm{an}}(o_L, K)$$

$$f \mapsto (\pi_L)_!(f)(a) := \begin{cases} f(\pi_L^{-1}a) & \text{if } a \in \pi_L o_L, \\ 0 & \text{otherwise,} \end{cases}$$

and then, via the Fourier transform, induces an operator $\psi_L^{\mathfrak{X}}$ on $\mathcal{O}_K(\mathfrak{X})$. One checks that for Dirac distributions we have

$$\psi_L^D(\delta_a) = \begin{cases} \delta_{\pi_L^{-1}a} & \text{if } a \in \pi_L o_L, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Together with Lemma 4.1.1 this implies the following lemma.

Lemma 4.1.3. *Let $R_0 \subseteq o_L$ be a set of representatives for the non-zero cosets in $o_L/\pi_L o_L$. Then*

$$\ker(\psi_L^{\mathfrak{X}}) = \bigoplus_{a \in R_0} \mathrm{ev}_a \cdot \varphi_L(\mathcal{O}_K(\mathfrak{X})).$$

We also recall the resulting projection formula

$$\psi_L^{\mathfrak{X}}(\varphi_L(F_1)F_2) = F_1 \psi_L^{\mathfrak{X}}(F_2) \quad \text{for any } F_1, F_2 \in \mathcal{O}_K(\mathfrak{X}).$$

Sometimes it will be useful to view $\psi_L^{\mathfrak{X}}$ as a normalized trace operator. Since $\mathcal{O}_K(\mathfrak{X})$ is a free module over $\varphi_L(\mathcal{O}_K(\mathfrak{X}))$ of rank q, we have the corresponding trace map

$$\mathrm{trace}_{\mathcal{O}_K(\mathfrak{X})/\varphi_L(\mathcal{O}_K(\mathfrak{X}))} : \mathcal{O}_K(\mathfrak{X}) \to \varphi_L(\mathcal{O}_K(\mathfrak{X})).$$

Remark 4.1.4. $\psi_L^{\mathfrak{X}} = \frac{1}{q}\varphi_L^{-1} \circ \mathrm{trace}_{\mathcal{O}_K(\mathfrak{X})/\varphi_L(\mathcal{O}_K(\mathfrak{X}))}.$

Proof. Since the functions ev_a generate a dense subspace in $\mathcal{O}_K(\mathfrak{X})$ (cf. [75, Lem. 3.1], the proof of which remains valid for general K by [61, Cor. 4.2.6 and Thm. 11.3.5]), it suffices, by the continuity of all operators involved, to check the asserted equality on the functions ev_a. As before we choose a set of representatives $R \subseteq o_L$ for the cosets $o_L/\pi_L o_L$, so that the functions ev_c, for $c \in R$, form a basis of $\mathcal{O}_K(\mathfrak{X})$ over $\varphi_L(\mathcal{O}_K(\mathfrak{X}))$. *Case 1:* Let $a \in o_L^\times$. Then $\psi_L^{\mathfrak{X}}(\mathrm{ev}_a) = 0$ by (4.1). On the other

hand, $\mathrm{ev}_a \cdot \mathrm{ev}_c = \mathrm{ev}_{a+c} \in \mathrm{ev}_{c'} \cdot \varphi_L(\mathcal{O}_K(\mathfrak{X}))$ for some $c \neq c' \in R$. Hence, the matrix of multiplication by ev_a with regard to our choice of basis has only zero entries on the diagonal. This means that $\mathrm{trace}_{\mathcal{O}_K(\mathfrak{X})/\varphi_L(\mathcal{O}_K(\mathfrak{X}))}(\mathrm{ev}_a) = 0$. *Case 2*: Let $a \in \pi_L \mathit{o}_L$. Then $\psi_L^{\mathfrak{X}}(\mathrm{ev}_a) = \mathrm{ev}_{\pi_L^{-1}a}$. On the other hand, the matrix of multiplication by ev_a now is the diagonal matrix with constant entry $\mathrm{ev}_a = \varphi_L(\mathrm{ev}_{\pi_L^{-1}a})$. We see that $\frac{1}{q}\varphi_L^{-1}(\mathrm{trace}_{\mathcal{O}_K(\mathfrak{X})/\varphi_L(\mathcal{O}_K(\mathfrak{X}))}(\mathrm{ev}_a)) = \frac{1}{q}\varphi_L^{-1}(q\varphi_L(\mathrm{ev}_{\pi_L^{-1}a})) = \mathrm{ev}_{\pi_L^{-1}a}$.   ∎

In order to establish a formula for the composition $\varphi_L \circ \psi_L^{\mathfrak{X}}$, we let $\mathfrak{X}[\pi_L] := \ker(\mathfrak{X} \xrightarrow{\pi_L^*} \mathfrak{X})$. Then $\mathfrak{X}[\pi_L](\mathbb{C}_p)$ is the character group of the finite group $\mathit{o}_L/\pi_L \mathit{o}_L$. The points in $\mathfrak{X}[\pi_L](\mathbb{C}_p)$ are defined over some finite extension K_1/K. For any $\zeta \in \mathfrak{X}[\pi_L](\mathbb{C}_p)$ we have the continuous translation operator

$$\mathcal{O}_{K_1}(\mathfrak{X}) \to \mathcal{O}_{K_1}(\mathfrak{X})$$
$$F \mapsto (\zeta F)(\chi) := F(\chi\zeta).$$

Proposition 4.1.5. *The following holds true:*

(i)   *For any $F \in \mathcal{O}_{K_1}(\mathfrak{X})$ we have*

$$[\mathit{o}_L : \pi_L \mathit{o}_L] \cdot \varphi_L \circ \psi_L^{\mathfrak{X}}(F) = \sum_{\zeta \in \mathfrak{X}[\pi_L](\mathbb{C}_p)} \zeta F.$$

(ii) $\varphi_L(\mathfrak{O}_K(\mathfrak{X})) = \{F \in \mathfrak{O}_K(\mathfrak{X}) : \zeta F = F \text{ for any } \zeta \in \mathfrak{X}[\pi_L](\mathbb{C}_p)\}.$

Proof. (i) Again it suffices to consider any $F = \mathrm{ev}_a$. We compute

$$\left(\sum_{\zeta} \zeta \mathrm{ev}_a\right)(\chi) = \sum_{\zeta} \mathrm{ev}_a(\chi\zeta) = \chi(a)\sum_{\zeta} \zeta(a)$$

$$= \begin{cases} [\mathit{o}_L : \pi_L \mathit{o}_L] \cdot \chi(a) & \text{if } a \in \pi_L \mathit{o}_L, \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} [\mathit{o}_L : \pi_L \mathit{o}_L] \cdot \mathrm{ev}_a(\chi) & \text{if } a \in \pi_L \mathit{o}_L, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$\varphi_L\big(\psi_L^{\mathfrak{X}}(\mathrm{ev}_a)\big) = \varphi_L\left(\begin{cases} \mathrm{ev}_{\pi_L^{-1}a} & \text{if } a \in \pi_L \mathit{o}_L, \\ 0 & \text{otherwise} \end{cases}\right) = \begin{cases} \mathrm{ev}_a & \text{if } a \in \pi_L \mathit{o}_L, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If $\zeta F = F$ for any $\zeta \in \mathfrak{X}[\pi_L](\mathbb{C}_p)$, then $\varphi_L(\psi_L^{\mathfrak{X}}(F)) = F$ by (i). On the other hand,

$$\big(\zeta \varphi_L(F)\big)(\chi) = \varphi_L(F)(\chi\zeta) = F\big(\pi_L^*(\chi)\pi_L^*(\zeta)\big) = F\big(\pi_L^*(\chi)\big) = \varphi_L(F)(\chi).$$ ∎

We have observed in the above proof that the functions ev_a, for $a \in o_L$, generate a dense subspace of $\mathcal{O}_K(\mathfrak{X})$. Considering the topological decomposition

$$\begin{aligned}
\mathcal{O}_K(\mathfrak{X}) &= \varphi_L\big(\mathcal{O}_K(\mathfrak{X})\big) \oplus \mathcal{O}_K(\mathfrak{X})^{\psi_L^{\mathfrak{X}}=0}, \\
F &= \varphi_L\big(\psi_L^{\mathfrak{X}}(F)\big) + \big(F - \varphi_L\big(\psi_L^{\mathfrak{X}}(F)\big)\big),
\end{aligned} \tag{4.2}$$

we see, using (4.1), that the ev_a for $a \in \pi_L o_L$, resp., the ev_u for $u \in o_L^\times$, generate a dense subspace of $\varphi_L(\mathcal{O}_K(\mathfrak{X}))$, resp., of $\mathcal{O}_K(\mathfrak{X})^{\psi_L^{\mathfrak{X}}=0}$. In view of Lemma 4.1.2, the obvious formula $u_*(\mathrm{ev}_a) = \mathrm{ev}_{ua}$ together with the fact that the Dirac distributions δ_u, for $u \in o_L^\times$, generate a dense subspace of $D(o_L^\times, K)$ then implies that the decomposition (4.2) is $D(o_L^\times, K)$-invariant.

Lemma 4.1.6 (Mellin transform). *The natural inclusion $D(o_L^\times, K) \hookrightarrow D(o_L, K)$ combined with the Fourier isomorphism induces the map*

$$\mathfrak{M} : D(o_L^\times, K) \xrightarrow{\cong} D(o_L, K)^{\psi_L^D=0} \cong \mathcal{O}_K(\mathfrak{X})^{\psi_L^{\mathfrak{X}}=0}$$
$$\lambda \mapsto \lambda(\delta_1) \,\widehat{=}\, \lambda(\mathrm{ev}_1)$$

which is a topological isomorphism of $D(o_L^\times, K)$-modules.

Proof. The disjoint decomposition into open sets $o_L = \pi_L o_L \cup o_L^\times$ induces the linear topological decomposition $D(o_L, K) = \varphi_L(D(o_L, K)) \oplus D(o_L^\times, K)$. The assertion follows by comparing this with the decomposition (4.2).[1]   ∎

The Robba ring

We recall a few facts from [8] about the analytic structure of the character variety $\mathfrak{X}$. As a general convention, all *radii r* which will occur throughout the paper are *assumed to lie in* $(0, 1) \cap p^{\mathbb{Q}}$. Let $\mathbf{B}_1(r)$, resp., $\mathbf{B}(r)$, denote the $\mathbb{Q}_p$-affinoid disk of radius r around 1, resp., around 0, and let $\mathbf{B}_1^-(r)$ be the open disk of radius r around 1. We put

$$\mathfrak{X}_0(r) := \mathbf{B}_1(r) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p) \quad \text{and} \quad \mathfrak{X}(r) := \mathfrak{X} \cap \mathfrak{X}_0(r)_{/L}.$$

These are affinoid subgroups of $\mathfrak{X}_0$ and $\mathfrak{X}$, respectively, which are respected by the action of the monoid $o_L \setminus \{0\}$. Since $\mathfrak{X}(r) \hookrightarrow \mathfrak{X}_0(r)_{/L}$ is a closed immersion of affinoid varieties, the restriction map between the affinoid algebras $\mathcal{O}_K(\mathfrak{X}_0(r)) \twoheadrightarrow \mathcal{O}_K(\mathfrak{X}(r))$ is a strict surjection of Banach algebras. The families $\{\mathfrak{X}_0(r)\}_r$, resp., $\{\mathfrak{X}(r)\}_r$, form an increasing admissible covering of $\mathfrak{X}_0$, resp., $\mathfrak{X}$, which exhibits the latter as a quasi-Stein space. Hence, $\mathcal{O}_K(\mathfrak{X}_0(r))$, resp., $\mathcal{O}_K(\mathfrak{X}(r))$, is the completion of $\mathcal{O}_K(\mathfrak{X}_0)$, resp., $\mathcal{O}_K(\mathfrak{X})$, in the supremum norm $|\ |_{\mathfrak{X}_0(r)}$, resp., $|\ |_{\mathfrak{X}(r)}$.

---

[1]The map $D(o_L^\times, K) \to D(G, K)$ sending λ to $\lambda(\delta_1)$ is the inclusion map since $\delta_u(\delta_1) = \delta_u$.

The structure of the affinoid variety $\mathfrak{X}(r_0)$ is rather simple for any radius $r_0 < p^{-\frac{d}{p-1}}$. Then (cf. [8, Lem. 1.16]) the map

$$\begin{aligned}\mathbf{B}(r_0)_{/L} &\xrightarrow{\cong} \mathfrak{X}(r_0) \\ y &\mapsto \chi_y(a) := \exp(ay)\end{aligned} \tag{4.3}$$

is an isomorphism of L-affinoid groups. Taking, somewhat unconventionally, $\exp -1$ as coordinate function on $\mathbf{B}(r_0)$, we may view $\mathcal{O}_K(\mathbf{B}(r_0))$ as the Banach algebra of all power series $f = \sum_{i \geq 0} c_i (\exp -1)^i$ such that $c_i \in K$ and $\lim_{i \to \infty} |c_i| r_0^i = 0$; the norm is $|f|_{\mathbf{B}(r_0)} := \max_i |c_i| r_0^i$. Since $\exp -1$ corresponds under the above isomorphism to the function $\mathrm{ev}_1 - 1$ on $\mathfrak{X}(r_0)$, we deduce that

$$\mathcal{O}_K\big(\mathfrak{X}(r_0)\big) = \Big\{ f = \sum_{i \geq 0} c_i (\mathrm{ev}_1 - 1)^i : c_i \in K \text{ and } \lim_{i \to \infty} |c_i| r_0^i = 0 \Big\}$$

is a Banach algebra with the supremum norm $|f|_{\mathfrak{X}(r_0)} = \max_i |c_i| r_0^i$.

Next we need to explain the admissible open subdomains $\mathfrak{X}_I$ of $\mathfrak{X}$, where the $I \subseteq (0, 1)$ are certain intervals (cf. [8, §2.1]). First of all, we have the admissible open subdomains

$$\mathfrak{X}_{(r,1)} := \mathfrak{X} \setminus \mathfrak{X}(r).$$

To introduce the relevant affinoid subdomains, we also need the open disk $\mathbf{B}_1^-(r)$ of radius r around 1. This allows us to first define the admissible open subdomains $\mathfrak{X}_0^-(r) := (\mathbf{B}_1^-(r) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p))_{/L}$ and $\mathfrak{X}^-(r) := \mathfrak{X} \cap \mathfrak{X}_0^-(r)$ of $\mathfrak{X}_0$ and $\mathfrak{X}$, respectively. For $r \leq s$ we then have the admissible open subdomains

$$\mathfrak{X}_0[r,s] := \mathfrak{X}_0(s) \setminus \mathfrak{X}_0^-(r) \subseteq \mathfrak{X}_0 \quad \text{and} \quad \mathfrak{X}_{[r,s]} := \mathfrak{X}(s) \setminus \mathfrak{X}^-(r) = \mathfrak{X} \cap \mathfrak{X}_0[r,s] \subseteq \mathfrak{X}.$$

We recall that the $\mathfrak{X}_{[r,s]}$ are actually affinoid varieties. There are the obvious inclusions $\mathfrak{X}_{[r,s]} \subseteq \mathfrak{X}(s)$ and $\mathfrak{X}_{[r,s]} \subseteq \mathfrak{X}_{(r',1)}$ provided $r' < r$. Moreover, $\mathfrak{X}_{(r',1)}$ is the increasing admissible union of the $\mathfrak{X}_{[r,s]}$ for $r' < r \leq s < 1$. Hence,

$$\mathcal{O}_K(\mathfrak{X}_{(r',1)}) = \varprojlim_{r' < r \leq s < 1} \mathcal{O}_K(\mathfrak{X}_{[r,s]}),$$

which exhibits the Fréchet algebra structure of the left side.

We point out that these subdomains $\mathfrak{X}_I$ all are invariant under $o_L^\times$. Their behavior with respect to π_L^* is more complicated. We recall from [8, Lem. 2.11] that, for any radius $p^{-\frac{dp}{p-1}} \leq r < 1$, we have

$$(\pi_L^*)^{-1}(\mathfrak{X}_{(r,1)}) \subseteq \mathfrak{X}_{(r^{1/p},1)}.$$

It is technically necessary in the following to sometimes only work with a smaller set of radii. We put

$$S_0 := [p^{-\frac{d}{e} - \frac{d}{p-1}}, p^{-\frac{d}{p-1}}) \cap p^{\mathbb{Q}} \subseteq [p^{-\frac{dp}{p-1}}, p^{-\frac{d}{p-1}}),$$

$S_n := S_0^{\frac{1}{p^n}}$ for $n \geq 1$, and $S_\infty := \bigcup_{n \geq 1} S_n$. Note that the sets S_n are pairwise disjoint. The point is (cf. [8, Prop. 1.20]) that for $s \in S_\infty$ we know that $\mathfrak{X}(s)$ becomes isomorphic to a closed disk over $\mathbb{C}_p$. Let s_n for $n \geq 0$ denote the left boundary point of the set S_n. Then we have the following result (cf. [8, Prop. 2.1]).

Proposition 4.1.7. *For any $n \geq 0$ the rigid variety $\mathfrak{X}_{(s_n,1)}$ is quasi-Stein with respect to the admissible covering $\{\mathfrak{X}_{[r,s]}\}$, where $s_n < r \leq s < 1$, $r \in S_n$, and $s \in \bigcup_{m \geq n} S_m$. In particular, the affinoid algebra $\mathcal{O}_K(\mathfrak{X}_{[r,s]})$ is the completion of $\mathcal{O}_K(\mathfrak{X}_{(s_n,1)})$ with respect to the supremum norm $|\,|_{\mathfrak{X}_{[r,s]}}$.*

Obviously, with $\mathfrak{X}$ each $\mathfrak{X}_{(s_n,1)}$ is one dimensional and smooth. But, in order to be able to apply later on Serre duality to the spaces $\mathfrak{X}_{(s_n,1)}$, we need to show that they are actually Stein spaces. This means that we have to check that the admissible covering in Proposition 4.1.7 has the property that $\mathfrak{X}_{[r',s']}$ is relatively compact in $\mathfrak{X}_{[r,s]}$ over L (cf. [14, §9.6.2]) for any $r < r' \leq s' < s$. We simply write $U \Subset X$ for an affinoid subdomain U being relatively compact over L in an L-affinoid variety X.

Lemma 4.1.8. *Let $U \subseteq X \subseteq X'$ be affinoid subdomains of the affinoid variety X'; we then have the following:*

(i)   *if $U \Subset X$, then $U \Subset X'$;*

(ii) *suppose that $U = U_1 \cup \cdots \cup U_m$ is an affinoid covering; if $U_i \Subset X$ for any $1 \leq i \leq m$, then $U \Subset X$.*

Proof. Let $A \to B$ be the homomorphism of affinoid algebras which induces the inclusion $U = \mathrm{Sp}(B) \subseteq X = \mathrm{Sp}(A)$. It is not difficult to see that the property $U \Subset X$ is equivalent to the homomorphism $A \to B$ being inner with regard to L in the sense of [9, Def. 2.5.1]. Therefore, (i), resp., (ii), is a special case of Corollary 2.5.5, resp., Lemma 2.5.10, in [9]. ∎

Proposition 4.1.9. *$\mathfrak{X}_{(s_n,1)}$, for any $n \geq 0$, is a Stein space.*

Proof. By Proposition 4.1.7, we already know that the $\mathfrak{X}_{(s_n,1)}$ are quasi-Stein. Hence, it remains to show that $\mathfrak{X}_{[r',s']} \Subset \mathfrak{X}_{[r,s]}$ for any $r < r' \leq s' < s$. Looking first at $\mathfrak{X}_0$, let $\mathfrak{B}_1[r,s] \subseteq \mathfrak{B}_1$ denote the affinoid annulus of inner radius r and outer radius s. Fixing coordinate functions $z_1, \ldots, z_d$ on $\mathfrak{X}_0$, we have the admissible open covering

$$\mathfrak{X}_0[r,s] = \bigcup_{i=1}^{d} \mathfrak{X}_0^{(i)}[r,s] \quad \text{with } \mathfrak{X}_0^{(i)}[r,s] := \{x \in \mathfrak{X}_0(s) : |z_i(x)| \geq r\}.$$

The affinoid varieties of this covering have the direct product structure

$$\mathfrak{X}_0^{(i)}[r,s] = \mathfrak{B}_1(s) \times \cdots \mathfrak{B}_1(s) \times \mathfrak{B}_1[r,s] \times \mathfrak{B}_1(s) \times \cdots \mathfrak{B}_1(s)$$

with the annulus being the ith factor. It immediately follows that $\mathfrak{X}_0^{(i)}[r',s'] \Subset \mathfrak{X}_0^{(i)}[r,s]$ (cf. [14, Lem. 9.6.2.1]). Since relative compactness is preserved by passing to closed

subvarieties, we deduce that

$$\mathfrak{X} \cap \mathfrak{X}_0^{(i)}[r', s'] \Subset \mathfrak{X} \cap \mathfrak{X}_0^{(i)}[r, s] \quad \text{for any } 1 \leq i \leq d.$$

Applying now Lemma 4.1.8, we conclude first that $\mathfrak{X} \cap \mathfrak{X}_0^{(i)}[r', s'] \Subset \mathfrak{X}_0[r, s]$ and then that $\mathfrak{X}_{[r',s']} \Subset \mathfrak{X}_{[r,s]}$. ∎

We finally recall that the Robba ring of $\mathfrak{X}$ over K is defined as the locally convex inductive limit $\varinjlim_{\mathfrak{Y}} \mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{Y})$, where $\mathfrak{Y}$ runs over all affinoid subdomains of $\mathfrak{X}$. Since any such $\mathfrak{Y}$ is contained in some $\mathfrak{X}(r)$, we have

$$\mathcal{R}_K(\mathfrak{X}) = \varinjlim_{n \geq 0} \mathcal{O}_K(\mathfrak{X}_{(s_n, 1)}),$$

and we view $\mathcal{R}_K(\mathfrak{X})$ as the locally convex inductive limit of the Fréchet algebras $\mathcal{O}_K(\mathfrak{X}_{(s_n, 1)})$. By [8, Prop. 1.20], the system $\mathfrak{X}_{(s_n, 1)/\mathbb{C}_p}$ is isomorphic to a decreasing system of one-dimensional annuli. This implies the following:

- $\mathcal{R}_K(\mathfrak{X})$ is the increasing union of the rings $\mathcal{O}_K(\mathfrak{X}_{(s_n, 1)})$ and contains $\mathcal{O}_K(\mathfrak{X})$;
- each $\mathcal{O}_K(\mathfrak{X}_{(s_n, 1)})$ and $\mathcal{R}_K(\mathfrak{X})$ are integral domains.

The action of the monoid $o_L \setminus \{0\}$ on $\mathcal{O}_K(\mathfrak{X})$ extends naturally to a continuous action on $\mathcal{R}_K(\mathfrak{X})$ (cf. [8, Lem. 2.12]). In fact, this action extends further uniquely to a separately continuous action of $D(o_L^\times, K)$-action on $\mathcal{R}_K(\mathfrak{X})$. This is a special case of the later Proposition 4.3.10 which implies that we will have such an action on any L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_K(\mathfrak{X})$. Via the isomorphism $\chi_{\mathrm{LT}} : \Gamma_L \xrightarrow{\cong} o_L^\times$ we later on will view this as a $D(\Gamma_L, K)$-action.

In order to extend the ψ-operator to the Robba ring, we need the following fact.

Lemma 4.1.10. *The morphism* $\pi_L^* : \mathfrak{X} \to \mathfrak{X}$ *is finite, faithfully flat, and étale.*

Proof. The character variety $\mathfrak{X}'$ of the subgroup $\pi_L o_L \subseteq o_L$ is isomorphic to $\mathfrak{X}$ via

$$\mathfrak{X} \xrightarrow{\cong} \mathfrak{X}'$$
$$\chi \mapsto \chi'(\pi_L a) := \chi(a).$$

We have the commutative diagram

$$\begin{array}{ccc} \mathfrak{X} & & \\ \pi_L^* \downarrow & \searrow^{\chi \mapsto \chi | \pi_L o_L} & \\ \mathfrak{X} & \xrightarrow[\chi \mapsto \chi']{\cong} & \mathfrak{X}'. \end{array}$$

The oblique arrow is finite and faithfully flat by the proof of [29, Prop. 6.4.5]. For its étaleness it remains to check that all its fibers are unramified. This can be done after

base change to $\mathbb{C}_p$. Then, since this arrow is a homomorphism of rigid groups, all fibers are isomorphic. But the fiber in the trivial character of $\pi_L o_L$ is isomorphic to $\mathrm{Sp}(\mathbb{C}_p[o_L/\pi_L o_L]) \cong \mathrm{Sp}(\mathbb{C}_p^q)$. It follows that π_L^* has these properties as well. ∎

Since the subsequent reasoning will be needed again in the next section in an analogous situation, we proceed in an axiomatic way. *Suppose* that the following hold:

- $\rho : \mathfrak{Y} \to \mathfrak{Z}$ is a finite and faithfully flat morphism of quasi-Stein spaces over K. In particular, the induced map $\rho^* : \mathcal{O}_K(\mathfrak{Z}) \to \mathcal{O}_K(\mathfrak{Y})$ is injective. Moreover, the finiteness of ρ implies that the preimage under ρ of any affinoid subdomain in $\mathfrak{Z}$ is an affinoid subdomain in $\mathfrak{Y}$ (cf. [14, Prop. 9.4.4.1 (i)]) and hence that ρ^* is continuous.

- $\mathcal{O}_K(\mathfrak{Y})$ is finitely generated free as a $\rho^*(\mathcal{O}_K(\mathfrak{Z}))$-module. Fix a corresponding basis $f_1, \ldots, f_h \in \mathcal{O}_K(\mathfrak{Y})$.

Proposition 4.1.11. *For any admissible open subset $\mathfrak{U} \subseteq \mathfrak{Z}$ we have*

$$\mathcal{O}_K\big(\rho^{-1}(\mathfrak{U})\big) = \mathcal{O}_K(\mathfrak{Y}) \otimes_{\mathcal{O}_K(\mathfrak{Z})} \mathcal{O}_K(\mathfrak{U})$$

is free with basis $f_1, \ldots, f_h$ over $\mathcal{O}_K(\mathfrak{U})$.

Proof. Since ρ is finite, $\rho_* \mathcal{O}_{\mathfrak{Y}}$ is a coherent $\mathcal{O}_{\mathfrak{Z}}$-module by [14, Prop. 9.4.4.1 (ii)]. Gruson's theorem (cf. [8, Prop. 1.13]) then tells us that $\rho_* \mathcal{O}_{\mathfrak{Y}}$ is, in fact, a free $\mathcal{O}_{\mathfrak{Z}}$-module with basis $f_1, \ldots, f_h$. ∎

We observe that the definition of the Robba ring $\mathcal{R}_K(\mathfrak{X})$ above was completely formal and works precisely the same way for any quasi-Stein space. Hence, we have available the Robba rings $\mathcal{R}_K(\mathfrak{Y})$ and $\mathcal{R}_K(\mathfrak{Z})$. Since the morphism $\rho : \mathfrak{Y} \to \mathfrak{Z}$ is finite, the preimage under ρ of any affinoid subdomain in $\mathfrak{Z}$ is an affinoid subdomain in $\mathfrak{Y}$ (cf. [14, Prop. 9.4.4.1 (i)]). We note again that the preimage under ρ of any affinoid subdomain in $\mathfrak{Z}$ is an affinoid subdomain in $\mathfrak{Y}$. The injective map $\rho^* : \mathcal{O}_K(\mathfrak{Z}) \subseteq \mathcal{O}_K(\mathfrak{Y})$ therefore extends to a natural homomorphism of rings

$$\rho^* : \mathcal{R}_K(\mathfrak{Z}) \to \mathcal{R}_K(\mathfrak{Y}). \tag{4.4}$$

Remark 4.1.12. The homomorphism (4.4) is injective.

Proof. We fix an admissible covering $\mathfrak{Z} = \bigcup_{j \geq 1} \mathfrak{U}_j$ by an increasing sequence of affinoid subdomains $\mathfrak{U}_j \subseteq \mathfrak{Z}$. As ρ is a finite map, $\mathfrak{Y} = \bigcup_{j \geq 1} \rho^{-1}(\mathfrak{U}_j)$ again is an admissible covering by affinoid subdomains. It follows that

$$\mathfrak{R}_K(\mathfrak{Y}) = \varinjlim_j \mathfrak{O}_K\big(\mathfrak{Y} \setminus \rho^{-1}(\mathfrak{U}_j)\big),$$

and therefore it suffices to show the injectivity of the maps $\rho^* : \mathcal{O}_K(\mathfrak{Z} \setminus \mathfrak{U}_j) \to \mathcal{O}_K(\mathfrak{Y} \setminus \rho^{-1}(\mathfrak{U}_j))$. But this is clear since the map $\rho : \mathfrak{Y} \setminus \rho^{-1}(\mathfrak{U}_j) \to \mathfrak{Z} \setminus \mathfrak{U}_j$ is faithfully flat. ∎

Corollary 4.1.13. $\mathcal{R}_K(\mathcal{Y}) = \mathcal{O}_K(\mathcal{Y}) \otimes_{\mathcal{O}_K(\mathfrak{Z})} \mathcal{R}_K(\mathfrak{Z})$ *is free over* $\rho^*(\mathcal{R}_K(\mathfrak{Z}))$ *with the basis* $f_1, \ldots, f_h$. *In fact, the map*

$$\mathcal{R}_K(\mathfrak{Z})^h \xrightarrow{\cong} \mathcal{R}_K(\mathcal{Y})$$

$$(z_1, \ldots, z_h) \mapsto \sum_{i=1}^{h} \rho^*(z_i) f_i$$

is a homeomorphism.

Proof. By passing to locally convex limits this follows from Proposition 4.1.11 which says that the map

$$\mathcal{O}_K(\mathfrak{U})^h \xrightarrow{\cong} \mathfrak{O}_K(\rho^{-1}(\mathfrak{U}))$$

$$(z_1, \ldots, z_h) \mapsto \sum_{i=1}^{h} \rho^*(z_i) f_i$$

is a continuous bijection between Fréchet spaces and hence a homeomorphism by the open mapping theorem. ∎

By the Lemmata 4.1.1 and 4.1.10 the above applies to the morphism $\pi_L^* : \mathfrak{X} \to \mathfrak{X}$ and we obtain the following result.

Proposition 4.1.14. *Let* $R \subseteq o_L$ *be a set of representatives for the cosets in* $o_L/\pi_L o_L$. *Then the Robba ring* $\mathcal{R}_K(\mathfrak{X})$ *is a free module over* $\varphi_L(\mathcal{R}_K(\mathfrak{X}))$ *with basis* $\{\mathrm{ev}_a\}_{a \in R}$.

In particular, we have the trace map

$$\mathrm{trace}_{\mathcal{R}_K(\mathfrak{X})/\varphi_L(\mathcal{R}_K(\mathfrak{X}))} : \mathcal{R}_K(\mathfrak{X}) \to \varphi_L(\mathcal{R}_K(\mathfrak{X}))$$

and therefore may introduce the operator

$$\psi_L^{\mathfrak{X}} := \frac{1}{q} \varphi_L^{-1} \circ \mathrm{trace}_{\mathcal{R}_K(\mathfrak{X})/\varphi_L(\mathcal{R}_K(\mathfrak{X}))} : \mathcal{R}_K(\mathfrak{X}) \to \mathcal{R}_K(\mathfrak{X}).$$

Because of Remark 4.1.4, it extends the operator $\psi_L^{\mathfrak{X}}$ on $\mathcal{O}_K(\mathfrak{X})$, which justifies denoting it by the same symbol. By construction it is a left inverse of φ_L and satisfies the projection formula. Furthermore, as a consequence of Corollary 4.1.13, $\psi_L^{\mathfrak{X}}$ is continuous.

4.1.2 The multiplicative character variety and its Robba ring

In this section, we consider the multiplicative group $o_L^\times$ as a locally L-analytic group. We introduce the open subgroups $U_n := 1 + \pi_L^n o_L$ for $n \geq 1$. Correspondingly, we have the inclusion of distribution algebras $D(U_{n+1}, K) \subseteq D(U_n, K) \subseteq D(o_L^\times, K)$.

There is an integer $n_0 \geq 1$ such that, for any $n \geq n_0$, the logarithm series induces an isomorphism of locally L-analytic groups $\log : U_n \xrightarrow{\cong} \pi_L^n o_L$. We then introduce the isomorphisms $\ell_n := \pi_L^{-n} \log : U_n \xrightarrow{\cong} o_L$ together with the algebra isomorphisms

$$\ell_{n*} : D(U_n, K) \xrightarrow{\cong} D(o_L, K) \cong \mathcal{O}_K(\mathfrak{X})$$

which they induce.

As for o_L in the previous section, we have rigid analytic varieties (over L) of locally L-analytic characters $\mathfrak{X}^\times$ for $o_L^\times$ and $\mathfrak{X}_n^\times$ for U_n as well (cf. [74, Thm. 2.3, Lem. 2.4, Cor. 3.7] and [29, Props. 6.4.5 and 6.4.6]):

- $\ell_n^* : \mathfrak{X} \xrightarrow{\cong} \mathfrak{X}_n^\times$ is, for $n \geq n_0$, an isomorphism of group varieties.
- The restriction map $\rho_n : \mathfrak{X}^\times \to \mathfrak{X}_n^\times$ is a finite faithfully flat covering (cf. [29, proof of Prop. 6.4.5]).
- $\mathfrak{X}^\times$ and $\mathfrak{X}_n^\times$ are one-dimensional Stein spaces. (As group varieties, they are separated and equidimensional.)
- For $n \geq n_0$ the variety $\mathfrak{X}_n^\times$ is smooth and $\mathcal{O}_L(\mathfrak{X}_n^\times)$ is an integral domain.
- The Fourier transforms

$$D(o_L^\times, K) \xrightarrow{\cong} \mathcal{O}_K(\mathfrak{X}^\times) \quad \text{and} \quad D(U_n, K) \xrightarrow{\cong} \mathcal{O}_K(\mathfrak{X}_n^\times)$$

sending a distribution μ to the function $F_\mu(\chi) := \mu(\chi)$ are isomorphisms of Fréchet algebras.

As a consequence of the properties of the morphism $\rho := \rho_n : \mathfrak{X}^\times \to \mathfrak{X}_n^\times$, the homomorphism $\rho^* : \mathcal{O}_K(\mathfrak{X}_n^\times) \to \mathcal{O}_K(\mathfrak{X}^\times)$ is injective and extends to an injective homomorphism $\rho^* : \mathcal{R}_K(\mathfrak{X}_n^\times) \to \mathcal{R}_K(\mathfrak{X}^\times)$ (cf. Remark 4.1.12).

Lemma 4.1.15. *We have the following equalities:*

(i) $\mathcal{O}_K(\mathfrak{X}^\times) = \mathbb{Z}[o_L^\times] \otimes_{\mathbb{Z}[U_n]} \mathcal{O}_K(\mathfrak{X}_n^\times)$.

(ii) $\mathcal{R}_K(\mathfrak{X}^\times) = \mathcal{O}_K(\mathfrak{X}^\times) \otimes_{\mathcal{O}_K(\mathfrak{X}_n^\times)} \mathcal{R}_K(\mathfrak{X}_n^\times) = \mathbb{Z}[o_L^\times] \otimes_{\mathbb{Z}[U_n]} \mathcal{R}_K(\mathfrak{X}_n^\times)$.

Proof. (i) Let $u_1, \ldots, u_h \in o_L^\times$ be a set of representatives for the cosets of U_n in $o_L^\times$. We then have the decomposition into open subsets $o_L^\times = u_1 U_n \cup \cdots \cup u_h U_n$. It follows that

$$D(o_L^\times, K) = \delta_{u_1} D(U_n, K) \oplus \cdots \oplus \delta_{u_h} D(U_n, K) = \mathbb{Z}[o_L^\times] \otimes_{\mathbb{Z}[U_n]} D(U_n, K)$$

is, in particular, a free $D(U_n, K)$-module of rank $h = [o_L^\times : U_n]$. Using the Fourier isomorphism, we obtain that $\mathcal{O}_K(\mathfrak{X}^\times)$ is a free $\mathcal{O}_K(\mathfrak{X}_n^\times)$-module over the basis $\mathrm{ev}_{u_1}, \ldots, \mathrm{ev}_{u_h}$.

(ii) Because of (i), the assumptions before Proposition 4.1.11 are satisfied and the present assertion is a special case of Corollary 4.1.13. ∎

Lemma 4.1.16. *The morphism ρ is étale.*

Proof. This is the same argument as in the proof of Lemma 4.1.10. ∎

Corollary 4.1.17. $\mathfrak{X}^\times$ *is smooth.*

Proof. This follows from the lemma since $\mathfrak{X}_n^\times$ is smooth for $n \geq n_0$. ∎

Remark 4.1.18. If $n \geq m$, then all the above assertions hold analogously for the finite morphism $\rho_{m,n} : \mathfrak{X}_m^\times \to \mathfrak{X}_n^\times$. In particular, all $\mathfrak{X}_n^\times$ are smooth.

Suppose that $n \geq n_0$. Then, due to the isomorphism $\ell_n^* : \mathfrak{X} \xrightarrow{\cong} \mathfrak{X}_n^\times$, everything which was defined for and recalled about $\mathfrak{X}$ in Section 4.1.1 holds correspondingly for $\mathfrak{X}_n^\times$. In particular, we have the admissible open subdomains $\mathfrak{X}_n^\times(r)$, $\mathfrak{X}_{n,(r,1)}^\times$, and $\mathfrak{X}_{n,[r,s]}^\times$. For $n \geq m \geq n_0$ we have the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow[\cong]{\ell_m^*} & \mathfrak{X}_m^\times \\
{\scriptstyle (\pi_L^*)^{n-m}} \downarrow & & \downarrow {\scriptstyle \rho_{m,n}} \\
\mathfrak{X} & \xrightarrow[\cong]{\ell_n^*} & \mathfrak{X}_n^\times .
\end{array}
\tag{4.5}
$$

Lemma 4.1.19. *Let $n \geq n_0$ and $m \geq 0$; for any $p^{-\frac{dp}{p-1}} \leq r < 1$ the map $\rho_{n,n+me}^* : \mathcal{O}_K(\mathfrak{X}_{n+me}^\times) \to \mathcal{O}_K(\mathfrak{X}_n^\times)$ extends to an isometric homomorphism of Banach algebras*

$$
\left(\mathfrak{O}_K\big(\mathfrak{X}_{n+me}^\times(r)\big), |\ |_{\mathfrak{X}_{n+me}^\times(r)} \right) \to \left(\mathfrak{O}_K\big(\mathfrak{X}_n^\times(r^{\frac{1}{p^m}})\big), |\ |_{\mathfrak{X}_n^\times(r^{\frac{1}{p^m}})} \right).
$$

Proof. By the above commutative diagram (4.5) our assertion amounts to the statement that the map $(\pi_L^{me})^* : \mathfrak{X} \to \mathfrak{X}$ restricts to a surjection $\mathfrak{X}(r^{\frac{1}{p^m}}) \to \mathfrak{X}(r)$. In [74, Lem. 3.3], this is shown to be the case for the map $(p^m)^*$. But p^m and π_L^{me} differ by a unit $u \in o_L^\times$, and u^* preserves $\mathfrak{X}(r)$. ∎

4.1.3 Twisting

Consider any locally L-analytic group G and fix a locally L-analytic character $\chi : G \to L^\times$. Then multiplication by χ is a K-linear topological isomorphism

$$
C^{\mathrm{an}}(G, K) \xrightarrow[\cong]{\chi \cdot} C^{\mathrm{an}}(G, K).
$$

We denote the dual isomorphism by

$$
\mathrm{Tw}_\chi^D : D(G, K) \xrightarrow{\cong} D(G, K),
$$

i.e., $\mathrm{Tw}_\chi^D(\mu) = \mu(\chi-)$, and call it the twist by χ. For Dirac distributions we obtain $\mathrm{Tw}_\chi^D(\delta_g) = \chi(g)\delta_g$.

Suppose now that G is one of the groups o_L or $U_n \subseteq o_L^\times$ of the previous subsections, and let $\mathfrak{X}_G$ denote its character variety. Then χ is an L-valued point $z_\chi \in \mathfrak{X}_G(L)$. Using the product structure of the variety $\mathfrak{X}_G$, we similarly have the twist operator

$$\mathrm{Tw}_z^{\mathfrak{X}_G} : \mathfrak{O}_K(\mathfrak{X}_G) \xrightarrow{\cong} \mathcal{O}_K(\mathfrak{X}_G), \quad f \mapsto f(z-).$$

As any rigid automorphism multiplication by a rational point respects the system of affinoid subdomains and hence the system of their complements. Hence, $\mathrm{Tw}_z^{\mathfrak{X}_G}$ extends straightforwardly to an automorphism $\mathrm{Tw}_z^{\mathfrak{X}_G} : \mathfrak{R}_K(\mathfrak{X}_G) \xrightarrow{\cong} \mathcal{R}_K(\mathfrak{X}_G)$. The following properties are straightforward to check:

(i)   Under the Fourier isomorphism, Tw_χ^D and $\mathrm{Tw}_{z_\chi}^{\mathfrak{X}_G}$ correspond to each other.

(ii) $\mathrm{Tw}_{z_1}^{\mathfrak{X}_G} \circ \mathrm{Tw}_{z_2}^{\mathfrak{X}_G} = \mathrm{Tw}_{z_1 \cdot z_2}^{\mathfrak{X}_G}$.

(iii) If $\alpha : G_1 \xrightarrow{\cong} G_2$ is an isomorphism between two of our groups, then, for any $z \in \mathfrak{X}_{G_2}(L)$, the twist operators $\mathrm{Tw}_{\alpha^*(z)}^{\mathfrak{X}_{G_1}}$ and $\mathrm{Tw}_z^{\mathfrak{X}_{G_2}}$ correspond to each other under the isomorphism

$$\alpha_* : \mathcal{R}_K(\mathfrak{X}_{G_1}) \xrightarrow{\cong} \mathcal{R}_K(\mathfrak{X}_{G_2}).$$

4.1.4 The LT-isomorphism, part 1

We write $\mathbf{B}$ for the open unit ball over L. The Lubin–Tate formal o_L-module gives $\mathbf{B}$ an o_L-action via $(a, z) \mapsto [a](z)$. If $\mathcal{O}_K(\mathbf{B})$ is the ring of power series in Z with coefficients in K which converge on $\mathbf{B}(\mathbb{C}_p)$, then the above o_L-action on $\mathbf{B}$ induces an action of the monoid $o_L \setminus \{0\}$ on $\mathcal{O}_K(\mathbf{B})$ by $(a, F) \mapsto F \circ [a]$. Similarly, as before, we let φ_L denote the action of π_L. Next we consider the continuous operator

$$\mathrm{tr} : \mathcal{O}_K(\mathbf{B}) \to \mathcal{O}_K(\mathbf{B})$$
$$f(z) \mapsto \sum_{y \in \ker([\pi_L])} f(y +_{\mathrm{LT}} z).$$

Coleman has shown (cf. [21, Lem. 3] or [80, §2]) that $\mathrm{tr}(Z^i) \in \mathrm{im}(\varphi_L)$ for any $i \geq 0$. Hence, since φ_L is a homeomorphism onto its image, we have $\mathrm{im}(\mathrm{tr}) \subseteq \mathrm{im}(\varphi_L)$ and hence, since φ_L is injective, we may introduce the K-linear operator

$$\psi_L : \mathcal{O}_K(\mathbf{B}) \to \mathcal{O}_K(\mathbf{B}) \quad \text{such that} \quad \pi_L^{-1}\mathrm{tr} = \varphi_L \circ \psi_L.$$

One easily checks that ψ_L is equivariant for the $o_L^\times$-action and satisfies the projection formula $\psi_L(f_1 \varphi_L(f_2)) = \psi_L(f_1) f_2$ as well as $\psi_L \circ \varphi_L = \frac{q}{\pi_L}$.

Furthermore, we fix a generator η' of T_π' as o_L-module and denote by $\Omega = \Omega_{\eta'}$ the corresponding period. In the following we *assume that Ω belongs to K*. From [74, Thm. 3.6] we recall the LT-isomorphism

$$\kappa^* : \mathcal{O}_K(\mathfrak{X}) \xrightarrow{\cong} \mathcal{O}_K(\mathbf{B})$$
$$F \mapsto \big[z \mapsto F(\kappa_z)\big], \tag{4.6}$$

where $\kappa_z(a) = 1 + F_{\eta'}([a](z))$ with $1 + F_{\eta'}(Z) := \exp(\Omega \log_{\mathrm{LT}}(Z))$. It is an isomorphism of topological rings which is equivariant with respect to the action by the monoid $o_L \setminus \{0\}$ (as a consequence of [74, Prop. 3.1]). Moreover, Lemma 4.1.2 implies that the $o_L^\times$-action on $\mathcal{O}_K(\mathbf{B})$ extends to a jointly continuous $D(o_L^\times, K)$-module structure (by descent even for general K) and that the LT-isomorphism is an isomorphism of $D(o_L^\times, K)$-modules.

By the construction of the LT-isomorphism we have

$$\kappa^*(\mathrm{ev}_a) = \exp\big(a\Omega \log_{\mathrm{LT}}(Z)\big) \in o_{\mathbb{C}_p}[\![Z]\!] \quad \text{for any } a \in o_L.$$

Hence, Lemma 4.1.3 implies that

$$\kappa^*\big(\ker(\psi_L^{\mathfrak{x}})\big) = \sum_{a \in R_0} \exp\big(a\Omega \log_{\mathrm{LT}}(Z)\big)\varphi_L\big(\mathcal{O}_K(\mathbf{B})\big),$$

where $R_0 \subseteq o_L$ denotes a set of representatives for the non-zero cosets in $o_L/\pi_L o_L$. Using that $\log_{\mathrm{LT}}(Z_1 +_{\mathrm{LT}} Z_2) = \log_{\mathrm{LT}}(Z_1) + \log_{\mathrm{LT}}(Z_2)$, we compute

$$\mathrm{tr}\big(\exp\big(a\Omega \log_{\mathrm{LT}}(Z)\big)\big) = \sum_{y \in \ker([\pi_L])} \exp\big(a\Omega \log_{\mathrm{LT}}(y +_{\mathrm{LT}} Z)\big)$$

$$= \Big(\sum_{y \in \ker([\pi_L])} \exp\big(a\Omega \log_{\mathrm{LT}}(y)\big)\Big) \exp\big(a\Omega \log_{\mathrm{LT}}(Z)\big)$$

$$= \Big(\sum_{y \in \ker([\pi_L])} \kappa_y(a)\Big) \exp\big(a\Omega \log_{\mathrm{LT}}(Z)\big).$$

But the κ_y for $y \in \ker([\pi_L])$ are precisely the characters of the finite abelian group $o_L/\pi_L o_L$. Hence, $\sum_{y \in \ker([\pi_L])} \kappa_y(a) = 0$ for $a \in R_0$. It follows that $\kappa^*(\ker(\psi_L^{\mathfrak{x}})) = \ker(\psi_L)$. We conclude that under the LT-isomorphism ψ_L corresponds to $\frac{q}{\pi_L}\psi_L^{\mathfrak{x}}$ using the fact that we also have a decomposition

$$\mathcal{O}_K(\mathbf{B}) = \sum_{a \in o_L/\pi_L} \exp\big(a\Omega \log_{\mathrm{LT}}(Z)\big)\varphi_L\big(\mathcal{O}_K(\mathbf{B})\big). \tag{4.7}$$

In the following we denote by

$$\mathcal{M}_{\mathrm{LT}} : D(\Gamma_L, K) \xrightarrow{\cong} \mathcal{O}_K(\mathbf{B})^{\psi_L = 0}$$

the composite

$$D(\Gamma_L, K) \cong D(o_L^\times, K) \cong \mathcal{O}_K(\mathfrak{X})^{\psi_L^{\mathfrak{x}} = 0} \cong \mathcal{O}_K(\mathbf{B})^{\psi_L = 0},$$

where the first isomorphism is induced by the character $\chi_{\mathrm{LT}} : \Gamma_L \xrightarrow{\cong} o_L^\times$, the second one is the Mellin transform $\mathfrak{M}$ from Lemma 4.1.6, and the third one is the LT-isomorphism. By inserting the definitions we obtain the explicit formula

$$\mathfrak{M}_{\mathrm{LT}}(\lambda)(z) = \lambda(\kappa_z \circ \chi_{\mathrm{LT}}).$$

The following commutative diagram relates the pairing in [74, Lem. 4.6],

$$\{\,,\,\} : \mathcal{O}_K(\mathbf{B}) \times C^{\mathrm{an}}(o_L, K) \to K$$
$$(F, f) \mapsto \mu(f),$$

where $\mu \in D(o_L, K)$ is such that $\mu(\kappa_z) = F(z)$, to the construction of the above map $\mathfrak{M}_{\mathrm{LT}}$:

$$
\begin{array}{ccccc}
D(\Gamma_L, K) & \times & C^{\mathrm{an}}(\Gamma_L, K) & \xrightarrow{(\lambda, f) \mapsto \lambda(f)} & K \\
\downarrow{\scriptstyle\mathfrak{M}_{\mathrm{LT}}} & & \downarrow{\scriptstyle(\chi_{\mathrm{LT}})_*} & & \| \\
\mathcal{O}_K(\mathbf{B})^{\psi_L = 0} & \times & C^{\mathrm{an}}(o_L^\times, K) & & \| \\
\downarrow{\scriptstyle\subseteq} & & \downarrow{\scriptstyle\text{extension by } \mathbf{0}} & & \| \\
\mathcal{O}_K(\mathbf{B}) & \times & C^{\mathrm{an}}(o_L, K) & \xrightarrow{\{\,,\}} & K.
\end{array}
$$

Remark 4.1.20 ($\Omega \in K$). For any $F \in \mathcal{O}_K(\mathbf{B})^{\psi_L = 0}$ and any $f \in C^{\mathrm{an}}(o_L, K)$ such that $f|o_L^\times = 0$ we have $\{F, f\} = 0$.

Proof. We have seen above that under the LT-isomorphism ψ_L corresponds, up to a non-zero constant, to $\psi_L^{\mathfrak{X}}$ and hence further under the Fourier isomorphism to ψ_L^D. It therefore suffices to show that for any $\mu \in D(o_L, K)^{\psi_L^D = 0}$ we have $\mu(f) = 0$. For this we define $\tilde{f} := f(\pi_L -) \in C^{\mathrm{an}}(o_L, K)$ and note that $(\pi_L)_!(\tilde{f}) = f$. By the definition of ψ_L^D we therefore obtain, under our assumption on μ, that

$$\mu(f) = \mu(f) - \psi_L^D(\mu)(\tilde{f}) = \mu\big(f - (\pi_L)_!(\tilde{f})\big) = \mu(0) = 0. \qquad \blacksquare$$

Lemma 4.1.21 ($\Omega \in K$). *For any $F \in \mathcal{O}_K(\mathbf{B})^{\psi_L = 1}$ and $n \geq 0$ we have*

$$\mathfrak{M}_{\mathrm{LT}}^{-1}\big((1 - \tfrac{\pi_L}{q}\varphi_L)F\big)(\chi_{\mathrm{LT}}^n) = \Omega^{-n}\Big(1 - \tfrac{\pi_L^{n+1}}{q}\Big)(\partial_{\mathrm{inv}}^n F)_{|Z=0}.$$

Proof. Note that $(1 - \tfrac{\pi_L}{q}\varphi_L)F$ belongs to $\mathcal{O}_K(\mathbf{B})^{\psi_L = 0}$. Let $\mathrm{inc}_! \in C^{\mathrm{an}}(o_L, K)$ denote the extension by zero of the inclusion $o_L^\times \subseteq o_L$, and let $\mathrm{id} : o_L \to K$ be the identity function. Using the above commutative diagram, the assertion reduces to the equality

$$\big\{(1 - \tfrac{\pi_L}{q}\varphi_L)F, \mathrm{inc}_!^n\big\} = \Omega^{-n}\Big(1 - \tfrac{\pi_L^{n+1}}{q}\Big)(\partial_{\mathrm{inv}}^n F)_{|Z=0}.$$

By Remark 4.1.20 we may replace on the left-hand side the function $\mathrm{inc}_!^n$ by the function id^n. Next we observe that $x\,\mathrm{id}^n(x) = \mathrm{id}^{n+1}(x)$. Hence, by [74, Lem. 4.6 (8)],

i.e., $\{F, xf(x)\} = \{\Omega^{-1}\partial_{\mathrm{inv}}F, f\}$, and induction, the left-hand side is equal to

$$\{(1 - \tfrac{\pi_L}{q}\varphi_L)F, \mathrm{id}^n\}$$
$$= \{\Omega^{-n}\partial_{\mathrm{inv}}^n\big((1 - \tfrac{\pi_L}{q}\varphi_L)F\big), \mathrm{id}^0\}$$
$$= \{\Omega^{-n}\big(1 - \tfrac{\pi_L^{n+1}}{q}\varphi_L\big)(\partial_{\mathrm{inv}}^n F), \mathrm{id}^0\} \quad \text{since } \partial_{\mathrm{inv}}\varphi_L = \pi_L\varphi_L\partial_{\mathrm{inv}} \text{ by (2.3)}$$
$$= \Omega^{-n}\big(1 - \tfrac{\pi_L^{n+1}}{q}\varphi_L\big)(\partial_{\mathrm{inv}}^n F)_{|Z=0} \quad \text{since } \mathrm{id}^0 \text{ is the trivial character of } o_L$$
$$= \Omega^{-n}\big(1 - \tfrac{\pi_L^{n+1}}{q}\big)(\partial_{\mathrm{inv}}^n F)_{|Z=0} \quad \text{since } [\pi_L](0) = 0. \qquad \blacksquare$$

In the course of the previous proof we have seen that, for F in $\mathcal{O}_K(\mathbf{B})^{\psi_L=0}$ and $n \geq 0$,

$$\mathfrak{M}_{\mathrm{LT}}^{-1}(F)(\chi_{\mathrm{LT}}^n) = \Omega^{-n}(\partial_{\mathrm{inv}}^n F)_{|Z=0}. \tag{4.8}$$

Lemma 4.1.22 ($\Omega \in K$). *For any $F \in \mathcal{O}_K(\mathbf{B})^{\psi_L=0}$ and $n \geq 1$ we have*

$$\mathfrak{M}_{\mathrm{LT}}^{-1}(\log_{\mathrm{LT}} \cdot F)(\chi_{\mathrm{LT}}^n) = n\Omega^{-1}\mathfrak{M}_{\mathrm{LT}}^{-1}(F)(\chi_{\mathrm{LT}}^{n-1}).$$

Proof. First, using (2.3), observe that

$$\psi_L(\log_{\mathrm{LT}} \cdot F) = \psi_L(\pi_L^{-1}\varphi_L(\log_{\mathrm{LT}}) \cdot F) = \pi_L^{-1}\varphi_L(\log_{\mathrm{LT}})\psi_L(F) = 0.$$

Secondly, note that $\partial_{\mathrm{inv}} \log_{\mathrm{LT}} = 1$, i.e., $\partial_{\mathrm{inv}}^i \log_{\mathrm{LT}} = 0$ for $i \geq 2$; also $\log_{\mathrm{LT}}(0) = 0$. Using (4.8) twice, we have

$$\mathfrak{M}_{\mathrm{LT}}^{-1}(\log_{\mathrm{LT}} F)(\chi_{\mathrm{LT}}^n) = \Omega^{-n}\big(\partial_{\mathrm{inv}}^n(\log_{\mathrm{LT}} F)\big)_{|Z=0}$$
$$= \Omega^{-n}\Big(\sum_{i+j=n} \binom{n}{i}(\partial_{\mathrm{inv}}^i \log_{\mathrm{LT}})(\partial_{\mathrm{inv}}^j F)\Big)_{|Z=0}$$
$$= \Omega^{-n}n(\partial_{\mathrm{inv}}^{n-1}F)_{|Z=0}$$
$$= n\Omega^{-1}\mathfrak{M}^{-1}(F)(\chi_{\mathrm{LT}}^{n-1}). \qquad \blacksquare$$

For the rest of this section we *assume* not only that K contains Ω but also that the action of G_L on $\mathbb{C}_p$ leaves K invariant.

The LT-isomorphism is a topological ring isomorphism

$$K \,\hat{\otimes}_L\, \mathcal{O}_L(\mathfrak{X}) = \mathcal{O}_K(\mathfrak{X}) \cong \mathcal{O}_K(\mathbf{B}) = K \,\hat{\otimes}_L\, \mathcal{O}_L(\mathbf{B})$$

(cf. [8, Prop. 2.1.5 (ii)] for the outer identities).

On both sides we have the obvious coefficientwise G_L-action induced by the Galois-action on the tensor-factor K. We use the following notation:

- $\sigma \in G_L$ acting *coefficientwise* on $\mathcal{O}_K(\mathbf{B})$ is denoted by: $F \mapsto {}^\sigma F$; the corresponding fixed ring is $\mathcal{O}_K(\mathbf{B})^{G_L} = \mathcal{O}_L(\mathbf{B})$.

- The coefficientwise action on $\mathcal{O}_K(\mathfrak{X})$ transfers to the *twisted action* on $\mathcal{O}_K(\mathbf{B})$ by [74, before Cor. 3.8] given as $F \mapsto^{\sigma *} F := {}^\sigma F \circ [\tau(\sigma^{-1})]$; the corresponding fixed ring is $\mathcal{O}_K(\mathbf{B})^{G_L,*} = \mathcal{O}_L(\mathfrak{X}) = D(o_L, L)$.

Remark 4.1.23. Note that the $o_L \setminus \{0\}$-action and hence the $D(o_L^\times, L)$-module structure commute with both G_L-actions. Moreover, ψ_L commutes with the G_L-actions as well.

Recall that using the notation from [74, Lem. 4.6 (1), (2)] the function

$$1 + F_{a\eta'}(Z) = \exp\big(a\Omega_{\eta'} \log_{\mathrm{LT}}(Z)\big)$$

corresponds to the Dirac distribution δ_a of $a \in o_L$ under the Fourier isomorphism.

Lemma 4.1.24. *Let σ be in G_L, $t' \in T_\pi'$ and $a \in o_L$. Then*

(i) $\sigma(\Omega_{t'}) = \Omega_{\tau(\sigma)t'} = \Omega_{t'}\tau(\sigma)$ *and*

(ii) ${}^\sigma F_{a\eta'} = F_{a\eta'} \circ [\tau(\sigma)] = F_{a\tau(\sigma)\eta'}$.

Proof. (i) The Galois equivariance of the pairing $(\,,\,) : T_\pi' \otimes_{o_L} \mathbb{C}_p \to \mathbb{C}_p$ from [74, before Prop. 3.1] with $(t', x) = \Omega_{t'}x$ implies that

$$\sigma(\Omega_{t'}) = \Omega_{\sigma(t')} = \Omega_{\tau(\sigma)t'},$$

while the o_L-invariance of that pairing implies that the latter expression equals $\Omega_{t'}\tau(\sigma)$.

(ii) This is immediate from (i) and the definition of F_a taking equation (2.3) into account.   ∎

Proposition 4.1.25. *We equip the distribution rings $D(o_L, K)$ and $D(o_L^\times, K)$ with the G_L-action which is induced from the coefficientwise action on $\mathcal{O}_K(\mathbf{B})$ and $\mathcal{O}_K(\mathbf{B})^{\psi_L=0}$ via the LT-isomorphism and Mellin transform, respectively. Then the following holds true:*

(i)   *The isomorphism $\mathfrak{L} : D(o_L, K) \cong \mathcal{O}_K(\mathfrak{X}) \cong \mathcal{O}_K(\mathbf{B})$ composed of LT together with Fourier restricts to an isomorphism*

$$D(o_L, K)^{G_L, \tilde{*}} = \mathcal{O}_K(\mathfrak{X})^{G_L} \cong \mathcal{O}_K(\mathbf{B})^{G_L} = \mathcal{O}_L(\mathbf{B})$$

of $D(o_L^\times, L)$-modules.

(ii)   *The Mellin transform restricts to an isomorphism of $D(o_L^\times, L)$-modules*

$$D(o_L^\times, K)^{G_L,*} = \mathcal{O}_K(\mathfrak{X})^{G_L, \psi_L=0} \cong \mathcal{O}_L(\mathbf{B})^{\psi_L=0}.$$

Proof. (i) and (ii) follow from passing to the fixed vectors with respect to the coefficientwise G_L-action and Remark 4.1.23.   ∎

In order to express the $D(o_L^\times, L)$-module $D(o_L^\times, K)^{G_L, *}$ in the above proposition more explicitly, we describe the previous two actions on $\mathcal{O}_K(\mathbf{B})$ now on $D(o_L, K)$:

- The coefficientwise G_L-action on $D(o_L, K) = K \hat{\otimes}_L D(o_L, L)$, which corresponds to the twisted action on $\mathcal{O}_K(\mathbf{B})$, will be written as $\lambda \mapsto {}^\sigma\lambda$.

- The G_L-action given by $\lambda \mapsto \tau(\sigma)_*({}^\sigma\lambda)$ corresponds to the coefficientwise action on $\mathcal{O}_K(\mathbf{B})$.

Note that for $\lambda \in D(o_L^\times, K)$ we have $\tau(\sigma)_*(\lambda) = \delta_{\tau(\sigma)}\lambda$, where the right-hand side refers to the product of λ and the Dirac distribution $\delta_{\tau(\sigma)}$ in the ring $D(o_L^\times, K)$. Then we conclude that

$$D(o_L^\times, K)^{G_L, *} = \left\{ \lambda \in D(o_L^\times, K) \mid {}^\sigma\lambda = \delta_{\tau(\sigma)^{-1}}\lambda \text{ for all } \sigma \in G_L \right\}.$$

4.2 Consequences of Serre duality

Recall that in any quasi-separated rigid analytic variety the complement of any affinoid subdomain is admissible open (cf. [70, §3 Prop. 3 (ii)]). This applies in particular to quasi-Stein spaces since they are separated by definition. For a rigid analytic variety $\mathfrak{Y}$ we will denote by $\mathrm{Aff}(\mathfrak{Y})$ the set of all affinoid subdomains of $\mathfrak{Y}$.

We have seen that $\mathfrak{X}$, $\mathfrak{X}^\times$, and $\mathfrak{X}_n^\times$ for $n \geq 1$ all are one-(equi)dimensional smooth Stein spaces.

4.2.1 Cohomology with compact support

We slightly rephrase the definition of cohomology with compact support given in [90, §1] in the case of a Stein space $\mathfrak{Y}$ over L. For any abelian sheaf $\mathcal{F}$ on $\mathfrak{Y}$ and any $\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})$ we put

$$H_{\mathfrak{U}}^0(\mathcal{Y}, \mathfrak{F}) := \ker\left(\mathcal{F}(\mathfrak{Y}) \to \mathcal{F}(\mathfrak{Y} \setminus \mathfrak{U})\right).$$

This is a left exact functor in $\mathcal{F}$, and we denote by $H_{\mathfrak{U}}^*(\mathfrak{Y}, \mathcal{F})$ its right derived functors. Since quasi-Stein spaces have no coherent cohomology, the relative cohomology sequence (cf. [90, Lem. 1.3]) gives rise, for a coherent sheaf $\mathcal{F}$, to the exact sequence

$$0 \to H_{\mathfrak{U}}^0(\mathfrak{Y}, \mathcal{F}) \to \mathcal{F}(\mathfrak{Y}) \to \mathcal{F}(\mathfrak{Y} \setminus \mathfrak{U}) \to H_{\mathfrak{U}}^1(\mathfrak{Y}, \mathcal{F}) \to 0. \qquad (4.9)$$

We then define the cohomology with compact support as

$$H_c^*(\mathfrak{Y}, \mathcal{F}) := \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})} H_{\mathfrak{U}}^*(\mathfrak{Y}, \mathcal{F}).$$

Again, if $\mathcal{F}$ is a coherent sheaf, we obtain the exact sequence

$$0 \to H_c^0(\mathfrak{Y}, \mathcal{F}) \to \mathcal{F}(\mathfrak{Y}) \to \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})} \mathcal{F}(\mathfrak{Y} \setminus \mathfrak{U}) \to H_c^1(\mathfrak{Y}, \mathcal{F}) \to 0.$$

Suppose in the following that $j : \mathfrak{Y}_0 \to \mathfrak{Y}$ is an open immersion of Stein spaces (over L) which, for simplicity, we view as an inclusion.

Lemma 4.2.1. *For any* $\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y}_0)$ *the covering* $\mathfrak{Y} = (\mathfrak{Y} \setminus \mathfrak{U}) \cup \mathfrak{Y}_0$ *is admissible.*

Proof. This follows from [90, Lem. 1.1]. ∎

Lemma 4.2.2. *For any* $\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y}_0)$ *and any sheaf* $\mathcal{F}$ *on* $\mathfrak{Y}$ *the natural map*

$$H_{\mathfrak{U}}^*(\mathfrak{Y}, \mathcal{F}) \xrightarrow[\cong]{\mathrm{res}} H_{\mathfrak{U}}^*(\mathfrak{Y}_0, \mathcal{F})$$

is bijective.

Proof. Recall that for an injective sheaf on $\mathfrak{Y}$ its restriction to $\mathfrak{Y}_0$ is injective as well. Hence, by using an injective resolution, it suffices to proof the assertion for $* = 0$. *Injectivity*: Let $f \in H_{\mathfrak{U}}^0(\mathfrak{Y}, \mathcal{F})$ such that $f|\mathfrak{Y}_0 = 0$. Since $f|\mathfrak{Y} \setminus \mathfrak{U} = 0$ as well, it follows from Lemma 4.2.1 that $f = 0$. *Surjectivity*: Let $g \in H_{\mathfrak{U}}^0(\mathfrak{Y}_0, \mathcal{F})$ so that $g|\mathfrak{Y}_0 \setminus \mathfrak{U} = 0$. Using Lemma 4.2.1 again, we may define a preimage f of g by $f|\mathfrak{Y} \setminus \mathfrak{U} := 0$ and $f|\mathfrak{Y}_0 := g$. ∎

By passing to inductive limits we obtain the composed map

$$
\begin{aligned}
j_! : H_c^*(\mathfrak{Y}_0, \mathcal{F}) \\
= \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y}_0)} H_{\mathfrak{U}}^*(\mathfrak{Y}_0, \mathcal{F}) \xrightarrow[\cong]{\mathrm{res}^{-1}} \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y}_0)} H_{\mathfrak{U}}^*(\mathfrak{Y}, \mathcal{F}) \to \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})} H_{\mathfrak{U}}^*(\mathfrak{Y}, \mathcal{F}) \\
= H_c^*(\mathfrak{Y}, \mathcal{F}).
\end{aligned}
$$

For later purposes we have to analyze the following situation. Let

$$\mathfrak{U}_1 \subseteq \mathfrak{U}_2 \subseteq \cdots \subseteq \mathfrak{U}_n \subseteq \cdots \subseteq \mathfrak{Y} = \bigcup_n \mathfrak{U}_n$$

be a Stein covering. We *assume* that each admissible open subset $\mathfrak{Y}_n := \mathfrak{Y} \setminus \mathfrak{U}_n$ also is a Stein space. Since $\cdots \subseteq \mathfrak{Y}_n \subseteq \cdots \subseteq \mathfrak{Y}_1 \subseteq \mathfrak{Y}$, we then have the projective system

$$\cdots \to H_c^*(\mathfrak{Y}_n, \mathcal{F}) \to \cdots \to H_c^*(\mathfrak{Y}_1, \mathcal{F}) \to H_c^*(\mathfrak{Y}, \mathcal{F}).$$

By Lemma 4.2.2 we may rewrite it as the projective system

$$\cdots \to \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y}_n)} H_{\mathfrak{U}}^*(\mathfrak{Y}, \mathcal{F}) \to \cdots \to \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y}_1)} H_{\mathfrak{U}}^*(\mathfrak{Y}, \mathcal{F}) \to \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})} H_{\mathfrak{U}}^*(\mathfrak{Y}, \mathcal{F}).$$

$$(4.10)$$

Lemma 4.2.3. *In the above situation we assume in addition that $\mathcal{F}$ is a coherent sheaf and that the restriction maps $\mathcal{F}(\mathfrak{Y}) \to \mathcal{F}(\mathfrak{Y} \setminus \mathfrak{U})$ for any $\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})$ are injective. We then have*

$$\varprojlim_n H_c^1(\mathfrak{Y}_n, \mathfrak{F}) = \left(\varprojlim_n \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y}_n)} \mathfrak{F}(\mathfrak{Y} \setminus \mathfrak{U}) \right) / \mathcal{F}(\mathfrak{Y}). \tag{4.11}$$

Proof. This is immediate from (4.10) and the relative cohomology sequence. ∎

For coherent sheaves $\mathcal{F}$ all the above cohomology vector spaces carry natural locally convex topologies, which we briefly recall. The global sections $\mathcal{F}(\mathfrak{Y})$ and $\mathcal{F}(\mathfrak{Y} \setminus \mathfrak{U})$, for $\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})$, are Fréchet spaces. Using the relative cohomology sequence (4.9), we equip $H_{\mathfrak{U}}^1(\mathfrak{Y}, \mathcal{F})$ with the quotient topology from $\mathcal{F}(\mathfrak{Y} \setminus \mathfrak{U})$ (which might be non-Hausdorff) and then $H_c^1(\mathfrak{Y}, \mathcal{F})$ with the locally convex inductive limit topology (with regard to varying $\mathfrak{U}$).

Remark 4.2.4. Let

$$\begin{array}{ccc} M^0 & \xrightarrow{\alpha} & M^1 \\ \downarrow & & \downarrow \\ N^0 & \xrightarrow{\beta} & N^1 \end{array}$$

be a commutative diagram of Fréchet spaces such that the induced map $\mathrm{coker}(\alpha) \to \mathrm{coker}(\beta)$ is bijective; then the latter map is a topological isomorphism for the quotient topologies.

Proof. A more general statement can be found in [2, Chap. VII, Lem. 1.32]. ∎

Using this remark, we see that the bijection $H_{\mathfrak{U}}^1(\mathfrak{Y}, \mathcal{F}) \xrightarrow[\cong]{\mathrm{res}} H_{\mathfrak{U}}^1(\mathfrak{Y}_0, \mathcal{F})$ from Lemma 4.2.2 is a topological isomorphism. It follows that, in degree one at least, the above map $j_!$ is as well as the transition maps in the projective system (4.10) are continuous. Hence, both sides of (4.11) carry natural locally convex topologies by performing the operations $\varprojlim$ and $\varinjlim$ in the category of locally convex vector spaces.

Lemma 4.2.5. *The isomorphism (4.11) is topological.*

Proof. First of all, one checks that forming the projective limit on the right-hand side commutes with passing to the quotient space by $\mathcal{F}(\mathfrak{Y})$ (compare (ii) in the proof of [63, Thm. 4.3] for a more general statement). Secondly, as a special case of [15, II.28 Cor. 2], forming inductive limits commutes with passing to quotient spaces. This reduces us to $H_{\mathfrak{U}}^1(\mathfrak{Y} \setminus \mathfrak{U}, \mathcal{F}) = \mathcal{F}(\mathfrak{Y} \setminus \mathfrak{U})/\mathcal{F}(\mathfrak{Y})$ being a topological isomorphism, but which holds by definition. ∎

In the following, we compute (4.11) further in two concrete cases.

The open unit disk

Let $\mathbf{B} = \mathbf{B}_{[0,1)}$ denote the open unit disk over L. We recall our convention that all radii are assumed to lie in $(0, 1) \cap p^{\mathbb{Q}}$. For any radii $r \leq s$ we introduce the affinoid disk $\mathbf{B}_{[0,r]}$ as well as the open disk $\mathbf{B}_{[0,r)}$ of radius r around 0 and the affinoid annulus $\mathbf{B}_{[r,s]} := \{r \leq |x| \leq s\}$. We put $\mathbf{B}_{(r,1)} := \mathbf{B} \setminus \mathbf{B}_{[0,r]}$, which are Stein spaces. By the identity theorem for Laurent series the assumptions of Lemma 4.2.3 are satisfied for the structure sheaf $\mathcal{O} = \mathcal{O}_{\mathbf{B}}$ of $\mathbf{B}$. We first fix a radius r and compute the cohomology $H_c^1(\mathbf{B}_{(r,1)}, \mathcal{O})$. By Lemma 4.2.2 and the relative cohomology sequence we have

$$H_c^1(\mathbf{B}_{(r,1)}, \mathcal{O}) = \varinjlim_{\mathcal{U} \in \mathrm{Aff}(\mathbf{B}_{(r,1)})} H_{\mathcal{U}}^*(\mathbf{B}, \mathcal{O}) = \big(\varinjlim_{\mathcal{U} \in \mathrm{Aff}(\mathbf{B}_{(r,1)})} \mathcal{O}(\mathbf{B} \setminus \mathcal{U}) \big)/\mathcal{O}(\mathbf{B}).$$

Of course, it suffices to take the inductive limit over a cofinal sequence of larger and larger affinoid annuli in $\mathbf{B}_{(r,1)}$. For this we choose two sequences of radii $r < \cdots < r_m < \cdots < r_1, s_1 < \cdots < s_m < \cdots < 1$ with $(r_m)_m$ and $(s_m)_m$ converging to r and 1, respectively. Each space $\mathbf{B} \setminus \mathbf{B}_{[r_m,s_m]} = \mathbf{B}_{(s_m,1)} \dot\cup \mathbf{B}_{[0,r_m)}$ has two connected components. We see that

$$H_c^1(\mathbf{B}_{(r,1)}, \mathcal{O}) = \big(\varinjlim_{m \to \infty} \mathcal{O}(\mathbf{B}_{(s_m,1)}) \oplus \varinjlim_{m \to \infty} \mathcal{O}(\mathbf{B}_{[0,r_m)}) \big)/\mathcal{O}(\mathbf{B}).$$

As explained in Lemma 4.2.5 this is a topological equality. We observe that

$$\mathcal{R}_L = \mathcal{R}_L(\mathbf{B}) = \varinjlim_{m \to \infty} \mathcal{O}(\mathbf{B}_{(s_m,1)})$$

is the usual Robba ring (over L), whereas $\mathcal{O}^{\dagger}(\mathbf{B}_{[0,r]}) = \varinjlim_{m \to \infty} \mathcal{O}(\mathbf{B}_{[0,r_m)})$ is the ring of overconvergent analytic functions on $\mathbf{B}_{[0,r]}$. Hence,

$$H_c^1(\mathbf{B}_{(r,1)}, \mathcal{O}) = \big(\mathcal{R}_L \oplus \mathcal{O}^{\dagger}(\mathbf{B}_{[0,r]}) \big)/\mathcal{O}(\mathbf{B}).$$

Passing now to the projective limit with regard to $r \to 1$ of the continuous restriction maps $\mathcal{O}(\mathbf{B}) \to \mathcal{O}^{\dagger}(\mathbf{B}_{[0,r]}) \to \mathcal{O}(\mathbf{B}_{[0,r]})$, we observe that $\varprojlim_{r \to 1} \mathcal{O}^{\dagger}(\mathbf{B}_{[0,r]}) = \mathcal{O}(\mathbf{B})$ holds true topologically. We finally deduce that

$$\varprojlim_{r \to 1} H_c^1(\mathbf{B}_{(r,1)}, \mathcal{O}) = \varprojlim_{r \to 1} \big(\varinjlim_{\mathcal{U} \in \mathrm{Aff}(\mathbf{B}_{(r,1)})} \mathcal{O}(\mathbf{B} \setminus \mathcal{U}) \big)/\mathcal{O}(\mathbf{B}) \qquad (4.12)$$
$$= \big(\mathcal{R}_L \oplus \mathcal{O}(\mathbf{B}) \big)/\mathcal{O}(\mathbf{B}) \cong \mathcal{R}_L$$

as locally convex vector spaces.

The character variety $\mathfrak{X}$

Since $\mathfrak{X}_{/\mathbb{C}_p} \cong \mathbf{B}_{/\mathbb{C}_p}$ by [74], the injectivity of the restriction maps $\mathcal{O}(\mathfrak{X}) \to \mathcal{O}(\mathfrak{X} \setminus \mathcal{U})$ for any $\mathcal{U} \in \mathrm{Aff}(\mathfrak{X})$ follows from the corresponding fact for $\mathbf{B}$, which we saw already. According to Proposition 4.1.9, the admissible open subdomains $\mathfrak{X}_{(s_n,1)}$ of $\mathfrak{X}$ are Stein

spaces. In order to compute their cohomology with compact support in the structure sheaf $\mathcal{O} = \mathcal{O}_{\mathfrak{X}}$, we fix an $n \geq 0$. We choose a sequence of radii $r_{n+1} > r_{n+2} > \cdots > s_n$ in S_n converging to s_n. Furthermore, we observe that the increasing sequence $(s_m)_{m>n}$ in S_∞ converges to 1. By Proposition 4.1.7 we then have the Stein covering

$$\mathfrak{X}_{(s_n,1)} = \bigcup_{m>n} \mathfrak{X}_{[r_m, s_m]}.$$

Hence, by Lemma 4.2.2, the relative cohomology sequence, and the explanation in Lemma 4.2.5, we have the topological equality

$$H_c^1(\mathfrak{X}_{(s_n,1)}, \mathfrak{O}) = \left(\varinjlim_{m \to \infty} \mathcal{O}(\mathfrak{X} \setminus \mathfrak{X}_{[r_m, s_m]}) \right)/\mathcal{O}(\mathfrak{X}).$$

The obvious set theoretic decomposition

$$\mathfrak{X} \setminus \mathfrak{X}_{[r_m, s_m]} = \mathfrak{X} \setminus \left(\mathfrak{X}(s_m) \setminus \mathfrak{X}^-(r_m) \right) = \left(\mathfrak{X} \setminus \mathfrak{X}(s_m) \right) \dot{\cup} \mathfrak{X}^-(r_m) = \mathfrak{X}_{(s_m,1)} \dot{\cup} \mathfrak{X}^-(r_m)$$

is in fact the decomposition of the space $\mathfrak{X} \setminus \mathfrak{X}_{[r_m, s_m]}$ into its connected components. This can be checked after base change to $\mathbb{C}_p$, where, by [8, Prop. 1.20 and proof of Prop. 2.1], the setting becomes isomorphic to the setting for the open unit disk which we discussed in the previous section. Entirely in the same way as in the previous subsection, it follows now that

$$H_c^1(\mathfrak{X}_{(s_n,1)}, \mathfrak{O}) = \left(\mathcal{R}_L(\mathfrak{X}) \oplus \mathcal{O}^\dagger(\mathfrak{X}(s_n)) \right)/\mathcal{O}(\mathfrak{X}),$$

where $\mathfrak{O}^\dagger(\mathfrak{X}(s_n)) := \varinjlim_{m \to \infty} \mathfrak{X}^-(r_m)$, and then

$$\varprojlim_{n \to \infty} H_c^1(\mathfrak{X}_{(s_n,1)}, \mathcal{O}) = \varprojlim_{n \to \infty} \left(\varinjlim_{m \to \infty} \mathcal{O}(\mathfrak{X} \setminus \mathfrak{X}_{[r_m, s_m]}) \right)/\mathcal{O}(\mathfrak{X})$$
$$= \left(\mathcal{R}_L(\mathfrak{X}) \oplus \mathcal{O}(\mathfrak{X}) \right)/\mathcal{O}(\mathfrak{X}) \cong \mathcal{R}_L(\mathfrak{X}) \quad (4.13)$$

as locally convex vector spaces.

4.2.2 Serre duality for Stein spaces

In the following, the continuous dual of a locally convex vector space is always equipped with the strong topology.

The Serre duality for smooth Stein spaces is established in [11, 20]. Let $\mathfrak{Y}$ be a one-(equi)dimensional smooth Stein space over L.

Theorem 4.2.6. *For any coherent sheaf $\mathcal{F}$ on $\mathfrak{Y}$ we have the following:*

(i) $H_c^1(\mathfrak{Y}, \mathcal{F})$ *is a complete reflexive Hausdorff space.*

(ii) $\mathrm{Hom}_{\mathfrak{Y}}(\mathcal{F}, \Omega_{\mathfrak{Y}}^1) = H^0(\mathfrak{Y}, \underline{\mathrm{Hom}}_{\mathfrak{Y}}(\mathcal{F}, \Omega_{\mathfrak{Y}}^1))$, *being the global sections of another coherent sheaf, is a reflexive Fréchet space strictly of countable type (cf. [61, Def. 4.2.3]).*

(iii)   *There is a canonical trace map*

$$\mathrm{tr}_{\mathfrak{Y}} : H^1_c(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}}) \to L$$

such that the Yoneda pairing

$$H^1_c(\mathfrak{Y}, \mathcal{F}) \times \mathrm{Hom}_{\mathfrak{Y}}(\mathcal{F}, \Omega^1_{\mathfrak{Y}}) \to H^1_c(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}})$$

composed with the trace map induces isomorphisms of topological vector spaces

$$\mathrm{Hom}^{\mathrm{cont}}_L(H^1_c(\mathfrak{Y}, \mathcal{F}), L) \xrightarrow{\cong} \mathrm{Hom}_{\mathfrak{Y}}(\mathcal{F}, \Omega^1_{\mathfrak{Y}}),$$
$$\mathrm{Hom}^{\mathrm{cont}}_L(\mathrm{Hom}_{\mathfrak{Y}}(\mathcal{F}, \Omega^1_{\mathfrak{Y}}), L) \xrightarrow{\cong} H^1_c(\mathfrak{Y}, \mathcal{F})$$

which are natural in $\mathcal{F}$.

Proof. See [11, Thm. 7.1][2] and [20, Thm. 4.21] (as well as [90, Prop. 3.6]). ∎

In the special case of the structure sheaf $\mathcal{F} = \mathcal{O}_{\mathfrak{Y}}$, the above assertion gives us $\mathrm{Hom}_{\mathfrak{Y}}(\mathcal{O}_{\mathfrak{Y}}, \Omega^1_{\mathfrak{Y}}) = \Omega^1_{\mathfrak{Y}}(\mathfrak{Y})$ for trivial reasons. On the other hand, the relative cohomology sequence implies that

$$H^1_c(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) = \mathcal{R}_L(\mathfrak{Y})/\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}). \tag{4.14}$$

Hence, we have the following consequence of Serre duality.

Corollary 4.2.7. *Serre duality gives rise to an isomorphism of topological vector spaces*
$$\mathrm{Hom}^{\mathrm{cont}}_L\left(\mathfrak{R}_L(\mathfrak{Y})/\mathfrak{D}_{\mathfrak{Y}}(\mathfrak{Y}), L\right) \xrightarrow{\cong} \Omega^1_{\mathfrak{Y}}(\mathfrak{Y}).$$

Lemma 4.2.8. *Let* $\alpha : \mathfrak{Y} \to \mathfrak{Y}'$ *be a finite étale morphism of one-dimensional smooth Stein spaces over* L. *We then have, for any coherent sheaf* $\mathcal{F}$ *on* $\mathfrak{Y}'$, *the commutative diagram of Serre duality pairings*

$$
\begin{array}{ccccccc}
H^1_c(\mathfrak{Y}, \alpha^*\mathcal{F}) & \times & \mathrm{Hom}_{\mathfrak{Y}}(\alpha^*\mathcal{F}, \Omega^1_{\mathfrak{Y}}) & \longrightarrow & H^1_c(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}}) & \xrightarrow{\mathrm{tr}_{\mathfrak{Y}}} & L \\
\cong \uparrow & & \downarrow & & \downarrow & & \| \\
H^1_c(\mathfrak{Y}', \alpha_*\alpha^*\mathcal{F}) & \times & \mathrm{Hom}_{\mathfrak{Y}'}(\alpha_*\alpha^*\mathcal{F}, \Omega^1_{\mathfrak{Y}'}) & \longrightarrow & H^1_c(\mathfrak{Y}', \Omega^1_{\mathfrak{Y}'}) & \xrightarrow{\mathrm{tr}_{\mathfrak{Y}'}} & L \\
\uparrow & & \downarrow & & \| & & \| \\
H^1_c(\mathfrak{Y}', \mathcal{F}) & \times & \mathrm{Hom}_{\mathfrak{Y}'}(\mathcal{F}, \Omega^1_{\mathfrak{Y}'}) & \longrightarrow & H^1_c(\mathfrak{Y}', \Omega^1_{\mathfrak{Y}'}) & \xrightarrow{\mathrm{tr}_{\mathfrak{Y}'}} & L.
\end{array}
$$

---

[2]This reference depends on the results in article [10], which unfortunately contains the following gaps. Firstly, in the proof of Lemma 4.2.2, Beyer quotes a result of Bosch concerning the connectedness of formal fibers without verifying the required assumptions. This is repaired by [54, Thm. 22, Cor. 23]. Secondly, Beyer claims implicitly and without proof that *special affinoid wide-open spaces* are *affinoid wide-open spaces* in the sense of [10, Def. 4.1.1, Rem. 4.1.2]. This crucial ingredient has now been shown explicitly in [54, §2.5].

Proof. The vertical arrows in the lower part of the diagram are induced by the adjunction homomorphism $\mathscr{F} \to \alpha_* \alpha^* \mathscr{F}$. It is commutative by the naturality of Serre duality in the coherent sheaf.

For the upper part we consider more generally a coherent sheaf $\mathscr{G}$ on $\mathfrak{Y}$ and check the commutativity of the diagram

$$
\begin{array}{ccccccc}
H_c^1(\mathfrak{Y}, \mathscr{G}) & \times & \mathrm{Hom}_{\mathfrak{Y}}(\mathscr{G}, \Omega_{\mathfrak{Y}}^1) & \longrightarrow & H_c^1(\mathfrak{Y}, \Omega_{\mathfrak{Y}}^1) & \xrightarrow{\mathrm{tr}_{\mathfrak{Y}}} & L \\
\cong \uparrow & & \downarrow {\scriptstyle f \mapsto \alpha_*(f)} & & \cong \uparrow & & \| \\
H_c^1(\mathfrak{Y}', \alpha_* \mathscr{G}) & \times & \mathrm{Hom}_{\mathfrak{Y}'}(\alpha_* \mathscr{G}, \alpha_* \Omega_{\mathfrak{Y}}^1) & \longrightarrow & H_c^1(\mathfrak{Y}', \alpha_* \Omega_{\mathfrak{Y}}^1) & & \| \\
\| & & \downarrow & & \downarrow & & \| \\
H_c^1(\mathfrak{Y}', \alpha_* \mathscr{G}) & \times & \mathrm{Hom}_{\mathfrak{Y}'}(\alpha_* \mathscr{G}, \Omega_{\mathfrak{Y}'}^1) & \longrightarrow & H_c^1(\mathfrak{Y}', \Omega_{\mathfrak{Y}'}^1) & \xrightarrow{\mathrm{tr}_{\mathfrak{Y}'}} & L.
\end{array}
$$

Here the second and third lower vertical arrows are induced by the relative trace map $\mathrm{tr}_\alpha : \alpha_* \Omega_{\mathfrak{Y}}^1 \to \Omega_{\mathfrak{Y}'}^1$ (see below). The commutativity of the Yoneda pairings (before applying the horizontal trace maps) is a trivial consequence of functoriality properties. That the first and third upper vertical arrows are isomorphisms follows from the fact that for a finite morphism the functor α_* is exact on quasi-coherent sheaves. This reduces us to showing that the diagram

$$
\begin{array}{ccc}
 & H_c^1(\mathfrak{Y}, \Omega_{\mathfrak{Y}}^1) & \\
\cong \nearrow & & \searrow {\scriptstyle \mathrm{tr}_{\mathfrak{Y}}} \\
H_c^1(\mathfrak{Y}', \alpha_* \Omega_{\mathfrak{Y}}^1) & & L \\
\searrow {\scriptstyle H_c^1(\mathfrak{Y}', \mathrm{tr}_\alpha)} & & \nearrow {\scriptstyle \mathrm{tr}_{\mathfrak{Y}'}} \\
 & H_c^1(\mathfrak{Y}', \Omega_{\mathfrak{Y}'}^1) &
\end{array}
\tag{4.15}
$$

is commutative.

For the convenience of the reader we briefly explain the definition of the relative trace map tr_α. But first we need to recall that any coherent $\alpha_* \mathcal{O}_{\mathfrak{Y}}$-module $\mathcal{M}$ can naturally be viewed (cf. [35, Prop. I.9.2.5]) as a coherent $\mathcal{O}_{\mathfrak{Y}}$-module $\widetilde{\mathcal{M}}$ such that $\alpha_* \widetilde{\mathcal{M}} = \mathcal{M}$ (for any open affinoid subdomain $\mathfrak{V} \subseteq \mathfrak{Y}'$ one has $\widetilde{\mathcal{M}}(\alpha^{-1}(\mathfrak{V})) = \mathcal{M}(\mathfrak{V})$). Since α is étale, we have (cf. [57, Thm. 25.1]) that

$$
(\alpha_* \mathfrak{D}_{\mathfrak{Y}} \otimes_{\mathfrak{D}_{\mathfrak{Y}'}} \Omega_{\mathfrak{Y}'}^1)^{\sim} \xrightarrow{\cong} \Omega_{\mathfrak{Y}}^1.
$$

Since α is finite flat, the natural map

$$
\underline{\mathrm{Hom}}_{\mathfrak{Y}'}(\alpha_* \mathcal{O}_{\mathfrak{Y}}, \mathcal{O}_{\mathfrak{Y}'}) \otimes_{\mathcal{O}_{\mathfrak{Y}'}} \Omega_{\mathfrak{Y}'}^1 \xrightarrow{\cong} \underline{\mathrm{Hom}}_{\mathfrak{Y}'}(\alpha_* \mathcal{O}_{\mathfrak{Y}}, \Omega_{\mathfrak{Y}'}^1)
$$

is an isomorphism. Finally, since α is finite étale, the usual trace pairing is non-degenerate and induces an isomorphism[3]

$$\underline{\mathrm{Hom}}_{\mathcal{Y}'}(\alpha_* \mathfrak{D}_{\mathcal{Y}}, \mathfrak{D}_{\mathcal{Y}'}) \xrightarrow{\cong} \alpha_* \mathcal{O}_{\mathcal{Y}}.$$

The relative trace map is now defined to be the composite map

$$\alpha_* \Omega^1_{\mathcal{Y}} \cong \alpha_*(\alpha_* \mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{Y}'}} \Omega^1_{\mathcal{Y}'})^{\sim} \cong \alpha_*(\underline{\mathrm{Hom}}_{\mathcal{Y}'}(\alpha_* \mathcal{O}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{Y}'}) \otimes_{\mathcal{O}_{\mathcal{Y}'}} \Omega^1_{\mathcal{Y}'})^{\sim}$$

$$\cong \alpha_* \underline{\mathrm{Hom}}_{\mathcal{Y}'}(\alpha_* \mathcal{O}_{\mathcal{Y}}, \Omega^1_{\mathcal{Y}'})^{\sim} = \underline{\mathrm{Hom}}_{\mathcal{Y}'}(\alpha_* \mathcal{O}_{\mathcal{Y}}, \Omega^1_{\mathcal{Y}'}) \xrightarrow{f \mapsto f(1)} \Omega^1_{\mathcal{Y}'}.$$

The commutativity of (4.15) is shown in detail in [54] and should also be a consequence of [1, Prop. 6.5.1 (2)] upon showing that their general construction boils down to the above description of the relative trace map. ∎

We make the last lemma more explicit for the structure sheaf. Let $\rho : \mathcal{Y} \to \mathfrak{Z}$ be a finite, faithfully flat, and étale morphism of one-dimensional smooth Stein spaces over L such that $\mathcal{O}_{\mathcal{Y}}(\mathcal{Y})$ is finitely generated free as an $\mathcal{O}_{\mathfrak{Z}}(\mathfrak{Z})$-module. Fix a basis $f_1, \ldots, f_h \in \mathcal{O}_{\mathcal{Y}}(\mathcal{Y})$. Going through the proof of Lemma 4.2.8, one checks that on global sections the relative trace map is given by

$$\mathrm{tr}_\rho : \Omega^1_{\mathcal{Y}}(\mathcal{Y}) = \mathcal{O}_{\mathcal{Y}}(\mathcal{Y}) \otimes_{\mathcal{O}_{\mathfrak{Z}}(\mathfrak{Z})} \Omega^1_{\mathfrak{Z}}(\mathfrak{Z}) \to \Omega^1_{\mathfrak{Z}}(\mathfrak{Z})$$

$$\omega = \sum_{i=1}^{h} f_i \otimes \omega_i \mapsto \sum_{i=1}^{h} \mathrm{trace}_{\mathcal{O}_{\mathcal{Y}}(\mathcal{Y})/\mathcal{O}_{\mathfrak{Z}}(\mathfrak{Z})}(f_i)\omega_i. \tag{4.16}$$

Hence, we have the commutative diagram of duality pairings

$$
\begin{array}{ccc}
\mathrm{Hom}^{\mathrm{cont}}_L(\mathcal{R}_L(\mathcal{Y})/\mathcal{O}_{\mathcal{Y}}(\mathcal{Y}), L) & \xrightarrow{\cong} & \Omega^1_{\mathcal{Y}}(\mathcal{Y}) \\
\Big\downarrow {\scriptstyle \mathrm{Hom}(\rho^*, L)} & & \Big\downarrow {\scriptstyle \mathrm{tr}_\rho} \\
\mathrm{Hom}^{\mathrm{cont}}_L(\mathcal{R}_L(\mathfrak{Z})/\mathcal{O}_{\mathfrak{Z}}(\mathfrak{Z}), L) & \xrightarrow{\cong} & \Omega^1_{\mathfrak{Z}}(\mathfrak{Z}).
\end{array}
\tag{4.17}
$$

It remains to explicitly compute the relative trace map in the cases of interest to us. But first we observe that, by the explanation at the end of Section 2.3 in [8], the sheaf of differentials $\Omega^1_{\mathcal{Y}}$ on a smooth one-dimensional Stein group variety $\mathcal{Y}$ is a free $\mathcal{O}_{\mathcal{Y}}$-module. Furthermore, if $\mathcal{Y}$ is one of our character varieties, say of the group G, then by the construction before Definition 1.27 in [8] we have the embedding

$$L = \mathrm{Lie}(G) \to \mathcal{O}_{\mathcal{Y}}(\mathcal{Y})$$
$$\mathfrak{x} \mapsto \big[\chi \mapsto d\chi(\mathfrak{x})\big]$$

and the function $\log_{\mathcal{Y}}$ defined as the image of $1 \in L = \mathrm{Lie}(G)$.

---

[3]Cf. [82, §49.3].

Remark 4.2.9. The function $\log_{\mathfrak{X}}$ corresponds under the LT-isomorphism κ to the function $\Omega \log_{\mathrm{LT}}$ by the commutative diagram after [74, Lem. 3.4]. In particular, $d \log_{\mathfrak{X}}$ corresponds to $\Omega d \log_{\mathrm{LT}}$ and $\partial_{\mathrm{inv}}^{\mathfrak{X}}$ to $\frac{1}{\Omega} \partial_{\mathrm{inv}}$, where $df = \partial_{\mathrm{inv}}^{\mathfrak{X}} d \log_{\mathfrak{X}}$ defines the invariant derivation on $\mathcal{O}_K(\mathfrak{X})$ similarly as for ∂_{inv} in (2.2).

Proposition 4.2.10. *We have explicit formulae for the relative trace map in the following cases:*

(i)   *For $\pi_L^* : \mathfrak{X} \to \mathfrak{X}$ we have $\Omega^1_{\mathfrak{X}}(\mathfrak{X}) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) d \log_{\mathfrak{X}}$ and*

$$\mathrm{tr}_{\pi_L^*}(f d \log_{\mathfrak{X}}) = \frac{q}{\pi_L} \psi_L^{\mathfrak{X}}(f) d \log_{\mathfrak{X}} .$$

(ii) *For $n \geq n_0$ and $\ell_n^* : \mathfrak{X} \xrightarrow{\cong} \mathfrak{X}_n^{\times}$ we have $\Omega^1_{\mathfrak{X}_n^{\times}}(\mathfrak{X}_n^{\times}) = \mathcal{O}_{\mathfrak{X}_n^{\times}}(\mathfrak{X}_n^{\times}) d \log_{\mathfrak{X}_n^{\times}}$ and*

$$\mathrm{tr}_{\ell_n^*}(f d \log_{\mathfrak{X}}) = \pi_L^n (\ell_n^*)_*(f) d \log_{\mathfrak{X}_n^{\times}} .$$

(iii) *For $n \geq m \geq 1$ and $\rho_{m,n} : \mathfrak{X}_m^{\times} \to \mathfrak{X}_n^{\times}$ we have $\Omega^1_{\mathfrak{X}_m^{\times}}(\mathfrak{X}_m^{\times}) = \mathcal{O}_{\mathfrak{X}_m^{\times}}(\mathfrak{X}_m^{\times}) d \log_{\mathfrak{X}_m^{\times}}$ and*

$$\mathrm{tr}_{\rho_{m,n}}(f d \log_{\mathfrak{X}_m^{\times}}) = q^{n-m} f_1 d \log_{\mathfrak{X}_n^{\times}} \quad \textit{if } f = \sum_{i=1}^{h} \mathrm{ev}_{u_i} \rho_{m,n}^*(f_i),$$

where $u_1 = 1, u_2, \ldots, u_h \in U_m$ are representatives for the cosets of U_n in U_m (with $h := q^{n-m}$).

(iv) *For $n \geq 1$ and $\rho_n : \mathfrak{X}^{\times} \to \mathfrak{X}_n^{\times}$ we have $\Omega^1_{\mathfrak{X}^{\times}}(\mathfrak{X}^{\times}) = \mathcal{O}_{\mathfrak{X}^{\times}}(\mathfrak{X}^{\times}) d \log_{\mathfrak{X}^{\times}}$ and*

$$\mathrm{tr}_{\rho_n}(f d \log_{\mathfrak{X}^{\times}}) = (q-1) q^{n-1} f_1 d \log_{\mathfrak{X}_n^{\times}} \quad \textit{if } f = \sum_{i=1}^{h} \mathrm{ev}_{u_i} \rho_n^*(f_i),$$

where $u_1 = 1, u_2, \ldots, u_h \in U_n$ are representatives for the cosets of U_n in $o_L^{\times}$ (with $h := (q-1) q^{n-1}$).

(v)   *For the multiplication $\mu_{\chi} : \mathfrak{X}^{\times} \xrightarrow{\cong} \mathfrak{X}^{\times}$ by a fixed point $\chi \in \mathfrak{X}^{\times}(L)$ we have*

$$\mathrm{tr}_{\mu_{\chi}}(f d \log_{\mathfrak{X}^{\times}}) = \mu_{\chi *}(f) d \log_{\mathfrak{X}^{\times}} = \mu_{\chi^{-1}}^*(f) d \log_{\mathfrak{X}^{\times}} .$$

Proof. All subsequent computations start, of course, from the formula (4.16) for the relative trace map.

(i) The assumptions are satisfied by Lemmata 4.1.1 and 4.1.10. As explained at the end of Section 2.3 in [8], the sheaf of differentials $\Omega^1_{\mathfrak{X}}$ on $\mathfrak{X}$ is a free $\mathcal{O}_{\mathfrak{X}}$-module of rank one with basis the global differential $d \log_{\mathfrak{X}}$. By [8, Lem. 1.28 (ii)] we have $\varphi_L(\log_{\mathfrak{X}}) = \pi_L \log_{\mathfrak{X}}$. The formula for $\mathrm{tr}_{\pi_L^*}$ now follows from Remark 4.1.4.

(ii) The assumptions are trivially satisfied. The map $d\ell_n : L = \mathrm{Lie}(U_n) \to L = \mathrm{Lie}(o_L)$ is multiplication by π_L^{-n}. It follows that the isomorphism $(\ell_n^*)^* : \mathcal{O}_{\mathfrak{X}_n^{\times}}(\mathfrak{X}_n^{\times}) \to \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ sends $\log_{\mathfrak{X}_n^{\times}}$ to $\pi_L^{-n} \log_{\mathfrak{X}}$. This implies the assertions.

(iii) The assumptions are satisfied by Remark 4.1.18. The inclusion $U_n \hookrightarrow U_m$ of an open subgroup induces the identity map on the Lie algebras. It follows that $\rho_{m,n}^*(\log_{\mathfrak{X}_n^\times}) = \log_{\mathfrak{X}_m^\times}$. Since $\rho_{m,n}$ is étale, we first may apply this with some $n \geq n_0$ and, using (ii), deduce that $\Omega^1_{\mathfrak{X}_m^\times}(\mathfrak{X}_m^\times) = \mathcal{O}_{\mathfrak{X}_m^\times}(\mathfrak{X}_m^\times)d\log_{\mathfrak{X}_m^\times}$ for any $m \geq 1$. The formula for $\mathrm{tr}_{\rho_{m,n}}$ follows by the same argument as in the proof of Remark 4.1.4.

(iv) The argument is the same as the one for (iii).

(v) The assumptions are trivially satisfied. Using that $d\chi(1) = \frac{d}{dt}\chi(\exp(t))_{|t=0}$, we check that

$$\log_{\mathfrak{X}^\times}(\chi_1\chi_2) = \log_{\mathfrak{X}^\times}(\chi_1) + \log_{\mathfrak{X}^\times}(\chi_2)$$

holds true. We deduce $\mu_\chi^*(d\log_{\mathfrak{X}^\times}) = d\log_{\mathfrak{X}^\times}$ and hence the formula for tr_{μ_χ}. ∎

We briefly remark on the case where our Stein space is the open unit disk **B** around zero. Then $\mathcal{R}_L(\mathbf{B})$ is the usual Robba ring of all Laurent series $f(Z) = \sum_{i \in \mathbb{Z}} c_i Z^i$ with coefficients $c_i \in L$ which converge in some annulus near 1. Analogously to (4.14), we have

$$H^1_c(\mathbf{B}, \Omega^1_{\mathbf{B}}) = \mathcal{R}_L(\mathbf{B})dZ/\mathcal{O}_{\mathbf{B}}(\mathbf{B})dZ$$

and the trace map sends $\sum_{i \in \mathbb{Z}} c_i Z^i dZ$ to its residue which is the coefficient c_{-1} (cf. [11, §3.1]).

4.2.3 Duality for boundary sections

First, we recall another functoriality property of Serre duality.

Proposition 4.2.11. *Let $j : \mathfrak{Y}_0 \to \mathfrak{Y}$ be an open immersion of one-(equi)dimensional smooth Stein spaces over L, and let $\mathcal{F}$ be a coherent sheaf on $\mathfrak{Y}$. Then the diagram*

$$
\begin{array}{ccccccc}
H^1_c(\mathfrak{Y}, \mathcal{F}) & \times & \mathrm{Hom}_{\mathfrak{Y}}(\mathcal{F}, \Omega^1_{\mathfrak{Y}}) & \longrightarrow & H^1_c(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}}) & \xrightarrow{\mathrm{tr}_{\mathfrak{Y}}} & L \\
{\scriptstyle j_!}\uparrow & & {\scriptstyle \mathrm{res}}\downarrow & & {\scriptstyle j_!}\uparrow & & \parallel \\
H^1_c(\mathfrak{Y}_0, \mathcal{F}) & \times & \mathrm{Hom}_{\mathfrak{Y}_0}(\mathcal{F}, \Omega^1_{\mathfrak{Y}_0}) & \longrightarrow & H^1_c(\mathfrak{Y}_0, \Omega^1_{\mathfrak{Y}_0}) & \xrightarrow{\mathrm{tr}_{\mathfrak{Y}_0}} & L
\end{array}
$$

is commutative.

Proof. The commutativity of the Yoneda pairing (before applying trace maps) is immediate from the functoriality of the cohomology with compact support in the coefficient sheaf. The assertion that $\mathrm{tr}_{\mathfrak{Y}} \circ j_! = \mathrm{tr}_{\mathfrak{Y}_0}$ holds true is shown in [90, Thm. 3.7]. ∎

In order to combine the above functoriality property with Lemma 4.2.3 in the case of the structure sheaf $\mathcal{F} = \mathcal{O}_{\mathfrak{Y}}$, we first recall the setting of that lemma.

(1) $\mathfrak{Y} = \bigcup_n \mathfrak{U}_n$ is a Stein covering of the Stein space $\mathfrak{Y}$ such that the $\mathfrak{Y}_n = \mathfrak{Y} \setminus \mathfrak{U}_n$ are Stein spaces as well. In particular, $\mathcal{R}_L(\mathfrak{Y}) = \varinjlim_n \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}_n)$ with the locally convex inductive limit topology.

(2) The restriction maps $\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}) \to \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y} \setminus \mathfrak{U})$ are injective for any $\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})$.

(3) The inductive system of Fréchet spaces

$$\Omega^1_{\mathfrak{Y}}(\mathfrak{Y}_1) \to \cdots \to \Omega^1_{\mathfrak{Y}}(\mathfrak{Y}_n) \to \Omega^1_{\mathfrak{Y}}(\mathfrak{Y}_{n+1}) \to \cdots$$

is regular (cf. [61, Def. 11.1.3 (ii)]). By [61, Thm. 11.2.4 (ii)] the locally convex inductive limit

$$\Omega^1_{\mathcal{R}_L(\mathfrak{Y})} := \varinjlim_n \Omega^1_{\mathfrak{Y}}(\mathfrak{Y}_n) = \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})} \Omega^1_{\mathfrak{Y}}(\mathfrak{Y} \setminus \mathfrak{U})$$

is a locally convex Hausdorff space.

Proposition 4.2.12. *In the above setting we have a natural topological isomorphism*

$$\mathrm{Hom}^{\mathrm{cont}}_L(\Omega^1_{\mathcal{R}_L(\mathfrak{Y})}, L) \cong \Big(\varprojlim_n \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathcal{Y}_n)} \mathfrak{D}_{\mathcal{Y}}(\mathcal{Y} \setminus \mathfrak{U}) \Big)/\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})$$

(where the left-hand side is equipped with the strong topology of bounded convergence).

Proof. The asserted isomorphism is the composite of the isomorphisms

$$\mathrm{Hom}^{\mathrm{cont}}_L(\Omega^1_{\mathcal{R}_L(\mathfrak{Y})}, L) = \mathrm{Hom}^{\mathrm{cont}}_L(\varinjlim_n \Omega^1_{\mathfrak{Y}}(\mathfrak{Y}_n), L) \xrightarrow{\cong} \varprojlim_n \mathrm{Hom}^{\mathrm{cont}}_L(\Omega^1_{\mathfrak{Y}}(\mathfrak{Y}_n), L)$$

$$= \varprojlim_n H^1_c(\mathfrak{Y}_n, \mathcal{O}_{\mathfrak{Y}})$$

$$= \Big(\varprojlim_n \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathcal{Y}_n)} \mathfrak{D}_{\mathcal{Y}}(\mathcal{Y} \setminus \mathfrak{U}) \Big)/\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}).$$

The isomorphism in the first line comes from [61, Thm. 11.1.13]. The equality in the second, resp., third, line is a consequence of Theorem 4.2.6 and Proposition 4.2.11, resp., Lemmata 4.2.3 and 4.2.5. ∎

We now evaluate this latter result in the same concrete cases as in Section 4.2.1.

The open unit disk

First of all, the sheaf of differentials $\Omega^1_{\mathbf{B}}$ on the open unit disk $\mathbf{B}$ is a free $\mathcal{O}_{\mathbf{B}}$-module of rank one. Hence, by choosing, for example, the global differential dZ for a coordinate function Z, as a basis, we obtain a topological isomorphism $\mathcal{R}_L(\mathbf{B}) \cong \Omega^1_{\mathcal{R}_L(\mathbf{B})}$ as $\mathcal{R}_L(\mathbf{B})$-modules. The regularity assumption in (3) above therefore is reduced to the corresponding property for $\mathcal{R}_L(\mathbf{B})$, which is established in the proof of [8, Prop. 2.6 (i)]. Hence, Proposition 4.2.12 is available. By combining its assertion with (4.12) we obtain a natural topological isomorphism

$$\mathrm{Hom}^{\mathrm{cont}}_L\big(\mathcal{R}_L(\mathbf{B}), L\big) \cong \mathrm{Hom}^{\mathrm{cont}}_L(\Omega^1_{\mathcal{R}_L(\mathbf{B})}, L) \cong \mathcal{R}_L(\mathbf{B}). \tag{4.18}$$

This shows that $\mathscr{R}_L(\mathbf{B})$ is topologically self-dual. By going through the definitions and using the explicit description of the trace map in this case as the usual residue map (end of Section 4.2.2) one checks that this self-duality comes from the pairing

$$\mathscr{R}_L(\mathbf{B}) \times \mathscr{R}_L(\mathbf{B}) \to L$$
$$\big(f_1(Z), f_2(Z)\big) \mapsto \text{residue of } f_1(Z) f_2(Z) dZ.$$

This latter form of the result was known (cf. [64, 67]) before Serre duality in rigid analysis was established. In this paper, it is more natural to use the self-duality given by the pairing

$$\langle\,,\,\rangle_{\mathbf{B}} : \mathscr{R}_L(\mathbf{B}) \times \mathscr{R}_L(\mathbf{B}) \to L$$
$$(f_1, f_2) \mapsto \text{residue of } f_1 f_2 d\log_{\mathrm{LT}}.$$

We will denote by $\mathrm{res}_{\mathbf{B}} : \Omega^1_{\mathscr{R}_L(\mathbf{B})} \to L$ the linear form which corresponds to $1 \in \mathscr{R}_L(\mathbf{B})$ under the second isomorphism in (4.18).

The character variety $\mathfrak{X}$

We recall that the sheaf of differentials $\Omega^1_{\mathfrak{X}}$ on $\mathfrak{X}$ is a free $\mathcal{O}_{\mathfrak{X}}$-module of rank one with basis the global differential $d\log_{\mathfrak{X}}$. Hence, again we have a topological isomorphism $\mathscr{R}_L(\mathfrak{X}) \cong \Omega^1_{\mathscr{R}_L(\mathfrak{X})}$ as $\mathscr{R}_L(\mathfrak{X})$-modules. The regularity assumption in (3) above therefore holds by [8, Prop. 2.6 (i)]. Hence, Proposition 4.2.12 is available. By combining its assertion with (4.13) we obtain a natural topological isomorphism

$$\mathrm{Hom}^{\mathrm{cont}}_L(\mathscr{R}_L(\mathfrak{X}), L) \cong \mathrm{Hom}^{\mathrm{cont}}_L(\Omega^1_{\mathscr{R}_L(\mathfrak{X})}, L) \cong \mathscr{R}_L(\mathfrak{X}). \tag{4.19}$$

This shows that $\mathscr{R}_L(\mathfrak{X})$ is topologically self-dual. Let $\mathrm{res}_{\mathfrak{X}} : \Omega^1_{\mathscr{R}_L(\mathfrak{X})} \to L$ be the linear form which corresponds to $1 \in \mathscr{R}_L(\mathfrak{X})$ under the above isomorphism. Then, as a consequence of the naturality of the Yoneda pairing, this self-duality comes from the pairing[4]

$$\langle\,,\,\rangle_{\mathfrak{X}} : \mathscr{R}_L(\mathfrak{X}) \times \mathscr{R}_L(\mathfrak{X}) \to L$$
$$(f_1, f_2) \mapsto \mathrm{res}_{\mathfrak{X}}(f_1 f_2 d\log_{\mathfrak{X}}). \tag{4.20}$$

Next we consider $\mathfrak{X}_n^{\times}$ for some $n \geq n_0$, where $n_0 \geq 1$ is the integer from Section 4.1.2. We then have the isomorphism of Stein group varieties $\ell_n^* : \mathfrak{X} \xrightarrow{\cong} \mathfrak{X}_n^{\times}$. Hence, all we have established for $\mathfrak{X}$ holds true correspondingly for $\mathfrak{X}_n^{\times}$. In particular, we have a natural topological isomorphism

$$\mathrm{Hom}^{\mathrm{cont}}_L(\mathscr{R}_L(\mathfrak{X}_n^{\times}), L) \cong \mathrm{Hom}^{\mathrm{cont}}_L(\Omega^1_{\mathscr{R}_L(\mathfrak{X}_n^{\times})}, L) \cong \mathscr{R}_L(\mathfrak{X}_n^{\times}).$$

---

[4]Warning: If $L = \mathbb{Q}_p$, then $\mathfrak{X} = \mathbf{B}_1 \cong \mathbf{B}$ with the latter isomorphism given by $z \mapsto z - 1$; but the self-dualities (4.18) and (4.19) do not correspond to each other since in this case $d\log_{\mathfrak{X}}$ corresponds to $d\log(1 + Z) = \frac{1}{1+Z} dZ$.

Let $\mathrm{res}_{\mathfrak{X}_n^\times} : \Omega^1_{\mathcal{R}_L(\mathfrak{X}_n^\times)} \to L$ be the linear form which corresponds to $1 \in \mathcal{R}_L(\mathfrak{X}_n^\times)$ under this isomorphism. We obtain that $\mathcal{R}_L(\mathfrak{X}_n^\times)$ is topologically self-dual with regard to the pairing

$$\langle\ ,\ \rangle_{\mathfrak{X}_n^\times} : \mathcal{R}_L(\mathfrak{X}_n^\times) \times \mathcal{R}_L(\mathfrak{X}_n^\times) \to L$$
$$(f_1, f_2) \mapsto \mathrm{res}_{\mathfrak{X}_n^\times}(f_1 f_2 d \log_{\mathfrak{X}_n^\times}).$$

It follows from [90, Thm. 3.7] that the diagram

$$
\begin{array}{ccc}
\Omega^1_{\mathcal{R}_L(\mathfrak{X})} & & \\
& \searrow^{\mathrm{res}_{\mathfrak{X}}} & \\
(\ell_n^*)_* \downarrow \cong & & L \\
& \nearrow_{\mathrm{res}_{\mathfrak{X}_n^\times}} & \\
\Omega^1_{\mathcal{R}_L(\mathfrak{X}_n^\times)} & &
\end{array}
$$

is commutative. But in the proof of Proposition 4.2.10 (ii) we have already seen that $(\ell_n^*)_*(\log_{\mathfrak{X}}) = \pi_L^n \log_{\mathfrak{X}_n^\times}$. Therefore, the diagram of pairings

$$
\begin{array}{ccccccc}
\mathcal{R}_L(\mathfrak{X}) & \times & \mathcal{R}_L(\mathfrak{X}) & \xrightarrow{\langle\,,\rangle_{\mathfrak{X}}} & L \\
(\ell_n^*)^* \uparrow \cong & & \pi_L^n(\ell_n^*)_* \downarrow \cong & & \parallel \\
\mathcal{R}_L(\mathfrak{X}_n^\times) & \times & \mathcal{R}_L(\mathfrak{X}_n^\times) & \xrightarrow{\langle\,,\rangle_{\mathfrak{X}_n^\times}} & L
\end{array}
\tag{4.21}
$$

is commutative. Alternatively, we could have used the following observation.

Remark 4.2.13. Let $\rho : \mathcal{Y} \to \mathcal{Z}$ be one of the morphisms in Proposition 4.2.10. Then Proposition 4.1.11 applies, and it follows that, for any admissible open subset $\mathcal{U} \subseteq \mathcal{Z}$ which is Stein, the relative trace map $\mathrm{tr}_{\rho|\rho^{-1}(\mathcal{U})}$ is given by the same formula as for tr_ρ.

In the case of the morphisms π_L^* and $\rho_{m,n}$ for $n \geq m \geq n_0$, this immediately leads to the following equalities of pairings.

Lemma 4.2.14. *We have the following:*

(i) $\langle \varphi_L(f_1), f_2 \rangle_{\mathfrak{X}} = \frac{q}{\pi_L} \langle f_1, \psi_L^{\mathfrak{X}}(f_2) \rangle_{\mathfrak{X}}$ *for any* $f_1, f_2 \in \mathcal{R}_L(\mathfrak{X})$.

(ii) *Let* $n \geq m \geq n_0$ *and let* $u_1 = 1, u_2, \ldots, u_h \in U_m$ *be representatives for the cosets of* U_n *in* U_m *(with* $h := q^{n-m}$*); for any* $f' \in \mathcal{R}_L(\mathfrak{X}_n^\times)$ *and any* $f \in \mathcal{R}_L(\mathfrak{X})_m^\times$ *of the form* $f = \sum_{i=1}^h \mathrm{ev}_{u_i} \rho_{m,n}^*(f_i)$ *(cf. Remark 4.1.18) we have*

$$\langle \rho_{m,n}^*(f'), f \rangle_{\mathfrak{X}_m^\times} = q^{n-m} \langle f', f_1 \rangle_{\mathfrak{X}_n^\times}.$$

The multiplicative character variety $\mathfrak{X}^\times$

We fix an $n \geq n_0$ for the moment as well as representatives $u_1 = 1, u_2, \ldots, u_h \in o_L^\times$ for the cosets of U_n in $o_L^\times$ (with $h := (q-1)q^{n-1}$). Recalling from Lemma 4.1.15 (ii) that

$$\mathcal{R}_L(\mathfrak{X}^\times) = \mathbb{Z}[o_L^\times] \otimes_{\mathbb{Z}[U_n]} \rho_n^*\big(\mathcal{R}_L(\mathfrak{X}_n^\times)\big),$$

we may write any $f \in \mathcal{R}_L(\mathfrak{X}^\times)$ as $f = \sum_{i=1}^{h} \mathrm{ev}_{u_i}\, \rho_n^*(f_i)$ with uniquely determined $f_i \in \mathcal{R}_L(\mathfrak{X}_n^\times)$. We now define $\mathrm{res}_{\mathfrak{X}^\times} : \Omega^1_{\mathcal{R}_L(\mathfrak{X}^\times)} \to L$ by

$$\mathrm{res}_{\mathfrak{X}^\times}(f d\, \log_{\mathfrak{X}^\times}) := (q-1)q^{n-1}\, \mathrm{res}_{\mathfrak{X}_n^\times}(f_1 d\, \log_{\mathfrak{X}_n^\times}) \tag{4.22}$$

and then the pairing

$$\langle\,,\,\rangle_{\mathfrak{X}^\times} : \mathcal{R}_L(\mathfrak{X}^\times) \times \mathcal{R}_L(\mathfrak{X}^\times) \to L$$
$$(f_1, f_2) \mapsto \mathrm{res}_{\mathfrak{X}^\times}(f_1 f_2). \tag{4.23}$$

These definitions are obviously independent of the choice of the representatives u_i. Moreover, due to Lemma 4.2.14 (ii), they are independent of the choice of n as well and we have

$$\langle \rho_n^*(f'), f \rangle_{\mathfrak{X}^\times} = (q-1)q^{n-1}\langle f', f_1 \rangle_{\mathfrak{X}_n^\times} \tag{4.24}$$

for any $f' \in \mathcal{R}_L(\mathfrak{X}_n^\times)$ and any $f = \sum_{i=1}^{(q-1)q^{n-1}} \mathrm{ev}_{u_i}\, \rho_n^*(f_i) \in \mathcal{R}_L(\mathfrak{X})^\times$, where $u_i \in o_L^\times$ runs through representatives for the cosets of U_n in $o_L^\times$. The topological self-duality of $\mathcal{R}_L(\mathfrak{X}_n^\times)$ easily implies that this pairing makes $\mathcal{R}_L(\mathfrak{X}^\times)$ topological self-dual.

Lemma 4.2.15. *The twist morphism* $\mu_\chi : \mathfrak{X}^\times \to \mathfrak{X}^\times$, *for any* $\chi \in \mathfrak{X}^\times(L)$, *satisfies*

$$\langle \mu_\chi^*(f_1), \mu_\chi^*(f_2) \rangle_{\mathfrak{X}^\times} = \langle f_1, f_2 \rangle_{\mathfrak{X}^\times} \quad \textit{for any } f_1, f_2 \in \mathcal{R}_L(\mathfrak{X}^\times).$$

Proof. The assertion immediately reduces to checking the equality $\mathrm{res}_{\mathfrak{X}^\times} \circ \mu_\chi^* = \mathrm{res}_{\mathfrak{X}^\times}$. Obviously, there are twist morphisms on $\mathfrak{X}_n^\times$ as well. One easily checks that

$$\mu_\chi^* \circ \rho_n^* = \rho_n^* \circ \mu_{\chi|U_n}^*$$

and that

$$\mu_\chi^*(\mathrm{ev}_u) = \chi(u)\, \mathrm{ev}_u \quad \text{for any } u \in o_L^\times.$$

Using Lemma 4.1.15 (ii), we write an $f \in \mathcal{R}_L(\mathfrak{X}^\times)$ as $f = \sum_{i=1}^{h} \mathrm{ev}_{u_i}\, \rho_n^*(f_i)$ and compute

$$\mu_\chi^*(f) = \sum_{i=1}^{h} \mu_\chi^*(\mathrm{ev}_{u_i})\mu_\chi^*\big(\rho_n^*(f)\big) = \sum_{i=1}^{h} \mathrm{ev}_{u_i}\, \rho_n^*\big(\chi(u_i)\mu_{\chi|U_n}^*(f_i)\big).$$

This shows that $\mu_\chi^*(f)_1 = \mu_{\chi|U_n}^*(f_1)$. This further reduces the claim to showing that $\mathrm{res}_{\mathfrak{X}_n^\times} \circ \mu_{\chi|U_n}^* = \mathrm{res}_{\mathfrak{X}_n^\times}$. But this follows from [90, Thm. 3.7] or, alternatively, from a version of Proposition 4.2.10 (v) for $\mathcal{R}_L(\mathfrak{X}_n^\times)$. ∎

Of course, everything in this entire Section 4.2 remains valid over any complete extension field K of L contained in $\mathbb{C}_p$. Moreover, our constructions above are compatible under (complete) base change: Let $\mathfrak{Y}_K$ denote the base change of $\mathfrak{Y}$ over L

to K (and similarly for affinoids). Then we obtain a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{R}_L(\mathfrak{Y}) & \times & \Omega^1_{\mathcal{R}_L(\mathfrak{Y})} & \longrightarrow & L \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{R}_K(\mathfrak{Y}_K) & \times & \Omega^1_{\mathcal{R}_K(\mathfrak{Y}_K)} & \longrightarrow & K.
\end{array}
\tag{4.25}
$$

Indeed, it is shown in [54] that Serre duality is compatible with base change in the sense that there is the following commutative diagram for any n, in which the horizontal lines are the Serre dualities over L and K, respectively:

$$
\begin{array}{ccccc}
H^1_c(\mathfrak{Y}_n, \mathcal{O}_{\mathfrak{Y}}) & \times & \Omega^1_{\mathfrak{Y}}(\mathfrak{Y}_n) & \longrightarrow & L \\
\downarrow & & \downarrow & & \downarrow \\
H^1_c(\mathfrak{Y}_{n,K}, \mathcal{O}_{\mathfrak{Y}_K}) & \times & \Omega^1_{\mathfrak{Y}_K}(\mathfrak{Y}_{n,K}) & \longrightarrow & K.
\end{array}
$$

Hence, taking limits as in the proof of Proposition 4.2.12, the claim follows upon observing that also the relative cohomology sequence (4.9) is compatible with base change. By inserting $1 \in \mathcal{R}_L(\mathfrak{Y}) \subseteq \mathcal{R}_K(\mathfrak{Y}_K)$ into the pairings of (4.25), we see that in any example discussed above the residue maps $\mathrm{res}_{\mathfrak{Y}}$ are compatible under base change as well as the pairings $\langle\,,\,\rangle_{\mathfrak{Y}}$. Since $\mathcal{R}_K(\mathfrak{Y}_K) \cong K \,\widehat{\otimes}_{L,\iota}\, \mathcal{R}_L(\mathfrak{Y})$ (and hence $\Omega^1_{\mathcal{R}_K(\mathfrak{Y}_K)} \cong K \,\widehat{\otimes}_{L,\iota}\, \Omega^1_{\mathcal{R}_L(\mathfrak{Y})}$) by [8, Cor. 2.8] with respect to the (completed) inductive tensor product, we see that

$$
\mathrm{res}_{\mathfrak{Y}_K} = K \,\widehat{\otimes}_{L,\iota}\, \mathrm{res}_{\mathfrak{Y}}
\tag{4.26}
$$

and that we have the commutative diagram

$$
\begin{array}{ccc}
K\widehat{\otimes}_{L,\iota}\mathrm{Hom}^{\mathrm{cont}}_L\big(\mathcal{R}_L(\mathfrak{Y}),L\big) \to \mathrm{Hom}^{\mathrm{cont}}_K\big(K\widehat{\otimes}_{L,\iota}\mathcal{R}_L(\mathfrak{Y}),K\big) \overset{\cong}{\to} \mathrm{Hom}^{\mathrm{cont}}_K\big(\mathcal{R}_K(\mathfrak{Y}_K),K\big) \\
\cong \downarrow \mathrm{id}_K\widehat{\otimes}_{L,\iota}\mathrm{duality}_{\mathfrak{Y}_L} \qquad\qquad\qquad\qquad \cong \downarrow \mathrm{duality}_{\mathfrak{Y}_K} \\
K \,\widehat{\otimes}_{L,\iota}\, \mathcal{R}_L(\mathfrak{Y}) \xrightarrow{\qquad\qquad \cong \qquad\qquad} \mathcal{R}_K(\mathfrak{Y}_K).
\end{array}
$$

4.3 (φ_L, Γ_L)-modules

As before, we let $L \subseteq K \subseteq \mathbb{C}_p$ be a complete intermediate field, and we denote by o_K its ring of integers.

4.3.1 The usual Robba ring

In Sections 4.2.2 and 4.2.3, we already had introduced the usual Robba ring $\mathcal{R} = \mathcal{R}_K = \mathcal{R}_K(\mathbf{B})$ of the Stein space $\mathbf{B}_{/K}$ in connection with Serre duality. We briefly

review its construction in more detail. The ring of K-valued global holomorphic functions[5] $\mathcal{O}_K(\mathbf{B})$ on $\mathbf{B}$ is the Fréchet algebra of all power series in the variable Z with coefficients in K which converge on the open unit disk $\mathbf{B}(\mathbb{C}_p)$. The Fréchet topology on $\mathcal{O}_K(\mathbf{B})$ is given by the family of norms

$$\left| \sum_{i \geq 0} c_i Z^i \right|_r := \max_i |c_i| r^i \quad \text{for } 0 < r < 1.$$

In the commutative integral domain $\mathcal{O}_K(\mathbf{B})$, we have the multiplicative subset $Z^{\mathbb{N}} = \{Z^j : j \in \mathbb{N}\}$, so that we may form the corresponding localization $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$. Each norm $|\ |_r$ extends to this localization $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$ by setting

$$\left| \sum_{i \gg -\infty} c_i Z^i \right|_r := \max_i |c_i| r^i.$$

The Robba ring $\mathcal{R} \supseteq \mathcal{O}_K(\mathbf{B})$ is constructed as follows. For any $s > 0$, resp., any $0 < r \leq s$, in $p^{\mathbb{Q}}$ let $\mathbf{B}_{[0,s]}$, resp., $\mathbf{B}_{[r,s]}$, denote the affinoid disk around 0 of radius s, resp., the affinoid annulus of inner radius r and outer radius s, over K. For $I = [0, s]$ or $[r, s]$ we denote by

$$\mathcal{R}^I := \mathcal{R}^I_K(\mathbf{B}) := \mathcal{O}_K(\mathbf{B}_I)$$

the affinoid K-algebra of $\mathbf{B}_I$. The Fréchet algebra $\mathcal{R}^{[r,1)} := \varprojlim_{r < s < 1} \mathcal{R}^{[r,s]}$ is the algebra of (infinite) Laurent series in the variable Z with coefficients in K which converge on the half-open annulus $\mathbf{B}_{[r,1)} := \bigcup_{r < s < 1} \mathbf{B}_{[r,s]}$. The Banach algebra $\mathcal{R}^{[0,s]}$ is the completion of $\mathcal{O}_K(\mathbf{B})$ with respect to the norm $|\ |_s$. The Banach algebra $\mathcal{R}^{[r,s]}$ is the completion of $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$ with respect to the norm $|\ |_{r,s} := \max(|\ |_r, |\ |_s)$. It follows that the Fréchet algebra $\mathcal{R}^{[r,1)}$ is the completion of $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$ in the locally convex topology defined by the family of norms $(|\ |_{r,s})_{r < s < 1}$. Finally, the Robba ring is $\mathcal{R} = \bigcup_{0 < r < 1} \mathcal{R}^{[r,1)}$.

Remark 4.3.1. $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$ is dense in $\mathcal{R}_K(\mathbf{B})$.

Let $p^{-\frac{d}{(q-1)e}} < r \leq s < 1$. Then we have a surjective map

$$\begin{aligned} \mathbf{B}_{[r,s]} &\to \mathbf{B}_{[r^q, s^q]} \\ z &\mapsto [\pi_L](z) \end{aligned} \tag{4.27}$$

according to [33, proof of Lem. 2.6].[6] It induces a ring homomorphism

$$\varphi_L^{[r^q, s^q]} : \mathcal{R}^{[r^q, s^q]} \to \mathcal{R}^{[r,s]} \tag{4.28}$$

---

[5]In the notation from [25, §1.2], this is the ring $\mathcal{R}^+$.

[6]The proof there is only written for the *special* Lubin–Tate group, i.e., such that $[\pi_L] = X^q + \pi_L X$, but generalizes easily by using the fact that $[\pi_L] = X^q + \pi_L X f(X)$ with $f(X) \in o_L[\![X]\!]^{\times}$.

which is isometric with respect to the supremum norms, i.e.,

$$\left|\varphi_L^{[r^q, s^q]}(f)\right|_{[r,s]} = |f|_{[r^q, s^q]} \quad \text{for any } f \in \mathcal{R}^{[r^q, s^q]}.$$

In particular, by taking first inverse and then direct limits we obtain a continuous ring homomorphism $\varphi_L : \mathcal{R} \to \mathcal{R}$. We shall often omit the interval in $\varphi_L^{[r,s]}$ and just write φ_L.

Similarly, we obtain a continuous Γ_L-action on $\mathcal{R}$: According to loc. cit., we have a bijective map

$$\mathbf{B}_{[r,s]} \to \mathbf{B}_{[r,s]}$$
$$z \mapsto [\chi_{\mathrm{LT}}(\gamma)](z)$$

for any $\gamma \in \Gamma_L$, whence we obtain an isometric isomorphism

$$\gamma : \mathcal{R}^{[r,s]} \to \mathcal{R}^{[r,s]}$$

with respect to the supremum norms, i.e., $|\gamma(f)|_{[r,s]} = |f|_{[r,s]}$ for any $f \in \mathcal{R}^{[r,s]}$.

Finally, we extend the operator ψ_L to $\mathcal{R}$: For $y \in \ker([\pi_L])$ we have the isomorphism

$$\mathbf{B}_{[r,s]} \to \mathbf{B}_{[r,s]}$$
$$z \mapsto z +_{\mathrm{LT}} y$$

of affinoid varieties because $|z +_{\mathrm{LT}} y| = |z + y| = |z|$. The latter equality comes from $|z| \geq r > p^{-\frac{d}{(q-1)e}} = q^{-\frac{1}{q-1}} = |y|$ for $y \neq 0$. Setting $\mathrm{tr}(f) := \sum_{y \in \ker([\pi_L])} f(z +_{\mathrm{LT}} y)$, we obtain a norm decreasing linear map $\mathrm{tr} : \mathcal{R}^{[r,s]} \to \mathcal{R}^{[r,s]}$. We claim that the image of tr is contained in the (closed) image of the isometry $\varphi_L^{[r^q, s^q]}$, whence there is a norm decreasing map

$$\psi_{\mathrm{Col}} : \mathcal{R}^{[r,s]} \to \mathcal{R}^{[r^q, s^q]},$$

such that $\varphi_L \circ \psi_{\mathrm{Col}} = \mathrm{tr}$. Indeed, by continuity it suffices to show that $\mathrm{tr}(Z^i)$ belongs to the image for any $i \in \mathbb{Z}$. For $i \geq 0$, Coleman has shown that $\mathrm{tr}(Z^i) = \varphi_L(\psi_{\mathrm{Col}}(Z^i))$ with $\psi_{\mathrm{Col}}(Z^i) \in o_L[\![Z]\!] \subseteq \mathcal{R}^{[r^q, s^q]}$, see [80, §2]. For $i < 0$, we calculate

$$\varphi_L\big(Z^i \psi_{\mathrm{Col}}([\pi_L](Z)^{-i} Z^i)\big) = \varphi_L(Z^i) \Big(\sum_{y \in \ker([\pi_L])} [\pi_L](Z)^{-i} Z^i \Big)(Z +_{\mathrm{LT}} y)$$

$$= \varphi_L(Z^i) \Big(\sum_{y \in \ker([\pi_L])} [\pi_L]\big((Z +_{\mathrm{LT}} y)\big)^{-i} (Z +_{\mathrm{LT}} y)^i \Big)$$

$$= \varphi_L(Z^i) \Big(\sum_{y \in \ker([\pi_L])} \varphi_L(Z)^{-i} (Z +_{\mathrm{LT}} y)^i \Big)$$

$$= \sum_{y \in \ker([\pi_L])} (Z +_{\mathrm{LT}} y)^i = \mathrm{tr}(Z^i),$$

whence the claim follows. We put $\psi_L^{[r,s]} := \frac{1}{\pi_L} \psi_{\mathrm{Col}} : \mathcal{R}^{[r,s]} \to \mathcal{R}^{[r^q, s^q]}$ which induces the continuous operator $\psi_L : \mathcal{R} \to \mathcal{R}$ by taking first inverse limits and then direct limits. By definition of tr the operators $\psi_L^{[r,s]}$ and hence ψ_L satisfy the projection formula. We shall often omit the interval in $\psi_L^{[r,s]}$ and just write ψ_L.

As in Section 4.1.4, we fix a generator η' of the dual Tate module T'_π and denote by Ω the corresponding period. For the rest of this subsection we *assume* in addition that K contains Ω. Following Colmez in the notation, we introduce the power series $\eta(a, Z) := \exp(a\Omega \log_{\mathrm{LT}}(Z)) \in o_K[\![Z]\!]$ for $a \in o_L$. As noted in Section 4.1.4, the power series $\eta(a, Z)$ is nothing else than the image under the LT-isomorphism κ^* of the holomorphic function $\mathrm{ev}_a \in \mathcal{O}_K(\mathfrak{X})$. Generalizing the equality (4.7), we have the following decompositions of Banach spaces:

$$\mathcal{R}^{[r,s]} = \bigoplus_{a \in o_L/\pi_L^n} \varphi_L^n(\mathcal{R}^{[r^q, s^q]})\eta(a, Z) \tag{4.29}$$

and hence

$$\mathcal{R} = \bigoplus_{a \in o_L/\pi_L^n} \varphi_L^n(\mathcal{R})\eta(a, Z) \tag{4.30}$$

of LF-spaces using the formula

$$r = \left(\tfrac{\pi_L}{q}\right)^n \sum_a \varphi_L^n \psi_L^n\big(\eta(-a, Z)r\big)\eta(a, Z). \tag{4.31}$$

This can easily be reduced by induction on n to the case $n = 1$. Using the definition of tr and the orthogonality relations for the characters κ_y for $y \in \ker([\pi_L])$, the formula follows and, moreover, defines a continuous inverse to the continuous map

$$\mathbb{Z}[o_L] \otimes_{\mathbb{Z}[\pi_L o_L]} \mathcal{R}^{[r^q, s^q]} \xrightarrow{\cong} \mathcal{R}^{[r,s]}$$
$$a \otimes f \mapsto a\varphi_L(f).$$

Inductively, we obtain canonical isomorphisms

$$\mathbb{Z}[o_L] \otimes_{\mathbb{Z}[\pi_L^n o_L]} \mathcal{R}^{[r^{q^n}, s^{q^n}]} \xrightarrow{\cong} \mathcal{R}^{[r,s]}$$
$$a \otimes f \mapsto a\varphi_L^n(f). \tag{4.32}$$

Moreover, immediately from the definitions, we have

$$\varphi_L\big(\eta(a, Z)\big) = \eta(\pi_L a, Z), \tag{4.33}$$

$$\sigma\big(\eta(a, Z)\big) = \eta\big(\chi_{\mathrm{LT}}(\sigma)a, Z\big) \quad \text{for } \sigma \in \Gamma_L, \tag{4.34}$$

$$\psi_L\big(\eta(a, Z)\big) = \begin{cases} \frac{q}{\pi_L}\eta\big(\frac{a}{\pi_L}, Z\big) & \text{for } a \in \pi_L o_L, \\ 0 & \text{otherwise.} \end{cases} \tag{4.35}$$

Remark 4.3.2. We have $\psi_L = \frac{1}{\pi_L}\varphi_L^{-1} \circ \mathrm{trace}_{\mathcal{R}/\varphi_L(\mathcal{R})}$.

Proof. Both maps tr and trace$_{\mathcal{R}/\varphi_L(\mathcal{R})}$ are easily seen to be $\varphi_L(\mathcal{R})$-linear and to be multiplication by q on $\varphi_L(\mathcal{R})$. Hence, by (4.30), it suffices to compare their values on the elements $\eta(a, Z)$. By (4.35) we have

$$\mathrm{tr}\left(\eta(a, Z)\right) = \begin{cases} q\varphi_L\big(\eta(\pi_L^{-1}a, Z)\big) & \text{if } a \in \pi_L o_L, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, a similar computation as in the proof of Remark 4.1.4 shows that trace$_{\mathcal{R}/\varphi_L(\mathcal{R})}$ has exactly the same values. But $\psi_L = \frac{1}{\pi_L}\varphi_L^{-1} \circ \mathrm{tr}$. ∎

For uniformity of notation we put $\psi_L^{\mathbf{B}} := \frac{\pi_L}{q}\psi_L = \frac{1}{q}\varphi_L^{-1} \circ \mathrm{trace}_{\mathcal{R}/\varphi_L(\mathcal{R})}$.

4.3.2 The LT-isomorphism, part 2

We *assume* throughout this subsection that Ω is contained in K. Since the map $\kappa : \mathbf{B} \xrightarrow{\cong} \mathfrak{X}$ is an isomorphism of rigid varieties, it preserves the systems of affinoid subdomains on both sides. Hence, the LT-isomorphism (4.6) extends to a topological isomorphism

$$\kappa^* : \mathcal{R}_K(\mathfrak{X}) \xrightarrow{\cong} \mathcal{R}_K(\mathbf{B}).$$

In order to have a uniform notation, we usually write from now on

$$\mathcal{R}_K^I(\mathfrak{X}) := \mathcal{O}_K\big(\kappa(\mathbf{B}_I)\big)$$

for any closed interval $I \subseteq (0, 1)$ so that we have the isomorphism of Banach algebras

$$\kappa^* : \mathcal{R}_K^I(\mathfrak{X}) \xrightarrow{\cong} \mathcal{R}_K^I(\mathbf{B}).$$

We warn the reader that only for specific closed intervals I there is another closed interval I' given by a complicated but explicit rule such that $\kappa(\mathbf{B}_I) = \mathfrak{X}_{I'}$. The precise statement can be worked out from [8, Prop. 1.20].

In the following, we list a few compatibilities under this extended LT-isomorphism.

First of all, under this isomorphism, the $\Gamma_L \cong o_L^\times$-action and the maps φ_L on both sides correspond to each other (cf. [8, §2.2]). Then it follows from Remarks 4.1.4 and 4.3.2 that the operators $\psi_L^{\mathfrak{X}}$ (defined at the end of Section 4.1.1) and $\psi_L^{\mathbf{B}}$ (defined in the previous section) also correspond under κ^*.

Secondly, as a consequence of [90, Thm. 3.7], we have the commutative diagram

$$
\begin{array}{c}
\Omega^1_{\mathcal{R}_K(\mathfrak{X})} \\
\kappa^* \Big\downarrow \cong \qquad\qquad \xrightarrow{\ \mathrm{res}_{\mathfrak{X}}\ } \quad K. \\
\Omega^1_{\mathcal{R}_K(\mathbf{B})} \qquad \mathrm{res}_{\mathbf{B}}
\end{array}
\tag{4.36}
$$

This combined with Remark 4.2.9 implies the explicit formula

$$\mathrm{res}_{\mathfrak{X}}(f d \log_{\mathfrak{X}}) = \Omega \, \mathrm{res}_{\mathbf{B}}\left(\kappa^*(f) g_{\mathrm{LT}} dZ\right). \tag{4.37}$$

4.3.3 φ_L-modules

Let $\mathfrak{Y}$ be either $\mathfrak{X}$ or $\mathbf{B}$ and $\mathcal{R} := \mathcal{R}_K(\mathfrak{Y})$. Henceforth, we will use the operator $\psi_L := \frac{q}{\pi_L} \psi_L^{\mathfrak{Y}}$ on $\mathcal{R}$. We also put

$$q_{\mathfrak{Y}} := \begin{cases} p & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ q & \text{if } \mathfrak{Y} = \mathbf{B}. \end{cases}$$

Definition 4.3.3. A φ_L-module M over $\mathcal{R}$ is a finitely generated free $\mathcal{R}$-module M equipped with a semilinear endomorphism φ_M such that the $\mathcal{R}$-linear map

$$\varphi_M^{\mathrm{lin}} : \mathcal{R} \otimes_{\mathcal{R}, \varphi_L} M \xrightarrow{\cong} M$$
$$f \otimes m \mapsto f \varphi_M(m)$$

is bijective.

Technically important is the following fact, which for $\mathfrak{X}$ is part of the proof of [8, Prop. 2.24]. The proof for $\mathbf{B}$ is entirely analogous. It allows us to extend the above maps and decompositions from the previous sections to φ_L-modules. For $r > 0$ we introduce the intervals

$$I(r, \mathfrak{Y}) := \begin{cases} (r, 1) & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ [r, 1) & \text{if } \mathfrak{Y} = \mathbf{B}. \end{cases}$$

Proposition 4.3.4. *Let M be a φ_L-module M over $\mathcal{R}$. There exists a radius*

$$r_0 \geq \begin{cases} p^{-\frac{dp}{p-1}} & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ p^{-\frac{dq}{(q-1)e}} & \text{if } \mathfrak{Y} = \mathbf{B} \end{cases}$$

and a finitely generated free $\mathcal{O}_K(\mathfrak{Y}_{I(r_0, \mathfrak{Y})})$-module M_0 equipped with a semilinear continuous homomorphism

$$\varphi_{M_0} : M_0 \to \mathcal{O}_K(\mathfrak{Y}_{I(r_0, \mathfrak{Y})^{1/q_{\mathfrak{Y}}}}) \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I(r_0, \mathfrak{Y})})} M_0$$

such that the induced $\mathcal{O}_K(\mathfrak{Y}_{I(r_0, \mathfrak{Y})^{1/q_{\mathfrak{Y}}}})$-linear map

$$\varphi_{M_0}^{\mathrm{lin}} : \mathcal{O}_K(\mathfrak{Y}_{I(r_0, \mathfrak{Y})^{1/q_{\mathfrak{Y}}}}) \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I(r_0, \mathfrak{Y})}), \varphi_L} M_0 \xrightarrow{\cong} \mathcal{O}_K(\mathfrak{Y}_{I(r_0, \mathfrak{Y})^{1/q_{\mathfrak{Y}}}}) \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I(r_0, \mathfrak{Y})})} M_0$$

is an isomorphism and such that

$$\mathcal{R} \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I(r_0, \mathfrak{Y})})} M_0 = M$$

with $\varphi_L \otimes \varphi_{M_0}$ and φ_M corresponding to each other.

The continuity condition for the φ_{M_0}, of course, refers to the product topology on $M_0 \cong (\mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})}))^d$.

In the following, we fix a φ_L-module M over $\mathcal{R}$ and a pair (r_0, M_0) as in Proposition 4.3.4. For any $r_0 \leq r' < 1$ and any closed interval $I = [r, s] \subseteq I(r', \mathfrak{Y})$ we then have the finitely generated free modules

$$M^{I(r',\mathfrak{Y})} := \mathcal{O}_K(\mathfrak{Y}_{I(r',\mathfrak{Y})}) \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})})} M_0 \quad \text{over } \mathcal{R}^{[r,1)},$$

$$M^I := \mathcal{O}_K(\mathfrak{Y}_I) \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I(r',\mathfrak{Y})})} M^{I(r',\mathfrak{Y})} \quad \text{over } \mathcal{O}_K(\mathfrak{Y}_I).$$

They satisfy

$$M^{I(r',\mathfrak{Y})} = \varprojlim_{s>r} M^I \quad \text{and} \quad M = \varinjlim_{r'} M^{I(r',\mathfrak{Y})}. \tag{4.38}$$

We equip M^I with the Banach norm $|\ |_{M^I}$ given by the maximum norm with respect to any fixed basis (the induced topology does not depend on the choice of basis) which is submultiplicative with respect to scalar multiplication and the norm $|\ |_I$ on $\mathcal{O}_K(\mathfrak{Y}_I)$.

Furthermore, base change with $\mathcal{O}_K(\mathfrak{Y}_{I^{1/q_\mathfrak{Y}}})$ over $\mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})^{1/q_\mathfrak{Y}}})$ induces isomorphisms of Banach spaces

$$\varphi_{\text{lin}}^I = \mathcal{O}_K(\mathfrak{Y}_{I^{1/q_\mathfrak{Y}}}) \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})^{1/q_\mathfrak{Y}}})} \varphi_{M_0}^{\text{lin}} :$$

$$\mathfrak{O}_K(\mathfrak{Y}_{I^{1/q_\mathfrak{Y}}}) \otimes_{\mathfrak{O}_K(\mathfrak{Y}_I),\varphi_L} M^I \xrightarrow{\cong} M^{I^{1/q_\mathfrak{Y}}}$$

and hence injective, continuous maps

$$\varphi^I : M^I \to M^{I^{1/q_\mathfrak{Y}}}$$

by restriction.

Assuming that $I^{q_\mathfrak{Y}} \subseteq I(r', \mathfrak{Y})$, we define the additive, K-linear, continuous map $\psi^I : M^I \to M^{I^{q_\mathfrak{Y}}}$ as the composite

$$\psi^I : M^I \xrightarrow{(\varphi_{\text{lin}}^{I^{q_\mathfrak{Y}}})^{-1}} \mathcal{O}_K(\mathfrak{Y}_I) \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I^{q_\mathfrak{Y}}}),\varphi_L} M^{I^{q_\mathfrak{Y}}} \to M^{I^{q_\mathfrak{Y}}}, \tag{4.39}$$

where the last map sends $f \otimes m$ to $\psi^I(f)m$. By construction, it satisfies the projection formulae

$$\psi^I\big(\varphi^{I^{q_\mathfrak{Y}}}(f)m\big) = f\psi^I(m) \quad \text{and} \quad \psi^I\big(g\varphi^{I^{q_\mathfrak{Y}}}(m')\big) = \psi^I(g)m', \tag{4.40}$$

for any $f \in \mathcal{O}_K(\mathfrak{Y}_{I^{q_\mathfrak{Y}}})$, $g \in \mathcal{O}_K(\mathfrak{Y}_I)$, and $m \in M^I$, $m' \in M^{I^{q_\mathfrak{Y}}}$ as well as the formula

$$\psi^I \circ \varphi^{I^{q_\mathfrak{Y}}} = \frac{q_\mathfrak{Y}}{\pi_L} \cdot \text{id}_{M^{I^{q_\mathfrak{Y}}}}.$$

Using Proposition 4.1.14 in case $\mathfrak{Y} = \mathfrak{X}$, resp., the decomposition (4.29) in case $\mathfrak{Y} = \mathbf{B}$ (under the assumption that Ω is contained in K), combined with (iterates of) φ_{lin}^I gives rise to decompositions

$$M^{I^{\frac{1}{q_{\mathfrak{Y}}^n}}} = \begin{cases} \bigoplus_{a \in (o_L/\pi_L^n)} \text{ev}_a \, \varphi_L^n(M^I) & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \bigoplus_{a \in (o_L/\pi_L^n)} \eta(a, Z)\varphi_L^n(M^I) & \text{if } \mathfrak{Y} = \mathbf{B} \text{ and } \Omega \in K \end{cases} \tag{4.41}$$

of Banach spaces and

$$M = \begin{cases} \bigoplus_{a \in (o_L/\pi_L^n)} \text{ev}_a \, \varphi_L^n(M) & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \bigoplus_{a \in (o_L/\pi_L^n)} \eta(a, Z)\varphi_L^n(M) & \text{if } \mathfrak{Y} = \mathbf{B} \text{ and } \Omega \in K \end{cases} \tag{4.42}$$

of LF-spaces, again given by the formula

$$m = \begin{cases} \left(\frac{\pi_L}{q}\right)^n \sum_a \varphi_M \psi_M \,(\text{ev}_{-a}\, m)\, \text{ev}_a & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \left(\frac{\pi_L}{q}\right)^n \sum_a \varphi_M \psi_M \big(\eta(-a, Z)m\big)\eta(a, Z) & \text{if } \mathfrak{Y} = \mathbf{B} \text{ and } \Omega \in K. \end{cases}$$

4.3.4 The Robba ring of a group

Recall that $L_n = L(\ker([\pi_L^n]))$. We set

$$\Gamma_n := \text{Gal}(L_\infty/L_n) = \ker\left(\Gamma_L \xrightarrow{\chi_{\text{LT}}} o_L^\times \to (o_L/\pi_L^n)^\times\right).$$

Also recall from Section 4.1.2 the notation $U_n := 1 + \pi_L^n o_L$ for $n \geq 1$ and the isomorphisms $\log : U_n \xrightarrow{\cong} \pi_L^n o_L$ and $\ell_n = \pi_L^{-n} \log : U_n \xrightarrow{\cong} o_L$ for $n \geq n_0$, where $n_0 \geq 1$ is minimal among n such that $\log : 1 + \pi_L^n o_L \to \pi_L^n o_L$ and $\exp : \pi_L^n o_L \to 1 + \pi_L^n o_L$ are mutually inverse isomorphisms.

Obviously, χ_{LT} restricts to isomorphisms $\Gamma_n \cong U_n$ for any $n \geq 1$. Consider the composed maps

$$\hat{\ell} := \log \circ \chi_{\text{LT}} : \Gamma_L \to L \quad \text{and} \quad \hat{\ell}_n := \ell_n \circ \chi_{\text{LT}} : \Gamma_n \xrightarrow{\cong} o_L \quad \text{for } n \geq n_0.$$

The latter isomorphisms induce isomorphisms of Fréchet algebras $D(\Gamma_n, K) \xrightarrow{\cong} D(o_L, K)$.

Because of the isomorphisms $\Gamma_L \cong o_L^\times$ and $\Gamma_n \cong U_n$, the formalism of character varieties and corresponding Robba rings applies to the groups Γ_L and Γ_n as well, giving us the corresponding character varieties $\mathfrak{X}_{\Gamma_L}$ and $\mathfrak{X}_{\Gamma_n}$, and the results of Section 4.1.2 transfer to this setting. To make a clear distinction, we put $\mathcal{R}_K(\Gamma_L) := \mathcal{R}_K(\mathfrak{X}_{\Gamma_L})$ and $\mathcal{R}_K(\Gamma_n) := \mathcal{R}_K(\mathfrak{X}_{\Gamma_n})$ and call them the Robba rings of the groups Γ_L and Γ_n. Clearly, the Lubin–Tate character χ_{LT} induces topological ring isomorphisms

$$\mathcal{R}_K(\Gamma_L) \xrightarrow{\cong} \mathcal{R}_K(\mathfrak{X}^\times) \quad \text{and} \quad \mathcal{R}_K(\Gamma_n) \xrightarrow{\cong} \mathcal{R}_K(\mathfrak{X}_n^\times) \text{ for } n \geq 1. \tag{4.43}$$

If Γ denotes any of these groups, then we will very often view, via the Fourier isomorphism, $K[\Gamma] \subseteq D(\Gamma, K)$ as subrings of $\mathcal{R}_K(\Gamma)$. In particular, we consider elements $\gamma \in \Gamma$ as elements of the Robba ring writing them in any of the forms $\gamma \,\hat{=}\, \delta_\gamma \,\hat{=}\, \mathrm{ev}_\gamma$.

Let $n \geq m \geq 1$. The inclusions $\iota_n : \Gamma_n \hookrightarrow \Gamma_L$ and $\iota_{n,m} : \Gamma_n \hookrightarrow \Gamma_m$ induce, by the transfer of the results in Section 4.1.2, ring monomorphisms $\iota_{n*} : \mathcal{R}_K(\Gamma_n) \hookrightarrow \mathcal{R}_K(\Gamma_L)$ and $\iota_{n,m*} : \mathcal{R}_K(\Gamma_n) \hookrightarrow \mathcal{R}_K(\Gamma_m)$. More precisely, we have (by Lemma 4.1.15 and Remark 4.1.18) topological ring isomorphisms

$$\mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_n]} \mathcal{R}_K(\Gamma_n) \xrightarrow{\ \cong\ } \mathcal{R}_K(\Gamma_L), \tag{4.44}$$

$$\mathbb{Z}[\Gamma_m] \otimes_{\mathbb{Z}[\Gamma_n]} \mathcal{R}_K(\Gamma_n) \xrightarrow{\ \cong\ } \mathcal{R}_K(\Gamma_m). \tag{4.45}$$

Here the left-hand sides are viewed as free $\mathcal{R}_K(\Gamma_n)$-modules endowed with the product topology.

We also note that, for $n \geq m \geq n_0$, the commutative diagram

$$
\begin{array}{ccc}
\Gamma_n & \xrightarrow{\ \hat{\ell}_n\ } & o_L \\
{\scriptstyle \iota_{n,m}}\downarrow & & \downarrow{\scriptstyle \pi_L^{n-m}} \\
\Gamma_m & \xrightarrow{\ \hat{\ell}_m\ } & o_L
\end{array}
$$

induces the commutative diagrams

$$
\begin{array}{ccccc}
D(\Gamma_n, K) & \xrightarrow{\ \hat{\ell}_{n*}\ } & D(o_L, K) & \xrightarrow[\text{Fourier}]{\cong} & \mathcal{O}_K(\mathfrak{X})_{\varphi_L^{n-m}} \\
{\scriptstyle \iota_{n,m*}}\uparrow & & {\scriptstyle (\pi_L^{n-m})_*}\downarrow & & \downarrow \\
D(\Gamma_m, K) & \xrightarrow{\ \hat{\ell}_{m*}\ } & D(o_L, K) & \xrightarrow[\text{Fourier}]{\cong} & \mathcal{O}_K(\mathfrak{X})
\end{array}
$$

and

$$
\begin{array}{ccc}
\mathcal{R}_K(\Gamma_n) & \xrightarrow[\cong]{\ \hat{\ell}_{n*}\ } & \mathcal{R}_K(\mathfrak{X})^{\varphi_L^{n-m}} \\
{\scriptstyle \iota_{n,m*}}\uparrow & & \downarrow \\
\mathcal{R}_K(\Gamma_m) & \xrightarrow[\cong]{\ \hat{\ell}_{m*}\ } & \mathcal{R}_K(\mathfrak{X}).
\end{array}
\tag{4.46}
$$

For the rest of this subsection we *assume* that Ω is contained in K. Let $n \geq n_0$. We then have the isomorphisms of rigid varieties

$$\mathbf{B} \xrightarrow[\kappa]{\ \sim\ } \mathfrak{X} \xrightarrow[\hat{\ell}_n^*]{\ \sim\ } \mathfrak{X}_{\Gamma_n}.$$

For any closed interval $I \subseteq (0, 1)$ we therefore have the affinoid subdomain $\hat{\ell}_n^* \circ \kappa(\mathbf{B}_I)$ in $\mathfrak{X}_{\Gamma_n}$ and we may introduce the Banach algebra $\mathcal{R}_K^I(\Gamma_n) := \mathcal{O}_K(\hat{\ell}_n^* \circ \kappa(\mathbf{B}_I))$.

By its very construction the diagram

$$
\begin{CD}
\mathcal{R}_K^{I^{q^{n-m}}}(\Gamma_n) @>{\hat{\ell}_{n*}}>{\cong}> \mathcal{R}_K^{I^{q^{n-m}}}(\mathfrak{X}) @>{\kappa^*}>{\cong}> \mathcal{R}_K^{I^{q^{n-m}}}(\mathbf{B}) \\
@V{\iota_{n,m*}}VV @VV{\varphi_L^{n-m}}V @VV{\varphi_L^{n-m}}V \\
\mathcal{R}_K^{I}(\Gamma_m) @>{\hat{\ell}_{m*}}>{\cong}> \mathcal{R}_K^{I}(\mathfrak{X}) @>{\kappa^*}>{\cong}> \mathcal{R}_K^{I}(\mathbf{B}),
\end{CD}
$$

for $n \geq m \geq n_0$, is commutative. Together with (4.32) it implies the canonical isomorphism

$$
\mathbb{Z}[\Gamma_m] \otimes_{\mathbb{Z}[\Gamma_n]} \mathcal{R}_K^{I^{q^{n-m}}}(\Gamma_n) \xrightarrow{\cong} \mathcal{R}_K^{I}(\Gamma_m). \tag{4.47}
$$

We will denote the composite of Fourier and LT-isomorphism by

$$
\mathfrak{L} : D(o_L, K) \xrightarrow{\cong} \mathcal{O}_K(\mathfrak{X}) \xrightarrow{\cong} \mathcal{O}_K(\mathbf{B}).
$$

Recall that $\mathcal{O}_K(\mathbf{B})$ is a space of certain power series in the variable Z. We put

$$
X := \mathfrak{L}^{-1}(Z) \in D(o_L, K) \quad \text{and} \quad Y_n := \hat{\ell}_{n*}^{-1}(X) \in D(\Gamma_n, K) \text{ for } n \geq n_0.
$$

In this way, we can express elements in these distribution algebras as power series in these variables. This will later on be an important technical tool for our proofs.

As an immediate consequence of Remark 4.3.1, we find the following.

Remark 4.3.5. The subsequent inclusions are dense:

(i) $D(o_L, K)_{Z^{\mathbb{N}}} \subseteq \mathcal{R}_K(\mathfrak{X})$ and

(ii) $D(\Gamma_n, K)_{Y_n^{\mathbb{N}}} \subseteq \mathcal{R}_K(\Gamma_n)$ for $n \geq n_0$.

4.3.5 Locally $\mathbb{Q}_p$-analytic versus locally L-analytic distribution algebras

We fix a $\mathbb{Z}_p$-basis $h_1 = 1, \ldots, h_d$ of o_L and set $b_i := h_i - 1$ and, for any multi-index $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$, $\mathbf{b^k} := \prod_{i=1}^d b_i^{k_i} \in \mathbb{Z}_p[o_L]$. We write $D_{\mathbb{Q}_p}(G, K)$ for the algebra of K-valued locally $\mathbb{Q}_p$-analytic distributions on a $\mathbb{Q}_p$-Lie group G. Any $\lambda \in D_{\mathbb{Q}_p}(o_L, K)$ has a unique convergent expansion $\lambda = \sum_{\mathbf{k} \in \mathbb{N}_0^d} \alpha_{\mathbf{k}} \mathbf{b^k}$ with $\alpha_{\mathbf{k}} \in K$ such that, for any $0 < r < 1$, the set $\{\alpha_{\mathbf{k}} r^{\diamond |\mathbf{k}|}\}_{\mathbf{k} \in \mathbb{N}_0^d}$ is bounded, where $\diamond := 2$ if $p = 2$ and $\diamond := 1$ otherwise. The completion with respect to the norm

$$
\|\lambda\|_{\mathbb{Q}_p, r} := \sup_{\mathbf{k} \in \mathbb{N}_0^d} |\alpha_{\mathbf{k}}| r^{\diamond |\mathbf{k}|}
$$

for $0 < r < 1$ is denoted by

$$
D_{\mathbb{Q}_p, r}(o_L, K) = \left\{ \sum_{\mathbf{k} \in \mathbb{N}_0^d} \alpha_{\mathbf{k}} \mathbf{b^k} \mid \alpha_{\mathbf{k}} \in K \text{ and } |\alpha_{\mathbf{k}}| r^{\diamond |\mathbf{k}|} \to 0 \text{ as } |\mathbf{k}| \to \infty \right\}.
$$

By [68, Prop. 2.1], the group o_L satisfies the hypothesis (HYP) of [76] with p-valuation ω satisfying $\omega(h_i) = \diamond$. Thus, by [76, Thm. 4.5], restricting to the subfamily $q^{-e} < r < 1$, $r \in p^{\mathbb{Q}}$, the norms $\| \ \|_{\mathbb{Q}_p, r}$ are multiplicative. If not otherwise specified, we denote by $V \otimes_K W$ the projective tensor product of locally convex K-vector spaces V, W.

Lemma 4.3.6. *Let*

$$0 \to V \to W \to X \to 0$$

be a strict exact sequence of locally convex topological K-vector spaces with W metrizable and X Hausdorff, and then

(i)   *the sequence of the associated Hausdorff completed spaces*

$$0 \to \widehat{V} \to \widehat{W} \to \widehat{X} \to 0$$

   is again strict exact,

(ii)   *for a complete valued field extension F of K the associated sequence of completed base extension*

$$0 \to F \,\widehat{\otimes}_K V \to F \,\widehat{\otimes}_K W \to F \,\widehat{\otimes}_K X \to 0$$

   is again strict exact,

(iii)   *if W is a K-Banach space, V a closed subspace with induced norm and $X = W/V$ endowed with the quotient norm, then in* (ii) *the quotient norm coincides with the tensor product norm on $F \,\widehat{\otimes}_K X$.*

Proof. By [15, I.17 §2] with W also V, X, and all their completions are metrizable. Hence, the first statement follows from [17, IX.26 Prop. 5]. For the second statement we first obtain the exact sequence

$$0 \to F \otimes_K V \to F \otimes_K W \to F \otimes_K X \to 0$$

of metrizable locally convex spaces (cf. [61, Thm. 10.3.13]). The first non-trivial map is strict by [61, Thm. 10.3.8]. Regarding the strictness of the second map, one easily checks that $F \otimes_K W / F \otimes_K V$ endowed with the quotient topology satisfies the universal property of the projective tensor product $F \otimes_K X$. Now apply (i). The third item is contained in [36, §3, n° 2, Thm. 1], see also [91, Thm. 4.28]. $\blacksquare$

The kernel of the surjection of Fréchet spaces $D_{\mathbb{Q}_p}(o_L, K) \twoheadrightarrow D(o_L, K)$ is generated as a closed ideal by $\mathfrak{a} := \ker(L \otimes_{\mathbb{Q}_p} \mathrm{Lie}_{\mathbb{Q}_p}(o_L) \xrightarrow{a \otimes \mathfrak{x} \mapsto a\mathfrak{x}} \mathrm{Lie}_L(o_L))$. For $K = L$ this is [68, Lem. 5.1]. As seen in the proof of Lemma 4.1.2, we have

$$K \,\widehat{\otimes}_L D_{\mathbb{Q}_p}(o_L, L) = D_{\mathbb{Q}_p}(o_L, K) \quad \text{and} \quad K \,\widehat{\otimes}_L D(o_L, L) = D(o_L, K).$$

Hence, the assertion for general K follows from Lemma 4.3.6 (ii). We write $D_r(o_L, K)$ for the completion of $D(o_L, K)$ with respect to the quotient norm $\| \ \|_r$ of $\| \ \|_{\mathbb{Q}_p, r}$. By the proof of [76, Prop. 3.7] we have the exact sequence of K-Banach algebras

$$0 \to \hat{\mathfrak{a}}_r \to D_{\mathbb{Q}_p, r}(o_L, K) \to D_r(o_L, K) \to 0, \qquad (4.48)$$

where $\hat{\mathfrak{a}}_r$ denotes the closed ideal generated by $\mathfrak{a}$. Moreover, the K-Banach algebras $D_r(o_L, K)$ realize a Fréchet–Stein structure on $D(o_L, K)$. For convenience we set $r_m := q^{-\frac{\diamond e}{p^m}}$ for $m \geq 0$. We, of course, have

$$D(o_L, K) = \varprojlim_m D_{r_m}(o_L, K).$$

Moreover, according to [69, Cor. 5.13], one has

$$D_{r_m}(o_L, K) = \mathbb{Z}[o_L] \otimes_{\mathbb{Z}[p^m o_L]} D_{r_0}(p^m o_L, K).$$

We have corresponding results and will be using analogous notation for groups isomorphic to o_L. This applies, in particular, to Γ_n for any $n \geq n_0$. Note that $\Gamma_n^{p^m} = \Gamma_{n+me}$.

4.3.6 (φ, Γ)-modules

We recall the definition of as well as a few known facts about (φ_L, Γ_L)-modules (cf. [8]). Let $\mathfrak{Y}$ be either $\mathfrak{X}$ or $\mathbf{B}$ and $\mathcal{R} := \mathcal{R}_K(\mathfrak{Y})$. Any (φ_L, Γ_L)-module M over $\mathcal{R}$ is, by definition, in particular an $\mathcal{R}$-module with a semilinear action of the group Γ_L. Our aim in this section is to show that these two structures on M give rise to a module structure on M under the "group" Robba ring $\mathcal{R}_K(\Gamma_L)$.

Definition 4.3.7. A (φ_L, Γ_L)-module M over $\mathcal{R}$ is a φ_L-module M (see Definition 4.3.3) equipped with a semilinear continuous action of Γ_L which commutes with the endomorphism φ_M. We shall write $\mathcal{M}(\mathcal{R})$ for the category of (φ_L, Γ_L)-modules over $\mathcal{R}$.

The continuity condition for the Γ_L-action on M, of course, refers to the product topology on $M \cong \mathcal{R}^d$.

According to [8, Prop. 2.25], the Γ_L-action on a (φ_L, Γ_L)-module M is differentiable so that the derived action of the Lie algebra $\mathrm{Lie}(o_L^\times)$ on M is available.

Definition 4.3.8. The (φ_L, Γ_L)-module M over $\mathcal{R}$ is called L-analytic if the derived action $\mathrm{Lie}(\Gamma_L) \times M \to M$ is L-bilinear, i.e., if the induced action $\mathrm{Lie}(\Gamma_L) \to \mathrm{End}(M)$ of the Lie algebra $\mathrm{Lie}(\Gamma_L)$ of Γ_L is L-linear (and not just $\mathbb{Q}_p$-linear). We shall write $\mathcal{M}^{\mathrm{an}}(\mathcal{R})$ for the category of L-analytic (φ_L, Γ_L)-modules over $\mathcal{R}$.

In [8], a (φ_L, Γ_L)-module M over $\mathcal{R}$ is only required to be projective instead of free as in our definition. Since throughout this paper we are exclusively interested in L-analytic modules, that makes no difference as by [8, Thm. 3.17] any L-analytic (φ_L, Γ_L)-module M is actually a free $\mathcal{R}$-module.

We have the following variant of Proposition 4.3.4 (cf. [8, Prop. 2.24]).

Proposition 4.3.9. *Let M be a (φ_L, Γ_L)-module over $\mathcal{R}$. Then there exists a model (M_0, r_0) as in Proposition 4.3.4 equipped with a semilinear continuous action of Γ_L such that*

$$\mathcal{R} \otimes_{\mathcal{R}^{[r_0, 1)}} M_0 = M$$

respects the Γ_L-actions (acting diagonally on the left-hand side).

From now on in this subsection *we fix a (φ, Γ)-module M over $\mathcal{R}$ and a pair (r_0, M_0) as in Proposition 4.3.9*. We then have available the objects introduced after Proposition 4.3.4. But now the finitely generated free modules $M^{I(r', \mathfrak{Y})}$ and M^I are each in addition equipped with a semilinear continuous Γ_L-action, compatible with the identities (4.38). Moreover, the Γ_L-actions commute with the ψ^I-operators (4.39), and the decompositions (4.42) and (4.41) are Γ_L-equivariant.

Assume henceforth in this subsection that M is an *L-analytic (φ_L, Γ_L)-module* over $\mathcal{R}$.

Proposition 4.3.10. *The Γ_L-action on M extends uniquely to a separately continuous action of the locally L-analytic distribution algebra $D(\Gamma_L, K)$ of Γ_L with coefficients in K. If $M \xrightarrow{f} N$ is a homomorphism of L-analytic (φ_L, Γ_L)-modules, then f is $D(\Gamma_L, K)$-equivariant with regard to this action.*

Proof. First of all, we observe that the Dirac distributions generate a dense L-subspace in $D(\Gamma_L, L)$ by [75, Lem. 3.1]. Since $\Gamma_L \cong o_L^\times$, we have seen in the proof of Lemma 4.1.2 that $D(\Gamma_L, K) = K \hat{\otimes}_L D(\Gamma_L, L)$. Hence, the Dirac distributions also generate a dense K-linear subspace of $D(\Gamma_L, K)$. Therefore, the extended action is unique provided it exists.

Our assertion is easily reduced to the analogous statement concerning the Banach spaces M^I for a closed interval $I = [r, s]$. From [8, Props. 2.16 and 2.17] we know that the Γ_L-action on M^I is locally $\mathbb{Q}_p$-analytic. But since we assume M to be L-analytic, it is actually locally L-analytic (cf. the addendum to [8, Prop. 2.25] and the argument at the end of the proof of [8, Prop. 2.17]).

For our purpose we show more generally the existence, for any K-Banach space W, of a continuous K-linear map

$$I : \mathcal{C}^{an}(\Gamma_L, W) \to \mathcal{L}_b\big(D(\Gamma_L, K), W\big)$$

satisfying $I(f)(\delta_g) = f(g)$. Note that this map, if it exists, is unique by our initial observation. Recall (cf. [72, §12]) that the locally convex vector space $\mathcal{C}^{an}(\Gamma_L, W)$

is the locally convex inductive limit of finite products of Banach spaces of the form $B \,\widehat{\otimes}_K\, W$ with a Banach space B, and that its strong dual $D(\Gamma_L, K)$ is the corresponding projective limit of the finite sums of dual Banach spaces B'. We therefore may construct the map I as the inductive limit of finite products of maps of the form

$$B \,\widehat{\otimes}_K\, W \to \mathcal{L}_b(B', W)$$
$$x \otimes y \mapsto \big[\ell \mapsto \ell(x)y\big].$$

Since B as a Banach space is barreled, this map is easily seen to be continuous (cf. the argument in the proof of [71, Lem. 9.9]).

Now suppose that W carries a locally L-analytic Γ_L-action (e.g., $W = M^I$). For $y \in W$ let $\rho_y(g) := gy$ denote the orbit map in $\mathcal{C}^{\mathrm{an}}(\Gamma_L, W)$. We then define

$$D(\Gamma_L, K) \times W \to W$$
$$(\mu, y) \mapsto I(\rho_y)(\mu).$$

Due to our initial observation, the proof of [75, Prop. 3.2], that the above is a separately continuous module structure, remains valid even though K is not assumed to be spherically complete.

By [8, Rem. 2.20], the homomorphism f is continuous and hence the $D(\Gamma_L, K)$-equivariance of f follows from the Γ_L-invariance by the argument in the first paragraph of this proof. ∎

Recall that M^I, for each $I = [r, s]$ with $r \geq r_0$, bears a natural Γ_L-action. Now, for each $n \geq 1$, we will define a different action of Γ_n on $M^{[r,s]}$, which is motivated by Lemma 4.3.11 below and which is crucial for analyzing the structure of $M^{\psi_M = 0}$ in the next subsection. To this end, consider for each $\gamma \in \Gamma_n$ the operator $H_n(\gamma)$ on $M^{[r,s]}$ defined by

$$H_n(\gamma)(m) := \begin{cases} \mathrm{ev}_{\pi_L^{-n}(\chi_{\mathrm{LT}}(\gamma)-1)}\, \gamma m & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \eta\big(\pi_L^{-n}(\chi_{\mathrm{LT}}(\gamma) - 1), Z\big)\gamma m & \text{if } \mathfrak{Y} = \mathbf{B} \text{ and } \Omega \in K. \end{cases}$$

Note that, since Γ_n acts on $\mathcal{O}_K(\mathfrak{Y})$ via χ_{LT} and the $o_L^\times$-action, we may form the skew group ring $\mathcal{O}_K(\mathfrak{Y})[\Gamma_n]$, which due to the semilinear action of Γ_L on M maps into the K-Banach algebra $\mathcal{E}nd_K(M^I)$ of continuous K-linear endomorphisms of M^I, endowed with the operator norm $\|\,\|_{M^I}$. Hence, we obtain the ring homomorphism

$$H_n : K[\Gamma_n] \to \mathcal{O}_K(\mathfrak{Y})[\Gamma_n] \to \mathcal{E}nd_K(M^I)$$
$$\gamma \mapsto \begin{cases} \mathrm{ev}_{\pi_L^{-n}(\chi_{\mathrm{LT}}(\gamma)-1)}\, \gamma & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \eta\big(\pi_L^{-n}\big(\chi_{\mathrm{LT}}(\gamma) - 1\big), Z\big)\gamma & \text{if } \mathfrak{Y} = \mathbf{B} \text{ and } \Omega \in K. \end{cases}$$

The next lemma holds true in both cases. We spell it out only in the **B**-case since we technically need it only there.

Lemma 4.3.11. *Suppose that Ω is contained in K, and let $n \geq m \geq 1$.*

(i)   *We have for all $\sigma \in \Gamma_n$*

$$\sigma\big(\eta(1, Z)\varphi_L^n(y)\big) = \eta(1, Z)\varphi_L^n\big(H_n(\sigma)(y)\big),$$

i.e., the isomorphisms

$$M \xrightarrow{\cong} \eta(1, Z)\varphi_L^n(M),$$
$$M^{[r,s]} \xrightarrow{\cong} \eta(1, Z)\varphi_L^n(M^{[r,s]})$$
$$y \mapsto \eta(1, Z)\varphi_L^n(y)$$

are Γ_n-equivariant with respect to the natural action on the right-hand side and the action via H_n on the left-hand side.

(ii)   *The map*

$$\mathbb{Z}[\Gamma_m] \otimes_{\mathbb{Z}[\Gamma_n], H_n} M^{[r,s]} \to M^{[r^{1/q^{n-m}}, s^{1/q^{n-m}}]}$$
$$\gamma \otimes y \mapsto \eta\big(\tfrac{\chi_{\mathrm{LT}}(\gamma)-1}{\pi_L^m}, Z\big)\varphi_M^{n-m}(\gamma y)$$

is a homeomorphism of Banach spaces, where the left-hand side is viewed as the direct sum of Banach spaces $\bigoplus_{\gamma \in \Gamma_m/\Gamma_n} \gamma \otimes M^{[r,s]}$. Moreover, the map is Γ_m-equivariant with respect to the H_m-action on the right-hand side.

(iii)   *If $H_n : K[\Gamma_n] \to \mathcal{E}nd_K(M^I)$ extends to a continuous ring homomorphism $R_K^I(\Gamma_n) \to \mathcal{E}nd_K(M^I)$, then $H_m : K[\Gamma_m] \to \mathcal{E}nd_K(M^{I^{1/q^{n-m}}})$ similarly extends to a continuous homomorphism*

$$R_K^{I^{1/q^{n-m}}}(\Gamma_m) \to \mathcal{E}nd_K(M^{I^{1/q^{n-m}}}).$$

If the first extension is unique, so is the second one.

Proof. (i) Setting $b := \frac{\chi_{\mathrm{LT}}(\sigma)-1}{\pi_L^n}$, we calculate

$$\sigma\big(\eta(1, Z)\varphi_L^n(m)\big) = \sigma(\eta(1, Z))\varphi_L^n(\sigma m)$$
$$= \eta(1 + \pi_L^n b, Z)\varphi_L^n(\sigma m)$$
$$= \eta(1, Z)\eta(\pi^n b, Z)\varphi_L^n(\sigma m)$$
$$= \eta(1, Z)\varphi_L^n\big(\eta(b, Z)\sigma m\big),$$

where we used the multiplicativity of η in the first variable in the third and (4.33) in the last equality.

(ii) By a straightforward computation one first checks that the map is well defined. The bijectivity follows from (4.41) using the isomorphism $1 + \pi_L^m o_L/1 + \pi_L^n o_L \xrightarrow{\cong} o_L/\pi_L^{n-m} o_L$, $\gamma \mapsto \frac{\chi_{\mathrm{LT}}(\gamma)-1}{\pi_L^m}$ and that $M^{[r,s]} = \gamma M^{[r,s]}$.

(iii) Base change induces the $R_K^{I^{1/q^{n-m}}}(\Gamma_m)$-action on

$$
\begin{aligned}
R_K^{I^{1/q^{n-m}}}(\Gamma_m) \otimes_{R_K^I(\Gamma_n), H_n} M^I &\cong \mathbb{Z}[\Gamma_m] \otimes_{\mathbb{Z}[\Gamma_n]} R_K^I(\Gamma_n) \otimes_{R_K^I(\Gamma_n), H_n} M^I \\
&\cong \mathbb{Z}[\Gamma_m] \otimes_{\mathbb{Z}[\Gamma_n], H_n} \otimes M^I \\
&\cong M^{I^{1/q^{n-m}}},
\end{aligned}
\tag{4.49}
$$

where we used (4.47) and (ii). The continuity is easily checked by considering "matrix entries" which are built by composites of the original continuous map by other continuous transformations. Here we use that the identifications (4.47) and (4.49) are homeomorphisms when we endow the left-hand side with the maximum norm. Finally, the claim regarding uniqueness follows from (4.49) as the action of Γ_m is already determined by the original H_m. ∎

For the rest of this subsection we *assume that Ω is contained in K* and we will work exclusively in the **B**-case, i.e., $\mathcal{R} = \mathcal{R}_K(\mathbf{B})$ and $\mathcal{R}^I = \mathcal{R}_K^I(\mathbf{B})$. The consequences for the $\mathfrak{X}$-case will be given in Section 4.3.8.

There is a natural ring homomorphism $\mathcal{R}^I \to \mathcal{E}nd_K(M^I)$ by assigning to $f \in \mathcal{R}^I$ the operator which multiplies by f, and which we denote by the same symbol f. Part (iii) of the following remark means that this ring homomorphism has operator norm 1.

Remark 4.3.12. We have the following:

(i) $\sup_{x \in o_L} |\eta(x, Z) - 1|_I < 1$ and $|\eta(x, Z)|_I = 1$ for all $x \in o_L$,[7]

(ii) $|\eta(px, Z) - 1|_I \leq \max\{|\eta(x, Z) - 1|_I^p, \frac{1}{q^e}|\eta(x, Z) - 1|_I\}$, where the right-hand side is equal to $|\eta(x, Z) - 1|_I^p$ if $|\eta(x, Z) - 1|_I \geq q^{-\frac{e}{p-1}}$, and

(iii) $|f|_I = \|f\|_{M^I}$ for all $f \in \mathcal{R}^I$.

Proof. It is known (cf. [74]) that $\eta(x, Z) = \eta(1, [x](Z))$ belongs to $1 + Zo_{\mathbb{C}_p}[\![Z]\!]$, whence we have, for any $x \in o_L$, that $|\eta(x, Z) - 1|_I < 1$ from the definition of $|\ |_I$, and (i) follows from the fact that the map $o_L \to \mathbb{R}, x \mapsto |\eta(x, Z) - 1|_I$ is continuous with compact source. Affirmation (ii) is a consequence of the expansion

$$
\begin{aligned}
\eta(px, Z) - 1 &= \big(\eta(x, Z) - 1 + 1\big)^p - 1 \\
&= \big(\eta(x, Z) - 1\big)^p + \sum_{k=1}^{p-1} \binom{p}{k} \big(\eta(x, Z) - 1\big)^k
\end{aligned}
$$

and $\left|\binom{p}{k}\right| = q^{-e}$ for $k = 1, \ldots, p - 1$. (iii) follows from the submultiplicativity of $|\ |_I$ plus the fact that $1 \in \mathcal{R}^I$, which implies the statement on $M \cong (\mathcal{R}^I)^m$. ∎

---

[7]$|\eta(x, Z) - 1|_I = |\eta(1, Z) - 1|_I < 1$ for all $x \in o_L^\times$ because any $x \in o_L^\times$ induces an isomorphism $[x](-)$ of B_I.

Remark 4.3.12 allows us to fix a natural number $m_0 = m_0(r_0)$ such that for all $m \geq m_0$ we have that

$$\left|\eta(x, Z) - 1\right|_I < \mathfrak{r}_m \text{ for all } x \in o_L \text{ and } \left|\eta(x, Z) - 1\right|_I \leq \mathfrak{r}_0 \text{ for all } x \in p^m o_L,$$
$$r_0^{1/q} < \mathfrak{r}_m, \tag{4.50}$$

for any of the intervals $I = [r_0, r_0]$, $[r_0, r_0^{1/q}]$ and $[r_0^{1/q}, r_0^{1/q}]$. In the following, let I always denote one of those intervals.

Lemma 4.3.13. *Let $\varepsilon > 0$ be arbitrary. Then there exists $n_1 \gg 0$ such that, for any $n \geq n_1$, the operator norm $\| \ \|_{M^I}$ on M^I satisfies*

$$\|\gamma - 1\|_{M^I} \leq \varepsilon \quad \text{for all } \gamma \in \Gamma_n. \tag{4.51}$$

Proof. We first prove the statement for the module $M = \mathcal{R}$. For the convenience of the reader we adopt the proof of [41, Lem. 5.2]. First note that for any fixed $f \in R^I$ by the continuity of the action of Γ_L there exists an open normal subgroup H of Γ_L such that

$$\left|(\gamma - 1)f\right|_I < \varepsilon |f|_I \tag{4.52}$$

holds for all $\gamma \in H$. So we may assume that the latter inequality holds for Z and Z^{-1} simultaneously. Using the twisted Leibniz rule

$$(\gamma - 1)(gf) = (\gamma - 1)(g)f + \gamma(g)(\gamma - 1)(f)$$

and induction, we get (4.52) for all powers $Z^{\mathbb{Z}}$. Since the latter form an orthogonal basis, the claim follows using that $|\gamma(g)|_I = |g|_I$ for any $\gamma \in H, g \in \mathcal{R}^I$. If $M \cong \bigoplus_{i=1}^d \mathcal{R}e_i$ and $m = \sum f_i e_i$, we may assume that

$$\left|(\gamma - 1)e_i\right|_{M^I} < \varepsilon |e_i|_{M^I} \tag{4.53}$$

holds for $1 \leq i \leq d$, and apply the same Leibniz rule to $f_i e_i$ instead of gf, whence the result follows, noting that $|e_i|_{M^I} = 1$ by the definition of the maximum norm and that $|\gamma(e_i)|_{M^I} = |e_i|_{M^I} = 1$ for any $\gamma \in H$ and $1 \leq i \leq d$ as a consequence of (4.53). ∎

We fix $n_1 = n_1(r_0) \geq n_0$ such that the Lemma holds for $\varepsilon = \mathfrak{r}_0$. Then, for any $n \geq n_1, m \geq m_0$, the above H_n extends to continuous ring homomorphisms

$$\tilde{H}_n : D_{\mathbb{Q}_p, \mathfrak{r}_m}(\Gamma_n, K) \to \mathcal{E}nd_K(M^I)$$

$$\sum_{\mathbf{k} \in \mathbb{N}_0^d} \alpha_{\mathbf{k}} \hat{\ell}_{n,*}^{-1}(\mathbf{b})^{\mathbf{k}} \mapsto \sum_{k \geq 0} \alpha_{\mathbf{k}} \prod_{i=1}^d H_n\left(\hat{\ell}_{n,*}^{-1}(b_i)\right)^{k_i}$$

and

$$\widetilde{\mathbb{H}}_n := \widetilde{H}_n \circ \hat{\ell}_{n,*}^{-1} : D_{\mathbb{Q}_p, \mathfrak{r}_m}(o_L, K) \xrightarrow{\hat{\ell}_{n,*}^{-1}} D_{\mathbb{Q}_p, \mathfrak{r}_m}(\Gamma_n, K) \to \mathcal{E}nd_K(M^I)$$

$$\sum_{\mathbf{k} \in \mathbb{N}_0^d} \alpha_{\mathbf{k}} \mathbf{b}^{\mathbf{k}} \mapsto \sum_{k \geq 0} \alpha_{\mathbf{k}} \prod_{i=1}^d H_n\big(\hat{\ell}_{n,*}^{-1}(b_i)\big)^{k_i}.$$

Indeed, we have

$$H_n\big(\hat{\ell}_{n,*}^{-1}(b_i)\big) = \eta\bigg(\frac{\hat{\ell}_n^{-1}(h_i) - 1}{\pi_L^n}, Z\bigg) - 1 + \eta\bigg(\frac{\hat{\ell}_n^{-1}(h_i) - 1}{\pi_L^n}, Z\bigg)\big(\hat{\ell}_n^{-1}(h_i) - 1\big)$$

and since

$$\bigg\| \eta\bigg(\frac{\hat{\ell}_n^{-1}(h_i) - 1}{\pi_L^n}, Z\bigg) - 1 + \eta\bigg(\frac{\hat{\ell}_n^{-1}(h_i) - 1}{\pi_L^n}, Z\bigg)\big(\hat{\ell}_n^{-1}(h_i) - 1\big) \bigg\|_{M^I}$$

$$\leq \max\bigg\{ \bigg\| \eta\bigg(\frac{\hat{\ell}_n^{-1}(h_i) - 1}{\pi_L^n}, Z\bigg) - 1 \bigg\|_{M^I}, \bigg\| \eta\bigg(\frac{\hat{\ell}_n^{-1}(h_i) - 1}{\pi_L^n}, Z\bigg)\big(\hat{\ell}_n^{-1}(h_i) - 1\big) \bigg\|_{M^I} \bigg\}$$

$$\leq \mathfrak{r}_m$$

by (4.51), (4.50) and Remark 4.3.12 (i), the above defining sum converges with respect to the operator norm. Moreover, we have

$$\big\| \widetilde{\mathbb{H}}_n(\lambda) \big\|_{M^I} \leq \sup_{\mathbf{k}} |\alpha_{\mathbf{k}}| \mathfrak{r}_m^{|\mathbf{k}|} = \|\lambda\|_{\mathbb{Q}_p, \mathfrak{r}_m} \tag{4.54}$$

for all $\lambda \in D_{\mathbb{Q}_p, \mathfrak{r}_m}(o_L, K)$; i.e., the operator norm of $\widetilde{\mathbb{H}}_n$ is less than or equal to 1.

Since M is assumed to be L-analytic, $\widetilde{H}_n$ factorizes over the desired ring homomorphism

$$H_n : \big(D(\Gamma_n, K) \subseteq \big) D_{\mathfrak{r}_m}(\Gamma_n, K) \to \mathcal{E}nd_K(M^I)$$

and $\widetilde{\mathbb{H}}_n$ over

$$\mathbb{H}_n : \big(D(o_L, K) \subseteq \big) D_{\mathfrak{r}_m}(o_L, K) \to \mathcal{E}nd_K(M^I)$$

by (4.48). As $D_{\mathfrak{r}_m}(o_L, K)$ carries the quotient norm of $D_{\mathbb{Q}_p, \mathfrak{r}_m}(o_L, K)$, we obtain from (4.54)

$$\big\| \mathbb{H}_n(\lambda) \big\|_{M^I} \leq \inf_{\tilde{\lambda}, \mathrm{pr}(\tilde{\lambda}) = \lambda} \|\lambda\|_{\mathbb{Q}_p, \mathfrak{r}_m} = \|\lambda\|_{\mathfrak{r}_m} \tag{4.55}$$

for all $\lambda \in D_{\mathfrak{r}_m}(o_L, K)$; i.e., the operator norm of $\mathbb{H}_n$ is again less than or equal to 1. By a similar, but simpler, reasoning one shows the following lemma.

Lemma 4.3.14. *The isomorphism $\mathfrak{L} : D(o_L, K) \cong \mathcal{O}_K(\mathbf{B}), \delta_a \mapsto \eta(a, Z)$ (composed of LT and Fourier) induces, for all $m \geq m_0$, a commutative diagram of continuous maps*

$$
\begin{array}{c}
D_{\mathbb{Q}_p, \mathfrak{r}_m}(o_L, K) \\
\text{pr} \downarrow \qquad \searrow \\
D_{\mathfrak{r}_m}(o_L, K) \xrightarrow{\ \mathfrak{L}\ } \mathcal{R}^I
\end{array}
$$

with operator norms less than or equal to 1. Moreover, the operator norm of the scalar *action via $\mathfrak{L}$*

$$
\text{scal} : \big(D(o_L, K) \subseteq \big) D_{\mathfrak{r}_m}(o_L, K) \xrightarrow{\mathfrak{L}} \mathcal{R}^I \to \mathcal{E}nd_K(M^I) \tag{4.56}
$$

is also bounded by 1, see Remark 4.3.12 (iii).

Remark 4.3.15. The maps $\widetilde{H}_n$ and $\widetilde{\mathbb{H}}_n$, as well as H_n and $\mathbb{H}_n$, are uniquely determined by their restriction to $K[\Gamma_n]$ and $K[o_L]$, respectively, because these group algebras are dense in the sources $D_{\mathbb{Q}_p, \mathfrak{r}_m}(\Gamma_n, K)$, $D_{\mathfrak{r}_m}(\Gamma_n, K)$ and $D_{\mathbb{Q}_p, \mathfrak{r}_m}(o_L, K)$, $D_{\mathfrak{r}_m}(o_L, K)$, respectively.

Applying our convention before Remark 4.3.12, we usually abbreviate $\text{scal}(\mu)$ by $\mathfrak{L}(\mu)$ for $\mu \in D(o_L, K)$ below when we refer to this scalar action on M^I. For the proof of Theorem 4.3.20 below it will be crucial to compare the two actions scal and $\mathbb{H}_n$ of $D(o_L, K)$ on M^I.

Finally, for $n \geq n_1$, we obtain similar maps for the original (multiplicative) action of $\Gamma_n \subseteq \Gamma$ on M^I:

$$
D_{\mathbb{Q}_p, \mathfrak{r}_m}(\Gamma_n, K) \to \mathcal{E}nd_K(M^I)
$$

$$
\sum_{\mathbf{k} \in \mathbb{N}_0^d} \alpha_{\mathbf{k}} \hat{\ell}_{n,*}^{-1}(\mathbf{b})^{\mathbf{k}} \mapsto \sum_{k \geq 0} \alpha_{\mathbf{k}} \prod_{i=1}^d \hat{\ell}_{n,*}^{-1}(b_i)^{k_i}
$$

with operator norm bounded by 1.

A special case of the following lemma was pointed out to us by Rustam Steingart.

Lemma 4.3.16. *Let $m \in \mathbb{N}$ be arbitrary. Setting $u_n(a) := \frac{\exp(\pi_L^n a) - 1}{\pi_L^n a}$ for $a \in o_L \setminus \{0\}$ and $u_n(0) = 1$, there exist $n_2 = n_2(m)$ such that $u_n(a) \in 1 + \pi_L^m o_L$ for all $a \in o_L$ and $n \geq n_2$.*

Proof. This is easily checked using $v_p(n!) \leq \frac{n}{p-1}$. ∎

In order to distinguish Dirac distributions for elements γ in the *multiplicative* group Γ_n from those for elements a in the *additive* group o_L, we often shall write $\delta_\gamma^\times$ in contrast to δ_a.

Lemma 4.3.17. *Let $0 < \varepsilon < 1$ be arbitrary and $\Delta = \sum_k c_k(\delta_{a_k} - 1) \in D(o_L, K)$ a finite sum with $a_k \in o_L$. Then there exists $n_3 = n_3(\varepsilon, \Delta, r_0)$ such that for all $n \geq n_3$ it holds that*

$$\left\| \mathfrak{L}(\Delta) - \mathbb{H}_n(\Delta) \right\|_{M^I} < \varepsilon.$$

Proof. Put $\xi := \sup_k |c_k|$ and choose $\varepsilon' \leq \varepsilon$ such that $\varepsilon'\xi < \varepsilon$. Then choose $m_1 \geq m_0$ such that

$$\left\| \delta_\gamma^\times - 1 \right\|_{M^I} < \varepsilon' \quad \text{and} \quad \left\| \delta_\gamma^\times - 1 \right\|_{\mathcal{R}^I} < \varepsilon'$$

for all $\gamma \in \Gamma_{m_1}$ (see Lemma 4.3.13). Now according to Lemma 4.3.16, we choose $n_3 := n_2(m_1) \geq m_1$. Observing that for $a \in o_L$

$$\mathbb{H}_n(\delta_a) = \mathfrak{L}(\delta_{u_n(a)a}) \circ \delta_{\hat{\ell}_n^{-1}(a)}^\times,$$

we estimate, for $n \geq n_3$,

$$\left\| \mathfrak{L}(\Delta) - \mathbb{H}_n(\Delta) \right\|_{M^I}$$

$$= \left\| \sum_k c_k \left\{ \mathfrak{L}(\delta_{a_k}) - 1 - \left(\mathfrak{L}(\delta_{u_n(a_k)a_k}) \circ \delta_{\hat{\ell}_n^{-1}(a_k)}^\times - 1 \right) \right\} \right\|_{M^I}$$

$$= \left\| \sum_k c_k \left\{ \mathfrak{L}(\delta_{a_k}) - 1 - \left(\mathfrak{L}(\delta_{u_n(a_k)a_k}) \circ \left(\delta_{\hat{\ell}_n^{-1}(a_k)}^\times - 1 \right) + \mathfrak{L}(\delta_{u_n(a_k)a_k}) - 1 \right) \right\} \right\|_{M^I}$$

$$= \left\| \sum_k c_k \left\{ \mathfrak{L}(\delta_{a_k}) - 1 - \left(\mathfrak{L}(\delta_{u_n(a_k)a_k}) \circ \left(\delta_{\hat{\ell}_n^{-1}(a_k)}^\times - 1 \right) + \delta_{u_n(a_k)}^\times \left(\mathfrak{L}(\delta_{a_k}) - 1 \right) \right) \right\} \right\|_{M^I}$$

$$= \left\| \sum_k c_k \left\{ \left(1 - \delta_{u_n(a_k)}^\times \right) \left(\mathfrak{L}(\delta_{a_k}) - 1 \right) - \left(\mathfrak{L}(\delta_{u_n(a_k)a_k}) \circ \left(\delta_{\hat{\ell}_n^{-1}(a_k)}^\times - 1 \right) \right) \right\} \right\|_{M^I}$$

$$\leq \sup_k \left(|c_k| \max \left\{ \left\| \left(1 - \delta_{u_n(a_k)}^\times \right) \left(\mathfrak{L}(\delta_{a_k}) - 1 \right) \right\|_{M^I}, \left\| \mathfrak{L}(\delta_{u_n(a_k)a_k}) \circ \left(\delta_{\hat{\ell}_n^{-1}(a_k)}^\times - 1 \right) \right\|_{M^I} \right\} \right)$$

$$\leq \varepsilon' \sup_k |c_k| = \varepsilon'\xi < \varepsilon,$$

where we used for the last line the estimate

$$\left\| \left(1 - \delta_{u_n(a_k)}^\times \right) \left(\mathfrak{L}(\delta_{a_k}) - 1 \right) \right\|_{M^I} = \left| \left(1 - \delta_{u_n(a_k)}^\times \right) \left(\mathfrak{L}(\delta_{a_k}) - 1 \right) \right|_I$$

$$\leq \left\| \left(1 - \delta_{u_n(a_k)}^\times \right) \right\|_{\mathcal{R}^I} \left| \mathfrak{L}(\delta_{a_k}) - 1 \right|_I$$

$$\leq \varepsilon' \left| \eta(a_k, Z) - 1 \right|_I \leq \varepsilon'$$

(by Remark 4.3.12 (i), (iii) and due to the choice of m_1 and n_3) for the first term as well as the estimate

$$\left\| \mathfrak{L}(\delta_{u_n(a_k)a_k}) \circ \left(\delta_{\hat{\ell}_n^{-1}(a_k)}^\times - 1 \right) \right\|_{M^I} \leq \left| \eta\left(u_n(a_k)a_k, Z \right) \right|_I \left\| \delta_{\hat{\ell}_n^{-1}(a_k)}^\times - 1 \right\|_{M^I}$$

$$= \left\| \delta_{\hat{\ell}_n^{-1}(a_k)}^\times - 1 \right\|_{M^I} < \varepsilon'$$

for the second term (again by Remark 4.3.12 (i), (iii) and due to the choice of m_1 and $n_3 \geq m_1$). ∎

Lemma 4.3.18. *Let $0 < \varepsilon < 1$ be arbitrary and $\mu \in D(o_L, K)$ any element. Then there exists $\Delta = \sum_k c_k(\delta_{a_k} - 1) \in D(o_L, K)$ a finite sum with $a_k \in o_L$ such that $\|\mu - \Delta\|_{\mathrm{r}_{m_0}} < \varepsilon$. Moreover, for $n_3 = n_3(\varepsilon, \Delta, r_0)$ from the previous lemma and all $n \geq n_3$, we have*

$$\big\| \mathfrak{L}(\mu) - \mathbb{H}_n(\mu) \big\|_{M^I} < \varepsilon.$$

In particular, if $\mathfrak{L}(\mu)$ is invertible as an operator in $\mathcal{E}nd_K(M^I)$ or equivalently invertible as an element of $\mathcal{R}^I$,[8] then firstly there exists $n_4 = n_4(\mu, r_0)$ such that

$$\big\| \mathfrak{L}(\mu) - \mathbb{H}_n(\mu) \big\|_{M^I} < |\mathfrak{L}(\mu)^{-1}|_I^{-1} \quad \text{and} \quad \big\| \mathbb{H}_n(\mu)^{-1} - \mathfrak{L}(\mu)^{-1} \big\|_{M^I} < |\mathfrak{L}(\mu)^{-1}|_I$$

for any $n \geq n_4$ and secondly the operator $\mathbb{H}_n(\mu)$ is invertible, too.

Proof. The existence of such Δ is clear because such elements form a dense subset of $D(o_L, K)$ in the Fréchet topology (as noted at the beginning of the proof of Proposition 4.3.10). Consider the estimation for $n \geq n_3$

$$\big\| \mathfrak{L}(\mu) - \mathbb{H}_n(\mu) \big\|_{M^I}$$
$$\leq \max\left(\big\| \mathfrak{L}(\mu - \Delta) \big\|_{M^I}, \big\| \mathfrak{L}(\Delta) - \mathbb{H}_n(\Delta) \big\|_{M^I}, \big\| \mathbb{H}_n(\mu - \Delta) \big\|_{M^I} \right) < \varepsilon,$$

where we use the estimate

$$\big\| \mathfrak{L}(\mu - \Delta) \big\|_{M^I} = \big| \mathfrak{L}(\mu - \Delta) \big|_I \leq \| \mu - \Delta \|_{\mathrm{r}_{m_0}} < \varepsilon$$

by (4.56) for the first, Lemma 4.3.17 for the second, and (4.55) for the last term.

Now suppose that $\mathfrak{L}(\mu)$ as an operator on M^I is invertible. We choose suitable $\varepsilon \leq \|\mathfrak{L}(\mu)^{-1}\|_{M^I}^{-1}$, Δ accordingly and put $n_4 = n_3(\varepsilon, \Delta, r_0)$. Then, for $n \geq n_4$, we have

$$\big\| 1 - \mathfrak{L}(\mu)^{-1} \mathbb{H}_n(\mu) \big\|_{M^I} = \big\| \mathfrak{L}(\mu)^{-1} \big(\mathfrak{L}(\mu) - \mathbb{H}_n(\mu) \big) \big\|_{M^I} < 1,$$

whence $\sum_{k \geq 0} (1 - \mathfrak{L}(\mu)^{-1} \mathbb{H}_n(\mu))^k$ converges in $\mathcal{E}nd_K(M^I)$ and

$$\mathbb{H}_n(\mu)^{-1} := \left(\sum_{k \geq 0} \big(1 - \mathfrak{L}(\mu)^{-1} \mathbb{H}_n(\mu) \big)^k \right) \mathfrak{L}(\mu)^{-1}$$

is the inverse of $\mathbb{H}_n(\mu) = \mu(1 - (1 - \mathfrak{L}(\mu)^{-1} \mathbb{H}_n(\mu)))$.

Furthermore,

$$\big\| \mathbb{H}_n(\mu)^{-1} - \mathfrak{L}(\mu)^{-1} \big\|_{M^I} = \left\| \left(\sum_{k \geq 1} \big(1 - \mathfrak{L}(\mu)^{-1} \mathbb{H}_n(\mu) \big)^k \right) \mathfrak{L}(\mu)^{-1} \right\|_{M^I}$$
$$\leq \sup_{k \geq 1} \big\| 1 - \mathfrak{L}(\mu)^{-1} \mathbb{H}_n(\mu) \big\|_{M^I}^k \big| \mathfrak{L}(\mu)^{-1} \big|_I < \big| \mathfrak{L}(\mu)^{-1} \big|_I. \quad \blacksquare$$

---

[8] M^I is a free $\mathcal{R}^I$-module on which $\mathfrak{L}(\mu)$ acts via the diagonal matrix with all diagonal entries equal to $\mathfrak{L}(\mu)$.

Note that the above lemma applies to the variable X and from now on we set $n_4 := n_4(X, r_0)$. In view of Lemma 4.3.11 (iii), the following lemma is crucial for the main result Theorem 4.3.20 of this section.

Lemma 4.3.19. *For $n \geq n_4$,*

(i)   *the map $\Theta_n : D(\Gamma_n, K) \xrightarrow{H_n} \mathcal{E}nd_K(M^I)$ extends uniquely to a continuous ring homomorphism*

$$\mathcal{R}_K^I(\Gamma_n) \to \mathcal{E}nd_K(M^I).$$

If $M \xrightarrow{f} N$ is a homomorphism of L-analytic (φ_L, Γ_L)-modules, then $f^I : M^I \to N^I$ is $\mathcal{R}_K^I(\Gamma_n)$-equivariant with regard to this action.

(ii) *M^I is a free $\mathcal{R}_K^I(\Gamma_n)$-module of rank $\mathrm{rk}_{\mathcal{R}} M$. Any basis as $\mathcal{R}^I$-module also is a basis as $\mathcal{R}_K^I(\Gamma_n)$-module.*

(iii) *The natural maps*

$$\mathcal{R}_K^{[r_0,r_0]}(\Gamma_n) \otimes_{\mathcal{R}_K^{[r_0,r_0^{\frac{1}{q}}]}(\Gamma_n)} M^{[r_0,r_0^{\frac{1}{q}}]} \xrightarrow{\cong} M^{[r_0,r_0]},$$

$$\mathcal{R}_K^{[r_0^{\frac{1}{q}},r_0^{\frac{1}{q}}]}(\Gamma_n) \otimes_{\mathcal{R}_K^{[r_0,r_0^{\frac{1}{q}}]}(\Gamma_n)} M^{[r_0,r_0^{\frac{1}{q}}]} \xrightarrow{\cong} M^{[r_0^{\frac{1}{q}},r_0^{\frac{1}{q}}]}$$

are isomorphisms.

Proof. (i) Inductively, for $n \geq n_4$, we obtain from Lemma 4.3.18 – by expressing $(\mathbb{H}_n(\mu)^{\pm})^k - (\mathcal{L}(\mu)^{\pm})^k$ as $\sum_{l=1}^{k} \binom{k}{l}(\mathbb{H}_n(\mu)^{\pm} - \mathcal{L}(\mu)^{\pm})^l (\mathcal{L}(\mu)^{\pm})^{k-l}$ – that

$$\left\| \mathbb{H}_n(\mu)^k - \mathcal{L}(\mu)^k \right\|_{M^I} \leq \begin{cases} \left| \mathcal{L}(\mu) \right|_I^k & \text{for } k \geq 0, \\ \left| \mathcal{L}(\mu)^{-1} \right|_I^{-k} \leq \left| \mathcal{L}(\mu) \right|_I^k \leq \left| \mathcal{L}(\mu)^k \right|_I & \text{for } k < 0 \end{cases} \quad (4.57)$$

for all $k \in \mathbb{Z}$. Evaluating for $\mu = X$, it follows that if $\sum_{k \in \mathbb{Z}} a_k Z^k \in \mathcal{R}^I$ with $a_i \in K$, then $\sum_{k \in \mathbb{Z}} a_k \mathbb{H}_n(X)^k$ converges in $\mathcal{E}nd_K(M^I)$ because

$$\left\| a_k \mathbb{H}_n(X)^k \right\|_{M^I} \leq \max \left\{ \left\| a_k \left(\mathbb{H}_n(X)^k - \mathcal{L}(\mu)^k \right) \right\|_{M^I}, \left\| a_k \mathcal{L}(\mu)^k \right\|_{M^I} \right\}$$

$$\leq \begin{cases} |a_k| |Z|_I^k & \text{for } k \geq 0, \\ |a_k| |Z^{-1}|_I^{-k} & \text{for } k < 0 \end{cases}$$

goes to zero for k going to $\pm\infty$. In other words, we have extended the continuous ring homomorphism Θ_n to a continuous ring homomorphism

$$\mathcal{R}^I \to \mathcal{E}nd_K(M^I), \quad Z \mapsto \mathbb{H}_n(X).$$

As by definition $\kappa^* \circ \hat{\ell}_{n,*}$ extends to a continuous ring isomorphism $\mathcal{R}_K^I(\Gamma_n) \xrightarrow{\cong} \mathcal{R}_K^I(\mathbf{B}) = \mathcal{R}^I$, we have constructed a continuous ring homomorphism

$$\mathcal{R}_K^I(\Gamma_n) \to \mathcal{E}nd_K(M^I)$$

as claimed.

The uniqueness is a consequence of the fact that $\mathcal{R}_K^I(\Gamma_n)$ is the completion of the localization $D(\Gamma_n, K)_{Y_n^{\mathbb{N}}}$ by a certain norm, for which the extended map is continuous.

Concerning functoriality, observe that the maps f and f^I are automatically continuous by [8, Rem. 2.20] (with respect to the canonical topologies). Without loss of generality, we may assume that the estimates of Lemma 4.3.18 hold for M and N simultaneously. By the invariance under the distribution algebra and $\mathcal{R}$-linearity of f, the map f^I is compatible with respect to the operators $\mathbb{H}_n(X)^{\pm}$ of M^I and N^I. By continuity this extends to arbitrary elements of $\mathcal{R}_K^I(\Gamma_n)$.

(ii) follows in the same way as in [42]: Recall that (e_k) denotes an $\mathcal{R}^I$-basis of M^I and consider the maps

$$\Phi : \bigoplus_{k=1}^m \mathcal{R}_K^I(\mathbf{B}) \cong M^I, \quad (f_k) \mapsto \sum_{k=1}^m f_k e_k,$$

$$\Phi' : \bigoplus_{k=1}^m \mathcal{R}_K^I(\Gamma_n) \to M^I, \quad (f_k) \mapsto \sum_{k=1}^m f_k(e_k),$$

and

$$\Upsilon : \bigoplus_{k=1}^m \mathcal{R}_K^I(\mathbf{B}) \xrightarrow{\cong} \bigoplus_{k=1}^m \mathcal{R}_K^I(\Gamma_n),$$

which in each component is given by $(\kappa^* \circ \hat{\ell}_{n,*})^{-1}$. Then we have from (4.57) that

$$\left| \Phi' \circ \Upsilon \circ \Phi^{-1}(m) - m \right|_I < |m|_I,$$

i.e.,

$$\| \Phi' \circ \Upsilon \circ \Phi^{-1} - \mathrm{id} \|_I < 1,$$

whence with Φ and Υ also Φ' is an isomorphism because $\Phi' \circ \Upsilon \circ \Phi^{-1}$ is invertible by the usual argument using the geometric series.

(iii) The base change property follows from the fact that Φ' is compatible with changing the interval.   ∎

Theorem 4.3.20. *Suppose that Ω is contained in K.*

(i)   *Let J be any of the intervals*

$$[r_0, r_0]^{1/q^n} \quad \text{or} \quad [r_0, r_0^{1/q}]^{1/q^n} \quad \text{for } n \geq 0.$$

> Then the Γ_{n_4}-action on M^J via H_{n_4} can be uniquely extended to a continuous $\mathcal{R}_K^J(\Gamma_{n_4})$-module structure. Moreover, M^J is a finitely generated free $\mathcal{R}_K^J(\Gamma_{n_4})$-module; any $\mathcal{R}^{[r_0,1)}$-basis of M_0 is also an $\mathcal{R}_K^J(\Gamma_{n_4})$-basis of M^J. If $M \xrightarrow{f} N$ is a homomorphism of L-analytic (φ_L, Γ_L)-modules, then f^J : $M^J \to N^J$ is $\mathcal{R}_K^J(\Gamma_{n_4})$-equivariant with regard to this action.

(ii) The Γ_1-action on M via H_1 extends uniquely to a separately continuous $\mathcal{R}_K(\Gamma_1)$-module structure. Moreover, M is a finitely generated free $\mathcal{R}_K(\Gamma_1)$-module; any $\mathcal{R}^{[r_0,1)}$-basis of M_0 is also an $\mathcal{R}_K(\Gamma_1)$-basis of M. If $M \xrightarrow{f} N$ is a homomorphism of L-analytic (φ_L, Γ_L)-modules, then f is $\mathcal{R}_K(\Gamma_1)$-equivariant with regard to this action.

Proof. (i) From Lemma 4.3.19 we obtain, for any $n \geq n_4$, the H_n-action of $\mathcal{R}_K(\Gamma_n)$ on M^I for the original three intervals I. Using Lemma 4.3.11 (iii), we deduce the H_{n_4}-action of $\mathcal{R}_K^{I^{1/q^{n-n_4}}}(\Gamma_{n_4})$-action on $M^{I^{1/q^{n-n_4}}}$. The asserted properties of these actions follow from the same lemmata.

(ii) By the uniqueness part in (i) we may glue the $\mathcal{R}_K^J(\Gamma_{n_4})$-actions on the M^J to a continuous $\mathcal{R}_K^{[r_0,1)}(\Gamma_{n_4})$-action on $M^{[r_0,1)}$. By Remark 4.3.5 (ii) it is uniquely determined by the Γ_{n_4}-action. Therefore, we may vary r_0 now and obtain in the inductive limit a separately continuous H_{n_4}-action of $\mathcal{R}_K(\Gamma_{n_4})$ on M. Using (4.45) and Lemma 4.3.11 (ii), we deduce the separately continuous H_1-action of $\mathcal{R}_K(\Gamma_1)$ on M. Again by Remark 4.3.5 this action is uniquely determined by the Γ_1-action. The remaining assertions follow from the corresponding ones in (i). ∎

4.3.7 The structure of $M^{\psi_M=0}$

We still *assume that* Ω *is contained in* K and let M be an L-analytic (φ_L, Γ)-module over $\mathcal{R} = \mathcal{R}_K(\mathbf{B})$. We want to show that $M^{\psi_L=0}$ carries a natural $\mathcal{R}_K(\Gamma_L)$-action extending the action of $D(\Gamma_L, K)$.

From (4.42) and using formula (4.35) and (4.34) we have

$$M^{\psi_L=0} = \bigoplus_{a\in(o_L/\pi_L)^\times} \eta(a,Z)\varphi_M(M) = \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_1]} \left(\eta(1,Z)\varphi_M(M)\right). \quad (4.58)$$

Theorem 4.3.21. *The Γ_L action on M extends to a unique separately continuous $\mathcal{R}_K(\Gamma_L)$-action on $M^{\psi_L=0}$ (with respect to the LF-topology on $\mathcal{R}_K(\Gamma_L)$ and the subspace topology on $M^{\psi_L=0}$); moreover, the latter is a free $\mathcal{R}_K(\Gamma_L)$-module of rank $\mathrm{rk}_{\mathcal{R}} M$. If $e_1, \ldots, e_r$ is a basis of M over $\mathcal{R}$, an $\mathcal{R}_K(\Gamma_L)$-basis of $M^{\psi_L=0}$ is given by $\eta(1, Z)\varphi_M(e_1), \ldots, \eta(1, Z)\varphi_M(e_r)$. If $M \xrightarrow{f} N$ is a homomorphism of L-analytic (φ_L, Γ_L)-modules, then $M \xrightarrow{f^{\psi_L=0}} N$ is $\mathcal{R}_K(\Gamma_L)$-equivariant with regard to this action.*

Proof. By Lemma 4.3.11 (i) we can transfer the $\mathcal{R}_K(\Gamma_1)$-action on M from Theorem 4.3.20 (ii) to the space $\eta(1, Z)\varphi_M(M)$. Note that the resulting action is separately continuous for the subspace topology of $\eta(1, Z)\varphi_M(M)$ because the map $\varphi_L : M \to M$ is a homeomorphism onto its image. The latter is a consequence of the existence of the continuous operator ψ_L and the relation $\psi_L \circ \varphi_L = \frac{q}{\pi_L}\mathrm{id}_M$. Finally, because of (4.44) and (4.58), the $\mathcal{R}_K(\Gamma_1)$-action extends to the asserted $\mathcal{R}_K(\Gamma_L)$-action. Similarly as before, since Γ_L spans a dense subspace of $D(\Gamma_L, K)$, the uniqueness of the action follows from Remark 4.3.5. ∎

4.3.8 Descent

For the proof of Theorem 4.3.21 we had to work over a field K containing the period Ω since only then we were able to write elements in $\mathcal{R}_K(\Gamma_L)$ or rather $\mathcal{R}_K(\Gamma_n)$ as certain Laurent series in one variable Y_n by means of the Lubin–Tate isomorphism $\mathcal{R}_K(\mathfrak{X}) \cong \mathcal{R}_K(\mathbf{B})$, which in general does not exist over L. In this section, we are going to explore to which extent the structure theorem over K descends to L. We shall consider two situations; i.e., we now start with an L-analytic (φ_L, Γ)-module M over $\mathcal{R}_L(\mathfrak{X})$ or $\mathcal{R}_L(\mathbf{B})$, respectively. Thus, in what follows let $\mathfrak{Y}$ be either $\mathfrak{X}$ or $\mathbf{B}$ and $\mathcal{R} = \mathcal{R}_L(\mathfrak{Y})$. Then we consider the functor

$$\mathcal{M}^{\mathrm{an}}\big(\mathcal{R}_L(\mathfrak{Y})\big) \to \mathcal{M}^{\mathrm{an}}\big(\mathcal{R}_K(\mathfrak{Y})\big)$$
$$M \mapsto M_K := \mathcal{R}_K(\mathfrak{Y}) \otimes_{\mathcal{R}_L(\mathfrak{Y})} M \cong K \,\widehat{\otimes}_{\iota,L}\, M,$$

where the last isomorphism and the well-definedness of the functor are established in [8, Lem. 2.23]. Moreover, there is a natural action of G_L on both $\mathcal{R}_K(\mathfrak{Y}) \cong K \,\widehat{\otimes}_{\iota,L}\, \mathcal{R}_L(\mathfrak{Y})$ and M_K via the first tensor factor (and the identity on the second). We have

$$\mathcal{R}_K(\mathfrak{Y})^{G_L} \cong \mathcal{R}_L(\mathfrak{Y}) \tag{4.59}$$

by [8, Prop. 2.7 (iii)], whence also

$$(M_K)^{G_L} = M$$

because M is finitely generated free over $\mathcal{R}_L(\mathfrak{Y})$ (by definition or [8, Thm. 3.17]) and hence M_K has a G_L-invariant basis over $\mathcal{R}_K(\mathfrak{Y})$.

Since the φ_L-operator on $\mathcal{R}_K(\mathfrak{Y})$ is induced from that on $\mathcal{R}_L(\mathfrak{Y})$, it commutes with the action of G_L. Similarly, one checks that this action commutes with the operator ψ_L of $\mathcal{R}_K(\mathfrak{Y})$. Indeed, by Proposition 4.1.14 there exists a G_L-invariant basis of $\mathcal{R}_K(\mathfrak{Y})$ over $\varphi_L(\mathcal{R}_K(\mathfrak{Y}))$, whence the trace commutes with the G_L-action. From this and the construction of the operator ψ_M, one derives easily that also ψ_M commutes with the G_L-action. As a consequence, we obtain natural isomorphisms

$$M^{\psi_M=0} \cong \big((M_K)^{G_L}\big)^{\psi_M=0} \cong \big((M_K)^{\psi_M=0}\big)^{G_L}. \tag{4.60}$$

Since the Lubin–Tate isomorphism $\mathcal{R}_K(\mathfrak{X}) \cong \mathcal{R}_K(\mathbf{B})$ respects the (φ_L, Γ_L)-module structure, Theorem 4.3.21 applies for both choices of $\mathfrak{Y}$; i.e., we obtain a separately continuous action

$$\mathcal{R}_K(\Gamma_L) \times (M_K)^{\psi=0} \to (M_K)^{\psi=0}. \tag{4.61}$$

Moreover, if $M = \bigoplus_{i=1}^{r} \mathcal{R}_L(\mathfrak{Y})e_i$, then the families $(\eta(1, Z)\varphi_M(e_i))$ and $(\mathrm{ev}_1 \varphi_M(e_i))$ form bases of $(M_K)^{\psi=0}$ as $\mathcal{R}_K(\Gamma_L)$-modules in case $\mathbf{B}$ and $\mathfrak{X}$, respectively. Therefore, we consider next a natural G_L-action on $\mathcal{R}_K(\Gamma_L)$ and show that (4.61) is G_L-equivariant. For the first aim we use the canonical isomorphisms (4.44) and (4.46)

$$\mathcal{R}_K(\Gamma_L) \xleftarrow{\cong} \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_n]} \mathcal{R}_K(\Gamma_n) \xrightarrow[\hat{\ell}_{n*}]{\cong} \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_n]} \mathcal{R}_K(\mathfrak{X})$$

to extend the G_L-action from $\mathcal{R}_K(\mathfrak{X})$ to $\mathcal{R}_K(\Gamma_L)$; clearly, we obtain from (4.59) and the fact that the isomorphism

$$\mathcal{R}_K(\Gamma_n) \xrightarrow[\cong]{\ell_{n*}} \mathcal{R}_K(\mathfrak{X})$$

is defined over L that

$$\mathcal{R}_K(\Gamma_L)^{G_L} \cong \mathcal{R}_L(\Gamma_L). \tag{4.62}$$

For the second aim we prove the following lemma.

Lemma 4.3.22. *The action* (4.61) *is G_L-equivariant.*

Proof. We fix $\sigma \in G_L$ and define a second separately continuous action

$$\mathcal{R}_K(\Gamma_L) \times (M_K)^{\psi=0} \to (M_K)^{\psi=0}$$

by sending (λ, x) to $\sigma^{-1}(\sigma(\lambda)(\sigma(x)))$ (using that σ and ψ_L commute and that σ is a homeomorphism). By the uniqueness statement of Theorem 4.3.21, it suffices to show that the new and original actions coincide on $\Gamma \times (M_K)^{\psi=0}$. We shall show that these actions coincide even as actions $\Gamma_L \times M_K \to M_K$: For $\gamma \in \Gamma$, $f \in \mathcal{R}_K(\mathfrak{Y})$, and $m \in M$ we calculate

$$\begin{aligned}
\sigma^{-1}\big(\sigma(\gamma)\big(\sigma(f \otimes m)\big)\big) &= \sigma^{-1}\big(\gamma\big(\sigma(f) \otimes m\big)\big) \\
&= \sigma^{-1}\big(\gamma\big(\sigma(f)\big) \otimes \gamma(m)\big) \\
&= \sigma^{-1}\big(\sigma\big(\gamma(f)\big)\big) \otimes \gamma(m) \\
&= \gamma(f) \otimes \gamma(m) \\
&= \gamma(f \otimes m).
\end{aligned}$$

Here we used firstly that σ acts trivially on γ (or rather ev_γ) as they are already defined over L (via the Fourier transformation) and secondly that the G_L- and Γ_L-actions commute. Since this equality holds for all $\sigma \in G_L$, the claim follows. ∎

Taking G_L-invariants of (4.61) therefore induces – upon using (4.60) and (4.62) – the following separately continuous action:

$$\mathcal{R}_L(\Gamma_L) \times M^{\psi=0} \to M^{\psi=0},$$

which extends the Γ_L-action.

Theorem 4.3.23. *We have the following:*

(i) *The Γ_L-action on M (in $\mathfrak{M}^{\mathrm{an}}(\mathcal{R}_L(\mathfrak{X}))$ or $\mathfrak{M}^{\mathrm{an}}(\mathcal{R}_L(\mathbf{B}))$) extends to a separately continuous $\mathcal{R}_L(\Gamma_L)$-action on $M^{\psi_L=0}$ (with respect to the LF-topology on $\mathcal{R}_L(\Gamma_L)$ and the subspace topology on $M^{\psi_L=0}$). If $M \xrightarrow{f} N$ is a homomorphism of L-analytic (φ_L, Γ_L)-modules, then $M \xrightarrow{f^{\psi_L=0}} N$ is $\mathcal{R}_L(\Gamma_L)$-equivariant with regard to this action.*

(ii) *If $\mathfrak{Y} = \mathfrak{X}$, then $M^{\psi_L=0}$ is a free $\mathcal{R}_L(\Gamma_L)$-module of rank $\mathrm{rk}_{\mathcal{R}} M$. More precisely, if $e_1, \ldots, e_r$ is a basis of M over $\mathcal{R}_L(\mathfrak{X})$, then an $\mathcal{R}_L(\Gamma_L)$-basis of $M^{\psi_L=0}$ is given by $\mathrm{ev}_1 \varphi_M(e_1), \ldots, \mathrm{ev}_1 \varphi_M(e_r)$.*

Proof. It is easy to check that also the $\mathcal{R}_L(\Gamma_L)$-equivariance of $f^{\psi_L=0}$ follows by descent. Therefore, only (ii) remains to be shown. But this is an immediate consequence of the fact noted above, that the family $(\mathrm{ev}_1 \varphi_M(e_i))$ forms a G_L-invariant basis of $(M_K)^{\psi=0}$ as $\mathcal{R}_K(\Gamma_L)$-module, by just taking G_L-invariants again. ∎

Remark 4.3.24. As for possible generalizations, we note the following:

(i) For each complete intermediate field $L \subseteq K' \subseteq \mathbb{C}_p$ we obtain an analogous structure theorem for $(M_{K'})^{\psi=0}$ over $\mathcal{R}_{K'}(\Gamma_L)$ by replacing L by K' everywhere in the above reasoning.

(ii) Since for $\mathfrak{Y} = \mathbf{B}$ the basis $(\eta(1, Z) \varphi_M(e_i))$ of $(M_K)^{\psi=0}$ as $\mathcal{R}_K(\Gamma_L)$-module is visibly not G_L-invariant, the analogue of Theorem 4.3.23 (ii) cannot hold true in this case.

4.3.9 The Mellin transform and twists

Extending the Mellin transform from Lemma 4.1.6, we introduce the map

$$\mathcal{M} : \mathfrak{R}_K(\Gamma_L) \xrightarrow{\cong} \mathcal{R}_K(\mathfrak{X})^{\psi_L=0}, \quad \lambda \mapsto \lambda(\mathrm{ev}_1),$$

which is an isomorphism by Theorem 4.3.23. If $\Omega \in K$, then its composite with the LT-isomorphism is the isomorphism

$$\mathcal{M}_{\mathrm{LT}} : \mathcal{R}_K(\Gamma_L) \xrightarrow{\cong} \mathcal{R}_K(\mathbf{B})^{\psi_L=0}, \quad \lambda \mapsto \lambda(\eta(1, Z)).$$

Recall the twist operators Tw_χ from Section 4.1.3.

Lemma 4.3.25. *The diagram*

$$
\begin{array}{ccc}
\mathcal{R}_K(\Gamma_L) & \xrightarrow{\;\mathfrak{M}\;} & \mathcal{R}_K(\mathfrak{X})^{\psi_L=0} \\
{\scriptstyle \mathrm{Tw}_{\chi_{\mathrm{LT}}}}\Big\downarrow & & \cong\;\Big\downarrow {\scriptstyle \partial^{\mathfrak{X}}_{\mathrm{inv}}} \\
\mathcal{R}_K(\Gamma_L) & \xrightarrow{\;\mathfrak{M}\;} & \mathcal{R}_K(\mathfrak{X})^{\psi_L=0}
\end{array}
\tag{4.63}
$$

is commutative; in particular, the right-hand vertical map is an isomorphism.

Proof. The commutativity can be checked after base change. Assuming $\Omega \in K$, the diagram corresponds by Remark 4.2.9 to the diagram

$$
\begin{array}{ccc}
\mathcal{R}_K(\Gamma_L) & \xrightarrow{\;\mathfrak{M}_{\mathrm{LT}}\;} & \mathcal{R}_K(\mathbf{B})^{\psi_L=0} \\
{\scriptstyle \mathrm{Tw}_{\chi_{\mathrm{LT}}}}\Big\downarrow & & \cong\;\Big\downarrow {\scriptstyle \frac{1}{\Omega}\partial_{\mathrm{inv}}} \\
\mathcal{R}_K(\Gamma_L) & \xrightarrow{\;\mathfrak{M}_{\mathrm{LT}}\;} & \mathcal{R}_K(\mathbf{B})^{\psi_L=0}.
\end{array}
\tag{4.64}
$$

Now, the corresponding result for $\mathcal{R}_K(\Gamma_L)$ replaced by $D(\Gamma_L, K)$ is implicitly given in Sections 4.1.3 and 4.1.4. Section 1.2.4 of [25] establishes, for $\gamma \in \Gamma_L$, the relation $\partial_{\mathrm{inv}} \circ \gamma = \chi_{\mathrm{LT}}(\gamma)\gamma \circ \partial_{\mathrm{inv}}$ as operators on $\mathcal{R}_K(\mathbf{B})$. It follows by K-linearity and continuity that the relation of operators $\partial_{\mathrm{inv}} \circ \lambda = \mathrm{Tw}_{\chi_{\mathrm{LT}}}(\lambda) \circ \partial_{\mathrm{inv}}$ holds for all $\lambda \in D(\Gamma_L, K)$. By continuity of the action of $\mathcal{R}_K(\Gamma_L) = \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_n]} \mathcal{R}_K(\Gamma_n)$ on $\mathcal{R}_K(\mathbf{B})^{\psi_L=0}$ it suffices to check the compatibility for the element Y_n^{-1}, where $Y_n \in D(\Gamma_n, K)$, for $n \gg 0$, has been defined at the end of Section 4.3.4. Using that $\mathrm{Tw}_{\chi_{\mathrm{LT}}}$ is multiplicative and that $\partial_{\mathrm{inv}}(\eta(1, Z)) = \Omega\eta(1, Z)$, the claim follows from the relation

$$
\begin{aligned}
\mathrm{Tw}_{\chi_{\mathrm{LT}}}(Y_n^{-1})\eta(1, Z) &= \mathrm{Tw}_{\chi_{\mathrm{LT}}}(Y_n)^{-1}\tfrac{1}{\Omega}\partial_{\mathrm{inv}}\big(Y_n Y_n^{-1}\eta(1, Z)\big) \\
&= \mathrm{Tw}_{\chi_{\mathrm{LT}}}(Y_n)^{-1}\mathrm{Tw}_{\chi_{\mathrm{LT}}}(Y_n)\tfrac{1}{\Omega}\partial_{\mathrm{inv}}\big(Y_n^{-1}\eta(1, Z)\big) \\
&= \tfrac{1}{\Omega}\partial_{\mathrm{inv}}\big(Y_n^{-1}\eta(1, Z)\big). \qquad\blacksquare
\end{aligned}
$$

Lemma 4.3.26. *Assume $\Omega \in K$ and let n_1 be as in Lemma 4.3.13. Then, for $n \geq n_1$, the map $\mathfrak{M}_{\mathrm{LT}}$ induces isomorphisms*

$$
\mathcal{R}_K(\Gamma_n) \cong \varphi_L^n\big(\mathcal{R}_K(\mathbf{B})\big)\eta(1, Z)\big(\subseteq \mathcal{R}_K(\mathbf{B})^{\psi_L=0}\big)
$$

of $\mathcal{R}_K(\Gamma_n)$-modules and

$$
D(\Gamma_n, K) \cong \varphi_L^n\big(\mathcal{O}_K(\mathbf{B})\big)\eta(1, Z)\big(\subseteq \mathcal{O}_K(\mathbf{B})^{\psi_L=0}\big)
$$

of $D(\Gamma_n, K)$-modules.

Proof. By taking limits the first isomorphism follows from Lemma 4.3.19 (ii) in combination with Lemma 4.3.11 (i), both applied to $M^I = \mathcal{R}_K(\mathbf{B}_I)$. The isomorphism of the latter restricts visibly to the isomorphism $\mathcal{O}_K(\mathbf{B}_I) \cong \varphi_L^n(\mathcal{O}_K(\mathbf{B}_I))\eta(1, Z)$, while $\mathcal{O}_K(\mathbf{B}_I)$ is a free $\mathcal{O}_K(\hat{\ell}_n^* \circ \kappa(\mathbf{B}_I))$-module with basis 1 by an obvious analogue of the former reference. Hence, we obtain the second isomorphism by the same reasoning. ∎

4.4 Explicit elements

There are two sources for explicit elements in the distribution algebras $D(o_L^\times, L)$ and $D(U_n, L)$, where in this section we fix an $n \geq n_1$; i.e., $\log : U_n \xrightarrow{\cong} \pi_L^n o_L$ is an isomorphism. First of all, we have, for any group element $u \in o_L^\times$, resp., $u \in U_n$, the Dirac distribution δ_u in $D(o_L^\times, L)$, resp., in $D(U_n, L)$. As in Section 4.1.1, the corresponding holomorphic function $F_{\delta_u} = \mathrm{ev}_u$ is the function of evaluation in u.

Lemma 4.4.1. *We have the following:*

   (i)   *Let $u \in o_L^\times$ be any element not of finite order; then the zeros of the function $\mathrm{ev}_u - 1$ on $\mathfrak{X}^\times$ are exactly the characters χ of finite order such that $\chi(u) = 1$.*

   (ii) *For any $1 \neq u \in U_n$ the zeros of the function $\mathrm{ev}_u - 1$ on $\mathfrak{X}_n^\times$ all have multiplicity one.*

Proof. (i) Obviously, the zeros of $\mathrm{ev}_u - 1$ are the characters χ such that $\chi(u) = 1$. On the other hand, consider any locally L-analytic character $\chi : o_L^\times \to \mathbb{C}_p^\times$. Its kernel $H := \ker(\chi)$ is a closed locally L-analytic subgroup of $o_L^\times$. Hence, its Lie algebra $\mathrm{Lie}(H)$ is an L-subspace of $\mathrm{Lie}(o_L^\times) = L$. We see that either $\mathrm{Lie}(H) = L$, in which case H is open in $o_L^\times$ and hence χ is a character of finite order, or $\mathrm{Lie}(H) = 0$, in which case H is zero dimensional and hence is a finite subgroup of $o_L^\times$. If $\chi(u) = 1$, then, by our assumption on u, the second case cannot happen.

(ii) (We will recall the concept of multiplicity further below.) Because of the isomorphism $\mathfrak{X}_n^\times \cong \mathfrak{X}$, it suffices to prove the corresponding assertion in the additive case. Let $0 \neq a \in o_L$ and let $\chi \in \mathfrak{X}(\mathbb{C}_p)$ be a character of finite order such that $\chi(a) = 1$. By [74] we have an isomorphism between $\mathfrak{X}_{/\mathbb{C}_p}$ and the open unit disk $\mathbf{B}_{/\mathbb{C}_p}$. Let $z \in \mathbf{B}(\mathbb{C}_p)$ denote the image of χ under this isomorphism. By [74, Prop. 3.1 and Eq. $(\diamond\diamond)$], the function $\mathrm{ev}_a - 1$ corresponds under this isomorphism to the holomorphic function on $\mathbf{B}(\mathbb{C}_p)$ given by the formal power series

$$F_{at_0'}(Z) = \exp\left(g\Omega \log_{\mathrm{LT}}(Z)\right) - 1,$$

where $\Omega \neq 0$ is a certain period. By assumption we have $F_{at_0'}(z) = 0$. On the other hand, the formal derivative of this power series is

$$\frac{d}{dZ} F_{gt_0'}(Z) = g\Omega g_{\mathrm{LT}}(Z)\left(F_{gt_0'}(Z) + 1\right).$$

Since $g_{\mathrm{LT}}(Z)$ is a unit in $o_L[\![Z]\!]$, we see that z is not a zero of this derivative. It follows that z has multiplicity one as a zero of $F_{at'_0}(Z)$.   ∎

The other source comes from the Lie algebra $\mathrm{Lie}(U_n) = \mathrm{Lie}(o_L^\times) = L$. We have the element

$$\nabla := 1 \in \mathrm{Lie}(o_L^\times) = L.$$

On the other hand, there is the L-linear embedding (cf. [75, §2])

$$\mathrm{Lie}(U_n) \rightarrow D(U_n, L)$$
$$\mathfrak{x} \mapsto \left[f \mapsto \tfrac{d}{dt} f\left(\exp_{U_n}(t\mathfrak{x})\right)_{|t=0}\right],$$

which composed with the Fourier isomorphism becomes the map

$$\mathrm{Lie}(U_n) \rightarrow \mathcal{O}_L(\mathfrak{X}_n^\times)$$
$$\mathfrak{x} \mapsto \left[\chi \mapsto d\chi(\mathfrak{x})\right].$$

On the one hand, we therefore may and will view ∇ always as a distribution on U_n or $o_L^\times$. On the other hand, using the formula before [8, Lem. 1.28], one checks that the function F_∇ (corresponding to ∇ via the Fourier isomorphism) on $\mathfrak{X}_n^\times$ is explicitly given by

$$F_\nabla(\chi) = \pi_L^{-n} \log\left(\chi\left(\exp(\pi_L^n)\right)\right). \tag{4.65}$$

Lemma 4.4.2. *The zeros of the function F_∇ on $\mathfrak{X}_n^\times$ are precisely the characters of finite order each with multiplicity one.*

Proof. Once again because of the isomorphism $\mathfrak{X}_n^\times \cong \mathfrak{X}$, it suffices to prove the corresponding assertion in the additive case. This is done in [8, Lem. 1.28].   ∎

To recall from [8, §1.1] the concept of multiplicity used above and to explain a divisibility criterion in these rings of holomorphic functions, we let $\mathfrak{Y}$ be any one-dimensional smooth rigid analytic quasi-Stein space over L such that $\mathcal{O}_L(\mathfrak{Y})$ is an integral domain. Under these assumptions, the local ring in a point y of the structure sheaf $\mathcal{O}_\mathfrak{Y}$ is a discrete valuation ring. Let $\mathfrak{m}_y$ denote its maximal ideal. The divisor $\mathrm{div}(f)$ of any non-zero function $f \in \mathcal{O}_L(\mathfrak{Y})$ is defined to be the function $\mathrm{div}(f) : \mathfrak{Y} \rightarrow \mathbb{Z}_{\geq 0}$ given by $\mathrm{div}(f)(y) = n$ if and only if the germ of f in y lies in $\mathfrak{m}_y^n \setminus \mathfrak{m}_y^{n+1}$. By Lemma 1.1 in loc. cit. for any affinoid subdomain $\mathfrak{Z} \subseteq \mathfrak{Y}$ the set

$$\left\{x \in \mathfrak{Z} \mid \mathrm{div}(f)(x) > 0\right\} \text{ is finite.} \tag{4.66}$$

Lemma 4.4.3. *For any two non-zero functions $f_1, f_2 \in \mathcal{O}_L(\mathfrak{Y})$, we have $f_2 \in f_1 \mathcal{O}_L(\mathfrak{Y})$ if and only if $\mathrm{div}(f_2) \geq \mathrm{div}(f_1)$.*

Proof. We consider the principal ideal $f_1 \mathcal{O}_L(\mathfrak{Y})$. As a consequence of [8, Props. 1.6 and 1.4], we have

$$f_1 \mathcal{O}_L(\mathfrak{Y}) = \{ f \in \mathcal{O}_L(\mathfrak{Y}) \setminus \{0\} : \operatorname{div}(f) \geq \operatorname{div}(f_1) \} \cup \{0\}. \qquad \blacksquare$$

We now apply these results to exhibit a few more explicit elements in the distribution algebra $D(U_n, L)$, which will be used later on.

Lemma 4.4.4. *For any $1 \neq u \in U_n$ the fraction $\frac{\nabla}{\delta_u - 1}$ is a well-defined element in the integral domain $D(U_n, L)$.*

Proof. By the Fourier isomorphism we may equivalently establish that the fraction $\frac{F_\nabla}{\operatorname{ev}_u - 1}$ exists in $\mathcal{O}_L(\mathfrak{X}_u^\times)$. But for this we only need to combine the Lemmata 4.4.1 to 4.4.3. $\blacksquare$

The next elements will only lie in the Robba ring of U_n. Since $\mathfrak{X}_n^\times \cong \mathfrak{X}$, we deduce from Proposition 4.1.7 and the subsequent discussion that there is an admissible covering $\mathfrak{X}_n^\times = \bigcup_{j \geq 1} \mathfrak{V}_{n,j}$ by an increasing sequence $\mathfrak{V}_{n,1} \subseteq \cdots \subseteq \mathfrak{V}_{n,j} \subseteq \cdots$ of affinoid subdomains $\mathfrak{V}_{n,j}$ with the following properties:

- The system $(\mathfrak{X}_n^\times \setminus \mathfrak{V}_{n,j})_{/\mathbb{C}_p}$ is isomorphic to an increasing system of one-dimensional annuli. This implies that

  – $\mathcal{R}_L(\mathfrak{X}_n^\times)$ is the increasing union of the rings $\mathcal{O}_L(\mathfrak{X}_n^\times \setminus \mathfrak{V}_{n,j})$ and contains $\mathcal{O}_L(\mathfrak{X}_n^\times)$;

  – each $\mathcal{O}_L(\mathfrak{X}_n^\times \setminus \mathfrak{V}_{n,j})$ and $\mathcal{R}_L(\mathfrak{X}_n^\times)$ are integral domains.

- Each $\mathfrak{X}_n^\times \setminus \mathfrak{V}_{n,j}$ is a one-dimensional smooth quasi-Stein space.

In particular, the $\mathcal{O}_L(\mathfrak{X}_n^\times \setminus \mathfrak{V}_{n,j})$ are naturally Fréchet algebras, and we may view $\mathcal{R}_L(\mathfrak{X}_n^\times)$ as their locally convex inductive limit. We also conclude that Lemma 4.4.3 applies to each $\mathfrak{X}_n^\times \setminus \mathfrak{V}_{n,j}$.

We now fix a basis $b = (b_1, \ldots, b_d)$ of U_n as a $\mathbb{Z}_p$-module such that $b_i \neq 1$ for any $1 \leq i \leq d$.

Proposition 4.4.5. *The fraction*

$$\Xi_b := \frac{F_\nabla^{d-1}}{\prod_{i=1}^d (\operatorname{ev}_{b_i} - 1)}$$

is well defined in the Robba ring $\mathcal{R}_L(\mathfrak{X}_n^\times)$.

Proof. The zeros of the fraction $\frac{F_\nabla}{\operatorname{ev}_{b_i} - 1} \in \mathcal{O}_L(\mathfrak{X}_n^\times)$ are precisely those finite-order characters which are non-trivial on b_i. Hence, if we fix $1 \leq j \leq d$, then the product $\prod_{i \neq j} \frac{F_\nabla}{\operatorname{ev}_{b_i} - 1}$ still has a zero in any finite-order character which is non-trivial on b_i for at least one $i \neq j$. On the other hand, the zeros of $\operatorname{ev}_{b_j} - 1$ are those finite-order

characters which are trivial on b_j (and they have multiplicity one). Since only the trivial character is trivial on all $b_1, \ldots, b_d$, we see that all zeros of $\mathrm{ev}_{b_j} - 1$ with the exception of the trivial character occur also as zeros of the product $\frac{F_\nabla^{d-1}}{\prod_{i \neq j}(\mathrm{ev}_{b_i} - 1)}$. It follows that the asserted fraction Ξ_b exists in $\mathcal{O}_L(\mathfrak{X}_n^\times \setminus \mathfrak{V}_{n,j})$ provided j is large enough so that the trivial character is a point in $\mathfrak{V}_{n,j}$. Since $(\prod_{i=1}^d (\mathrm{ev}_{b_i} - 1)) \Xi_b = F_\nabla^{d-1}$ and $\mathcal{R}_L(\mathfrak{X}_\Gamma^\times)$ is an integral domain, we see the independence of j. ∎

In fact, the proof of Proposition 4.4.5 shows that Ξ_b is a meromorphic function on $\mathfrak{X}_n^\times$ with a single pole at the trivial character, which moreover is a simple pole. We abbreviate $\ell(b) := \prod_{i=1}^d \log(b_i)$.

Proposition 4.4.6. *For any other basis $b' = (b'_1, \ldots, b'_d)$ of U_n as a $\mathbb{Z}_p$-module with $b'_i \neq 1$ we have*

$$\ell(b') \Xi_{b'} - \ell(b) \Xi_b \in \mathcal{O}_L(\mathfrak{X}_n^\times).$$

Proof. We only have to check that the asserted difference does not any longer have a pole at the trivial character. Both $\Xi_{b'}^{-1}$ and Ξ_b^{-1} are uniformizers in the local ring $\mathcal{O}_1$ of $\mathfrak{X}_n^\times$ in the trivial character. Hence, we have in $\mathcal{O}_1$ an equality of the form

$$\frac{\Xi_{b'}}{\Xi_b} = x + \Xi_b^{-1} \cdot G$$

with some $x \in L$ and $G \in \mathcal{O}_1$. Our assertion amounts to the claim that $x = \prod_i \frac{\log(b_i)}{\log(b'_i)}$. To compute x, we use (4.3) which leads to the open embedding

$$\mathbf{B}(r_0)_{/L} \xrightarrow{\cong} \mathfrak{X}(r_0) \xrightarrow{\subseteq} \mathfrak{X} \xrightarrow{\ell_n^*} \mathfrak{X}_n^\times$$

which maps y to the character $\chi_y(u) := \exp(\pi_L^{-n} \log(u) y)$ (and, in particular, 0 to the trivial character). Using (4.65), we see that F_∇ pulls back to the function $y \mapsto \pi_L^{-n} y$ on $\mathbf{B}(r_0)$. On the other hand, $\mathrm{ev}_{b_i} - 1$ pulls back to $y \mapsto \exp(\pi_L^{-n} \log(b_i) y) - 1$. Hence, Ξ_b pulls back to the meromorphic function

$$y \mapsto \frac{\pi_L^{-n(d-1)} y^{d-1}}{\prod_i \left(\exp\left(\pi_L^{-n} \log(b_i) y\right) - 1 \right)}.$$

Its germ at zero lies in

$$\frac{1}{\pi_L^{-n} \left(\prod_i \log(b_i) \right) y} (1 + y \mathcal{O}_0),$$

where $\mathcal{O}_0$ denotes the local ring of $\mathbf{B}(r_0)_{/L}$ in zero. It follows that the germ of the pull back of $\frac{\Xi_{b'}}{\Xi_b}$ lies in $\left(\prod_i \frac{\log(b_i)}{\log(b'_i)} \right)(1 + y \mathcal{O}_0)$. ∎

By Lemma 4.4.2 the function $F_\nabla \Xi_b$ is holomorphic on $\mathfrak{X}_n^\times$ and has no zero in the trivial character.

Lemma 4.4.7. *The value of $F_\nabla \Xi_b$ at the trivial character is $\ell(b)^{-1}$.*

Proof. We use the same strategy as in the previous proof. The function Θ_b pulls back to the function

$$y \mapsto \frac{\pi_L^{-nd} y^d}{\prod_i \left(\exp \left(\pi_L^{-n} \log(b_i) y \right) - 1 \right)}$$

on $\mathbf{B}(r_0)_{/L}$, and we have to compute its value at 0. But visibly the above right-hand side is a power series in y with constant term $\frac{1}{\prod_{i=1}^d \log(b_i)}$.   ∎

These last two facts suggest renormalizing our functions by setting

$$\bar{\Xi}_b := \ell(b) \Xi_b \quad \text{and} \quad \bar{\Theta}_b := F_\nabla \bar{\Xi}_b.$$

Choosing a field K containing Ω, we also let $\widetilde{\bar{\Xi}}_b$ denote the image of $\bar{\Xi}_b$ under the composite map

$$\mathcal{R}_L(\mathfrak{X}_n^\times) \xrightarrow{\ell_{n*}} \mathcal{R}_L(\mathfrak{X}) \subseteq \mathcal{R}_K(\mathfrak{X}) \xrightarrow{\kappa^*} \mathcal{R}_K(\mathbf{B}). \tag{4.67}$$

Remark 4.4.8. Suppose that K contains Ω. We have

$$\widetilde{\bar{\Xi}}_b = \frac{\ell(b)\left(\frac{\Omega}{\pi_L^n} \log_{\mathrm{LT}}(Z)\right)^{d-1}}{\prod_j \left(\exp \left(\log(b_j) \frac{\Omega}{\pi_L^n} \log_{\mathrm{LT}}(Z) \right) - 1 \right)},$$

and it follows from the proof of Proposition 4.4.5 that $Z\widetilde{\bar{\Xi}}_b$ belongs to $\mathcal{O}_K(\mathbf{B})$ with constant term $\left(\frac{\Omega}{\pi_L^n}\right)^{-1}$, whence

$$\widetilde{\bar{\Xi}}_b \equiv \frac{\pi_L^n}{\Omega Z} \quad \mod \mathcal{O}_K(\mathbf{B}).$$

Proof. One checks that the map (4.67) sends a distribution μ to the map

$$g_\mu(z) = \mu\left(\exp \left(\Omega \frac{\log(-)}{\pi_L^n} \log_{\mathrm{LT}}(z) \right) \right).$$

In particular, a Dirac distribution δ_a is sent to $\exp(\Omega \frac{\log(a)}{\pi_L^n} \log_{\mathrm{LT}}(z))$. Recall that the action of ∇ as distribution is given as sending a locally L-analytic function f to $-(\frac{d}{dt} f(\exp(-t)))_{|t=0}$, whence ∇ is sent to

$$\nabla\left(\exp \left(\Omega \frac{\log(-)}{\pi_L^n} \log_{\mathrm{LT}}(z) \right) \right) = -\left(\frac{d}{dt} \exp \left(\Omega \frac{\log(\exp(-t))}{\pi_L^n} \log_{\mathrm{LT}}(z) \right) \right)_{|t=0}$$

$$= \frac{\Omega}{\pi_L^n} \log_{\mathrm{LT}}(z). \qquad \blacksquare$$

Remark 4.4.9. Recall that Θ_b lies in $\mathcal{O}_L(\mathfrak{X}_n^\times)$ and therefore can be viewed, via the Fourier transform, as a distribution in $D(U_n, L) \subseteq D(o_L^\times, L)$. If K contains Ω and for sufficiently large n, the Mellin transform $\mathfrak{M}$ in Lemma 4.1.6 then satisfies

$$\kappa^* \circ \mathfrak{M}(\Theta_b) = \varphi_L^n(\xi_b)\eta(1, Z)$$

with

$$\xi_b \equiv \frac{\log_{LT}(Z)}{Z} \quad \mod \; \log_{LT}(Z)\mathcal{O}_K(\mathbf{B}).$$

Proof. Consider the element

$$F(X) = \frac{X}{\exp(X) - 1} = 1 + XQ(X)$$

with $Q(X) \in \mathbb{Q}_p[\![X]\!]$ and let $r > 0$ be such that $Q(X)$ converges on $|X| \le r$. We shall proof the claim within the Banach algebra $\mathcal{R}_K^I(\mathbf{B})$ for $I = [0, r]$ (which contains $\mathcal{O}_K(\mathbf{B})$), using that the actions on both rings are compatible. We assume for the operator norm that $\|\delta_{b_i} - 1\|_I < \min(p^{-\frac{1}{p-1}}, r)$ for all i (otherwise, we enlarge n according to Lemma 4.3.13). From [8, Cor. 2.3.2, proof of Lem. 2.3.1] it follows that $\nabla = \frac{\log(\delta_{b_i})}{\log(b_i)}$ as operators in the Banach algebra A of continuous linear endomorphisms of $\mathcal{R}_K^I(\mathbf{B})$ and

$$\exp\left(\log(b_i)\nabla\right) = \exp\left(\log(\delta_{b_i})\right) = \delta_{b_i}$$

in A. Moreover,

$$\left\|\log(\delta_{b_i})\right\|_I < \min(p^{-\frac{1}{p-1}}, r) \tag{4.68}$$

for all i, whence $\|\nabla\|_I < \min(p^{-\frac{1}{p-1}}, r)|\log(b_i)|$. Then, as operators in A, we have

$$\log(b_i)^{-1} + \nabla Q\left(\log(b_i)\nabla\right) = \log(b_i)^{-1} F\left(\log(b_i)\nabla\right)$$
$$= \frac{\nabla}{\exp\left(\log(b_i)\nabla\right) - 1}$$
$$= \frac{\nabla}{\delta_{b_i} - 1}.$$

Hence,

$$\Theta_b = \frac{\ell(b)\nabla^d}{\prod\left(\exp\left(\log(b_j)\nabla\right) - 1\right)} = 1 + \ell(b)\nabla g\left(\log(b_j)\nabla\right)$$

for some power series $g \in \mathcal{R}_K^I(\mathbf{B})$. It follows that

$$\kappa^* \circ \mathfrak{M}(\Theta_b) = \left(1 + \ell(b)\Omega \log_{LT}(Z) f(Z)\right)\eta(1, Z) \tag{4.69}$$

for some $f(Z) \in \mathcal{R}_K^I(\mathbf{B})$. Indeed, concerning the derived action, we have

$$\nabla\big(\eta(1, Z)\big) = \left(\tfrac{d}{dt} \exp\left(\Omega \exp(t) \log_{LT}(Z)\right)\right)_{|t=0} = \Omega \log_{LT}(Z)\eta(1, Z)$$

using from [8, end of §2.3] the fact that

$$\nabla \text{ acts as } \log_{\mathrm{LT}}(Z)\partial_{\mathrm{inv}} \text{ on } \mathcal{O}_K(\mathbf{B}), \tag{4.70}$$

and

$$\nabla\big(\Omega \log_{\mathrm{LT}}(Z)\big) = \Omega \log_{\mathrm{LT}}(Z).$$

Furthermore, we obtain inductively that

$$\nabla^i \eta(1, Z) = \left(\prod_{k=0}^{i-1} \big(\Omega \log_{\mathrm{LT}}(Z) + k\big) \right) \eta(1, Z)$$

for all $i \geq 0$. The convergence of $f(Z)$ can be deduced using the operator norm (4.68).

On the other hand, according to [7, Lem. 2.4.2], we have

$$\Theta_b \eta(1, Z) = \frac{\ell(b) \log_{\mathrm{LT}}(Z)}{\varphi_L^n(Z)} g(Z)$$

for some $g(Z) \in \mathcal{O}_K(\mathbf{B})$. Since the element $\Theta_b \eta(1, Z)$ lies in $(\mathcal{O}_K(\mathbf{B}))^{\psi_L=0}$, we conclude from

$$0 = \psi_L\left(\frac{\pi_L^{-1}\varphi_L\big(\log_{\mathrm{LT}}(Z)\big)}{\varphi_L^n(Z)} g(Z) \right) = \frac{\pi_L^{-1}\log_{\mathrm{LT}}(Z)}{\varphi_L^{n-1}(Z)} \psi_L\big(g(Z)\big)$$

that $g(Z)$ belongs to $(\mathcal{O}_K(\mathbf{B}))^{\psi_L=0}$, whence it is of the form

$$g(Z) = \sum_{a \in (o_L/\pi_L)^\times} \varphi_L\big(g_a(Z)\big)\eta(a, Z)$$

for some $g_a \in \mathcal{O}_K(\mathbf{B})$ by the analogue of (4.58) for $\mathcal{O}_K(\mathbf{B})$. From Lemma 4.3.26, we derive that, for some $a(Z) \in \mathcal{O}_K(\mathbf{B})$, we have

$$\Theta_b \eta(1, Z) = \ell(b)\varphi_L^n\big(a(Z)\big)\eta(1, Z).$$

Since the decomposition in (4.58) is direct, we conclude that

$$g(Z) = \varphi_L\big(g_1(Z)\big)\eta(1, Z) \quad \text{and} \quad \frac{\log_{\mathrm{LT}}(Z)}{\varphi_L^n(Z)}\varphi_L\big(g_1(Z)\big) = \varphi_L^n\big(a(Z)\big),$$

whence $\log_{\mathrm{LT}}(Z)$ divides $\varphi_L^n(a(Z)Z)$. Since φ_L^n sends the zeros of $\log_{\mathrm{LT}}(Z)$, i.e., the points in $LT(\pi_L) = \bigcup_k LT[\pi_L^k]$, surjectively onto itself, we conclude by Lemma 4.4.3 that $\log_{\mathrm{LT}}(Z)$ divides also $a(Z)Z$ in $\mathcal{O}_K(\mathbf{B})$ and that there exists $c(Z) \in \mathcal{O}_K(\mathbf{B})$ such that

$$\kappa^* \circ \mathfrak{M}(\Theta_b) = \ell(b)\varphi_L^n\Big(\frac{\log_{\mathrm{LT}}(Z)}{Z}c(Z)\Big)\eta(1, Z). \tag{4.71}$$

Comparing (4.71) with the first description (4.69) gives the claim as $c(0) = \ell(b)^{-1}$ because evaluation at 0 is compatible with the embedding $\mathcal{O}_K(\mathbf{B}) \subseteq \mathcal{R}_K^I(\mathbf{B})$ and $\frac{\log_{\mathrm{LT}}(Z)}{Z}(0) = 1$ by (2.1).   ∎

4.5 Pairings

In this section, we discuss various kinds of pairings. The starting point is Serre duality on $\mathfrak{X}$ which induces a (residue) pairing

$$\langle \, , \, \rangle_{\mathfrak{X}} : \mathcal{R}_L(\mathfrak{X}) \times \mathcal{R}_L(\mathfrak{X}) \to L,$$

as we have seen in (4.20). Similarly, Serre duality on $\mathfrak{X}^{\times}$ induces a pairing

$$\langle \, , \, \rangle_{\Gamma_L} : \mathcal{R}_L(\Gamma_L) \times \mathcal{R}_L(\Gamma_L) \to L \qquad (4.72)$$

for the Robba ring of Γ_L, which by definition is the Robba ring of its character variety $\mathfrak{X}_{\Gamma_L} \cong \mathfrak{X}^{\times}$ (induced by the isomorphism $\chi_{\mathrm{LT}} : \Gamma_L \to o_L^{\times}$) as constructed in (4.23). This pairing, as defined in Section 4.5.2, is actually already characterized by its restriction to $\mathcal{R}_L(\Gamma_n)$ for any $n \geq n_0$ and thus is by construction and the functoriality properties of Section 4.2 closely related to the pairing $\langle \, , \, \rangle_{\mathfrak{X}}$ using the "logarithm" $\mathcal{R}_L(\Gamma_n) \xrightarrow{(\ell_n)_*} \mathcal{R}_L(\mathfrak{X})$, see (4.21).

In contrast, the commutative diagram

$$
\begin{array}{ccc}
\mathcal{R}_L(\mathfrak{X}^{\times}) & \xrightarrow{\cdot d \log_{\mathfrak{X}^{\times}}} & \Omega^1_{\mathcal{R}_L(\mathfrak{X}^{\times})} \\
\Big\downarrow{\scriptstyle (-)(\mathrm{ev}_1 \, d \log_{\mathfrak{X}})} & & \Big\downarrow{\scriptstyle \mathrm{res}_{\mathfrak{X}^{\times}}} \\
& & L \\
& & \Big\uparrow{\scriptstyle \mathrm{res}_{\mathfrak{X}}} \\
(\Omega^1_{\mathcal{R}_L(\mathfrak{X})})^{\psi=0} & \xrightarrow{\mathrm{ev}_{-1} \cdot} & \Omega^1_{\mathcal{R}_L(\mathfrak{X})}
\end{array}
$$

from Theorem 4.5.12 in Section 4.5.3 relates the pairing $\langle \, , \, \rangle_{\Gamma_L}$ to the pairing $\langle \, , \, \rangle_{\mathfrak{X}}$ *in a highly non-trivial, non-obvious way*. The resulting description of $\langle \, , \, \rangle_{\Gamma_L}$ in (4.84) (resp., (4.86)) forms one main ingredient in the proof of the abstract reciprocity law in Theorem 4.5.32 in Section 4.5.5 below.

By contrast, based on the (generalized) residue pairings (4.73) in Section 4.5.1,

$$\{ \, , \, \} : \check{M} \times M \to L,$$

with $\check{M} := \mathrm{Hom}_{\mathcal{R}}(M, \Omega^1_{\mathcal{R}})$, the pairing (4.72) induces for any (analytic) (φ_L, Γ_L)-module M over $\mathcal{R}_L(\mathfrak{X})$ an Iwasawa pairing (4.88)

$$\{ \, , \, \}_{\mathrm{Iw}} : \check{M}^{\psi_L = \frac{q}{\pi_L}} \times M^{\psi_L = 1} \to D(\Gamma_L, L)$$

in Section 4.5.4, which behaves well with twisting (cf. Lemma 4.5.22).

By construction and the comparison isomorphism (4.91) for Kisin–Ren modules – the second main ingredient – the pairing $\{ \, , \, \}_{\mathrm{Iw}}$ is closely related to a pairing

$$[\, , \,] : \mathcal{R}_L(\mathfrak{X})^{\psi_L = 0} \otimes_L D_{\mathrm{cris},L}(V^*(1)) \times \mathcal{R}_L(\mathfrak{X})^{\psi_L = 0} \otimes_L D_{\mathrm{cris},L}(V(\tau^{-1})) \to \mathcal{R}_L(\Gamma_L)$$

induced from the natural pairing for $D_{\mathrm{cris},L}$. The precise relationship is the content of an abstract form of a reciprocity formula, see Theorem 4.5.32. As a consequence, we shall later derive a concrete reciprocity formula, i.e., the adjointness of Berger's and Fourquaux's big exponential map with our regulator map, see Theorem 5.2.1.

4.5.1 The residuum pairing for modules

Throughout our coefficient field K is a complete intermediate extension $L \subseteq K \subseteq \mathbb{C}_p$. Let $\mathfrak{Y}$ be either $\mathfrak{X}$ or $\mathbf{B}$ and $\mathcal{R} = \mathcal{R}_K(\mathfrak{Y})$. Consider the residuum map $\mathrm{res}_{\mathfrak{X}}$ defined after (4.19) and the residuum map

$$\mathrm{res}_{\mathbf{B}} : \Omega^1_{\mathcal{R}} \to K, \quad \sum_i a_i Z^i dZ \mapsto a_{-1}.$$

Recall that we are using the operator $\psi_L := \frac{q}{\pi} \psi_L^{\mathfrak{Y}}$ on $\mathcal{R}$.

Moreover, we define $\iota_* : \mathcal{R}_K(\Gamma_L) \to \mathcal{R}_K(\Gamma_L)$ to be the map which is induced by sending $\gamma \in \Gamma_L$ to its inverse γ^{-1}; i.e., the involution of the group induces an isomorphism on the multiplicative character variety, which in turn gives rise to ι_*. The corresponding involution on $\mathcal{R}_K(\Gamma_{n_0})$, also denoted by ι_*, satisfies the commutative diagram

$$
\begin{array}{ccc}
\mathcal{R}_K(\Gamma_{n_0}) & \xrightarrow[\cong]{\hat{\ell}_{n_0*}} & \mathcal{R}_K(\mathfrak{X}) \\
\iota_* \downarrow \cong & & \cong \downarrow \iota \\
\mathcal{R}_K(\Gamma_{n_0}) & \xrightarrow[\cong]{\hat{\ell}_{n_0*}} & \mathcal{R}_K(\mathfrak{X}),
\end{array}
$$

where the involution ι on $\mathcal{R}_K(\mathfrak{X})$ sends ev_x to ev_{-x}.

Setting $\check{M} := \mathrm{Hom}_{\mathcal{R}}(M, \Omega^1_{\mathcal{R}}) \cong \mathrm{Hom}_{\mathcal{R}}(M, \mathcal{R})(\chi_{\mathrm{LT}})$, for any finitely generated projective $\mathcal{R}$-module M, we obtain more generally the pairing

$$\{ , \} := \{ , \}_M : \check{M} \times M \to K, \quad (g, f) \mapsto \mathrm{res}_{\mathfrak{Y}}\big(g(f)\big), \tag{4.73}$$

which satisfies the following properties.

Lemma 4.5.1. *For M in $\mathcal{M}(\mathcal{R})$ we have the following:*

(i) $\{ , \}$ *identifies M and $\check{M}$ with the (strong) topological duals of $\check{M}$ and M, respectively,*

(ii) $\{\varphi_L(g), \varphi_L(f)\} = \frac{q}{\pi_L}\{g, f\}$ *for all $g \in \check{M}$ and $f \in M$,*

(iii) $\{\sigma(g), \sigma(f)\} = \{g, f\}$ *for all $g \in \check{M}$, $f \in M$, and $\sigma \in \Gamma_L$,*

(iv) $\{\psi_L(g), f\} = \{g, \varphi_L(f)\}$ *and* $\{\varphi_L(g), f\} = \{g, \psi_L(f)\}$ *for all $g \in \check{M}$ and $f \in M$.*

Proof. (i) follows from the discussion in Section 4.2.3. (ii) is a purely formal consequence of (iv). (iii) follows as in (4.36) with σ_* instead of κ_*. For (iv) we refer to Lemma 4.2.14. For $\mathfrak{Y} = \mathbf{B}$ see also [80, Prop. 3.17, Cor. 3.18, Prop. 3.19].   ∎

Convention. For coherence of our notation we set $\log_{\mathbf{B}} := \log_{\mathrm{LT}}$ although in general this is *not* the standard logarithm!

Proposition 4.5.2. *The pairing* $\langle\ ,\ \rangle_{\mathfrak{Y}} : \mathcal{R} \times \mathcal{R} \to K$, $(f, g) \mapsto \mathrm{res}_{\mathfrak{Y}}(fgd\log_{\mathfrak{Y}})$, *induces topological isomorphisms*

$$\mathrm{Hom}_{K,\mathrm{cont}}(\mathcal{R}, K) \cong \mathcal{R} \quad and \quad \mathrm{Hom}_{K,cts}(\mathcal{R}/\mathcal{O}_K(\mathfrak{Y}), K) \cong \mathcal{O}_K(\mathfrak{Y}).$$

Proof. See Section 4.2.3.   ∎

Remark 4.5.3. If we assume $\Omega \in K$, then these pairings can be compared via the LT-isomorphism κ. By (4.37) we have

$$\Omega\langle\kappa^*(f), \kappa^*(g)\rangle_{\mathbf{B}} = \langle f, g\rangle_{\mathfrak{X}}$$

for $f, g \in \mathcal{R}_K(\mathfrak{X})$.

Assume henceforth that M is an analytic (φ_L, Γ_L)-module over $\mathcal{R}$ and recall from Proposition 4.3.10 that the Γ_L-action on M extends continuously to a $D(\Gamma_L, K)$-module structure.

Corollary 4.5.4. *The isomorphism* $\check{M} \cong \mathrm{Hom}_{K,\mathrm{cont}}(M, K)$ *(induced by* $\{\ ,\ \}$*) is* $D(\Gamma_L, K)$*-linear.*

Proof. This follows from Lemma 4.5.1 (iii) since Γ_L generates a dense subspace of $D(\Gamma_L, K)$.   ∎

Since $\frac{\pi_L}{q}\psi_L \circ \varphi_L = \mathrm{id}_M$, we have a canonical decomposition $M = \varphi_L(M) \oplus M^{\psi_L=0}$. By Lemma 4.5.1 we see that $M^{\psi_L=0}$ is the exact orthogonal complement of $\varphi_L(\check{M})$; i.e., we obtain a canonical isomorphism

$$\check{M}^{\psi_L=0} \cong \mathrm{Hom}_{K,\mathrm{cont}}(M^{\psi_L=0}, K). \qquad (4.74)$$

Lemma 4.5.5. *The isomorphism* (4.74) *is* $\mathcal{R}_K(\Gamma_L)$*-equivariant; i.e., we have for all* $\check{m} \in \check{M}^{\psi_L=0}$, $m \in M^{\psi_L=0}$, *and* $\lambda \in \mathcal{R}_K(\Gamma_L)$ *that*

$$\{\lambda\check{m}, m\} = \{\check{m}, \iota_*(\lambda)m\}.$$

Proof. This is clear for $D(\Gamma_L, K)$ by Corollary 4.5.4. Without loss of generality, we may and do assume that Ω belongs to K. It then follows for the localization $D(\Gamma_L, K)_{Y_{n_1}^{\mathbb{N}}}$, where we use the notation and considerations from Section 4.3.6, especially Lemma 4.3.19 and its proof. Since $D(\Gamma_L, K)_{Y_{n_1}^{\mathbb{N}}}$ is dense in $\mathcal{R}_K(\Gamma_L)$, Remark 4.3.5, the assertion now is a consequence of the continuity property in Theorem 4.3.21.   ∎

4.5.2 The duality pairing $\langle\,,\,\rangle_{\Gamma_L}$ for the group Robba ring

Using the isomorphisms (4.43) induced by the Lubin–Tate character χ_{LT}, we now carry over structures concerning the (multiplicative) character varieties $\mathfrak{X}^\times$, $\mathfrak{X}_n^\times$ to those of the groups Γ_L, Γ_n. In particular, we use analogous notation res_{Γ_L}, res_{Γ_n}, $\log_{\Gamma_L}$, $\log_{\Gamma_n}$ for corresponding objects. In this sense we introduce and recall from (4.23) the pairing

$$\langle\,,\,\rangle_{\Gamma_L} : \mathcal{R}_K(\Gamma_L) \times \mathcal{R}_K(\Gamma_L) \to K \tag{4.75}$$

and similarly $\langle\,,\,\rangle_{\Gamma_n}$ from (4.21). This pairing is of the form

$$\langle\,,\,\rangle_{\Gamma_L} : \mathcal{R}_K(\Gamma_L) \times \mathcal{R}_K(\Gamma_L) \xrightarrow{\mathrm{mult}} \mathcal{R}_K(\Gamma_L) \xrightarrow{\varrho} K,$$

where

$$\varrho = \langle 1, -\rangle_{\Gamma_L} : \mathcal{R}_K(\Gamma_L) \to \Omega^1_{\mathcal{R}_K(\Gamma_L)} \xrightarrow{\mathrm{res}_{\Gamma_L}} K$$
$$f \mapsto f d\log_{\Gamma_L} \mapsto \mathrm{res}_{\Gamma_L}(f d\log_{\Gamma_L})$$

has also the following description – writing $\mathrm{pr}_{n,m}$ and similarly $\mathrm{pr}_{L,m}$ for the projection maps induced by (4.44), (4.45) –

$$\varrho : \mathcal{R}_K(\Gamma_L) = \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_{n_0}]} \mathcal{R}(\Gamma_{n_0}) \to K$$
$$f \mapsto \tfrac{q-1}{q}\left(\tfrac{q}{\pi_L}\right)^{n_0} \mathrm{res}_{\mathfrak{X}}\left(\hat{\ell}_{n_0*} \circ \mathrm{pr}_{L,n_0}(f) d\log_{\mathfrak{X}}\right) \tag{4.76}$$

with n_0 as defined at the beginning of Section 4.3.4. Indeed, using (4.22), (4.21) we obtain

$$\langle 1, f\rangle_{\Gamma_L} = \mathrm{res}_{\Gamma_L}(f d\log_{\Gamma_L})$$
$$= \tfrac{q-1}{q} q^{n_0} \mathrm{res}_{\Gamma_{n_0}}\left(\mathrm{pr}_{L,n_0}(f) d\log_{\Gamma_{n_0}}\right)$$
$$= \tfrac{q-1}{q}\left(\tfrac{q}{\pi_L}\right)^{n_0} \mathrm{res}_{\mathfrak{X}}\left(\hat{\ell}_{n_0*} \circ \mathrm{pr}_{L,n_0}(f) d\log_{\mathfrak{X}}\right)$$

because $(\ell_n^*)^*(1) = 1$.

The following properties follow immediately from the definition.

Lemma 4.5.6. *We have for all* $f, \lambda, \mu \in \mathcal{R}_K(\Gamma_L)$ *that*

(i) $\langle \lambda, f\mu\rangle_{\Gamma_L} = \langle f\lambda, \mu\rangle_{\Gamma_L}$,

(ii) $\langle \lambda, \mu\rangle_{\Gamma_L} = \langle \mu, \lambda\rangle_{\Gamma_L}$.

Remark 4.5.7. For $n \geq n_0$ we have the projection formula

$$\mathrm{pr}_{L,n}\left(\iota_{n,*}(x) y\right) = x\, \mathrm{pr}_{L,n}(y)$$

and (4.24) translates into

$$\langle \iota_{n,*}(x), y\rangle_{\Gamma_L} = (q-1)q^{n-1}\langle x, \mathrm{pr}_{L,n}(y)\rangle_{\Gamma_n} \tag{4.77}$$

for $x \in \mathcal{R}(\Gamma_n)$, $y \in \mathcal{R}(\Gamma_L)$ and the canonical inclusion $\mathcal{R}(\Gamma_n) \xrightarrow{\iota_{n,*}} \mathcal{R}(\Gamma_L)$. Analogous formulae hold for Γ_m with $n \geq m \geq n_0$ instead of Γ_L Lemma 4.2.14 (ii).

Remark 4.5.8 (Frobenius reciprocity). The projection map $\mathrm{pr}_{\Gamma_L, \Gamma_n} : \mathcal{R}_K(\Gamma_L) \to \mathcal{R}_K(\Gamma_n)$ induces an isomorphism

$$\mathrm{Hom}_{\mathcal{R}_K(\Gamma_L)} \left(N, \mathcal{R}_K(\Gamma_L) \right) \cong \mathrm{Hom}_{\mathcal{R}_K(\Gamma_n)} \left(N, \mathcal{R}_K(\Gamma_n) \right)$$

for any $\mathcal{R}_K(\Gamma_L)$-module N; the inverse sends f to the homomorphism given by $x \mapsto \sum_{g \in \Gamma_L / U} g \iota_{n,*} \circ f(g^{-1} x)$.

The following proposition translates the results at the end of Section 4.2.3 into the present setting.

Proposition 4.5.9. *The pairing $\langle \, , \, \rangle_{\Gamma_L} : \mathcal{R}_K(\Gamma_L) \times \mathcal{R}_K(\Gamma_L) \to K$ induces topological isomorphisms*

$$\mathrm{Hom}_{K,\mathrm{cont}} \left(\mathcal{R}_K(\Gamma_L), K \right) \cong \mathcal{R}_K(\Gamma_L),$$
$$\mathrm{Hom}_{K,\mathrm{cont}} \left(\mathcal{R}_K(\Gamma_L) / D(\Gamma_L, K), K \right) \cong D(\Gamma_L, K).$$

Proposition 4.5.10. *Assume $\Omega \in K$ and M in $\mathcal{M}(\mathcal{R})$. Then the map*

$$\mathrm{Hom}_{\mathcal{R}_K(\Gamma_L)} \left(M^{\psi_L = 0}, \mathcal{R}_K(\Gamma_L) \right)^{\iota} \xrightarrow{\cong} \mathrm{Hom}_{K,\mathrm{cont}}(M^{\psi_L = 0}, K) \xrightarrow[\;(4.74)\;]{\cong} \check{M}^{\psi_L = 0}$$
$$F \mapsto \rho \circ F \tag{4.78}$$

is an isomorphism of $\mathcal{R}_K(\Gamma_L)$-modules, where the superscript "ι" on the left-hand side indicates that $\mathcal{R}_K(\Gamma_L)$ acts through the involution ι_*.

Proof. According to Theorem 4.3.21, the $\mathcal{R}_K(\Gamma_L)$-module $M^{\psi_L = 0}$ is finitely generated free. Hence, it suffices to show that the map

$$\mathrm{Hom}_{\mathcal{R}_K(\Gamma_L)} \left(\mathcal{R}_K(\Gamma_L), \mathcal{R}_K(\Gamma_L) \right) \to \mathrm{Hom}_{K,\mathrm{cont}} \left(\mathcal{R}_K(\Gamma_L), K \right)$$
$$F \mapsto \rho \circ F$$

is bijective. But this map is nothing else than the duality isomorphism in Proposition 4.5.9. ∎

The following twist invariance is just Lemma 4.2.15.

Proposition 4.5.11. *Let U be Γ_L or Γ_n for $n \geq n_0$. Then, for all $\lambda, \mu \in \mathcal{R}(U)$ we have*

$$\left\langle \mathrm{Tw}_{\chi_{\mathrm{LT}}}(\mu), \mathrm{Tw}_{\chi_{\mathrm{LT}}}(\lambda) \right\rangle_U = \langle \mu, \lambda \rangle_U.$$

4.5.3 A residuum identity and an alternative description of $\langle \ , \ \rangle_{\Gamma_L}$

Let $\sigma_{-1} \in \Gamma_L$ be the element with $\chi_{\mathrm{LT}}(\sigma_{-1}) = -1$. Consider the continuous K-linear map

$$\varsigma : \mathcal{R}_K(\Gamma_L) \to K$$
$$\lambda \mapsto \mathrm{res}_{\mathfrak{X}}\big(\mathfrak{M}(\sigma_{-1})\mathfrak{M}^{\Omega^1}(\lambda)\big),$$

where $\mathcal{M}^{\Omega^1} : \mathfrak{R}_K(\Gamma_L) \xrightarrow{\cong} \Omega^1_{\mathcal{R}(\mathfrak{X})}{}^{\psi=0} \subseteq \Omega^1_{\mathcal{R}(\mathfrak{X})}$ sends λ to

$$\lambda(\mathrm{ev}_1 \, d \log_{\mathfrak{X}}) = \big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}(\lambda)(\mathrm{ev}_1)\big)d \log_{\mathfrak{X}}, \tag{4.79}$$

whence we also have

$$\mathrm{res}_{\mathfrak{X}}\big(\mathfrak{M}(\sigma_{-1})\mathfrak{M}^{\Omega^1}(\lambda)\big) = \mathrm{res}_{\mathfrak{X}}\big(\mathfrak{M}(\sigma_{-1})\mathfrak{M}\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}(\lambda)\big)d \log_{\mathfrak{X}}\big). \tag{4.80}$$

Recall the definition of ϱ from (4.75).

Theorem 4.5.12. *We have*

$$\varsigma = \frac{q}{q-1}\varrho;$$

i.e., the following identity for the residue map holds:

$$\Big(\frac{q}{\pi_L}\Big)^{n_0}\mathrm{res}_{\mathfrak{X}}\big(\hat{\ell}_{n_0*} \circ \mathrm{pr}_{L,n_0}(\lambda)d \log_{\mathfrak{X}}\big) = \mathrm{res}_{\mathfrak{X}}\big(\mathrm{ev}_{-1} \, \lambda(\mathrm{ev}_1 \, d \log_{\mathfrak{X}})\big)$$

for all $\lambda \in \mathcal{R}_K(\Gamma_L)$; i.e., the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{R}_K(\mathfrak{X}^\times) & \xrightarrow{\cdot d \log_{\mathfrak{X}} \times} & \Omega^1_{\mathcal{R}_K(\mathfrak{X}^\times)} \\
\downarrow{\scriptstyle (-)(\mathrm{ev}_1 \, d \log_{\mathfrak{X}})} & & \downarrow{\scriptstyle \mathrm{res}_{\mathfrak{X}^\times}} \\
& & K \\
& & \uparrow{\scriptstyle \mathrm{res}_{\mathfrak{X}}} \\
(\Omega^1_{\mathcal{R}_K(\mathfrak{X})})^{\psi=0} & \xrightarrow{\mathrm{ev}_{-1}} & \Omega^1_{\mathcal{R}_K(\mathfrak{X})}.
\end{array}
$$

Remark 4.5.13. Compare with [3, Props. 2.2.1, 3.2.1] where residue identities also play a crucial role in the proof of his reciprocity formula.

The proof of this theorem requires some preparation.

Lemma 4.5.14. *For all $\lambda \in \mathcal{R}_K(\Gamma_L)$ and $j \in \mathbb{Z}$ we have*

$$\varsigma\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}^j}(\lambda)\big) = \varsigma(\lambda).$$

Proof. For the proof we may and do assume that Ω belongs to K. Since then

$$\mathrm{res}_{\mathfrak{X}}\big(\partial_{\mathrm{inv}}^{\mathfrak{X}}(f)d\log_{\mathfrak{X}}\big) = \Omega\mathrm{res}_{\mathbf{B}}\big(\tfrac{1}{\Omega}\partial_{\mathrm{inv}}\big(\kappa^*(f)\big)d\log_{\mathrm{LT}}\big) = \mathrm{res}_{\mathbf{B}}\big(d\kappa^*(f)\big) = 0$$

for any f by Remark 4.2.9, (4.37), and [33, Prop. 2.12], the case $j = 1$ follows directly from (4.80) using with $g := \mathrm{Tw}_{\chi_{\mathrm{LT}}}(\lambda)$ that

$$\begin{aligned}
\partial_{\mathrm{inv}}^{\mathfrak{X}}\big(\mathfrak{M}(\sigma_{-1})\mathfrak{M}(g)\big) &= \partial_{\mathrm{inv}}^{\mathfrak{X}}\big(\mathfrak{M}(\sigma_{-1})\big)\mathfrak{M}(g) + \mathfrak{M}(\sigma_{-1})\partial_{\mathrm{inv}}^{\mathfrak{X}}\big(\mathfrak{M}(g)\big)\\
&\overset{(4.63)}{=} \mathfrak{M}\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}(\sigma_{-1})\big)\mathfrak{M}(g) + \mathfrak{M}(\sigma_{-1})\mathfrak{M}\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}(g)\big)\\
&= -\mathfrak{M}(\sigma_{-1})\mathfrak{M}(g) + \mathfrak{M}(\sigma_{-1})\mathfrak{M}\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}(g)\big).
\end{aligned}$$

From this the general case is immediate. ∎

Lemma 4.5.15. *Let* $\lambda \in D(\Gamma_L, \mathbb{C}_p)$ *with* $\mathrm{ev}_{\chi_{\mathrm{LT}}^j}(\lambda) = 0$ *for infinitely many* j*; then* $\lambda = 0$.

Proof. On the character variety the characters χ_{LT}^j correspond to points which converge to the trivial character. It follows that λ corresponds to the trivial function since otherwise its divisor of zeros would have only finitely many zeros in any disk with fixed radius strictly smaller than 1 by (4.66), which would contradict the assumptions. ∎

Now fix a $\mathbb{Z}_p$-basis $b = (b_1, \ldots, b_d)$ of U_{n_0} with all $b_i \neq 1$ and set $\ell^*(b) := \ell_{\Gamma}^*(b) := q^{-n_0}\ell(b) \in o_L^\times$ with $\Gamma := \Gamma_{n_0}$. According to Section 4.4, we may define the operator

$$\widehat{\Xi_b} := q^{-n_0}\chi_{\mathrm{LT}}^*(\bar{\Xi}_b) = \ell^*(b)\chi_{\mathrm{LT}}^*(\Xi_b)$$

in $\mathcal{R}_K(\Gamma)$. Let $\mathrm{aug} : D(\Gamma, K) \to K$ denote the augmentation map, induced by the trivial map $\Gamma \to \{1\}$.

Lemma 4.5.16. *The element* $\widehat{\Xi_b}$ *induces – up to the constant* q^{-n_0} *– the augmentation map*

$$\langle\widehat{\Xi_b}, -\rangle_{\Gamma_{n_0}} = q^{-n_0}\mathrm{aug} : D(\Gamma_{n_0}, K) \to K. \tag{4.81}$$

Moreover, we have

$$\varsigma(\widehat{\Xi_b}) = 1 = \frac{q}{q-1}\varrho(\widehat{\Xi_b}). \tag{4.82}$$

Proof. We may and do assume $\Omega \in K$ by compatibility of res with respect to (complete) base change (4.26). Since

$$\kappa^*\big(\hat{\ell}_{n_0*}(\widehat{\Xi_b})\big) = q^{-n_0}\tilde{\Xi}_b \equiv \frac{\pi_L^{n_0}}{q^{n_0}\Omega}\frac{1}{Z} \mod \mathcal{O}_K(\mathbf{B})$$

by Remark 4.4.8, one has for every $\lambda \in D(\Gamma, K)$

$$
\begin{aligned}
\langle \widehat{\Xi}_b, \lambda \rangle_\Gamma &\overset{(4.77)}{=} \frac{1}{(q-1)q^{n_0-1}} \langle \widehat{\Xi}_b, \lambda \rangle_{\Gamma_L} \\
&\overset{(4.76)}{=} q^{-n_0} \mathrm{res}_{\mathfrak{x}} \left(\left(\tfrac{q}{\pi_L} \right)^{n_0} \kappa^* \left(\hat{\ell}_{n_0 *}(\widehat{\Xi}_b \lambda) \right) d \log_{\mathfrak{x}} \right) \\
&\overset{(4.37)}{=} q^{-n_0} \mathrm{res}_{\mathbf{B}} \left(\Omega \left(\tfrac{q}{\pi_L} \right)^{n_0} \kappa^* \left(\hat{\ell}_{n_0 *}(\widehat{\Xi}_b \lambda) \right) g_{\mathrm{LT}} dZ \right) \\
&= q^{-n_0} \mathrm{res}_{\mathbf{B}} \left(\tfrac{1}{Z} \kappa^* \left(\hat{\ell}_{n_0 *}(\lambda) \right) g_{\mathrm{LT}} dZ \right) \\
&= q^{-n_0} \mathrm{aug}(\lambda),
\end{aligned}
$$

where we use for the last equation that $g_{\mathrm{LT}}(Z)$ has constant term 1 and the fact that the augmentation map corresponds via Fourier theory and the LT-isomorphism to the "evaluation at $Z = 0$" map. Taking $\lambda = 1$ we see that $\varrho(\widehat{\Xi}_b) = \langle \widehat{\Xi}_b, 1 \rangle_{\Gamma_L} = \frac{q-1}{q}$.

For the other equation of the second claim one has by definition of ς

$$
\begin{aligned}
\varsigma(\widehat{\Xi}_b) &\overset{(4.37)}{=} \Omega \ell^*(b) \mathrm{res}_{\mathbf{B}} \left(\kappa^* \left(\mathfrak{M}(\sigma_{-1}) \mathfrak{M} \left(\mathrm{Tw}_{\chi_{\mathrm{LT}}} (\chi_{\mathrm{LT}}^*(\Xi_b)) \right) \right) \right) d \log_{\mathrm{LT}} \\
&= \Omega \ell^*(b) \mathrm{res}_{\mathbf{B}} \left(\mathfrak{M}_{\mathrm{LT}}(\sigma_{-1}) \mathfrak{M}_{\mathrm{LT}} \left(\mathrm{Tw}_{\chi_{\mathrm{LT}}} (\chi_{\mathrm{LT}}^*(\Xi_b)) \right) d \log_{\mathrm{LT}} \right) \\
&= \ell^*(b) \mathrm{res}_{\mathbf{B}} \left(\mathfrak{M}_{\mathrm{LT}}(\sigma_{-1}) \log_{\mathrm{LT}}(Z) \partial_{\mathrm{inv}} \mathfrak{M}_{\mathrm{LT}} (\chi_{\mathrm{LT}}^*(\Xi_b)) \frac{d \log_{\mathrm{LT}}}{\log_{\mathrm{LT}}(Z)} \right) \\
&= \ell^*(b) \mathrm{res}_{\mathbf{B}} \left(\mathfrak{M}_{\mathrm{LT}}(\sigma_{-1}) \mathfrak{M}_{\mathrm{LT}} (\nabla \chi_{\mathrm{LT}}^*(\Xi_b)) \frac{d \log_{\mathrm{LT}}}{\log_{\mathrm{LT}}(Z)} \right) \\
&= \ell^*(b) \mathrm{res}_{\mathbf{B}} \left(\mathfrak{M}_{\mathrm{LT}}(\sigma_{-1}) \frac{\pi_L^{n_0} \log_{\mathrm{LT}}(Z)}{\varphi_L^{n_0}(Z \ell(b))} \eta(1, Z) \frac{d \log_{\mathrm{LT}}}{\log_{\mathrm{LT}}(Z)} \right) \\
&= \ell^*(b) \pi_L^{n_0} \mathrm{res}_{\mathbf{B}} \left(\eta(-1, Z) \frac{1}{\varphi_L^{n_0}(Z \ell(b))} \eta(1, Z) d \log_{\mathrm{LT}} \right) \\
&= \ell^*(b) \pi_L^{n_0} \mathrm{res}_{\mathbf{B}} \left(\eta(1 - 1, Z) \frac{1}{\varphi_L^{n_0}(Z \ell(b))} d \log_{\mathrm{LT}} \right) \\
&= \ell^*(b) \pi_L^{n_0} \mathrm{res}_{\mathbf{B}} \left(\varphi_L^{n_0} \left(\eta(0, Z) \frac{1}{Z \ell(b)} d \log_{\mathrm{LT}} \right) \right) \\
&= \frac{\ell^*(b)}{\ell(b)} \pi_L^{n_0} \left(\tfrac{q}{\pi_L} \right)^{n_0} \mathrm{res}_{\mathbf{B}} \left(\tfrac{1}{Z} g_{\mathrm{LT}} dZ \right) \\
&= 1,
\end{aligned}
$$

where we use (4.64) in the third equation, the fact that ∇ acts on $\mathcal{R}$ as $\log_{\mathrm{LT}}(Z) \partial_{\mathrm{inv}}$ (cf. (4.70)) in the fourth equation, Remark 4.4.9 for the fifth equation, Lemma 4.5.1 (iv) with $\psi_L(1) = \frac{q}{\pi_L}$ for the penultimate equation, and finally for the last equation that $g_{\mathrm{LT}}(Z)$ has constant term 1. ∎

Proof of Theorem 4.5.12. Since the equality can also be checked after base change by (4.26), we may and do assume that Ω belongs to K. Due to Proposition 4.5.9,

there exists $g \in D(\Gamma_L, K)$ such that $\varsigma(\lambda) = \langle g, \lambda \rangle_{\Gamma_L}$ for all $\lambda \in \mathcal{R}_K(\Gamma_L)$ because ς sends $D(\Gamma_L, K)$ to zero. We claim that

$$\mathrm{Tw}_{\chi_{\mathrm{LT}}^j}(g) = g \tag{4.83}$$

for all $j \in \mathbb{Z}$: By Proposition 4.5.11 and Lemma 4.5.14 we have

$$\begin{aligned}
\langle Tw_{\chi_{\mathrm{LT}}^j}(g), f \rangle_{\Gamma_L} &= \langle g, \mathrm{Tw}_{\chi_{\mathrm{LT}}^{-j}}(f) \rangle_{\Gamma_L} \\
&= \varsigma\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}^{-j}}(f)\big) \\
&= \varsigma(f) \\
&= \langle g, f \rangle_{\Gamma_L}
\end{aligned}$$

for all $f \in \mathcal{R}_K(\Gamma_L)$.

Now it follows from (4.83) combined with Lemma 4.5.15 that g is constant (and equal to $\mathrm{ev}_{\chi_{\mathrm{LT}}^0}(g)$), i.e., $\varsigma(-) = g\langle 1, - \rangle_{\Gamma_L} = g\varrho(-)$. Finally, it follows from (4.82) that $g = \frac{q}{q-1}$. ∎

Corollary 4.5.17. *The pairing $\frac{q}{q-1}\langle\,,\,\rangle_{\Gamma_L}$ makes the following diagram commutative:*

$$\begin{array}{ccccccc}
\mathcal{R}_K(\mathfrak{X})^{\psi_L=0} & \times & (\Omega^1_{\mathcal{R}(\mathfrak{X})})^{\psi_L=0} & \xrightarrow{\mathrm{mult}} & \Omega^1_{\mathcal{R}(\mathfrak{X})} & \xrightarrow{\mathrm{res}_{\mathfrak{X}}} & K \\
\uparrow{\scriptstyle \sigma_{-1}\mathfrak{M}\circ\iota_*} & & \uparrow{\scriptstyle \mathfrak{M}^{\Omega^1}} & & & & \| \\
\mathcal{R}_K(\Gamma_L) & \times & \mathcal{R}_K(\Gamma_L) & \xrightarrow{\quad\frac{q}{q-1}\langle\,,\,\rangle_{\Gamma_L}\quad} & & & K;
\end{array} \tag{4.84}$$

i.e., we have

$$\begin{aligned}
\tfrac{q}{q-1}\langle \mu, \lambda \rangle_{\Gamma_L} &= \big\{\mathfrak{M}\big(\sigma_{-1}\iota_*(\mu)\big), \mathfrak{M}^{\Omega^1}(\lambda)\big\}_{\Omega^1} \\
&= \mathrm{res}_{\mathfrak{X}}\big(\sigma_{-1}\mathfrak{M}\big(\iota_*(\mu)\big)\mathfrak{M}^{\Omega^1}(\lambda)\big) \\
&= \mathrm{res}_{\mathfrak{X}}\big(\mathfrak{M}\big(\sigma_{-1}\iota_*(\mu)\big)\mathfrak{M}\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}(\lambda)\big)d\log_{\mathfrak{X}}\big) \\
&= \mathrm{res}_{\mathfrak{X}}\big(\mathfrak{M}\big(\iota_*(\mu)\big)\mathfrak{M}\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}(\sigma_{-1}\lambda)\big)d\log_{\mathfrak{X}}\big).
\end{aligned} \tag{4.85}$$

Proof. By Theorem 4.5.12 and the definition of ς and of $\langle\,,\,\rangle_{\Gamma_L}$ we have

$$\begin{aligned}
\tfrac{q}{q-1}\langle \mu, \lambda \rangle_{\Gamma_L} &= \frac{q}{q-1}\langle 1, \mu\lambda \rangle_{\Gamma_L} \\
&= \big\{\mathfrak{M}(\sigma_{-1}), \mathfrak{M}^{\Omega^1}(\mu\lambda)\big\}_{\Omega^1} \\
&= \big\{\mathfrak{M}\big(\sigma_{-1}\iota_*(\mu)\big), \mathfrak{M}^{\Omega^1}(\lambda)\big\}_{\Omega^1},
\end{aligned} \tag{4.86}$$

where we use Lemma 4.5.5 for the last equation. ∎

Lemma 4.5.18. *We have for all $\lambda, \mu \in \mathcal{R}_K(\Gamma_L)$ that $\langle \lambda, \mu \rangle_{\Gamma_L} = -\langle \iota_*(\lambda), \iota_*(\mu) \rangle_{\Gamma_L}$.*

Proof. Using (4.85) for the first and third equation, property (3) in Section 4.1.3 applied to ι and the fact that $\mathrm{Tw}_{\chi_{\mathrm{LT}}}(\sigma_{-1}) = -\sigma_{-1}$ for the second equation, and Proposition 4.5.11 for the last one, we see that

$$
\begin{aligned}
\tfrac{q}{q-1}\langle \mu, \lambda \rangle_{\Gamma_L} &= \mathrm{res}_{\mathfrak{X}}\big(\mathfrak{M}\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}(\sigma_{-1}\lambda)\big)\mathfrak{M}\big(\iota_*(\mu)\big)d\log_{\mathfrak{X}}\big) \\
&= -\mathrm{res}_{\mathfrak{X}}\big(\mathfrak{M}\big(\sigma_{-1}\iota_*\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}^{-1}}(\iota_*(\lambda))\big)\big)\mathfrak{M}\big(\iota_*(\mu)\big)d\log_{\mathfrak{X}}\big) \\
&= -\tfrac{q}{q-1}\big\langle \mathrm{Tw}_{\chi_{\mathrm{LT}}^{-1}}(\iota_*(\lambda)), \mathrm{Tw}_{\chi_{\mathrm{LT}}^{-1}}(\iota_*(\mu))\big\rangle_{\Gamma_L} \\
&= -\tfrac{q}{q-1}\big\langle \iota_*(\lambda), \iota_*(\mu)\big\rangle_{\Gamma_L}. \qquad\blacksquare
\end{aligned}
$$

4.5.4 The Iwasawa pairing for (φ_L, Γ_L)-modules over the Robba ring

As before, let $\mathfrak{Y}$ be either $\mathfrak{X}$ or $\mathbf{B}$ and $\mathcal{R} = \mathcal{R}_K(\mathfrak{Y})$ and M in $\mathcal{M}^{\mathrm{an}}(\mathcal{R})$, where K is any complete intermediate extension $L \subseteq K \subseteq \mathbb{C}_p$. Using Proposition 4.5.9, we define the pairing

$$
\{\,,\,\}_{\mathrm{Iw}}^0 := \{\,,\,\}_{M,\mathrm{Iw}}^0 : \check{M}^{\psi_L=0} \times M^{\psi_L=0} \to \mathcal{R}_K(\Gamma_L),
$$

which is $\mathcal{R}_K(\Gamma_L)$-ι_*-sesquilinear in the sense that

$$
\lambda\{\check{m}, m\}_{\mathrm{Iw}}^0 = \{\lambda\check{m}, m\}_{\mathrm{Iw}}^0 = \{\check{m}, \iota_*(\lambda)m\}_{\mathrm{Iw}}^0 \tag{4.87}
$$

for all $\lambda \in \mathcal{R}_K(\Gamma_L)$ and $\check{m} \in \check{M}^{\psi_L=0}, m \in M^{\psi_L=0}$. This requires the commutativity of the diagram

$$
\begin{array}{ccccc}
\mathcal{R}_K(\Gamma_L) & \times & \check{M}^{\psi_L=0} \times M^{\psi_L=0} & \longrightarrow & K \\
\| & & {\scriptstyle \{,\}_{\mathrm{Iw}}^0}\Big\downarrow & & \| \\
\mathcal{R}_K(\Gamma_L) & \times & \mathcal{R}_K(\Gamma_L) & \xrightarrow{\ \langle,\rangle_{\Gamma_L}\ } & K,
\end{array}
$$

in which the upper line sends (f, x, y) to $\{f(x), y\}_M$, where the latter pairing is (4.73). Indeed, the property

$$
\{\lambda\check{m}, m\}_{\mathrm{Iw}}^0 = \{\check{m}, \iota_*(\lambda)m\}_{\mathrm{Iw}}^0
$$

follows from the corresponding property for $\{\,,\,\}_M$ by Lemma 4.5.5, while with regard to the second one

$$
\lambda\{\check{m}, m\}_{\mathrm{Iw}}^0 = \{\lambda\check{m}, m\}_{\mathrm{Iw}}^0
$$

we have for all $f \in \mathcal{R}_K(\Gamma_L)$

$$
\big\langle f, \{\lambda\check{m}, m\}_{\mathrm{Iw}}^0\big\rangle_{\Gamma_L} = \{f \cdot \lambda\check{m}, m\} = \big\langle \lambda f, \{\check{m}, m\}_{\mathrm{Iw}}^0\big\rangle_{\Gamma_L} = \big\langle f, \lambda\{\check{m}, m\}_{\mathrm{Iw}}^0\big\rangle_{\Gamma_L}
$$

by Lemma 4.5.6. Note that the pairing $\{\,,\,\}_{\mathrm{Iw}}^0$ induces the isomorphism in (4.78).

We set

$$\mathcal{C} := \left(\tfrac{\pi_L}{q}\varphi_L - 1\right) M^{\psi_L=1} \quad \text{and} \quad \check{\mathcal{C}} := (\varphi_L - 1)\check{M}^{\psi_L=\frac{q}{\pi_L}}$$

and we shall need the following lemma.

Lemma 4.5.19. *For $f \in D(\Gamma_L, K)$ we have $\{f \cdot (\varphi_L - 1)x, (\tfrac{\pi_L}{q}\varphi_L - 1)y\} = 0$ for all $x \in \check{M}^{\psi_L=\frac{q}{\pi_L}}$ and $y \in M^{\psi_L=1}$.*

Proof. The result follows by straightforward calculation using Lemma 4.5.1 (cf. [42, Lem. 4.2.7]). ∎

This lemma combined with the second statement of Proposition 4.5.9 implies that the restriction of $\{\ ,\ \}^0_{\mathrm{Iw}}$ to $\check{\mathcal{C}} \times \mathcal{C}$, which by abuse of notation we denote by the same symbol, is characterized by the commutativity of the diagram

$$
\begin{array}{ccccc}
\check{\mathcal{C}} \times \mathcal{C} & \times & \mathcal{R}_K(\Gamma_L)/D(\Gamma_L, K) & \longrightarrow & K \\
{\scriptstyle \{\ ,\ \}^0_{\mathrm{Iw}}}\Big\downarrow & & \Big\| & & \Big\| \\
D(\Gamma_L, K) & \times & \mathcal{R}_K(\Gamma_L)/D(\Gamma_L, K) & \xrightarrow{\langle\ ,\ \rangle_{\Gamma_L}} & K,
\end{array}
$$

in which the upper line sends (x, y, f) to $\{f(x), y\}_M$. In particular, it takes values in $D(\Gamma_L, K)$.

Finally, we obtain a $D(\Gamma_L, K)$-ι_*-sesquilinear pairing $\{\ ,\ \}_{\mathrm{Iw}} := \{\ ,\ \}_{M,\mathrm{Iw}}$ which by definition fits into the following commutative diagram:

$$
\begin{array}{ccccc}
\check{M}^{\psi_L=\frac{q}{\pi_L}} & \times & M^{\psi_L=1} & \xrightarrow{\{\,,\,\}_{M,Iw}} & D(\Gamma_L, K) \\
{\scriptstyle \varphi_L-1}\Big\downarrow & & {\scriptstyle \frac{\pi_L}{q}\varphi_L-1}\Big\downarrow & & \Big\| \\
\check{\mathcal{C}} & \times & \mathcal{C} & \xrightarrow{\{\,,\,\}^0_{M,Iw}} & D(\Gamma_L, K).
\end{array}
$$

Altogether we obtain the following theorem.

Theorem 4.5.20. *There is a $D(\Gamma_L, K)$-ι_*-sesquilinear pairing*

$$\{\ ,\ \}_{\mathrm{Iw}} : \check{M}^{\psi_L=\frac{q}{\pi_L}} \times M^{\psi_L=1} \to D(\Gamma_L, K). \tag{4.88}$$

It is characterized by the following equality:

$$\left\langle f, \{\check{m}, m\}_{\mathrm{Iw}}\right\rangle_{\Gamma_L} = \left\{f \cdot (\varphi_L - 1)\check{m}, \left(\tfrac{\pi_L}{q}\varphi_L - 1\right)m\right\} \tag{4.89}$$

for all $f \in \mathcal{R}_K(\Gamma_L)$, $\check{m} \in \check{M}$, $m \in M$.

Remark 4.5.21. For any $n \geq n_0$, we obtain in the same way as in (4.88) a $D(\Gamma_n, K)$-ι_*-sesquilinear pairing

$$\{\ ,\ \}_{\mathrm{Iw},\Gamma_n} : \check{M}^{\psi_L=\frac{q}{\pi_L}} \times M^{\psi_L=1} \to D(\Gamma_n, K).$$

It follows immediately from the definitions, the projection formulae (4.77), and the Frobenius reciprocity (Remark 4.5.8) that

$$\{\,,\,\}_{\mathrm{Iw},\Gamma_n} := (q-1)q^{n-1}\,\mathrm{pr}_{L,n}\circ\{\,,\,\}_{\mathrm{Iw}}.$$

If $\chi : \Gamma_L \to o_L^\times$ is any continuous character with representation module $W_\chi = o_L\eta_\chi$, then, for any (φ_L, Γ_L)-module M over $\mathcal{R}$, we have the twisted (φ_L, Γ_L)-module $M(\chi)$, where $M(\chi) := M \otimes_{o_L} W_\chi$ as $\mathcal{R}$-module, $\varphi_{M(\chi)}(m \otimes w) := \varphi_M(m) \otimes w$, and $\gamma(m \otimes w) := \gamma(m) \otimes \gamma(w) = \chi(\gamma) \cdot \gamma(m) \otimes w$ for $\gamma \in \Gamma_L$. It follows that

$$\psi_{M(\chi)}(m \otimes w) = \psi_M(m) \otimes w.$$

For the character χ_{LT} we take $W_{\chi_{\mathrm{LT}}} = T = o_L\eta$ and $W_{\chi_{\mathrm{LT}}^{-1}} = T^* = o_L\eta^*$ as representation module, where T^* denotes the o_L-dual with dual basis η^* of η.

Consider the $\mathcal{R}_K$-linear (but of course not $\mathcal{R}_K(\Gamma_L)$-linear) map

$$\mathrm{tw}_\chi : M \to M(\chi), \quad m \mapsto m \otimes \eta_\chi.$$

Lemma 4.5.22. *There is a commutative diagram*

$$
\begin{array}{ccccc}
\check{M}(\chi_{\mathrm{LT}}^{-j})^{\psi_L=\frac{q}{\pi_L}} & \times & M(\chi_{\mathrm{LT}}^{j})^{\psi_L=1} & \xrightarrow{\{\,,\,\}_{\mathrm{Iw}}} & D(\Gamma_L, \mathbb{C}_p) \\
\uparrow{\scriptstyle\mathrm{tw}_{\chi_{\mathrm{LT}}^{-j}}} & & \uparrow{\scriptstyle\mathrm{tw}_{\chi_{\mathrm{LT}}^{j}}} & & \uparrow{\scriptstyle\mathrm{Tw}_{\chi_{\mathrm{LT}}^{j}}} \\
\check{M}^{\psi_L=\frac{q}{\pi_L}} & \times & M^{\psi_L=1} & \xrightarrow{\{\,,\,\}_{\mathrm{Iw}}} & D(\Gamma_L, \mathbb{C}_p).
\end{array}
$$

Proof. We have for all $f \in \mathcal{R}_K(\Gamma_L)$,

$$
\begin{aligned}
\langle f, \{\mathrm{tw}&_{\chi_{\mathrm{LT}}^{-j}}(\check{m}), \mathrm{tw}_{\chi_{\mathrm{LT}}^{j}}(m)\}_{\mathrm{Iw}}\rangle_{\Gamma_L} \\
&= \{f \cdot ((\varphi_L - 1)\check{m} \otimes \eta^{\otimes -j}), (\tfrac{\pi_L}{q}\varphi_L - 1)m \otimes \eta^{\otimes j}\} \\
&= \{(\mathrm{Tw}_{\chi_{\mathrm{LT}}^{-j}}(f) \cdot (\varphi_L - 1)\check{m}) \otimes \eta^{\otimes -j}, (\tfrac{\pi_L}{q}\varphi_L - 1)m \otimes \eta^{\otimes j}\} \\
&= \{(\mathrm{Tw}_{\chi_{\mathrm{LT}}^{-j}}(f) \cdot (\varphi_L - 1)\check{m}), (\tfrac{\pi_L}{q}\varphi_L - 1)m\} \\
&= \langle \mathrm{Tw}_{\chi_{\mathrm{LT}}^{-j}}(f), \{\check{m}, m\}_{\mathrm{Iw}}\rangle_{\Gamma_L} \\
&= \langle f, \mathrm{Tw}_{\chi_{\mathrm{LT}}^{j}}(\{\check{m}, m\}_{\mathrm{Iw}})\rangle_{\Gamma_L},
\end{aligned}
$$

where we used Proposition 4.5.11 for the last equation. The second equation is clear for δ-distributions and hence extends by the uniqueness result of Theorem 4.3.21, cf. the proof of Theorem 4.3.20. ∎

4.5.5 The abstract reciprocity formula

We keep the notation from the preceding subsection and set $t_{\mathfrak{Y}} := \log_{\mathfrak{Y}}$.

Compatibility of the Iwasawa pairing under comparison isomorphisms

Let M, N be (not necessarily étale) L-analytic (φ_L, Γ_L)-modules over $\mathcal{R}$. We extend the action of Γ_L, φ_L, and ψ_L to the $\mathcal{R}[\frac{1}{t_{\mathfrak{Y}}}]$-module $M[\frac{1}{t_{\mathfrak{Y}}}]$ (and in the same way to $N[\frac{1}{t_{\mathfrak{Y}}}]$) as follows:[9]

$$\gamma \frac{m}{t_{\mathfrak{Y}}^k} := \frac{\gamma m}{\gamma t_{\mathfrak{Y}}^k} = \frac{\frac{\gamma m}{\chi_{\mathrm{LT}}^k(\gamma)}}{t_{\mathfrak{Y}}^k},$$

$$\varphi_L\left(\frac{m}{t_{\mathfrak{Y}}^k}\right) := \frac{\varphi_L(m)}{\varphi_L(t_{\mathfrak{Y}}^k)} = \frac{\frac{\varphi_L(m)}{\pi_L^k}}{t_{\mathfrak{Y}}^k},$$

$$\psi_L\left(\frac{m}{t_{\mathfrak{Y}}^k}\right) := \frac{\pi_L^k \psi_L(m)}{t_{\mathfrak{Y}}^k}.$$

Lemma 4.5.23. *We have the following:*

  (i) $(M[\frac{1}{t_{\mathfrak{Y}}}])^{\psi_L=0} = (M^{\psi_L=0})[\frac{1}{t_{\mathfrak{Y}}}] := \{\frac{m}{t_{\mathfrak{Y}}^k} | m \in M^{\psi_L=0},\ k \geq 0\}.$

  (ii) *The (separately continuous) $\mathcal{R}_K(\Gamma_L)$-action on $M^{\psi_L=0}$ extends to a (separately continuous with respect to direct limit topology) action of $\mathcal{R}_K(\Gamma_L)$ on $(M[\frac{1}{t_{\mathfrak{Y}}}])^{\psi_L=0}$.*

Proof. For (i) note that $0 = \psi_L(\frac{m}{t_{\mathfrak{Y}}^k}) = \frac{\pi_L^k \psi_L(m)}{t_{\mathfrak{Y}}^k}$ if and only if $\psi_L(m) = 0$. For (ii) take for any $f \in \mathcal{R}_K(\Gamma_L)$ the direct limit of the following commutative diagram:

$$
\begin{array}{ccccccccc}
M^{\psi_L=0} & \xrightarrow{t_{\mathfrak{Y}}} & M^{\psi_L=0} & \xrightarrow{t_{\mathfrak{Y}}} & \cdots & \xrightarrow{t_{\mathfrak{Y}}} & M^{\psi_L=0} & \xrightarrow{t_{\mathfrak{Y}}} & \cdots \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Tw}_{\chi_{\mathrm{LT}}^{-1}}(f)} & & & & \downarrow{\scriptstyle \mathrm{Tw}_{\chi_{\mathrm{LT}}^{-i}}(f)} & & \\
M^{\psi_L=0} & \xrightarrow{t_{\mathfrak{Y}}} & M^{\psi_L=0} & \xrightarrow{t_{\mathfrak{Y}}} & \cdots & \xrightarrow{t_{\mathfrak{Y}}} & M^{\psi_L=0} & \xrightarrow{t_{\mathfrak{Y}}} & \cdots
\end{array}
$$

This defines a (separately continuous) action. ∎

Now we assume that there is an isomorphism

$$c : \mathcal{R}\left[\tfrac{1}{t_y}\right] \otimes_{\mathcal{R}} M \xrightarrow{\cong} \mathcal{R}\left[\tfrac{1}{t_y}\right] \otimes_{\mathcal{R}} N$$

of (φ_L, Γ_L)-modules over $\mathcal{R}[\frac{1}{t_{\mathfrak{Y}}}]$.

---

[9] Since $t_{\mathfrak{Y}}^k = \varphi_L\left(\frac{t_{\mathfrak{Y}}^k}{\pi_L^k}\right)$, one checks that $\psi_L(t_{\mathfrak{Y}}^k m) = \frac{t_{\mathfrak{Y}}^k}{\pi_L^k} \psi_L(m)$ by the projection formula. In particular, the definition is independent of the chosen denominator.

Consider the composite map

$$\check{c} : \mathcal{R}\big[\tfrac{1}{t_y}\big] \otimes_{\mathcal{R}} \check{M} \cong \mathrm{Hom}_{\mathcal{R}[\frac{1}{t_y}]}\big(\mathcal{R}\big[\tfrac{1}{t_y}\big] \otimes_{\mathcal{R}} M, \mathcal{R}\big[\tfrac{1}{t_y}\big] \otimes_{\mathcal{R}} \Omega^1_{\mathcal{R}}\big)$$

$$\cong \mathrm{Hom}_{\mathcal{R}[\frac{1}{t_y}]}\big(\mathcal{R}\big[\tfrac{1}{t_y}\big] \otimes_{\mathcal{R}} N, \mathcal{R}\big[\tfrac{1}{t_y}\big] \otimes_{\mathcal{R}} \Omega^1_{\mathcal{R}}\big)$$

$$\cong \mathcal{R}\big[\tfrac{1}{t_y}\big] \otimes_{\mathcal{R}} \check{N},$$

where the second isomorphism is $(c^{-1})^*$.

Lemma 4.5.24. $c^{\psi_L=0}$ and $\check{c}^{\psi_L=0}$ are $\mathcal{R}_K(\Gamma_L)$-equivariant.

Proof. Consider, for $n \in \mathbb{Z}$, the (φ_L, Γ_L)-modules (!) $M_n := t_y^{-n} M$ over $\mathcal{R}$ and note that the inclusion $(M_n)^{\psi_L=0} \subseteq (M[\tfrac{1}{t_y}])^{\psi_L=0}$ is $\mathcal{R}_K(\Gamma_L)$-equivariant by construction of the action. Now, since M, N are finitely generated over $\mathcal{R}$, there exists $n_0 \geq 0$ such that c restricts to a homomorphism $c_0 : M \to N_{n_0}$ of (φ_L, Γ_L)-modules over $\mathcal{R}$, whence $c_0^{\psi_L=0} : M^{\psi_L=0} \to N_{n_0}^{\psi_L=0} \subseteq (N[\tfrac{1}{t_y}])^{\psi_L=0}$ is $\mathcal{R}_K(\Gamma_L)$-equivariant by the functoriality of Theorem 4.3.23 and similarly for the induced maps $c_n : M_n \to N_{n_0+n}$ for all $n \geq 0$. The equivariance for $c^{\psi_L=0}$ follows by taking direct limits.

Similarly, for some $n_0 \geq 0$, the inverse b of c induces homomorphisms $b_n : N_{-n_0-n} \to M_{-n}$ of (φ_L, Γ_L)-modules over $\mathcal{R}$ all $n \in \mathbb{Z}$. We obtain homomorphisms of (φ_L, Γ_L)-modules over $\mathcal{R}$

$$\check{c}_n : (\check{M})_n = \mathrm{Hom}_{\mathcal{R}}(M, t_y^{-n} \Omega^1_{\mathcal{R}})$$

$$\cong \mathrm{Hom}_{\mathcal{R}}(M_{-n}, \Omega^1_{\mathcal{R}})$$

$$\cong \mathrm{Hom}_{\mathcal{R}}(N_{-n_0-n}, \Omega^1_{\mathcal{R}})$$

$$\cong (\check{N})_{n_0+n},$$

where the third isomorphism is $(b_n)^*$. As above $(\check{c}_n)^{\psi_L=0}$ is $\mathcal{R}_K(\Gamma_L)$-equivariant and the claim follows by taking direct limits. ∎

Lemma 4.5.25. *The following diagram commutes on the vertical intersections:*

$$
\begin{array}{ccccc}
\check{M}^{\psi_L=0} & \times & M^{\psi_L=0} & \xrightarrow{\{\,,\,\}^0_{M,\mathrm{Iw}}} & \mathcal{R}_K(\Gamma_L) \\
\downarrow & & \downarrow & & \| \\
\big(\mathcal{R}\big[\tfrac{1}{t_y}\big] \otimes_{\mathcal{R}} \check{M}\big)^{\psi_L=0} & \times & \big(\mathcal{R}\big[\tfrac{1}{t_y}\big] \otimes_{\mathcal{R}} M\big)^{\psi_L=0} & & \| \\
\check{c}\downarrow\cong & & c\downarrow\cong & & \| \\
\big(\mathcal{R}\big[\tfrac{1}{t_y}\big] \otimes_{\mathcal{R}} \check{N}\big)^{\psi_L=0} & \times & \big(\mathcal{R}\big[\tfrac{1}{t_y}\big] \otimes_{\mathcal{R}} N\big)^{\psi_L=0} & & \| \\
\uparrow & & \uparrow & & \| \\
\check{N}^{\psi_L=0} & \times & N^{\psi_L=0} & \xrightarrow{\{\,,\,\}^0_{N,\mathrm{Iw}}} & \mathcal{R}_K(\Gamma_L);
\end{array}
$$

i.e., if $\check{m} \in \check{M}$, $m \in M$, $\check{n} \in \check{N}$, $n \in N$ with $\check{c}(\check{m}) = \check{n}$ and $c(m) = n$, then

$$\{\check{m}, m\}^0_{M,\mathrm{Iw}} = \{\check{n}, n\}^0_{N,\mathrm{Iw}}.$$

Proof. By definition of the Iwasawa pairings, we have for all $f \in \mathcal{R}_K(\Gamma_L)$

$$
\begin{aligned}
\langle f, \{\check{n}, n\}^0_{N,\mathrm{Iw}} \rangle_{\Gamma_L} &= \{f \cdot \check{n}, n\}_N \\
&= \{f \cdot \check{c}(\check{m}), c(m)\}_N \\
&= \{\check{c}(f \cdot \check{m}), c(m)\}_N \\
&= \mathrm{res}_{\mathfrak{Y}}\big(\check{c}(f \cdot \check{m})(c(m))\big) \\
&= \mathrm{res}_{\mathfrak{Y}}\big(((f \cdot \check{m}) \circ c^{-1})(c(m))\big) \\
&= \mathrm{res}_{\mathfrak{Y}}\big((f \cdot \check{m})(m)\big) \\
&= \{f \cdot \check{m}, m\}_M \\
&= \langle f, \{\check{m}, m\}^0_{M,\mathrm{Iw}} \rangle_{\Gamma_L},
\end{aligned}
$$

whence the claim. Here, we use the $\mathcal{R}_K(\Gamma_L)$-equivariance of $\check{c}$ in the third equality. ∎

Now let D be any φ_L-module over L of finite dimension, say d (with trivial Γ_L-action), and consider the (φ_L, Γ_L)-module $N := \mathcal{R} \otimes_L D$ over $\mathcal{R}$ (with diagonal actions). Since $N \cong \mathcal{R}^d$ as Γ_L-module, it is L-analytic. Moreover, we have $\check{N} \cong \Omega^1_{\mathcal{R}} \otimes D^*$ with $D^* = \mathrm{Hom}_L(D, L)$ being the dual φ_L-module. We set

$$\widetilde{\Omega} := \begin{cases} 1 & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \Omega & \text{if } \mathfrak{Y} = \mathbf{B} \text{ (and } \Omega \in K). \end{cases}$$

Lemma 4.5.26. *If $\mathfrak{Y} = \mathbf{B}$, we assume $\Omega \in K$. There is a commutative diagram*

$$
\begin{array}{ccccc}
(\Omega^1_{\mathcal{R}} \otimes D^*)^{\psi_L=0} & \times & (\mathcal{R} \otimes_L D)^{\psi_L=0} & \xrightarrow{\widetilde{\Omega}\frac{q-1}{q}\{\,,\,\}^0_{N,\mathrm{Iw}}} & \mathcal{R}_K(\Gamma_L) \\
{\scriptstyle \mathfrak{M}^{\Omega^1} \otimes \mathrm{id}} \big\uparrow {\scriptstyle \cong} & & {\scriptstyle \sigma_{-1}\mathfrak{M} \circ \iota_* \otimes \mathrm{id}} \big\uparrow {\scriptstyle \cong} & & \big\| \\
\mathcal{R}_K(\Gamma_L) \otimes_L D^* & \times & \mathcal{R}_K(\Gamma_L) \otimes_L D & \longrightarrow & \mathcal{R}_K(\Gamma_L),
\end{array}
$$

where the bottom line is the $\mathcal{R}_K(\Gamma_L)$-linear extension of the canonical pairing between D^* and D; i.e., it maps $(\lambda \otimes l, \mu \otimes d)$ to $\lambda \mu l(d)$.

Proof. Let $\check{d}_j$ and d_i be a basis of D^* and D, respectively, and $x = \sum_j \lambda_j \cdot \check{d}_j$ and $y = \sum_i \mu_i \cdot d_i$. Then, by definition of $\{\,,\,\}^0_{\mathrm{Iw}}$ we have for all $\lambda \in \mathcal{R}_K(\Gamma_L)$

$$
\begin{aligned}
&\big\langle \lambda, \{(\mathfrak{M}^{\Omega^1} \otimes \mathrm{id})(x), (\sigma_{-1}\mathfrak{M} \circ \iota_* \otimes \mathrm{id})(y)\}^0_{\mathrm{Iw}} \big\rangle_{\Gamma_L} \\
&\quad = \{(\lambda \mathfrak{M}^{\Omega^1} \otimes \mathrm{id})(x), (\sigma_{-1}\mathfrak{M} \circ \iota_* \otimes \mathrm{id})(y)\}
\end{aligned}
$$

$$= \left\{ \sum_j (\lambda \lambda_j) \cdot (\mathrm{ev}_1 \, d \log_{\mathfrak{Y}} \otimes \check{d}_j), \sum_i \iota_*(\mu_i) \cdot \mathrm{ev}_{-1} \otimes d_i \right\}$$

$$= \sum_{i,j} \left\{ (\lambda \lambda_j \mu_i) \cdot (\mathrm{ev}_1 \, d \log_{\mathfrak{Y}}) \otimes \check{d}_j, \mathrm{ev}_{-1} \otimes d_i \right\}$$

$$= \sum_{i,j} \mathrm{res}_{\mathfrak{Y}} \big(\check{d}_j(d_i) \, \mathrm{ev}_{-1}(\lambda \lambda_j \mu_i) \cdot (\mathrm{ev}_1 \, d \log_{\mathfrak{Y}}) \big)$$

$$= \sum_{i,j} \check{d}_j(d_i) \mathrm{res}_{\mathfrak{Y}} \big(\mathrm{ev}_{-1}(\lambda \lambda_j \mu_i) \cdot (\mathrm{ev}_1 \, d \log_{\mathfrak{Y}}) \big).$$

Here, for the third equation we used property (iii) in Lemma 4.5.1. On the other hand, we can pair the image $\sum_{i,j} \lambda_j \mu_i \check{d}_j(d_i)$ of (x, y) under the bottom pairing with λ using the description (4.86)

$$\frac{q}{q-1} \Big\langle \lambda, \sum_{i,j} \lambda_j \mu_i \check{d}_j(d_i) \Big\rangle_{\Gamma_L} = \sum_{i,j} \check{d}_j(d_i) \{ \mathfrak{M}(\sigma_{-1}), \mathfrak{M}^{\Omega^1}(\lambda \lambda_j \mu_i) \}$$

$$= \sum_{i,j} \check{d}_j(d_i) \mathrm{res}_{\mathfrak{X}} \big(\mathrm{ev}_{-1}(\lambda \lambda_j \mu_i) \cdot (\mathrm{ev}_1 \, d \log_{\mathfrak{X}}) \big),$$

whence comparing with the above gives the result for $\mathfrak{Y} = \mathfrak{X}$, using Proposition 4.5.9. If $\mathfrak{Y} = \mathbf{B}$, we obtain the factor Ω due to Remark 4.5.3. ∎

Definition 4.5.27. We write:

(i) $\mathrm{Rep}_L^{\mathrm{an}}(G_L)$ for the full subcategory of the category $\mathrm{Rep}_L(G_L)$ consisting of L-*analytic* representations V, i.e., $\mathbb{C}_p \otimes_{\sigma, L} V$ is the trivial semilinear $\mathbb{C}_p$-representation $\mathbb{C}_p^{\dim_L V}$ for all embeddings $\sigma : L \to \mathbb{C}_p$ different from the fixed inclusion $\iota : L \to \mathbb{C}_p$;

(ii) $\mathcal{M}^{\mathrm{an}, \mathrm{\acute{e}t}}(\mathcal{R})$ for the category of étale, analytic (φ_L, Γ_L)-modules over $\mathcal{R}$ (remember that an L-analytic (φ_L, Γ_L)-module M over $\mathcal{R}$ is called étale if it is semistable and of slope 0).

The following theorem is crucial.

Theorem 4.5.28. *There are equivalences of categories*

$$\mathrm{Rep}_L^{\mathrm{an}}(G_L) \leftrightarrow \mathcal{M}^{\mathrm{an}, \mathrm{\acute{e}t}}\big(\mathcal{R}_L(\mathbf{B}) \big)$$

$$V \mapsto D_{\mathrm{rig}}^\dagger(V)$$

and

$$\mathrm{Rep}_L^{\mathrm{an}}(G_L) \leftrightarrow \mathcal{M}^{\mathrm{an}, \mathrm{\acute{e}t}}\big(\mathcal{R}_L(\mathfrak{X}) \big)$$

$$V \mapsto D_{\mathrm{rig}}^\dagger(V)_{\mathfrak{X}}.$$

Proof. The first equivalence is Theorem D in [6]. By [8, Thm. 3.27] one has an equivalence of categories

$$\mathcal{M}^{\mathrm{an,\acute{e}t}}\big(\mathcal{R}_L(\mathbf{B})\big) \leftrightarrow \mathcal{M}^{\mathrm{an,\acute{e}t}}\big(\mathcal{R}_L(\mathfrak{X})\big) \tag{4.90}$$
$$M \mapsto M_{\mathfrak{X}}. \qquad\blacksquare$$

We recall the definition of the subring $\mathbf{B}_L^\dagger$ of $\mathcal{R}_L(\mathbf{B})$ by defining first $\widetilde{\mathbf{A}} := W(\mathbb{C}_p^\flat)_L$ and

$$\widetilde{\mathbf{A}}^\dagger := \Big\{ x = \sum_{n \geq 0} \pi_L^n [x_n] \in \widetilde{\mathbf{A}} : |\pi_L^n| |x_n|_\flat^r \xrightarrow{n \to \infty} 0 \text{ for some } r > 0 \Big\}.$$

Then we set $\boldsymbol{A}^\dagger := \widetilde{\boldsymbol{A}}^\dagger \cap \boldsymbol{A}$, $\boldsymbol{B}^\dagger := \boldsymbol{A}^\dagger[\frac{1}{\pi_L}]$ as well as $\boldsymbol{A}_L^\dagger := (\boldsymbol{A}^\dagger)^{H_L}$ and $\boldsymbol{B}_L^\dagger := (\boldsymbol{B}^\dagger)^{H_L}$.

It follows from the proof of [6, Thm. 10.1] that for $V \in \mathrm{Rep}_L^{\mathrm{an}}(G_L)$ we have $D_{\mathrm{rig}}^\dagger(V) = \mathcal{R}_L(\mathbf{B}) \otimes_{\mathbf{B}_L^\dagger} D^\dagger(V)$, where $D^\dagger(V)$ belongs to $\mathfrak{M}^{\mathrm{\acute{e}t}}(\mathbf{B}_L^\dagger)$. From the theory of Wach modules we actually know that $D_{\mathrm{LT}}(V)$ is even of finite height if V is crystalline in addition:

$$D^\dagger(V) = \mathbf{B}_L^\dagger \otimes_{\mathbf{A}_L^+} N(T) = \mathbf{B}_L^\dagger \otimes_{\mathbf{B}_L^+} N(V)$$

for any Galois stable o_L-lattice $T \subseteq V$. From the diagram in (3.2), we thus obtain the following diagram, in which the horizontal maps are equivalences of categories:

$$
\begin{array}{ccc}
\mathrm{Mod}_{\mathbf{B}_L^+}^{\varphi_L, \Gamma_L, \mathrm{an}} & \xrightarrow[\simeq]{\mathbf{B}_L \otimes_{\mathbf{B}_L^+} -} & \mathfrak{M}^{\mathrm{\acute{e}t,cris}}(\mathbf{B}_L) \\
{\scriptstyle \mathcal{O} \otimes_{\mathbf{B}_L^+} -} \Big\downarrow & {\scriptstyle N(-) \nwarrow} \quad {\scriptstyle \simeq} \Big\uparrow {\scriptstyle D_{\mathrm{LT}}(-)} \\
\mathrm{Mod}_{\mathcal{O}}^{\varphi_L, \Gamma_L, 0} & \underset[\mathcal{M} \circ D_{\mathrm{cris},L}]{\overset{V_L \circ D}{\underset{\simeq}{\rightleftarrows}}} & \mathrm{Rep}_L^{\mathrm{cris,an}}(G_L) \\
{\scriptstyle \mathcal{R}_L(\mathbf{B}) \otimes_{\mathcal{O}} -} \Big\downarrow & & \subseteq \Big\downarrow \\
\mathcal{M}\big(\mathcal{R}_L(\mathbf{B})\big)^{\mathrm{an,\acute{e}t}} & \xleftarrow[\simeq]{D^\dagger(V)} & \mathrm{Rep}_L^{\mathrm{an}}(G_L).
\end{array}
$$

Here $\mathfrak{M}^{\mathrm{\acute{e}t,cris}}(\mathbf{B}_L)$ denotes the essential image of $\mathrm{Rep}_L^{\mathrm{cris,an}}(G_L)$ under $D_{\mathrm{LT}}(-)$ in $\mathfrak{M}^{\mathrm{\acute{e}t}}(\mathbf{B}_L)$ with $\mathbf{B}_L := \mathbf{A}_L[\frac{1}{\pi_L}]$.

Now let T be an o_L-lattice in an L-linear continuous representation of G_L such that $V^*(1)$ (and hence $V(\tau^{-1})$) is L-analytic and crystalline. Then it follows from [44] and the discussion above that

$$M := D_{\mathrm{rig}}^\dagger\big(V(\tau^{-1})\big) = \mathcal{R}_L(\mathbf{B}) \otimes_{\mathcal{O}_K(\mathbf{B})} \mathcal{M}\big(D_{\mathrm{cris},L}\big(V(\tau^{-1})\big)\big)$$
$$= \mathcal{R}_L(\mathbf{B}) \otimes_{\mathbf{A}_L^+} N\big(T(\tau^{-1})\big)$$

as well as

$$\check{M} = D_{\mathrm{rig}}^\dagger\big(V^*(1)\big) = \mathcal{R}_L(\mathbf{B}) \otimes_{\mathcal{O}_K(\mathbf{B})} \mathcal{M}\big(D_{\mathrm{cris},L}\big(V^*(1)\big)\big)$$
$$= \mathcal{R}_L(\mathbf{B}) \otimes_{\mathbb{A}_L^+} N\big(T^*(1)\big)$$

and the comparison isomorphism in (3.16) induces isomorphisms

$$\mathrm{comp}_M : M\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \cong \mathfrak{R}_L(\mathbf{B})\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big)$$

and

$$\mathrm{comp}_{\widetilde{M}} : \check{M}\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \cong \mathfrak{R}_L(\mathbf{B})\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big).$$

By [8, §3.4 and §3.5] an analogue of Kisin–Ren modules exists for $\mathfrak{Y} = \mathfrak{X}$; i.e., if we take $M := D_{\mathrm{rig}}^\dagger(V(\tau^{-1}))_{\mathfrak{X}}$ and $\check{M} = D_{\mathrm{rig}}^\dagger(V^*(1))_{\mathfrak{X}}$, we obtain analogous comparison isomorphisms

$$\mathrm{comp}_M : M\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \cong \mathfrak{R}_L(\mathfrak{X})\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) \tag{4.91}$$

and

$$\mathrm{comp}_{\widetilde{M}} : \check{M}\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \cong \mathfrak{R}_L(\mathfrak{X})\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big),$$

which this time stem from [8, Prop. 3.42] by base change $\mathcal{R}_L(\mathfrak{X}) \otimes_{\mathcal{O}_K(\mathfrak{X})} -$ using the inclusion $\mathcal{O}_L(\mathfrak{X})[Z^{-1}] \subseteq \mathcal{R}_L(\mathfrak{X})[\tfrac{1}{t_{\mathfrak{Y}}}]$. Moreover, these comparison isomorphisms for $\mathbf{B}$ and $\mathfrak{X}$ are compatible with regard to the equivalence of categories (4.90) by [8, Thm. 3.48]. Note that for $c = \mathrm{comp}_M$ and $D = D_{\mathrm{cris},L}(V(\tau^{-1}))$ we have

$$\mathrm{comp}_{\widetilde{M}} = (\mathrm{comp}_{\Omega_{\mathcal{R}}^1} \otimes_L \mathrm{id}_{D^*}) \circ \check{c} \tag{4.92}$$

using the identifications $\Omega_{\mathcal{R}}^1 \cong \mathcal{R}(\chi_{\mathrm{LT}})$ and

$$D_{\mathrm{cris},L}\big(V^*(1)\big) \cong D^* \otimes D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}})\big).$$

We set $b := \mathrm{comp}_{\Omega_{\mathcal{R}}^1}(t_{\mathfrak{Y}}^{-1} d \log_{\mathrm{LT}}) = \tfrac{1}{t_{\mathfrak{Y}}} \otimes \eta \in D_0 := D_{\mathrm{cris},L}(L(\chi_{\mathrm{LT}}))$ and

$$\widetilde{\nabla} := \begin{cases} \nabla & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \dfrac{\nabla}{\Omega}, & \text{if } \mathfrak{Y} = \mathbf{B} \text{ (and } \Omega \in K). \end{cases}$$

Remark 4.5.29. As operators on $\mathcal{R}$, we have the equalities

$$\nabla = t_{\mathfrak{Y}} \partial_{\mathrm{inv}}^{\mathfrak{Y}} \quad \text{and} \quad \widetilde{\nabla} = t_{\mathfrak{Y}} \widetilde{\partial}_{\mathrm{inv}}^{\mathfrak{Y}},$$

where we define $\partial_{\mathrm{inv}}^{\mathbf{B}} := \partial_{\mathrm{inv}}$ and

$$\widetilde{\partial}_{\mathrm{inv}}^{\mathfrak{Y}} := \begin{cases} \partial_{\mathrm{inv}}^{\mathfrak{X}} & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \dfrac{\partial_{\mathrm{inv}}}{\Omega} & \text{if } \mathfrak{Y} = \mathbf{B} \text{ (and } \Omega \in K). \end{cases}$$

Indeed, for $\mathfrak{Y} = \mathbf{B}$ (4.70) grants these equalities of operators on the subring $\mathcal{O}_K(\mathbf{B})$. Concerning the ring $\mathcal{R}_K(\mathbf{B})$, we note that ∇ is acting as a continuous derivation as can be shown in the same way as in [44, Lem. 2.1.2], while for the operator $t_{\mathbf{B}}\partial_{\mathrm{inv}}$ this is clear anyway. Thus, the same equalities hold for $\mathcal{R}$, Remark 4.3.1. Indeed, on the localization $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$ it extends uniquely by the derivation property and then it extends uniquely by continuity to $\mathcal{R}_K(\mathbf{B})$. Regarding $\mathfrak{Y} = \mathfrak{X}$, note that all operators are defined over K. Since the equality can be checked over $\mathbb{C}_p$, the claim follows from Remark 4.2.9 and the previous case $\mathfrak{Y} = \mathbf{B}$.

Lemma 4.5.30. *Assuming $\Omega \in K$ if $\mathfrak{Y} = \mathbf{B}$, the following diagram commutes:*

$$
\begin{array}{ccc}
\Omega^1_{\mathcal{R}}[\frac{1}{t_{\mathfrak{Y}}}] \otimes D^* & \xrightarrow{\ \mathrm{comp}_{\Omega^1_{\mathcal{R}}} \otimes_L \mathrm{id}_{D^*}\ } & \mathcal{R}[\frac{1}{t_{\mathfrak{Y}}}] \otimes D^* \otimes D_0 \\[2mm]
\Big\uparrow & & \tilde{\nabla}\Big\uparrow \\[2mm]
(\Omega^1_{\mathcal{R}} \otimes_L D^*)^{\psi_L = 0} & & \mathcal{R}^{\psi_L=0} \otimes D^* \otimes D_0 \\[2mm]
\mathfrak{M}^{\Omega^1} \otimes \mathrm{id}_{D^*}\Big\uparrow \cong & & \mathfrak{M} \otimes \mathrm{id}_{D^* \otimes D_0}\Big\uparrow \cong \\[2mm]
\mathcal{R}_K(\Gamma_L) \otimes_L D^* & \xrightarrow{\ \mathrm{id}_{\mathcal{R}_K(\Gamma_L)} \otimes_L D^* \otimes b\ } & \mathcal{R}_K(\Gamma_L) \otimes_L D^* \otimes D_0.
\end{array}
$$

Proof. We first give the proof for $\mathfrak{Y} = \mathbf{B}$. Observe, since on D^* we have the identity throughout, that the commutativity of the above diagram follows from the commutativity of

$$
\begin{array}{ccc}
\mathcal{R}_K(\Gamma_L) \xrightarrow{\ \mathfrak{M}^{\Omega^1}\ } (\Omega^1_{\mathcal{R}})^{\psi_L=0} & \hookleftarrow & \Omega^1_{\mathfrak{R}}[\frac{1}{t_{\mathfrak{Y}}}]^{\psi_L=0} \\[2mm]
\cong\Big\uparrow & & \\[2mm]
\mathcal{R}(\chi_{\mathrm{LT}})^{\psi_L=0} & & \\[2mm]
\tilde{\partial}^{\mathfrak{Y}}_{\mathrm{inv}} \otimes t_{\mathfrak{Y}}\Big\uparrow & & \cong\Big\downarrow \mathrm{comp}_{\Omega^1_{\mathcal{R}}} \\[2mm]
\mathcal{R}_K(\Gamma_L) \xrightarrow{\ \mathfrak{M}\otimes b\ } \mathcal{R}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}})\big) & & \\[2mm]
\tilde{\nabla}\Big\downarrow & t_{\mathfrak{Y}}\tilde{\partial}^{\mathfrak{Y}}_{\mathrm{inv}}\otimes\mathrm{id}\Big\downarrow & \\[2mm]
\mathcal{R}_K(\Gamma_L) \xrightarrow{\ \mathfrak{M}\otimes b\ } \mathcal{R}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}})\big) & \hookrightarrow & \big(\mathfrak{R}[\frac{1}{t_{\mathfrak{Y}}}]\otimes_L D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}})\big)\big)^{\psi_L=0},
\end{array}
$$
(4.93)

where the map $\tilde{\partial}^{\mathfrak{Y}}_{\mathrm{inv}} \otimes t_{\mathfrak{Y}} : \mathcal{R} \otimes_L D_{\mathrm{cris},L}(L(\chi_{\mathrm{LT}})) \to \mathcal{R}(\chi_{\mathrm{LT}})$ sends $f \otimes \frac{1}{t_{\mathfrak{Y}}} \otimes \eta$ to $\tilde{\partial}^{\mathfrak{Y}}_{\mathrm{inv}} f \otimes \eta$ and the composite with the natural identification $\mathcal{R}(\chi_{\mathrm{LT}}) \cong \Omega^1$, which sends η to $d \log_{\mathrm{LT}}$, is the map $\frac{d}{\Omega} : \mathcal{R} \to \Omega^1_{\mathcal{R}}$ upon identifying $\mathcal{R} \otimes_L D_{\mathrm{cris},L}(L(\chi_{\mathrm{LT}}))$ with $\mathcal{R}$ by sending $f \otimes \frac{1}{t_{\mathfrak{Y}}} \otimes \eta$ to f. Remark 4.5.29 implies the commutativity of the left lower corner. For the upper left corner the commutativity follows from (4.64),

the easily checked identity $\widetilde{\partial}^{\mathfrak{Y}}_{\text{inv}} \eta(1, Z) = \eta(1, Z)$, and (4.79)

$$\mathfrak{M}^{\Omega^1}(\lambda) = \left(\text{Tw}_{\chi_{\text{LT}}}(\lambda) \cdot \eta(1, Z)\right) d \log_{\text{LT}}$$
$$= \left(\text{Tw}_{\chi_{\text{LT}}}(\lambda) \cdot \widetilde{\partial}^{\mathfrak{Y}}_{\text{inv}} \eta(1, Z)\right) d \log_{\text{LT}}$$
$$= \widetilde{\partial}^{\mathfrak{Y}}_{\text{inv}} \left(\lambda \cdot \eta(1, Z)\right) d \log_{\text{LT}}.$$

Finally, since $\eta(1, Z) \otimes b \in \mathcal{R}^{\psi_L=0} \otimes_L D_{\text{cris},L}(L(\chi_{\text{LT}}))$ is sent up to $\eta(1, Z) d \log_{\text{LT}}$ and down to $t_{\mathfrak{Y}} \eta(1, Z) \otimes b$, the compatibility with $\text{comp}_{\Omega^1_{\mathcal{R}}}$ is easily checked. The same proof works for $\mathfrak{Y} = \mathfrak{X}$ by using (4.63) instead of (4.64) and replacing $\eta(1, Z)$ and $\frac{d}{\Omega}$ by ev_1 and d, respectively. ∎

Now we introduce a pairing – if $\mathfrak{Y} = \mathbf{B}$ assuming $\Omega \in K$ as usual –

$$[\,,\,]_{D_{\text{cris},L}(V(\tau^{-1}))} :$$
$$\mathcal{R}^{\psi_L=0} \otimes_L D_{\text{cris},L}(V^*(1)) \times \mathcal{R}^{\psi_L=0} \otimes_L D_{\text{cris},L}(V(\tau^{-1})) \to \mathcal{R}_K(\Gamma_L)$$

which we will abbreviate as $[\,,\,] := [\,,\,]_{D_{\text{cris},L}(V(\tau^{-1}))}$, by requiring that the following diagram becomes commutative:

$$\begin{array}{ccccc}
\mathcal{R}^{\psi_L=0} \otimes D^* \otimes D_0 & \times & \mathcal{R}^{\psi_L=0} \otimes_L D & \xrightarrow{[\,,\,]} & \mathcal{R}_K(\Gamma_L) \\
\mathfrak{M} \otimes \text{id}_{D^* \otimes D_0} \Big\uparrow \cong & & \sigma_{-1} \mathfrak{M} \circ \iota_* \otimes \text{id} \Big\uparrow \cong & & \Big\| \\
\mathcal{R}_K(\Gamma_L) \otimes_L D^* \otimes D_0 & \times & \mathcal{R}_K(\Gamma_L) \otimes_L D & \longrightarrow & \mathcal{R}_K(\Gamma_L),
\end{array} \qquad (4.94)$$

where the bottom line sends $(\lambda \otimes l \otimes \beta b, \mu \otimes d)$ to $\lambda \mu \beta l(d)$.

Combining the Lemmata 4.5.26 and 4.5.30, we obtain the following lemma for $N = \mathcal{R} \otimes_L D_{\text{cris},L}(V(\tau^{-1}))$.

Lemma 4.5.31. $[\,,\,]_{D_{\text{cris},L}(V(\tau^{-1}))} = \frac{q-1}{q} \{\nabla(\text{comp}_{\Omega^1_{\mathcal{R}}} \otimes_L \text{id}_{D^*})^{-1}(-), -\}^0_{N,\text{Iw}}.$

Setting

$$M' := \text{comp}^{-1}\left(\mathcal{R}^{\psi_L=0} \otimes_L D_{\text{cris},L}(V(\tau^{-1}))\right),$$
$$\check{M}' := \text{comp}^{-1}\left(\mathcal{R}^{\psi_L=0} \otimes_L D_{\text{cris},L}(V^*(1))\right)$$

we obtain the subsequent result.

Theorem 4.5.32. *Assume* $\Omega \in K$ *if* $\mathfrak{Y} = \mathbf{B}$. *For all* $x \in \check{M}' \cap (\check{M}^{\psi_L=0})$ *and* $y \in M' \cap (M^{\psi_L=0})$ *it holds that*

$$\frac{q-1}{q} \{\nabla x, y\}^0_{\text{Iw}} = [x, y];$$

i.e., the following diagram commutes on the vertical intersections:

$$
\begin{array}{ccc}
\check{M}^{\psi_L=0} & \times & M^{\psi_L=0} \xrightarrow{\frac{q-1}{q}\nabla\{\,,\}^0_{M,Iw}} \mathcal{R}_K(\Gamma_L) \\
\downarrow & & \downarrow \quad\quad\quad\quad\quad \| \\
\left(\mathcal{R}\!\left[\tfrac{1}{t_y}\right]\otimes_{\mathcal{R}}\check{M}\right)^{\psi_L=0} & \times & \left(\mathcal{R}\!\left[\tfrac{1}{t_y}\right]\otimes_{\mathcal{R}} M\right)^{\psi_L=0} \\
\mathrm{comp}_{\check{M}}\downarrow\cong & & \mathrm{comp}_M\downarrow\cong \\
\left(\Re\!\left[\tfrac{1}{t_{\mathfrak{y}}}\right]\otimes_L D_{\mathrm{cris},L}\!\left(V^*(1)\right)\right)^{\psi_L=0}\!\!\times\!\left(\Re\!\left[\tfrac{1}{t_{\mathfrak{y}}}\right]\otimes_L D_{\mathrm{cris},L}\!\left(V(\tau^{-1})\right)\right)^{\psi_L=0} \\
\uparrow \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \uparrow \quad\quad\quad\quad\quad \| \\
\mathcal{R}^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}\!\left(V^*(1)\right) \quad\times\quad \mathcal{R}^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}\!\left(V(\tau^{-1})\right) \xrightarrow{[\,,\,]} \mathcal{R}_K(\Gamma_L).
\end{array}
$$

Proof. Combine Lemmata 4.5.25 and 4.5.31 using (4.92). ∎

Interpretation of the abstract reciprocity formula in terms of the $D_{\mathrm{cris},L}$-pairing

Assuming as usual that $\Omega \in K$ if $\mathfrak{Y} = \mathbf{B}$, the canonical pairing

$$
D_{\mathrm{cris},L}\!\left(V^*(1)\right) \times D_{\mathrm{cris},L}\!\left(V(\tau^{-1})\right) \to D_{\mathrm{cris},L}\!\left(L(\chi_{\mathrm{LT}})\right)
$$

extends to a pairing

$$
\mathcal{R}^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}\!\left(V^*(1)\right) \times \mathcal{R}^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}\!\left(V(\tau^{-1})\right)
$$
$$
\to \mathcal{R}^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}\!\left(L(\chi_{\mathrm{LT}})\right)
$$

denoted by $[\,,\,]_{\mathrm{cris}}$, by requiring that the following diagram is commutative (in which the lower one is induced by multiplication within $\mathcal{R}_K(\Gamma_L)$ and the natural duality paring on $D_{\mathrm{cris},L}$)

$$
\begin{array}{ccc}
\mathcal{R}^{\psi_L=0}\underset{L}{\otimes}D_{\mathrm{cris},L}\!\left(V^*(1)\right)\!\times\!\mathcal{R}^{\psi_L=0}\underset{L}{\otimes}D_{\mathrm{cris},L}\!\left(V(\tau^{-1})\right) & \xrightarrow{[\,,\,]_{\mathrm{cris}}} & \mathcal{R}^{\psi_L=0}\underset{L}{\otimes}D_{\mathrm{cris},L}\!\left(L(\chi_{\mathrm{LT}})\right) \\
\uparrow\mathfrak{M}\otimes\mathrm{id} & \uparrow\sigma_{-1}\mathfrak{M}\circ\iota_*\otimes\mathrm{id} & \uparrow\mathfrak{M}\otimes\mathrm{id} \\
\mathcal{R}_K(\Gamma_L)\underset{L}{\otimes}D_{\mathrm{cris},L}\!\left(V^*(1)\right)\times \mathcal{R}_K(\Gamma_L)\underset{L}{\otimes}D_{\mathrm{cris},L}\!\left(V(\tau^{-1})\right) & \to & \mathcal{R}_K(\Gamma_L)\underset{L}{\otimes}D_{\mathrm{cris},L}\!\left(L(\chi_{\mathrm{LT}})\right).
\end{array}
$$

Note that
$$
\mathrm{comp}\!\left([x,y]\cdot\mathrm{ev}_1\otimes(t_{\mathfrak{Y}}^{-1}\otimes\eta)\right) = [x,y]_{\mathrm{cris}}.
$$

Hence, using the diagram in (4.93), Theorem 4.5.32 is also equivalent to

$$
\mathrm{comp}\circ\mathfrak{M}^{\Omega^1}\circ\widetilde{\Omega}\,\tfrac{q-1}{q}\{x,y\}^0_{Iw} = \big[\mathrm{comp}(x),\mathrm{comp}(y)\big]_{\mathrm{cris}},
$$

i.e., the "commutativity" (whenever it makes sense) of the following diagram:

$$
\begin{array}{ccc}
D_{\mathrm{rig}}^{\dagger}\big(V^*(1)\big)^{\psi_L=\frac{q}{\pi_L}}\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \times D_{\mathrm{rig}}^{\dagger}\big(V(\tau^{-1})\big)^{\psi_L=1}\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] & \xrightarrow{\ \tilde{\Omega}\frac{q-1}{q}\{\,,\}_{\mathrm{Iw}}\ } & cR_K(\Gamma_L) \\[2mm]
{\scriptstyle 1-\varphi_L}\big\downarrow \qquad\qquad\quad {\scriptstyle 1-\frac{\pi_L}{q}\varphi_L}\big\downarrow & & \big\| \\[2mm]
D_{\mathrm{rig}}^{\dagger}\big(V^*(1)\big)^{\psi_L=0}\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \times D_{\mathrm{rig}}^{\dagger}\big(V(\tau^{-1})\big)^{\psi_L=0}\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] & \xrightarrow{\ \tilde{\Omega}\frac{q-1}{q}\{\,,\}_{\mathrm{Iw}}^0\ } & \mathcal{R}_K(\Gamma_L) \\[2mm]
\big\| \qquad\qquad\qquad\quad \big\| & & \big\downarrow{\scriptstyle \mathfrak{M}^{\Omega^1}} \\[2mm]
D_{\mathrm{rig}}^{\dagger}\big(V^*(1)\big)^{\psi_L=0}\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \times D_{\mathrm{rig}}^{\dagger}\big(V(\tau^{-1})\big)^{\psi_L=0}\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] & \xrightarrow{\ \tilde{\Omega}\frac{q-1}{q}\{\,,\}_{\mathrm{Iw}}^0\ } & \mathfrak{R}(\chi_{\mathrm{LT}})\big[\tfrac{1}{t_{\mathfrak{Y}}}\big]^{\psi_L=0}
\end{array}
$$

$$
\mathfrak{R}^{\psi_L=0}\big[\tfrac{1}{t_{\mathfrak{Y}}}\big]\otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big) \times \mathfrak{R}^{\psi_L=0}\big[\tfrac{1}{t_{\mathfrak{Y}}}\big]\otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big)
$$

with vertical maps labelled $\mathrm{comp}\ \big\downarrow\cong$ and $\cong\ \big\downarrow\,\mathrm{comp}$, and the dashed arrow $[\,,\,]_{\mathrm{cris}}$ into

$$
\mathfrak{R}^{\psi_L=0}\big[\tfrac{1}{t_{\mathfrak{Y}}}\big]\otimes_L D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}})\big)
$$

for $\mathfrak{Y} = \mathbf{B}$, while for $\mathfrak{Y} = \mathfrak{X}$ one has to decorate the $D_{\mathrm{rig}}^{\dagger}$s with index $\mathfrak{X}$.

Question. Is it possible to extend the definition of $[\,,\,]$ and $\{\,,\}$ to $(\check{M}[\tfrac{1}{t_{\mathfrak{Y}}}])^{\psi_L=0} \times (M[\tfrac{1}{t_{\mathfrak{Y}}}])^{\psi_L=0}$ by perhaps enlarging the target $\mathcal{R}_K(\Gamma_L)$ by an appropriate localization, which reflects the inversion of $t_{\mathfrak{Y}}$ somehow?

Chapter 5

Application

For the convenience of the reader we recall some definitions. Firstly, for $V \in \mathrm{Rep}_L(G_L)$ we put

$$D_{\mathrm{cris},L}(V) := (B_{\mathrm{cris}} \otimes_{L_0} V)^{G_L} \quad \text{and} \quad D_{\mathrm{dR},L}(V) := (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_L}.$$

The second definition will be used also in the more general form

$$D_{\mathrm{dR},L'}(V) := D_{\mathrm{dR},L'}(V_{|G_{L'}}) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_{L'}}$$

for L' any finite extension of L. Moreover, we write $D_{\mathrm{dR},L'}^0(V) := \mathrm{Fil}^0(D_{\mathrm{dR},L'}(V))$. Recall that $T_\pi = o_L \eta$ denotes the Tate module of the Lubin–Tate formal group LT. We set $V_\pi := L \otimes_{o_L} T_\pi \in \mathrm{Rep}_L(G_L)$. We write $\tau = \chi_{\mathrm{LT}}^{-1}\chi_{\mathrm{cyc}}$, $t_{\mathrm{LT}} = \log_{\mathrm{LT}}(\omega_{\mathrm{LT}}) \in B_{\mathrm{cris},L}$, and analogously $t_{\mathbb{Q}_p} := \log_{\widehat{\mathbb{G}}_m}(\omega_{\widehat{\mathbb{G}}_m})$ for the Lubin–Tate formal group $\widehat{\mathbb{G}}_m$. Then $\mathbf{d}_r := t_{\mathrm{LT}}^r t_{\mathbb{Q}_p}^{-1} \otimes (\eta^{\otimes -r} \otimes \eta_{\mathrm{cyc}})$, where η_{cyc} is a generator of the cyclotomic Tate module $\mathbb{Z}_p(1)$, is an L-basis of $D_{\mathrm{dR},L}^0(V_\pi^{\otimes -r}(1))$.

The generalized Iwasawa cohomology of $T \in \mathrm{Rep}_{o_L}(G_L)$ is defined by

$$H_{\mathrm{Iw}}^*(L_\infty/L, T) := \varprojlim_{L'} H^*(L', T),$$

where L' runs through the finite Galois extensions of L contained in L_∞ and the transition maps in the projective system are the cohomological corestriction maps. For $V := T \otimes_{o_L} L \in \mathrm{Rep}_L(G_L)$ we define

$$H_{\mathrm{Iw}}^*(L_\infty/L, V) := H_{\mathrm{Iw}}^*(L_\infty/L, T) \otimes_{o_L} L,$$

which is independent of the choice of T. As usual, we denote by

$$\mathrm{cor} : H_{\mathrm{Iw}}^*(L_\infty/L, T) \to H^*(L', T)$$

the projection map and analogously for rational coefficients.

5.1 The regulator map

Let T be in $\mathrm{Rep}_{o_L,f}^{\mathrm{cris}}(G_L)$ such that $T(\tau^{-1})$ belongs to $\mathrm{Rep}_{o_L,f}^{\mathrm{cris,an}}(G_L)$ with all Hodge–Tate weights in $[0, r]$, and such that $V := L \otimes_{o_L} T$ does not have any quotient isomorphic to $L(\tau)$. Then the assumptions of Lemma 3.3.6 are satisfied and we may define the regulator maps

$$\mathbf{L}_V : H_{\mathrm{Iw}}^1(L_\infty/L, T) \to D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big),$$

$$\mathcal{L}_V^0 : H^1_{\mathrm{Iw}}(L_\infty/L, T) \to \mathcal{O}_L(\mathbf{B})^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big),$$

$$\mathcal{L}_V : H^1_{\mathrm{Iw}}(L_\infty/L, T) \to D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}(V)$$

as (part of) the composite

$$
\begin{aligned}
H^1_{\mathrm{Iw}}(L_\infty/L, T) &\cong D_{\mathrm{LT}}\big(T(\tau^{-1})\big)^{\psi_L=1} \\
&= N\big(T(\tau^{-1})\big)^{\psi_{D_{\mathrm{LT}}(T(\tau^{-1}))}=1} \xrightarrow{(1-\frac{\pi_L}{q}\varphi_L)} \varphi_L^*\big(N\big(V(\tau^{-1})\big)\big)^{\psi_L=0} \\
&\hookrightarrow \mathcal{O}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) \subseteq \mathcal{O}_{\mathbb{C}_p}(\mathbf{B})^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) \\
&\xrightarrow{\mathfrak{M}^{-1}\otimes\mathrm{id}} D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) \to D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}(V)
\end{aligned}
\tag{5.1}
$$

using [80, Thm. 5.13], Lemma 3.3.6, the inclusion (3.17), and where the last map sends $\mu \otimes d \in D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}(V(\tau^{-1}))$ to $\mu \otimes d \otimes \mathbf{d}_1 \in D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}(V(\tau^{-1})) \otimes_L D_{\mathrm{cris},L}(L(\tau)) \cong D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}(V)$. Note that we have $D := D_{\mathrm{cris},L}(L(\tau)) = D^0_{\mathrm{dR},L}(L(\tau)) = L\mathbf{d}_1$ with $\mathbf{d}_1 = t_{\mathrm{LT}} t_{\mathbb{Q}_p}^{-1} \otimes (\eta^{\otimes -1} \otimes \eta_{\mathrm{cyc}})$, where $L(\chi_{\mathrm{LT}}) = L\eta$ and $L(\chi_{\mathrm{cyc}}) = L\eta_{\mathrm{cyc}}$.

Alternatively, in order to stress that the regulator is essentially the map $1 - \varphi_L$, one can rewrite this as

$$
\begin{aligned}
H^1_{\mathrm{Iw}}(L_\infty/L, T) &\cong D_{\mathrm{LT}}\big(V(\tau^{-1})\big)^{\psi_L=1} = N\big(T(\tau^{-1})\big)^{\psi_{D_{\mathrm{LT}}(T(\tau^{-1}))}=1} \\
&\hookrightarrow N\big(V(\tau^{-1})\big)^{\psi_{D_{\mathrm{LT}}(V(\tau^{-1}))}=1} \otimes_L D \xrightarrow{1-\varphi_L} \varphi_L^*\big(N\big(V(\tau^{-1})\big)\big)^{\psi_L=0} \otimes_L D \\
&\hookrightarrow \mathcal{O}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) \otimes_L D \subseteq \mathcal{O}_{\mathbb{C}_p}(\mathbf{B})^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}(V) \\
&\xrightarrow{\mathfrak{M}^{-1}\otimes\mathrm{id}} D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}(V),
\end{aligned}
$$

where the $\hookrightarrow$ in the second line sends n to $n \otimes \mathbf{d}_1$ and the φ_L now acts diagonally. By construction, this regulator map $\mathcal{L}_V$ takes values in $D(\Gamma_L, K)^{G_L,*} \otimes_L D_{\mathrm{cris},L}(V)$, where the twisted action of G_L on the distribution algebra is induced by the Mellin transform as in (ii) of Proposition 4.1.25.

We write $\nabla_{\mathrm{Lie}} \in \mathrm{Lie}(\Gamma_L)$ for the element in the Lie algebra of Γ_L corresponding to 1 under the identification $\mathrm{Lie}(\Gamma_L) = L$.

Proposition 5.1.1. *The regulator maps for V and $V(\chi_{\mathrm{LT}})$ – assuming that both representations satisfy the conditions above – are related by*

$$\mathcal{L}_{V(\chi_{\mathrm{LT}})}(x \otimes \eta) = \nabla_{\mathrm{Lie}} \cdot \big(\tfrac{1}{\Omega}\mathrm{Tw}_{\chi_{\mathrm{LT}}^{-1}}\big(\mathcal{L}_V(x)\big) \otimes t_{\mathrm{LT}}^{-1}\eta\big);$$

i.e., the following Γ_L-equivariant diagram commutes:

$$
\begin{array}{ccc}
H^1_{\mathrm{Iw}}\big(L_\infty/L, V(\chi_{\mathrm{LT}})\big) & \xrightarrow{\quad\mathcal{L}_{V(\chi_{\mathrm{LT}})}\quad} & D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}\big(V(\chi_{\mathrm{LT}})\big) \\
\cong \downarrow & & \uparrow \frac{\nabla_{\mathrm{Lie}}\mathrm{Tw}_{\chi^{-1}}}{\Omega}\otimes t_{\mathrm{LT}}^{-1} \\
H^1_{\mathrm{Iw}}(L_\infty/L, V) \otimes_{o_L} L(\chi_{\mathrm{LT}}) & \xrightarrow{\mathcal{L}_V \otimes L(\chi_{\mathrm{LT}})} & D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}(V) \otimes_L L(\chi_{\mathrm{LT}}).
\end{array}
$$

Proof. The proof is analogous to [51, Prop. 3.1.4]. Note that the period Ω enters due to (4.64). ∎

This twisting property can be used to drop the condition concerning the Hodge–Tate weights in the definition of the regulator map; i.e., upon replacing $D(\Gamma_L, \mathbb{C}_p)$ in the target by its total ring of quotients, one can extend the regulator map as usual to all T in $\mathrm{Rep}_{o_L, f}^{\mathrm{cris}}(G_L)$ such that $T(\tau^{-1})$ belongs to $\mathrm{Rep}_{o_L, f}^{\mathrm{cris,an}}(G_L)$.

In order to better understand the effect of twisting, we have the following lemma.

Lemma 5.1.2. *For $\mu \in D(\Gamma_L, K)$ we have*

$$\frac{1}{\Omega}(\nabla_{\mathrm{Lie}}\mathrm{Tw}_{\chi^{-1}})(\mu) = \mathfrak{M}^{-1}\big(t_{\mathrm{LT}}\mathfrak{M}(\mu)\big)$$

and for all $n \geq 1$

$$\big(\nabla_{\mathrm{Lie}}\mathrm{Tw}_{\chi^{-1}}(\mu)\big)(\chi_{\mathrm{LT}}^n) = n\mu(\chi_{\mathrm{LT}}^{n-1}).$$

Proof. The first claim follows by combining (5.5) with (4.64), while the second claim is just Lemma 4.1.22 applied to the first. ∎

One significance of regulator maps is that it should interpolate (dual) Bloch–Kato exponential maps. We shall prove such interpolation formulae in Section 5.2.4 by means of a reciprocity formula.

5.1.1 The basic example

Setting $U := \varprojlim_n o_{L_n}^\times$ with transition maps given by the norm, we have the maps

$$CW_r : U \otimes_{\mathbb{Z}} T_\pi^{\otimes -r} \to D_{\mathrm{dR}, L}^0\big(V_\pi^{\otimes -r}(1)\big) = L\mathbf{d}_r$$

$$u \otimes a\eta^{\otimes -r} \mapsto a\frac{1 - \pi_L^{-r}}{(r-1)!}\partial_{\mathrm{inv}}^r \log g_{u,\eta}(Z)_{|Z=0}\mathbf{d}_r$$

for all $r \geq 1$. Furthermore, $g_{u,\eta}$ denotes the Coleman power series (cf. [21, Thm. A, Cor. 17]) attached to $u = (u_n)_n$ and $\eta = (\eta_n)_n$ satisfying $g_{u,\eta}(\eta_n) = u_n$ for all $n \geq 1$.

We are looking for a map

$$\mathcal{L} : U \otimes_{\mathbb{Z}} T_\pi^* \to D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris}, L}\big(L(\tau)\big)$$

such that

$$\frac{\Omega^r}{r!}\frac{1 - \pi_L^{-r}}{1 - \frac{\pi_L^r}{q}}\mathcal{L}(u \otimes a\eta^*)(\chi_{\mathrm{LT}}^r) \otimes (t_{\mathrm{LT}}^{r-1} \otimes \eta^{\otimes -r+1}) = CW_r(u \otimes a\eta^{\otimes -r}) \quad (5.2)$$

for all $r \geq 1, u \in U, a \in o_L$. Recall that $D_{\mathrm{cris}, L}(L(\tau)) = L\mathbf{d}_1$ and note the equalities $D_{\mathrm{dR}, L}^0(V_\pi^{\otimes -r}(1)) = L\mathbf{d}_r = L\mathbf{d}_1 \otimes (t_{\mathrm{LT}}^{r-1} \otimes \eta^{\otimes -r+1})$.

Theorem 5.1.3 (A special case of Kato's explicit reciprocity law [80, Cor. 8.7]). *For* $r \geq 1$ *the diagram*

$$
\begin{array}{c}
U \otimes_{\mathbb{Z}} T_\pi^{\otimes -r} \\
{\scriptstyle -\kappa \otimes \mathrm{id}} \downarrow \qquad \qquad \searrow^{CW_r} \\
H_{\mathrm{Iw}}^1\big(L_\infty/L, T_\pi^{\otimes -r}(1)\big) \\
{\scriptstyle \mathrm{cor}} \downarrow \\
H^1\big(L, T_\pi^{\otimes -r}(1)\big) \xrightarrow{\ \exp^* \ } D_{\mathrm{dR},L}^0\big(V_\pi^{\otimes -r}(1)\big) = L\mathbf{d}_r
\end{array}
$$

commutes.

For the definition of the Kummer map κ see [80, §6], while for the dual Bloch–Kato exponential map $\exp^*$ we refer the reader to [12].

We set $\mathcal{L} = \mathfrak{L} \otimes \mathbf{d}_1$ with $\mathfrak{L}$ given as the composition

$$
\mathfrak{L} : U \otimes T_\pi^* \xrightarrow{\nabla} o_L[\![\omega_{\mathrm{LT}}]\!]^{\psi_L = 1} \xrightarrow{(1 - \frac{\pi_L}{q}\varphi)} \mathcal{O}_{\mathbb{C}_p}(\mathbf{B})^{\psi_L = 0}
$$
$$
\xrightarrow{\log_{\mathrm{LT}} \cdot} \mathfrak{O}_{\mathbb{C}_p}(\mathbf{B})^{\psi_L = 0} \xrightarrow{\mathfrak{M}^{-1}} D(\Gamma_L, \mathbb{C}_p),
$$

where the map ∇ has been defined in [80, §6] as the homomorphism

$$
\nabla : U \otimes_{\mathbb{Z}} T^* \to o_L[\![\omega_{\mathrm{LT}}]\!]^{\psi = 1}
$$
$$
u \otimes a\eta^* \mapsto a \frac{\partial_{\mathrm{inv}}(g_{u,\eta})}{g_{u,\eta}}(\omega_{\mathrm{LT}}).
$$

Note that due to the multiplication by $\log_{\mathrm{LT}}$, the maps $\mathcal{L}, \mathfrak{L}$ are not Γ_L-equivariant. Using Lemmata 4.1.21 and 4.1.22, we obtain

$$
\begin{aligned}
\mathfrak{L}(u \otimes a\eta^*)(\chi_{\mathrm{LT}}^r) &= a\mathfrak{M}^{-1}\big(\log_{\mathrm{LT}}\big(1 - \tfrac{\pi_L}{q}\varphi\big)\partial_{\mathrm{inv}}\log g_{u,\eta}\big)(\chi_{\mathrm{LT}}^r) \\
&= a\Omega^{-1} r \mathfrak{M}^{-1}\big(1 - \tfrac{\pi_L}{q}\varphi\big)(\partial_{\mathrm{inv}}\log g_{u,\eta})(\chi_{\mathrm{LT}}^{r-1}) \\
&= ar\Omega^{-r}\big(1 - \tfrac{\pi_L}{q}\pi_L^{r-1}\big)(\partial_{\mathrm{inv}}^{r-1}\partial_{\mathrm{inv}}\log g_{u,\eta})_{|Z=0} \\
&= ar\Omega^{-r}\big(1 - \tfrac{\pi_L^r}{q}\big)(\partial_{\mathrm{inv}}^{r-1}\partial_{\mathrm{inv}}\log g_{u,\eta})_{|Z=0};
\end{aligned} \tag{5.3}
$$

i.e., $\mathcal{L}$ satisfies (5.2), indeed. By construction and Proposition 4.1.25 the image of $\mathfrak{L}$ actually lies in the G_L-invariants:

$$
\mathfrak{L} : U \otimes_{\mathbb{Z}} T_\pi^* \to D(\Gamma_L, K)^{G_L} \otimes_L D_{\mathrm{cris},L}\big(L(\tau)\big).
$$

We claim that the composition

$$
\begin{array}{c}
U \otimes_{\mathbb{Z}} T_\pi^* \dashrightarrow \qquad \qquad \qquad \qquad \\
{\scriptstyle \kappa \otimes T_\pi^*} \downarrow \qquad \qquad \qquad \qquad \searrow \\
H_{\mathrm{Iw}}^1\big(L_\infty/L, o_L(\tau)\big) \xrightarrow{\ \mathfrak{L}_{L(\tau\chi_{\mathrm{LT}})} \otimes o_L(\chi_{\mathrm{LT}}^{-1}) \otimes t_{\mathrm{LT}} \ } D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}\big(L(\tau)\big)
\end{array} \tag{5.4}
$$

coincides with

$$\mathcal{L} : U \otimes_{\mathbb{Z}} T_\pi^* \to D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}\big(L(\tau)\big).$$

Indeed, from the commutativity of the diagram in Figure 5.1 (cf. with [51, Appx. C] for $L = \mathbb{Q}_p$) the above claim immediately follows by tensoring the diagram for $r = 1$ with $o_L(\chi_{\mathrm{LT}}^{-1})$ and then composing with the multiplication by t_{LT}. In this diagram, we use the following abbreviations:

$$\diamondsuit_r := \mathcal{L}_{L(\tau\chi_{\mathrm{LT}}^r)} \otimes \mathbf{d}_1^{\otimes -1} \text{ for any } r \geq 1,$$
$$e_r := t_{\mathrm{LT}}^{-r} \otimes \eta^{\otimes r} \in D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}}^r)\big),$$
$$\mathfrak{l}_i := t_{\mathrm{LT}}\partial_{\mathrm{inv}} - i,$$

and we recall that $\partial_{\mathrm{inv}} = \frac{d}{dt_{\mathrm{LT}}}$.
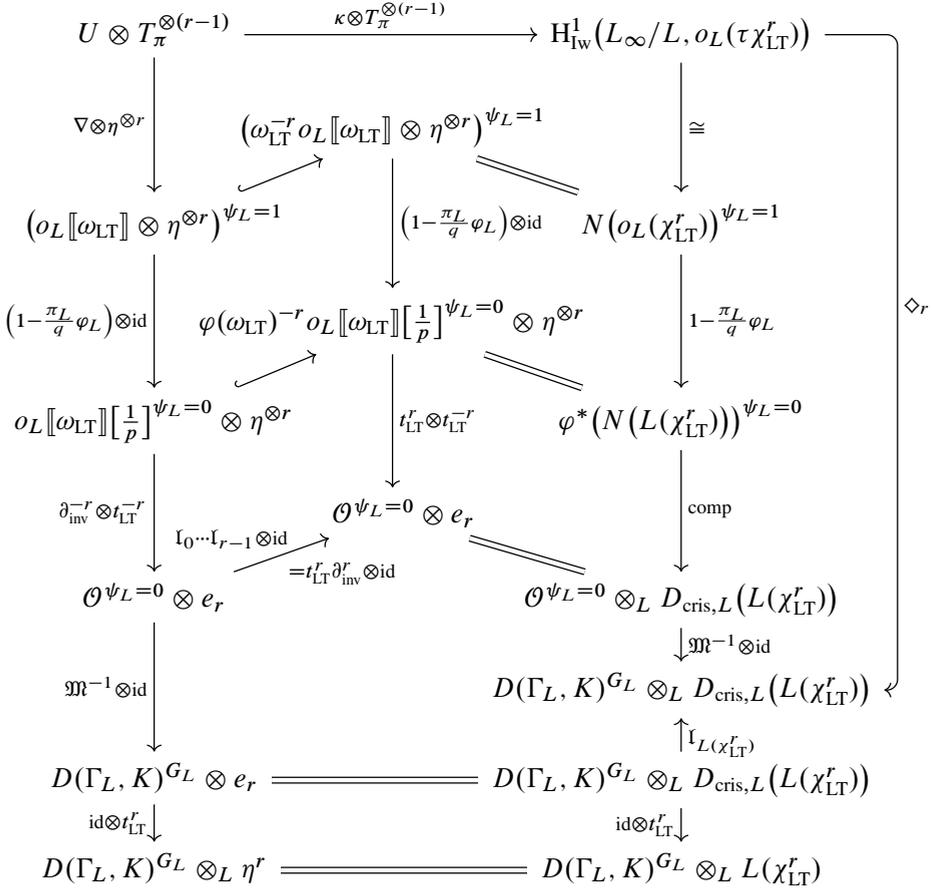
Figure 5.1. Comparison of $\mathcal{L}$ and $\mathcal{L}_{L(\tau\chi_{\mathrm{LT}})}$.

Note that we have

$$\mathfrak{M}^{-1}(\mathfrak{l}_0 f) = \lim_{\gamma \to 1} \frac{\delta_\gamma\big(\mathfrak{M}^{-1}(f)\big) - \mathfrak{M}^{-1}(f)}{\ell(\gamma)} = \nabla_{\mathrm{Lie}}\mathfrak{M}^{-1}(f), \qquad (5.5)$$

see [44, Lem. 2.1.4] for the fact that

$$\nabla_{\mathrm{Lie}} = t_{\mathrm{LT}}\partial_{\mathrm{inv}}$$

as operators on $\mathcal{O}$. By abuse of notation we thus also write

$$\mathfrak{l}_i := \nabla_{\mathrm{Lie}} - i$$

for the corresponding element in $D(\Gamma_L, K)$; compare [75, §2.3] for the action of $\mathrm{Lie}(\Gamma_L)$ on and its embedding into $D(\Gamma_L, K)$. Moreover, we set

$$\mathfrak{l}_{L(\chi_{\mathrm{LT}}^r)} := \prod_{i=0}^{r-1} \mathfrak{l}_i.$$

Note that ∂_{inv} is invertible on $\mathcal{O}^{\psi_L=0}$ by [33, Prop. 3.12]. Finally, the map

$$\mathrm{comp} : \varphi^*\big(N\big(o_L(\chi_{\mathrm{LT}}^r)\big)\big)^{\psi_L=0} \to \mathcal{O}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}}^r)\big)$$

is given by Corollary 3.1.14. Note that the commutativity of the upper part of the diagram in Figure 5.1 is an immediate consequence of [80, Thm. 6.2].

Inspired by Proposition 5.1.1, we define $\mathscr{L}_{L(\tau)}$ – since $L(\tau)$ does not satisfy the conditions from the beginning of this chapter while $L(\tau\chi_{\mathrm{LT}})$ does – as a twist of $\mathscr{L}_{L(\tau\chi_{\mathrm{LT}})}$ by requiring the commutativity of the diagram in Figure 5.2, which is possible due to the commutativity of the diagram in Figure 5.1.
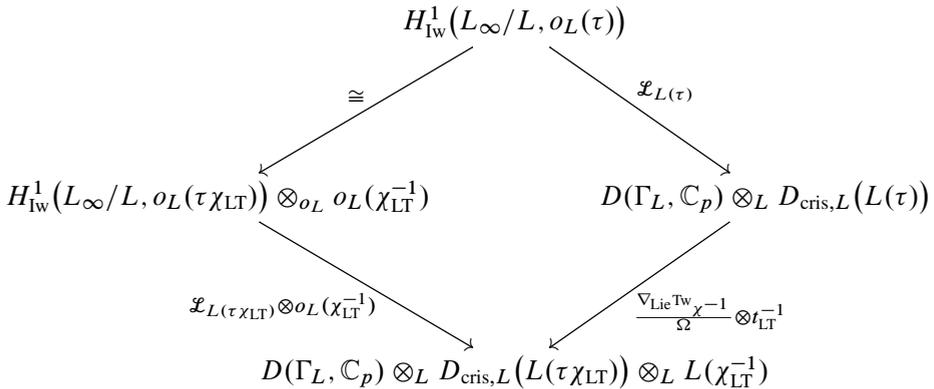
Figure 5.2. Comparison of $\mathscr{L}_{L(\tau)}$ and $\mathscr{L}_{L(\tau\chi_{\mathrm{LT}})}$.

Then
$$\mathcal{L} : U \otimes_{\mathbb{Z}} T_\pi^* \to D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}\big(L(\tau)\big)$$

also coincides with

$$U \otimes_{\mathbb{Z}} T_\pi^* \xrightarrow{\kappa \otimes T_\pi^*} H^1_{\mathrm{Iw}}\big(L_\infty/L, o_L(\tau)\big)$$

$$\xrightarrow{\big(\frac{1}{\Omega}\nabla_{\mathrm{Lie}}\mathrm{Tw}_{\chi^{-1}}\otimes\mathrm{id}\big)\circ\mathcal{L}_{L(\tau)}} D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}\big(L(\tau)\big) \quad (5.6)$$

by Proposition 5.1.1.

We refer the interested reader to [74, §5] for an example of a CM-elliptic curve E with supersingular reduction at p in which they attach to a norm-compatible sequence of elliptic units $e(\mathfrak{a})$ (in the notation of [27, §II 4.9]) a distribution $\mu(\mathfrak{a}) \in D(\Gamma_L, K)$ in [74, Prop. 5.2] satisfying a certain interpolation property with respect to the values of the attached (partial) Hecke-L-function. Without going into any detail concerning their setting and instead referring the reader to the notation in loc. cit., we just want to point out that up to twisting this distribution is the image of $\kappa(e(\mathfrak{a})) \otimes \eta^{-1}$ under the regulator map $\mathcal{L}_{L(\tau)}$:

$$\mathcal{L}_{L(\tau)}\big(\kappa\big(e(\mathfrak{a})\big) \otimes \eta^{-1}\big) = \Omega\mathrm{Tw}_{\chi_{\mathrm{LT}}}\big(\mu(\mathfrak{a})\big) \otimes \mathbf{d}_1.$$

Here, $L = \mathbf{K}_p = \mathbf{F}_\wp$ (in their notation) is the unique unramified extension of $\mathbb{Q}_p$ of degree 2, $\pi_L = p, q = p^2$, and the Lubin–Tate formal group is $\widehat{E}_\wp$, while $K = \widehat{L_\infty}$.

Indeed, we have a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad\quad\mathrm{Col}\quad\quad} & D(\Gamma_L, K) \\ {\scriptstyle\kappa(-)\otimes\eta^{-1}}\downarrow & & \downarrow{\scriptstyle\Omega\mathrm{Tw}_{\chi_{\mathrm{LT}}}\otimes\mathbf{d}_1} \\ H^1_{\mathrm{Iw}}\big(L_\infty/L, o_L(\tau)\big) & \xrightarrow{\quad\mathcal{L}_{L(\tau)}\quad} & D(\Gamma_L, K) \otimes_L D_{\mathrm{cris},L}\big(L(\tau)\big), \end{array} \quad (5.7)$$

where the Coleman map Col is given as the composite in the upper line of the following commutative diagram:

$$\begin{array}{ccccccc} U & \xrightarrow{\log g} & \mathcal{O}^{\psi_L=\frac{1}{\pi_L}} & \xrightarrow{1-\frac{\varphi_L}{p^2}} & \mathcal{O}^{\psi_L=0} \xrlongequal{\quad} \mathcal{O}^{\psi_L=0} & \xrightarrow{\mathfrak{M}^{-1}} & D(\Gamma_L, K) \\ \downarrow & & \downarrow{\scriptstyle\partial_{\mathrm{inv}}} & & \downarrow{\scriptstyle\partial_{\mathrm{inv}}} \quad\quad\quad \mathfrak{r}_0\downarrow & & \downarrow{\scriptstyle\nabla_{\mathrm{Lie}}} \\ U \otimes T_p^* & \xrightarrow{\nabla} & \mathcal{O}^{\psi_L=1} & \xrightarrow{1-\frac{\pi_L}{q}\varphi_L} & \mathcal{O}^{\psi_L=0} \xrightarrow{\log_{\mathrm{LT}}\cdot} \mathcal{O}^{\psi_L=0} & \xrightarrow{\mathfrak{M}^{-1}} & D(\Gamma_L, K), \end{array} \quad (5.8)$$

in which the second line is just $\mathcal{L}$. Then the commutativity of (5.7) follows by comparing (5.8) with (5.6). Finally, $\mathrm{Col}(e(\mathfrak{a})) = \mu(\mathfrak{a})(= \mathfrak{M}^{-1}(g_\mathfrak{a}(Z))$ in their notation) holds by construction in loc. cit. upon noting that on $\mathcal{O}^{\psi_L=\frac{1}{\pi_L}}$ the operator $1 - \frac{\pi}{p^2}\varphi_L \circ \psi_L$, which is used implicitly to define $g_\mathfrak{a}(Z)(= (1 - \frac{\pi}{p^2}\varphi_L \circ \psi_L)\log Q_\mathfrak{a}(Z))$, equals $1 - \frac{\varphi_L}{p^2}$.

5.2 Relation to Berger's and Fourquaux's big exponential map

Let V denote an L-analytic representation of G_L and take an integer $h \geq 1$ such that $\mathrm{Fil}^{-h} D_{\mathrm{cris},L}(V) = D_{\mathrm{cris},L}(V)$ and such that $D_{\mathrm{cris},L}(V)^{\varphi_L = \pi_L^{-h}} = 0$ holds. Under these conditions in [7], a big exponential map à la Perrin-Riou

$$\Omega_{V,h} : \big(\mathcal{O}^{\psi_L = 0} \otimes_L D_{\mathrm{cris},L}(V)\big)^{\Delta = 0} \to D_{\mathrm{rig}}^{\dagger}(V)^{\psi_L = \frac{q}{\pi_L}}$$

is constructed as follows: According to [7, Lem. 3.5.1], there is an exact sequence

$$0 \to \bigoplus_{k=0}^{h} t_{\mathrm{LT}}^k D_{\mathrm{cris},L}(V)^{\varphi_L = \pi_L^{-k}} \to \big(\mathcal{O} \otimes_{o_L} D_{\mathrm{cris},L}(V)\big)^{\psi_L = \frac{q}{\pi_L}}$$

$$\xrightarrow{1 - \varphi_L} \mathcal{O}^{\psi_L = 0} \otimes_L D_{\mathrm{cris},L}(V) \xrightarrow{\Delta} \bigoplus_{k=0}^{h} D_{\mathrm{cris},L}(V)/(1 - \pi_L^k \varphi_L) D_{\mathrm{cris},L}(V) \to 0,$$

where, for $f \in \mathcal{O} \otimes_L D_{\mathrm{cris},L}(V)$, we denote by $\Delta(f)$ the image of the map

$$\bigoplus_{k=0}^{h} (\partial_{\mathrm{inv}}^k \otimes \mathrm{id}_{D_{\mathrm{cris},L}(V)})(f)(0) \to \bigoplus_{k=0}^{h} D_{\mathrm{cris},L}(V)/(1 - \pi_L^k \varphi_L) D_{\mathrm{cris},L}(V).$$

Hence, if already $f \in (\mathcal{O}^{\psi_L = 0} \otimes_L D_{\mathrm{cris},L}(V))^{\Delta = 0}$, then there exists an element $y \in (\mathcal{O} \otimes_{o_L} D_{\mathrm{cris},L}(V))^{\psi_L = \frac{q}{\pi_L}}$ such that $f = (1 - \varphi_L)y$. Setting $\nabla_i := \nabla - i$ for any integer i, one observes that $\nabla_{h-1} \circ \cdots \circ \nabla_0$ annihilates $\bigoplus_{k=0}^{h-1} t_{\mathrm{LT}}^k D_{\mathrm{cris},L}(V)^{\varphi_L = \pi_L^{-k}}$, whence $\Omega_{V,h}(f) := \nabla_{h-1} \circ \cdots \circ \nabla_0(y)$ is well defined and belongs under the comparison isomorphism in (3.16) to $D_{\mathrm{rig}}^{\dagger}(V)^{\psi_L = \frac{q}{\pi_L}}$ by Proposition 3.1.13.

Note that

$$\big(\mathcal{O}^{\psi_L = 0} \otimes_L D_{\mathrm{cris},L}(V)\big)^{\Delta = 0} = \mathcal{O}^{\psi_L = 0} \otimes_L D_{\mathrm{cris},L}(V)$$

if $D_{\mathrm{cris},L}(V)^{\varphi_L = \pi_L^{-k}} = 0$ for all $0 \leq k \leq h$. If this does not hold for V itself, it does hold for $V(\chi_{\mathrm{LT}}^{-r})$ for r sufficiently large (with respect to the same h).

In the special case $L = \mathbb{Q}_p$, the above map becomes the exponential map due to Perrin-Riou and satisfies the following adjointness property with Loeffler's and Zerbes' regulator map, see [51, Prop. A.2.2], where the upper pairing and notation are introduced:

$$
\begin{array}{ccccc}
D(\Gamma, \mathbb{Q}_p) \otimes_{\Lambda_L} H_{\mathrm{Iw}}\big(\mathbb{Q}_p, V^*(1)\big) & \times & D(\Gamma, \mathbb{Q}_p) \otimes_{\Lambda_{\mathbb{Q}_p}} H_{\mathrm{Iw}}(\mathbb{Q}_p, V) & \to & D(\Gamma, \mathbb{Q}_p) \\
\Big\uparrow \Omega_{V^*(1),1} & & \gamma_{-1} \mathscr{L}_V \Big\downarrow & & \Big\| \\
D(\Gamma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{\mathrm{cris},\mathbb{Q}_p}\big(V^*(1)\big) & \times & D(\Gamma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{\mathrm{cris},\mathbb{Q}_p}(V) & \to & D(\Gamma, \mathbb{Q}_p).
\end{array}
$$

In fact, this is a variant of Perrin-Riou's reciprocity law comparing $\Omega_{V,h}$ with $\Omega_{V^*(1),1-h}$.

For $L \neq \mathbb{Q}_p$ the issue of L-analyticity requires that $V^*(1)$ is L-analytic for the construction of $\Omega_{V^*(1),1-h}$, which then implies that V is not L-analytic. Instead, our regulator map is available and the purpose of this subsection is to prove an analogue of the above adjointness for arbitrary L.

Theorem 5.2.1 (Reciprocity formula/Adjointness of big exponential and regulator map). *Assume that $V^*(1)$ is L-analytic with $\mathrm{Fil}^{-1} D_{\mathrm{cris},L}(V^*(1)) = D_{\mathrm{cris},L}(V^*(1))$ and $D_{\mathrm{cris},L}(V^*(1))^{\varphi_L = \pi_L^{-1}} = D_{\mathrm{cris},L}(V^*(1))^{\varphi_L = 1} = 0$. Then the following diagram consisting of $D(\Gamma_L, K)$-ι_*-sesquilinear pairings (in the sense of (4.87)) commutes:*

$$
\begin{array}{ccccc}
D_{\mathrm{rig}}^\dagger(V^*(1))^{\psi_L = \frac{q}{\pi_L}} & \times & D\big(V(\tau^{-1})\big)^{\psi_L = 1} & \xrightarrow{\frac{q-1}{q}\{\,,\,\}_{\mathrm{Iw}}} & D(\Gamma_L, \mathbb{C}_p) \\
\Big\uparrow{\scriptstyle \Omega_{V^*(1),1}} & & \Big\downarrow{\scriptstyle \mathscr{L}_V^0} & & \Big\| \\
\mathcal{O}^{\psi_L = 0} \otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big) & \times & \mathcal{O}^{\psi_L = 0} \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) & \xrightarrow{[\,,\,]} & D(\Gamma_L, \mathbb{C}_p).
\end{array}
$$

Note that the terms on the right-hand side of the pairings are all defined over L!

Proof. This follows from the abstract reciprocity law in Theorem 4.5.32 (where we set $M := D_{\mathrm{rig}}^\dagger(V(\tau^{-1}))$ as before). Indeed, assuming that $z \in \mathcal{O}^{\psi_L = 0} \otimes_L D_{\mathrm{cris},L}(V^*(1))$ and $y \in D(V(\tau^{-1}))^{\psi_L = 1}$, we have that $(1 - \frac{\pi_L}{q}\varphi_L)y \in M' \cap (M^{\psi_L = 0})$ (see (5.1)) and

$$
\mathrm{comp}^{-1}\big((1 - \varphi_L)x\big) \in \check{M}' \quad \text{for } x \in \big(\mathcal{O} \otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big)\big)^{\psi_L = \frac{q}{\pi_L}}
$$

such that $z = (1 - \varphi_L)x$. Moreover, $\mathrm{comp}^{-1}((1 - \varphi_L)x) \in \check{M}^{\psi_L = 0}$ by Proposition 3.1.13 as $V^*(1)$ is positive by assumption. Recall that $\mathrm{comp}^{-1}(\nabla x)$ is an element in $D_{\mathrm{rig}}^\dagger(V^*(1))^{\psi_L = \frac{q}{\pi_L}}$ again by Proposition 3.1.13. We thus obtain

$$
\begin{aligned}
\frac{q-1}{q}\big\{\mathrm{comp}^{-1}(\nabla x), y\big\}_{\mathrm{Iw}} &= \frac{q-1}{q}\big\{\nabla\mathrm{comp}^{-1}\big((1 - \varphi_L)x\big), \big(1 - \frac{\pi_L}{q}\varphi_L\big)y\big\}_{\mathrm{Iw}}^0 \\
&= \big[(1 - \varphi_L)x, \mathrm{comp}\big(\big(1 - \frac{\pi_L}{q}\varphi_L\big)y\big)\big].
\end{aligned}
$$

By definition of the big exponential and regulator map the latter is equivalent to

$$
\big\{\Omega_{V^*(1),1}(z), y\big\}_{\mathrm{Iw}} = \big[z, \mathscr{L}_V^0(y)\big]. \qquad \blacksquare
$$

We also could consider the following variant of the big exponential map (under the assumptions of the theorem):

$$
\boldsymbol{\Omega}_{V,h} : D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big) \to D_{\mathrm{rig}}^\dagger(V)^{\psi_L = \frac{q}{\pi_L}}
$$

by extending scalars from L to $\mathbb{C}_p$ and composing the original one with Ω^{-h} times[1]

$$
D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big) \xrightarrow{\mathcal{M} \otimes \mathrm{id}} \big(\mathcal{O}_K(\mathbf{B})\big)^{\psi_L = 0} \otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big).
$$

---

[1]This means to replace ∇ by $\frac{\nabla}{\Omega}$ in order to achieve twist invariance of the big exponential map, see the remark below.

Corollary 5.2.2 (Reciprocity formula/Adjointness of big exponential and regulator map). *Under the assumptions of the theorem, the following diagram of $D(\Gamma_L, K)$-ι_*-sesquilinear pairings commutes:*

$$
\begin{array}{ccc}
D^\dagger_{\mathrm{rig}}\big(V^*(1)\big)^{\psi_L=\frac{q}{\pi_L}} & \times & D\big(V(\tau^{-1})\big)^{\psi_L=1} \xrightarrow{\frac{q-1}{q}\{\,,\,\}_{\mathrm{Iw}}} D(\Gamma_L, \mathbb{C}_p) \\[4pt]
\Big\uparrow {\scriptstyle \boldsymbol{\Omega}_{V^*(1),1}} & & {\scriptstyle \frac{\sigma_{-1}\mathbf{L}_V}{\Omega}}\Big\downarrow \qquad\qquad \Big\|\; \Big\| \\[4pt]
D(\Gamma_L, \mathbb{C}_p)\otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big) & \times & D(\Gamma_L, \mathbb{C}_p)\otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) \xrightarrow{[\,,\,]^0} D(\Gamma_L, \mathbb{C}_p),
\end{array}
$$

where $[-,-]^0 = [\mathfrak{M}\otimes\mathrm{id}(-), \sigma_{-1}\mathfrak{M}\otimes\mathrm{id}(-)]$, *i.e.,*

$$
[\lambda\otimes\check{d}, \mu\otimes d]^0 \cdot \eta(1,Z)\otimes(t_{\mathrm{LT}}^{-1}\otimes\eta) = \lambda\iota_*(\mu)\cdot\eta(1,Z)\otimes[\check{d},d]_{\mathrm{cris}}, \qquad (5.9)
$$

where $D_{\mathrm{cris},L}(V^*(1))\times D_{\mathrm{cris},L}(V(\tau^{-1})) \xrightarrow{[\,,\,]_{\mathrm{cris}}} D_{\mathrm{cris},L}(L(\chi_{\mathrm{LT}}))$ *is the canonical pairing.*

Remark 5.2.3. By [7, Cor. 3.5.4], we have

$$
\Omega_{V,h}(x)\otimes\eta^{\otimes j} = \Omega_{V(\chi_{\mathrm{LT}}^j),h+j}(\partial_{\mathrm{inv}}^{-j}x\otimes t_{\mathrm{LT}}^{-j}\eta^{\otimes j})
$$

and $\mathfrak{l}_h\circ\Omega_{V,h} = \Omega_{V,h+1}$, whence we obtain the equality

$$
\boldsymbol{\Omega}_{V,h}(x)\otimes\eta^{\otimes j} = \boldsymbol{\Omega}_{V(\chi_{\mathrm{LT}}^j),h+j}\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}^{-j}}(x)\otimes t_{\mathrm{LT}}^{-j}\eta^{\otimes j}\big)
$$

and $\mathfrak{l}_h\circ\boldsymbol{\Omega}_{V,h} = \boldsymbol{\Omega}_{V,h+1}$.

5.2.1 Some homological algebra

Let $X\xrightarrow{f}Y$ be a morphism of cochain complexes. Its mapping cone $\mathrm{cone}(f)$ is defined as $X[1]\bigoplus Y$ with differential $d^i_{\mathrm{cone}(f)} := \begin{pmatrix} d^i_{X[1]} & 0 \\ f[1]^i & d^i_Y \end{pmatrix}$ (using column notation) and we define the mapping fiber of f as $\mathrm{Fib}(f) := \mathrm{cone}(f)[-1]$. Here the translation $X[n]$ of a complex X is given by $X[n]^i := X^{i+n}$ and $d^i_{X[n]} := (-1)^n d^{i+n}_X$. Alternatively, we may consider f as a double cochain complex concentrated horizontally in degree 0 and 1 and form the total complex (as in [82, Tag 012Z]). Then the associated total complex coincides with $\mathrm{Fib}(-f)$.

For a complex $(X^\bullet, d_X)$ of topological L-vector spaces we define its L-dual $((X^*)^\bullet, d_{X^*})$ to be the complex with

$$
(X^*)^i := \mathrm{Hom}_{L,\mathrm{cont}}(X^{-i}, L)
$$

and

$$
d_{X^*}(f) := (-1)^{\deg(f)-1} f\circ d_X.
$$

More generally, for two complexes $(X^\bullet, d_X)$ and $(Y^\bullet, d_Y)$ of topological L-vector spaces we define the complex $\operatorname{Hom}^\bullet_{L,\mathrm{cont}}(X^\bullet, Y^\bullet)$ by

$$\operatorname{Hom}^n_{L,\mathrm{cont}}(X^\bullet, Y^\bullet) = \prod_{i \in \mathbb{Z}} \operatorname{Hom}_{L,\mathrm{cont}}(X^i, Y^{i+n})$$

with differentials $df = d \circ f + (-1)^{\deg(f)-1} f \circ d$. Note that the canonical isomorphism

$$\operatorname{Hom}^\bullet(X^\bullet, Y^\bullet)[n] \xrightarrow{\cong} \operatorname{Hom}^\bullet\left(X^\bullet, Y^\bullet[n]\right)$$

does not involve any sign; i.e., it is given by the identity map in all degrees.

We also recall that the tensor product of two complexes $X^\bullet$ and $Y^\bullet$ is given by

$$(X^\bullet \otimes_L Y^\bullet)^i := \bigoplus_n X^n \otimes_L Y^{i-n}$$

and

$$d(x \otimes y) = dx \otimes y + (-1)^{\deg(x)} x \otimes dy.$$

The adjunction morphism on the level of complexes

$$\operatorname{adj} : \operatorname{Hom}^\bullet_{L,\mathrm{cont}}(X^\bullet \otimes_L Y^\bullet, Z^\bullet) \to \operatorname{Hom}^\bullet_{L,\mathrm{cont}}\left(Y^\bullet, \operatorname{Hom}^\bullet_{L,\mathrm{cont}}(X^\bullet, Z^\bullet)\right)$$

sends u to $(y \mapsto (x \mapsto (-1)^{\deg(x)\deg(y)} u(x \otimes y)))$. It is well defined and continuous with respect to the projective tensor product topology and the strong topology for the Homs. Furthermore, by definition we have the following commutative diagram:

$$
\begin{array}{ccc}
X^\bullet \otimes_L Y^\bullet & \xrightarrow{\quad u \quad} & L[-2] \\
{\scriptstyle \mathrm{id} \otimes \mathrm{adj}(u)} \downarrow & & \| \\
X^\bullet \otimes_L \operatorname{Hom}^\bullet_{L,\mathrm{cont}}\left(X^\bullet, L[-2]\right) & \xrightarrow{\quad \mathrm{ev}_2 \quad} & L[-2],
\end{array}
\tag{5.10}
$$

where ev_2 sends (x, f) to $(-1)^{\deg(x)\deg(f)} f(x)$.

Lemma 5.2.4. *Let $(\mathcal{C}^\bullet, d^\bullet)$ be a complex in the category of locally convex topological L-vector spaces.*

(i) *If $\mathcal{C}$ consists of Fréchet spaces and $h^i(\mathcal{C}^\bullet)$ is finite dimensional over L, then d^{i-1} is strict and has closed image.*

(ii) *If d^i is strict, then $h^{-i}(\mathcal{C}^*) \cong h^i(\mathcal{C})^*$.*

Proof. (i) Apply the argument from [13, §IX, Lem. 3.4] and use the open mapping theorem [71, Prop. 8.8]. (ii) If

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

forms part of the complex with B in degree i, one immediately obtains a map

$$\ker(\alpha^*)/\mathrm{im}(\beta^*) \to \big(\ker(\beta)/\mathrm{im}(\alpha)\big)^*,$$

where $\ker(\beta)$ carries the subspace topology and $\ker(\beta)/\mathrm{im}(\alpha)$ the quotient topology. Now use the Hahn–Banach theorem [71, Cor. 9.4] for the strict maps $B/\ker(\beta) \hookrightarrow C$ (induced from β) and $\ker(\beta) \hookrightarrow B$ in order to show that this map is an isomorphism. ∎

Definition 5.2.5. A locally convex topological vector space is called an LF-space if it is the direct limit of a countable family of Fréchet spaces, the limit being formed in the category of locally convex vector spaces.

Remark 5.2.6. If one replaces Fréchet spaces by LF-spaces, the situation of Lemma 5.2.4 is more subtle:

(i)   If $V \xrightarrow{\alpha} W$ is a continuous linear map of Hausdorff LF-spaces with finite-dimensional cokernel, then α is strict and has closed image by the same argument used in (i) of the previous lemma. However, since a closed subspace of an LF-space need not be an LF-space, we cannot achieve the same conclusion for complexes by this argument as $\ker(d^i)$ may fail to be an LF-space, whence one cannot apply the open mapping theorem, in general. But consider the following special situation. Assume that the complex $\mathcal{C}^\bullet$ consists of LF-spaces and $h^i(\mathcal{C}^\bullet)$ is finite dimensional. If moreover $\mathcal{C}^{i+1} = 0$, i.e., $\mathcal{C}^i = \ker(d^i)$, then d^{i-1} is strict and $h^{1-i}(\mathcal{C}^*) \cong h^{i-1}(\mathcal{C})^*$.

(ii) If d^i is not strict, the above proof still shows that we obtain a surjection $h^{-i}(\mathcal{C}^*) \twoheadrightarrow h^i(\mathcal{C})^*$.

However, for a special class of LF-spaces and under certain conditions we can say more about how forming duals and cohomology interacts.

Lemma 5.2.7. *Let* $(\mathcal{C}^\bullet, d^\bullet) = \varinjlim_r (\mathcal{C}_r^\bullet, d_r^\bullet)$ *be a complex in the category of locally convex topological L-vector spaces arising as regular inductive limit of complexes of Fréchet spaces; i.e., in each degree i the transition maps in the countable sequence $(\mathcal{C}_r^i)_r$ are injective and for each bounded subset $B \subseteq \mathcal{C}^i$ there exists an $r \geq 1$ such that B is contained in $\mathcal{C}_r^i$ and is bounded as a subset of the Fréchet space $\mathcal{C}_r^i$. Then,*

(i)   *we have topological isomorphisms* $(\mathcal{C}^\bullet)^* \cong \varprojlim_r (\mathcal{C}_r^\bullet)^*$,

(ii) *if, in addition,* $\varprojlim_{r \geq 0}^1 h^i((\mathcal{C}_r^\bullet)^*) = 0$ *for all i, we have a long exact sequence*

$$\cdots \to h^i\big((\mathcal{C}^\bullet)^*\big) \to \varprojlim_{r \geq 0} h^i\big((\mathcal{C}_r^\bullet)^*\big) \to h^{i-1}\big(\varprojlim_{r \geq 0}^1 (\mathcal{C}_r^\bullet)^*\big) \to h^{i+1}\big((\mathcal{C}^\bullet)^*\big) \to \cdots,$$

(iii)   *if, in addition to* (ii)*, the differentials $d_r^\bullet$ are strict, e.g., if all $h^i(\mathcal{C}_r^\bullet)$ have finite dimension over L, and $\varprojlim_{r\geq 0}^1 (\mathcal{C}_r^\bullet)^* = 0$, we have isomorphisms*

$$h^i\big((\mathcal{C}^\bullet)^*\big) \cong \varinjlim_{r\geq 0} h^{-i}(\mathcal{C}_r^\bullet)^*.$$

Proof. (i) is [61, Thm. 11.1.13] while (ii) and (iii) follow from (i) and [53, Chap. 3, Prop. 1] applied to the inverse system $((\mathcal{C}_r^\bullet)^*)_r$ combined with Lemma 5.2.4. ∎

5.2.2 Koszul complexes

In this paragraph, we restrict to the situation $U \cong \mathbb{Z}_p^d$ and fix topological generators $\gamma_1,\ldots,\gamma_d$ of U and we set $\Lambda := \Lambda(U)$. Furthermore, let M be any complete linearly topologized o_L-module with a continuous U-action. Then by [49, Thm. II.2.2.6] this actions extends to continuous Λ-action and one has

$$\mathrm{Hom}_{\Lambda,\mathrm{cont}}(\Lambda, M) = \mathrm{Hom}_\Lambda(\Lambda, M).$$

Consider the (homological) complexes $K_\bullet(\gamma_i) := [\Lambda \xrightarrow{\gamma_i - 1} \Lambda]$ concentrated in degrees 1 and 0 and define

$$K_\bullet := K_\bullet^U := K_\bullet(\gamma) := \bigotimes_{\substack{\Lambda \\ i=1}}^{d} K_\bullet(\gamma_i),$$

$$K^\bullet(M) := K_U^\bullet(M) := \mathrm{Hom}_\Lambda^\bullet(K_\bullet, M) \cong \mathrm{Hom}_\Lambda^\bullet(K_\bullet, \Lambda) \otimes_\Lambda M = K^\bullet(\Lambda) \otimes_\Lambda M,$$

$$K_\bullet(M) := K_\bullet \otimes_\Lambda M \quad \text{(homological complex)},$$

$$K_\bullet(M)^\bullet := (K_\bullet \otimes_\Lambda M)^\bullet \quad \text{(the associated cohomological complex)}.$$

If we want to indicate the dependence on $\gamma = (\gamma_1,\ldots,\gamma_d)$, we also write $K^\bullet(\gamma, M)$ instead of $K^\bullet(M)$ and similarly for other notation; moreover, we shall use the notation $\gamma^{-1} = (\gamma_1^{-1},\ldots,\gamma_d^{-1})$ and $\gamma^{p^n} = (\gamma_1^{p^n},\ldots,\gamma_d^{p^n})$. Note that in each degree these complexes consist of a direct sum of finitely many copies of M and will be equipped with the corresponding direct product topology.

The complex $K_\bullet$ will be identified with the exterior algebra complex $\bigwedge_\Lambda^\bullet \Lambda^d$ of the free Λ-module with basis $e_1,\ldots,e_d$, for which the differentials $d_q : \bigwedge_\Lambda^q \Lambda^d \to \bigwedge_\Lambda^{q-1} \Lambda^d$ with respect to the standard basis $e_{i_1,\ldots,i_q} = e_{i_1} \wedge \cdots \wedge e_{i_q}$, $1 \leq i_1 < \cdots < i_q \leq d$, are given by the formula

$$d_q(a_{i_1,\ldots,i_q}) = \sum_{k=1}^{q} (-1)^{k+1}(\gamma_{i_k} - 1)a_{i_1,\ldots,\widehat{i_k},\ldots,i_q}.$$

Then the well-known self-duality (compare [28, Prop. 17.15] although the claim there is not precisely the same) of the Koszul complex, i.e., the isomorphism of complexes

$$K_\bullet(\Lambda)^\bullet \cong K^\bullet(\Lambda)[d], \tag{5.11}$$

can be explicitly described in degree $-q$ as follows (by identifying $\bigwedge_\Lambda^d \Lambda^d = \Lambda e_1 \wedge \cdots \wedge e_d = \Lambda$):

$$\bigwedge_\Lambda^q \Lambda^d \xrightarrow{\alpha_{-q}} \mathrm{Hom}_\Lambda \left(\bigwedge_\Lambda^{d-q} \Lambda^d, \Lambda \right)$$

$$e_{i_1,\dots,i_q} \longmapsto \mathrm{sign}(I, J) e^*_{j_1,\dots,j_{d-q}},$$

where $e_1^*, \dots, e_d^*$ denotes the dual basis of $e_1, \dots, e_d$, the elements $e^*_{j_1,\dots,j_{d-q}} = e^*_{j_1} \wedge \cdots \wedge e^*_{j_{d-q}}$, $1 \le j_1 < \cdots < j_{d-q} \le d$, form a (dual) basis of $\mathrm{Hom}_\Lambda (\bigwedge_\Lambda^{d-q} \Lambda^d, \Lambda)$, the indices $J = (j_k)_k$ are complementary to $I = (i_n)_n$ in the following sense $\{i_1, \dots, i_q\} \cup \{j_1, \dots, j_{d-q}\} = \{1, \dots, d\}$, and $\mathrm{sign}(I, J)$ denotes the sign of the permutation $[i_1, \dots, i_q, j_1, \dots, j_{d-q}]$. Indeed, the verification that the induced diagram involving the differentials from cohomological degree $-q$ to $-q+1$

$$
\begin{array}{ccc}
\bigwedge_\Lambda^q \Lambda^d & \xrightarrow{\alpha_{-q}} & \mathrm{Hom}_\Lambda \left(\bigwedge_\Lambda^{d-q} \Lambda^d, \Lambda \right) \\
{\scriptstyle d_q} \downarrow & & \downarrow {\scriptstyle (-1)^d (-1)^{d-q-1} d^*_{d-q+1}} \\
\bigwedge_\Lambda^{q-1} \Lambda^d & \xrightarrow{\alpha_{-q+1}} & \mathrm{Hom}_\Lambda \left(\bigwedge_\Lambda^{d-q+1} \Lambda^d, \Lambda \right)
\end{array}
$$

commutes[2] relies on the observation that

$$\mathrm{sign}(I, J)\,\mathrm{sign}(I_{\hat{k}}, J_k)^{-1} = (-1)^{q-k+l-1},$$

where $I_{\hat{k}} := (i_1, \dots, \widehat{i_k}, \dots, i_q)$ denotes the sequence which results from I by omitting i_k, while $J_k = (j_1, \dots, j_{l-1}, i_k, j_l, \dots, i_{d-q})$ denotes the sequence which arises from J by inserting i_k at position l with regard to the strict increasing ordering: The permutations $[i_1, \dots, i_q, j_1, \dots, j_{d-q}]$ and $[i_1, \dots, \widehat{i_k}, \dots, i_q, j_1, \dots, j_{l-1}, i_k, j_l, \dots, j_{d-q}]$ differ visibly by $q - k + l - 1$ transpositions.

Now we assume that M is any complete locally convex L-vector space with continuous U-action such that its strong dual is again complete with continuous U-action. Then we obtain isomorphisms of complexes

$$
\begin{aligned}
K^\bullet(\gamma, M)^* &= \mathrm{Hom}^\bullet_{L,\mathrm{cont}} \left(\mathrm{Hom}^\bullet_\Lambda \left(K_\bullet(\gamma), \Lambda \right) \otimes_\Lambda M, L \right) \\
&\cong \mathrm{Hom}^\bullet_\Lambda \left(\mathrm{Hom}^\bullet_\Lambda \left(K_\bullet(\gamma^{-1}), \Lambda \right), \mathrm{Hom}_{L,\mathrm{cont}}(M, L) \right) \\
&\cong \mathrm{Hom}^\bullet_\Lambda \left(\mathrm{Hom}^\bullet_\Lambda \left(K_\bullet(\gamma^{-1}), \Lambda \right), \Lambda \right) \otimes_\Lambda \mathrm{Hom}_{L,\mathrm{cont}}(M, L) \\
&\cong K_\bullet(\gamma^{-1}, \Lambda)^\bullet \otimes_\Lambda \mathrm{Hom}_{L,\mathrm{cont}}(M, L) \\
&\cong K^\bullet(\gamma^{-1}, \Lambda)[d] \otimes_\Lambda M^* \\
&\cong K^\bullet(\gamma^{-1}, M^*)[d],
\end{aligned}
\tag{5.12}
$$

---

[2]The signs $(-1)^d$ and $(-1)^{d-q-1}$ result from the shift by d and the sign rule for complex homomorphisms, respectively.

where in the second line we use the adjunction morphism; the isomorphism in the fourth line being the biduality morphism (according to [58, (1.2.8)])

$$K_\bullet(\Lambda)^\bullet \xrightarrow{\cong} \mathrm{Hom}_\Lambda^\bullet \left(\mathrm{Hom}_\Lambda^\bullet(K_\bullet, \Lambda), \Lambda \right)$$
$$x \mapsto (-1)^i x^{**}$$

with the usual biduality of modules

$$K_\bullet(\Lambda)^i \xrightarrow{\cong} \mathrm{Hom}_\Lambda \left(\mathrm{Hom}_\Lambda(K_{-i}, \Lambda), \Lambda \right)$$
$$x \mapsto \left(x^{**} : f \mapsto f(x) \right)$$

involves a sign, while the isomorphism in the third last line stems from (5.11) together with Lemma 4.5.1 (i). Note that the isomorphism in the second last line does not involve any further signs by [58, (1.2.15)].

We finish this subsection by introducing restriction and corestriction maps concerning the change of group for Koszul complexes. To this end let $U_1 \subseteq U$ be the open subgroup generated by $\gamma_1^{p^n}, \dots, \gamma_d^{p^n}$. Then $\mathrm{Hom}_\Lambda^\bullet(-, M)$ applied to the tensor product of the diagrams

$$\begin{array}{ccc} \Lambda(U) & \xrightarrow{\gamma_i^{p^n}-1} & \Lambda(U) \\ \| & & \uparrow{\scriptstyle \sum_{k=0}^{p^n-1} \gamma_i^k} \\ \Lambda(U) & \xrightarrow{\gamma_i-1} & \Lambda(U) \end{array}$$

gives a map

$$\mathrm{cor}_U^{U_1} : K_{U_1}^\bullet(\gamma^{p^n})(M) \to K_U^\bullet(\gamma)(M)$$

which we call corestriction map and which is compatible under (5.20) below with the corestriction map on cocycles (for appropriate choices of representatives in the definition of the latter). Using the diagram

$$\begin{array}{ccc} \Lambda(U) & \xrightarrow{\gamma_i^{p^n}-1} & \Lambda(U) \\ {\scriptstyle \sum_{k=0}^{p^n-1} \gamma_i^k}\downarrow & & \| \\ \Lambda(U) & \xrightarrow{\gamma_i-1} & \Lambda(U) \end{array}$$

instead, one obtains the restriction map

$$\mathrm{res}_{U_1}^U : K_U^\bullet(\gamma)(M) \to K_{U_1}^\bullet(\gamma^{p^n})(M),$$

again compatible under (5.20) with the restriction map on cocycles.

5.2.3 Continuous and analytic cohomology

For any profinite group G and topological abelian group M with continuous G-action we write $\mathcal{C}^\bullet := \mathcal{C}^\bullet(G, M)$ for the continuous (inhomogeneous) cochain complex of G with coefficients in M and $H^*(G, M) := h^*(\mathcal{C}^\bullet(G, M))$ for continuous group cohomology. Note that $\mathcal{C}^0(G, M) = M$.

Assume that G is moreover an L-analytic group and $M = \varinjlim_s \varprojlim_r M^{[r,s]}$ with Banach spaces $M^{[r,s]}$, an LF space with a pro-L-analytic action of G, i.e., a locally analytic action on each $M^{[r,s]}$, which means that for all $m \in M^{[r,s]}$ there exists an open L-analytic subgroup $\Gamma_n \subseteq \Gamma$ in the notation of Section 4.3.4 such that the orbit map of m restricted to Γ_n is a power series of the form $g(m) = \sum_{k \geq 0} \ell(g)^k m_k$ for a sequence m_k of elements in $M^{[r,s]}$ with $\pi_L^{nk} m_k$ converging to zero. Following [25, §5], we write

$$\mathcal{C}_{\mathrm{an}}^\bullet := \mathcal{C}_{\mathrm{an}}^\bullet(G, M)$$

for the locally L-analytic cochain complex of G with coefficients in M and

$$H_{\mathrm{an}}^*(G, M) := h^*\big(\mathcal{C}_{\mathrm{an}}^\bullet(G, M)\big)$$

for locally L-analytic group cohomology. More precisely, if $\mathrm{Maps}_{\mathrm{loc}\, L-\mathrm{an}}(G, M^{[r,s]})$ denotes the space of locally L-analytic maps from G to $M^{[r,s]}$, then

$$C_{\mathrm{an}}^n(G, M) = \varinjlim_s \varprojlim_r \mathrm{Maps}_{\mathrm{loc}\, L-\mathrm{an}}(G^n, M^{[r,s]})$$

is the space of locally L-analytic functions (locally with values in $\varprojlim_r M^{[r,s]}$ for some s and such that the composite with the projection onto $M^{[r,s]}$ is locally L-analytic for all r). Note that again $\mathcal{C}_{\mathrm{an}}^0(G, M) = M$ and that there are canonical homomorphisms

$$\mathcal{C}_{\mathrm{an}}^\bullet(G, M) \hookrightarrow \mathcal{C}^\bullet(G, M),$$
$$H_{\mathrm{an}}^\bullet(G, M) \to H^\bullet(G, M).$$

Let f be any continuous endomorphism of M which commutes with the G-action. We define

$$H^0(f, M) := M^{f=1} \quad \text{and} \quad H^1(f, M) := M_{f=1}$$

as the kernel and cokernel of the map $M \xrightarrow{f-1} M$, respectively.

The endomorphism f induces an operator on $\mathcal{C}^\bullet$ or $\mathcal{C}_{\mathrm{an}}^\bullet$ and we denote by $\mathcal{T} := \mathcal{T}_{f,G}(M)$ and $\mathcal{T}^{\mathrm{an}} := \mathcal{T}_{f,G}^{\mathrm{an}}(M)$ the mapping fiber of $\mathcal{C}^\bullet(G, f)$ and $\mathcal{C}_{\mathrm{an}}^\bullet(G, f)$, respectively.

Again there are canonical homomorphisms

$$\mathcal{T}_{f,G}^{\mathrm{an}}(M) \hookrightarrow \mathcal{T}_{f,G}(M),$$
$$h^\bullet\big(\mathcal{T}_{f,G}^{\mathrm{an}}(M)\big) \to h^\bullet\big(\mathcal{T}_{f,G}(M)\big).$$

$$\tag{5.13}$$

For ? either empty or *an*, one of the corresponding double complex spectral sequences is

$$_I E_2^{i,j} = H^i\big(f, H_?^j(G, M)\big) \Longrightarrow h^{i+j}(\mathcal{T}^?).\,^3$$

It degenerates into the short exact sequences

$$0 \to H_?^{i-1}(G, M)_{f=1} \to h^i\big(\mathcal{T}_{f,G}^?(M)\big) \to H_?^i(G, M)^{f=1} \to 0.$$

In loc. cit. as well as in [7], analytic cohomology is also defined for the semi-groups $\Gamma_L \times \Phi$ and $\Gamma_L \times \Psi$ with $\Phi = \{\varphi_L^n \mid n \geq 0\}$ and $\Psi = \{(\frac{\pi}{q}\psi_L)^n \mid n \geq 0\}$ if M denotes an *L-analytic* (φ_L, Γ_L)-module over the Robba ring $\mathcal{R}$.

Remark 5.2.8. Any *L-analytic* (φ_L, Γ_L)-module M over the Robba ring $\mathcal{R}$ is a pro-*L*-analytic Γ_L-module by the discussion at the end of the proof of [8, Prop. 2.25], whence it is also an *L-analytic* $\Gamma_L \times \Phi$- and $\Gamma_L \times \Psi$-module as Φ and Ψ possess the discrete structure as *L*-analytic manifolds.

Proposition 5.2.9. *We have canonical isomorphisms*

$$h^i\big(\mathcal{T}_{\varphi_L,\Gamma_L}^{\mathrm{an}}(M)\big) \cong H_{\mathrm{an}}^i(\Gamma_L \times \Phi, M) \cong H_{\mathrm{an}}^i(\Gamma_L \times \Psi, M) \cong h^i\big(\mathcal{T}_{\frac{\pi}{q}\psi_L,\Gamma_L}^{\mathrm{an}}(M)\big)$$

and an exact sequence

$$0 \longrightarrow H_{\mathrm{an}}^1(\Gamma_L, M^{\psi_L = \frac{q}{\pi}}) \longrightarrow h^i\big(\mathcal{T}_{\frac{\pi}{q}\psi_L,\Gamma_L}^{\mathrm{an}}(M)\big)$$

(5.14)

$$\longrightarrow (M_{\psi_L = \frac{q}{\pi}})^{\Gamma_L} \longrightarrow H_{\mathrm{an}}^2(\Gamma_L, M^{\psi_L = \frac{q}{\pi}}) \longrightarrow h^2\big(\mathcal{T}_{\frac{\pi}{q}\psi_L,\Gamma_L}^{\mathrm{an}}(M)\big).$$

Proof. The isomorphism in the middle is [7, Cor. 2.2.3]. For the two outer isomorphisms we refer the reader to [87, Thm. 3.7.6]. The exact sequence is the extension [87, Thm. 5.1.5] of [7, Thm. 2.2.4]. ∎

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[3]Naively, one would expect that the second corresponding double complex spectral sequence looks like

$$_{II} E_2^{i,j} = H_?^i\big(G, H^j(f, M)\big) \Longrightarrow h^{i+j}(\mathcal{T}^?).$$

But this would require to first of all give sense to the required structure of $H^j(f, M)$ as topological/analytic G-module! In low degrees this can be achieved and we obtain an exact sequence

$$0 \to H_?^1(G, M^{f=1}) \to h^1(\mathcal{T}^?) \to (M_{f=1})^G \xrightarrow{\delta} H_?^2(G, M^{f=1}).$$

See [87–89]. If $M^{f=1}$ is again an LF-space with pro-*L*-analytic G-operation, one might be able to interpret the second spectral sequence in low degrees.

Note that, for an inclusion $U \subseteq U'$, the restriction and corestriction homomorphisms $\mathcal{C}^\bullet(U', M) \xrightarrow{\text{res}} \mathcal{C}^\bullet(U, M)$ and $\mathcal{C}^\bullet(U, M) \xrightarrow{\text{cor}} \mathcal{C}^\bullet(U', M)$ induce maps on $\mathcal{T}_{f,U'}(M) \xrightarrow{\text{res}} \mathcal{T}_{f,U}(M)$ and $\mathcal{T}_{f,U}(M) \xrightarrow{\text{cor}} \mathcal{T}_{f,U'}(M)$, respectively.

We write $\text{Ext}^1_{\mathfrak{C}}(A, B)$ for isomorphism classes of extensions of B by A in any abelian category $\mathfrak{C}$. Furthermore, we denote by $\mathfrak{M}_U(R)$ ($\mathfrak{M}^{\text{ét}}_U(R)$, $\mathfrak{M}^\dagger_U(R)$) the category of all (étale, overconvergent) (φ_L, U)-modules over R, respectively, and by $\text{Rep}^\dagger_L(G_{L^U_\infty})$ the category of overconvergent representations of $G_{L^U_\infty}$ consisting of those representations V of $G_{L^U_\infty}$ such that $\dim_{\mathbf{B}^\dagger_L} D^\dagger(V) = \dim_L V$ with $D^\dagger(V) := (\mathbf{B}^\dagger \otimes_L V)^{H_L}$.

Theorem 5.2.10. *Let V be in $\text{Rep}_L(G_L)$ and $U \subseteq \Gamma_L$ any open subgroup.*

(i)   *For $D(V)$ the corresponding (φ_L, Γ_L)-module over $\mathbf{B}_L$ we have canonical isomorphisms*

$$h^* = h^*_{U,V} : H^*(L^U_\infty, V) \xrightarrow{\cong} h^*\big(\mathcal{T}_{\varphi_L,U}\big(D(V)\big)\big) \tag{5.15}$$

which are functorial in V and compatible with restriction and corestriction.

(ii)   *If V is in addition overconvergent, there are isomorphisms*

$$h^0\big(\mathcal{T}_{\varphi_L,U}(D^\dagger_{\text{rig}}(V))\big) \cong V^{G_{L^U_\infty}}, \tag{5.16}$$

$$h^1\big(\mathcal{T}_{\varphi_L,U}(D^\dagger_{\text{rig}}(V))\big) \cong H^1_\dagger(L^U_\infty, V), \tag{5.17}$$

which are functorial in V and compatible with restriction and corestriction and where by definition $H^1_\dagger(L^U_\infty, V) \subseteq H^1(L^U_\infty, V)$ classifies the overconvergent extensions of L by V. In particular, these L-vector spaces have finite dimension.

(iii)   *If V is in addition L-analytic, then we have*

$$H^1_{\text{an}}(L^U_\infty, V) \xrightarrow{\cong} h^1\big(\mathcal{T}^{\text{an}}_{\varphi_L,U}\big(D^\dagger_{\text{rig}}(V)\big)\big),$$

where by definition[4] $H^1_{\text{an}}(L^U_\infty, V) \subseteq H^1_\dagger(L^U_\infty, V) \subseteq H^1(L^U_\infty, V)$ classifies the L-analytic extensions of L by V.

Proof. (i) is [46, Thm. 5.1.11] or [45, Thm. 5.1.11]. Statement (iii) is [7, Prop. 2.2.1] combined with Proposition 5.2.9, while (ii) follows from [33] (the reference literally only covers the case $U = \Gamma_L$, but the same arguments allow us to extend the result to general U) as follows: Firstly, by Lemma 5.2.11 below one has an isomorphism $h^1(\mathcal{T}_{\varphi_L,U}(D^\dagger_{\text{rig}}(V))) \cong \text{Ext}^1_{\mathfrak{M}_U(\mathcal{R}_L)}(\mathcal{R}_L, D^\dagger_{\text{rig}}(V))$. Then use the HN-filtration à la

---

[4]Note that the absolute Galois group of L^U_∞ is not L-analytic, so this group has not been defined earlier.

Kedlaya to see that any extension of étale (φ_L, U)-modules is étale again, whence

$$\mathrm{Ext}^1_{\mathcal{M}_U(\mathcal{R}_L)}\big(\mathcal{R}_L, D^\dagger_{\mathrm{rig}}(V)\big) = \mathrm{Ext}^1_{\mathcal{M}^{\text{ét}}_U(\mathcal{R}_L)}\big(\mathcal{R}_L, D^\dagger_{\mathrm{rig}}(V)\big)$$

and the latter group equals

$$\mathrm{Ext}^1_{\mathcal{M}^\dagger_U(\mathcal{R}_L)}\big(\mathcal{R}_L, D^\dagger_{\mathrm{rig}}(V)\big) \cong \mathrm{Ext}^1_{\mathrm{Rep}^\dagger_L(G_{L_\infty^U})}(L, V) = H^1_\dagger(L^U_\infty, V)$$

by [33, Props. 1.5 and 1.6]. For the claim in degree 0 one has to show that the inclusion $D^\dagger(V) \subseteq D^\dagger_{\mathrm{rig}}(V)$ induces an isomorphism on φ_L-invariants, which follows from [39, Hyp. 1.4.1, Prop. 1.2.6].[5]   ∎

Lemma 5.2.11. *Let M be in $\mathfrak{M}_U(\mathcal{R})$. Then we have a canonical isomorphism*

$$h^1\big(\mathfrak{T}_{\varphi_L, U}(M)\big) \cong \mathrm{Ext}^1_{\mathfrak{M}_U(\mathcal{R}_L)}(\mathcal{R}_L, M).$$

Proof. Starting with a class $z = [(c_1, -c_0)]$ in $h^1(\mathcal{T}_{\varphi_L, U}(M))$ with $c_1 \in C^1(M)$ and $c_0 \in C^0(M) = M$ (i.e., we work with *inhomogeneous* continuous cocycles) satisfying the cocycle property

$$\begin{aligned} c_1(\sigma\tau) &= \sigma c_1(\tau) + c_1(\sigma) \quad \text{for all } \sigma, \tau \in U, \\ (\varphi_L - 1)c_1(\tau) &= (\tau - 1)c_0 \quad \text{for all } \tau \in U, \end{aligned} \tag{5.18}$$

we define an extension of (φ_L, U)-modules

$$0 \to M \to E_c \to \mathcal{R}_L \to 0$$

with $E_c := M \times \mathcal{R}_L$ as $\mathcal{R}_L$-module, $g(m, r) := (gm + gr \cdot c_1(g), gr)$ for $g \in U$ and $\varphi_{E_c}((m, r)) := (\varphi_M(m) + \varphi_L(r)c_0, \varphi_L(r))$; note that this defines a (continuous) group action by the first identity in (5.18), while the U- and φ_L-action commute by the second identity in (5.18). If we change the representatives $(c_1, -c_0)$ by the coboundary induced by $m_0 \in M$, then sending $(0, 1)$ to $(-m_0, 1)$ induces an isomorphism of extensions from the first to the second one, whence our map is well defined.

Conversely, if E is any such extension, choose a lift $e \in E$ of $1 \in \mathcal{R}_L$ and define

$$c_1(\tau) := (\tau - 1)e \in M, \quad c_0 := (\varphi_E - 1)e,$$

which evidently satisfy the cocycle conditions (5.18). Choosing another lift $\tilde{e}$ leads to a cocycle which differs from the previous one by the coboundary induced by $\tilde{e} - e \in M$, whence the inverse map is well defined.

One easily verifies that these maps are mutually inverse to each other.   ∎

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[5]Since the *strong hypothesis* holds by [39, Hyp. 1.4.1, Prop. 1.2.6], we also obtain an isomorphism on the φ_L-coinvariants $H^1(\varphi_L, -)$. Then the second spectral sequence above or a similar argument via the Koszul complexes as in Corollary A.8 implies that the canonical base change map induces an isomorphism $h^*(\mathcal{T}_{\varphi_L, U}(D^\dagger(V))) \cong h^*(\mathcal{T}_{\varphi_L, U}(D^\dagger_{\mathrm{rig}}(V)))$. Cf. [50, proof of Prop. 2.7].

Question 5.2.12. Can one show that $h^2(\mathcal{T}_{\varphi_L,U}(D_{\mathrm{rig}}^\dagger(V)))$ is finite dimensional (and related to $H^2(L_\infty^U, V)$) and that the groups $h^i(\mathcal{T}_{\varphi_L,U}(D_{\mathrm{rig}}^\dagger(V)))$ vanish for $i \geq 3$?

Remark 5.2.13. By [33, Thm. 0.2, Rem. 5.21], it follows that the inclusions

$$H^1_{\mathrm{an}}(L_\infty^U, V) \subseteq H^1_\dagger(L_\infty^U, V) \subseteq H^1(L_\infty^U, V)$$

are in general strict. More precisely, the codimension of the left most term equals $([L_\infty^U : \mathbb{Q}_p] - 1) \dim_L V^{G_{L_\infty^U}}$.

Let us recall Tate's local duality in this context.

Proposition 5.2.14 (Local Tate duality). *Let V be an object in $\mathrm{Rep}_L(G_L)$, and K any finite extension of L. Then the cup product and the local invariant map induce perfect pairings of finite-dimensional L-vector spaces*

$$H^i(K, V) \times H^{2-i}\big(K, \mathrm{Hom}_{\mathbb{Q}_p}\big(V, \mathbb{Q}_p(1)\big)\big) \to H^2\big(K, \mathbb{Q}_p(1)\big) = \mathbb{Q}_p,$$
$$H^i(K, V) \times H^{2-i}\big(K, \mathrm{Hom}_L\big(V, L(1)\big)\big) \to H^2\big(K, L(1)\big) = L,$$

where $-(1)$ denotes the Galois twist by the cyclotomic character. In other words, there are canonical isomorphisms

$$H^i(K, V) \cong H^{2-i}\big(K, V^*(1)\big)^*.$$

Proof. This is well known. For lack of a reference (with proof) we sketch the second claim (the first being proved similarly). Choose a Galois stable o_L-lattice $T \subseteq V$ and denote by $_{\pi_L^n}A$ the kernel of multiplication by π_L^n on any o_L-module A. Observe that we have short exact sequences

$$0 \to H^i(K, T)/\pi_L^n \to H^i(K, T/\pi_L^n T) \to {}_{\pi_L^n}H^{i+1}(K, T) \to 0$$

for $i \geq 0$ and similarly for T replaced by $T^*(1) = \mathrm{Hom}_{o_L}(T, o_L(1))$. By [80, Prop. 5.7] (remember the normalization given there!) the cup product induces isomorphism

$$H^i(K, T/\pi_L^n T) \cong \mathrm{Hom}_{o_L}\big(H^{2-i}\big(K, T^*(1)/\pi_L^n T^*(1)\big), o_L/\pi_L^n\big)$$

such that we obtain altogether canonical maps

$$H^i(K, T)/\pi_L^n \to \mathrm{Hom}_{o_L}\big(H^{2-i}\big(K, T^*(1)\big)/\pi_L^n, o_L/\pi_L^n\big)$$
$$\cong \mathrm{Hom}_{o_L}\big(H^{2-i}\big(K, T^*(1)\big), o_L\big)/\pi_L^n.$$

Using that the cohomology groups are finitely generated o_L-modules and isomorphic to the inverse limits of the corresponding cohomology groups with coefficients modulo π_L^n, we see that the inverse limit of the above maps induces a surjective map

$$H^i(K, T) \twoheadrightarrow \mathrm{Hom}_{o_L}\big(H^{2-i}\big(K, T^*(1)\big), o_L\big)$$

with finite kernel, whence the claim after tensoring with L over o_L using the isomorphism $H^i(K, T) \otimes_{o_L} L \cong H^i(K, V)$ and analogously for $T^*(1)$. ∎

Now let W be an L-analytic representation of G_L and set

$$H^1_{/\dagger}\big(L^U_\infty, W^*(1)\big) := H^1_\dagger(L^U_\infty, W)^*,$$

which, by local Tate duality and Theorem 5.2.10, is a quotient of $H^1(L^U_\infty, W^*(1))$. By definition, the local Tate pairing induces a non-degenerate pairing

$$\langle\,,\,\rangle_{\text{Tate},L,\dagger} : H^1_\dagger(L^U_\infty, W) \times H^1_{/\dagger}\big(L^U_\infty, W^*(1)\big) \to H^2\big(L, L(1)\big) \cong L. \quad (5.19)$$

In order to compute this pairing more explicitly in certain situations, we shall use Koszul complexes. For this we have to assume first that U is torsion-free. Following [26, §4.2], we obtain for any complete linearly topologized o_L-module M with continuous U-action a quasi-isomorphism[6]

$$K^\bullet_U(M) \xrightarrow{\simeq} \mathcal{C}^\bullet(U, M) \quad (5.20)$$

which arises as follows: Let $X_\bullet := X_\bullet(U)$ and $Y_\bullet = Y_\bullet(U)$ denote the completed standard complex [49, §V.1.2.1], i.e., $X_n = \mathbb{Z}_p[\![U]\!]^{\hat\otimes(n+1)}$, and the standard complex computing group cohomology, i.e., $Y_n = \mathbb{Z}_p[U]^{\otimes(n+1)}$. Then, by [49, Lem. V.1.1.5.1], we obtain a diagram of complexes

$$
\begin{array}{ccc}
Y_\bullet(U) & \xrightarrow{\Delta} & Y_\bullet(U \times U) \cong Y_\bullet(U) \otimes_{\mathbb{Z}_p} Y_\bullet(U) \\
\downarrow & & \downarrow \\
X_\bullet(U) & \xrightarrow{\Delta} & X_\bullet(U \times U) \cong X_\bullet(U) \,\hat\otimes_{\mathbb{Z}_p}\, X_\bullet(U) \\
\downarrow & & \downarrow \\
K^U_\bullet & \xrightarrow{\Delta} & K^{U \times U}_\bullet \cong K^U_\bullet \,\hat\otimes_{\mathbb{Z}_p}\, K^U_\bullet,
\end{array}
\quad (5.21)
$$

which commutes up to homotopy (of filtered Λ-modules). Here the maps Δ are induced by the diagonal maps $U \to U \times U$, e.g.,

$$\mathbb{Z}_p[\![U]\!] \to \mathbb{Z}_p[\![U \times U]\!] \cong \mathbb{Z}_p[\![U]\!] \,\hat\otimes_{\mathbb{Z}_p}\, \mathbb{Z}_p[\![U]\!].$$

The first column induces a morphism

$$\text{Hom}_\Lambda(K^U_\bullet, M) \to \text{Hom}_{\Lambda,\text{cont}}\big(X_\bullet(U), M\big) \to \text{Hom}_{\mathbb{Z}_p[U],\text{cont}}\big(Y_\bullet(U), M\big),$$

which is (5.20). The upper line induces as usual the cup product on continuous group cohomology

$$H^r(U, M) \times H^s(U, N) \xrightarrow{\cup_U} H^{r+s}(U, M \otimes N)$$

---

[6]This quasi-isomorphism is unique up to homotopy, i.e., unique in the derived category of o_L-linear topological U-modules. We have not yet defined any topology on the cocycles nor do we know whether the references say anything about it! M is allowed to be any complete linearly topologized o_L-module with continuous U-action by [49, Prop. V.1.2.6].

via

$$\mathrm{Hom}_{\mathbb{Z}_p[U],\mathrm{cont}}\left(Y_\bullet(U), M\right) \times \mathrm{Hom}_{\mathbb{Z}_p[U],\mathrm{cont}}\left(Y_\bullet(U), N\right)$$
$$\overset{\times}{\to} \mathrm{Hom}_{\mathbb{Z}_p[U]\otimes\mathbb{Z}_p[U],\mathrm{cont}}\left(Y_\bullet(U) \otimes_{\mathbb{Z}_p} Y_\bullet(U), M \otimes N\right)$$
$$\overset{\Delta^*}{\longrightarrow} \mathrm{Hom}_{\mathbb{Z}_p[U],\mathrm{cont}}\left(Y_\bullet(U), M \otimes N\right).$$

The lower line induces analogously the Koszul product

$$K_U^r(M) \times K_U^s(N) \overset{\cup_K}{\longrightarrow} K_U^{r+s}(M \otimes N).$$

By diagram (5.21) both products are compatible with each other.

Let f be any continuous endomorphism of M which commutes with the U-action; it induces an operator on $K^\bullet(M)$ and we denote by

$$K_{f,U}(M) := \mathrm{cone}\big(K^\bullet(M) \overset{f-\mathrm{id}}{\longrightarrow} K^\bullet(M)\big)[-1]$$

the mapping fiber of $K^\bullet(f)$. Then the quasi-isomorphism (5.20) induces a quasi-isomorphism

$$K_{\varphi,U}(M) \overset{\cong}{\to} \mathcal{T}_{\varphi,U}(M).$$

Remark 5.2.15. By a standard procedure cup products can be extended to hyper-cohomology (defined via total complexes), we follow [58, (3.4.5.2)], but for the special case of a cone, see also [60, Prop. 3.1]. In particular, we obtain compatible cup products $\cup_K$ and $\cup_U$ for $K_{\varphi,U}(M)$ and $\mathcal{T}_{\varphi,U}(M)$, respectively.

Now we allow some arbitrary open subgroup $U \subseteq \Gamma_L$ and let $L' = L_\infty^U$. Note that we obtain a decomposition $U \cong \Delta \times U'$ with a subgroup $U' \cong \mathbb{Z}_p^d$ of U and Δ the torsion subgroup of U. By Lemma A.1 we obtain a canonical isomorphism

$$K_{\varphi,U'}(M^\Delta) \overset{\cong}{\to} \mathcal{T}_{\varphi,U}(M). \tag{5.22}$$

Now let M be a finitely generated projective $\mathcal{R}$-module M with continuous U-action. Then $M^* = \check{M}$ is again a finitely generated projective $\mathcal{R}$-module M with continuous U-action by Lemma 4.5.1 (i). Hence, M as well as M^Δ satisfies the assumptions of (5.12) and we have isomorphisms[7]

$$
\begin{aligned}
K_{\varphi,U}(M^\Delta)^* &\cong \mathrm{cone}\big(K^\bullet(M^\Delta)^* \overset{\varphi^*-1}{\longrightarrow} K^\bullet(M^\Delta)^*\big) \\
&= \mathrm{cone}\big(K^\bullet((M^\Delta)^*)[d] \overset{\varphi^*-1}{\longrightarrow} K^\bullet((M^\Delta)^*)[d]\big) \\
&= K_{\varphi^*,U}\big((M^*)_\Delta\big)[d+1] = K_{\psi,U}(\check{M}_\Delta)[d+1] \\
&= K_{\psi,U}(\check{M}^\Delta)[d+1].
\end{aligned}
\tag{5.23}
$$

The last isomorphism is induced by the canonical isomorphism $\check{M}^\Delta \cong \check{M}_\Delta$.

---

[7]For $X \overset{f}{\to} Y$ we have $\mathrm{cone}(f)^* \cong \mathrm{cone}(f^*)[-1]$, the isomorphism being realized by multiplying with $(-1)^i$ on $(X^*)^i$ and $\mathrm{cone}(f[n]) = \mathrm{cone}((-1)^n f)[n]$.

Now note that

$$D_{\mathrm{rig}}^{\dagger}(W)^{\vee} \cong D_{\mathrm{rig}}^{\dagger}(W^*(\chi_{\mathrm{LT}})) \tag{5.24}$$

for any L-analytic representation W by the fact that the functor $D_{\mathrm{rig}}^{\dagger}$ respects inner homs (cf. [79, Rem. 5.6] for the analogous case D_{LT}). Hence, the tautological pairing ev_2 from (5.10) together with the above isomorphism (5.23) induces the following pairing:

$$\cup_{K,\psi} : h^1\big(K_{\varphi,U'}\big(D_{\mathrm{rig}}^{\dagger}(W)^{\Delta}\big)\big) \times h^1\big(K_{\psi,U'}\big(D_{\mathrm{rig}}^{\dagger}(W^*(\chi_{\mathrm{LT}}))^{\Delta}\big)[d-1]\big) \to L.$$

Remark 5.2.16. For $U = U'$ and $M = D_{\mathrm{rig}}^{\dagger}(W)$, on the level of cochains, this pairing is given as follows:

$$\check{M} \oplus K^{d-1}(\check{M}) \times K^1(M) \oplus M \to L, \quad ((x,y),(x',y')) \mapsto \{y',x\} - y(x'),$$

where we again use that $K^{d-1}(\check{M}) \cong K^1(M)^*$ and where $\{\,,\,\}$ denotes the pairing (4.73). More generally, we have a diagram as in Figure 5.3.
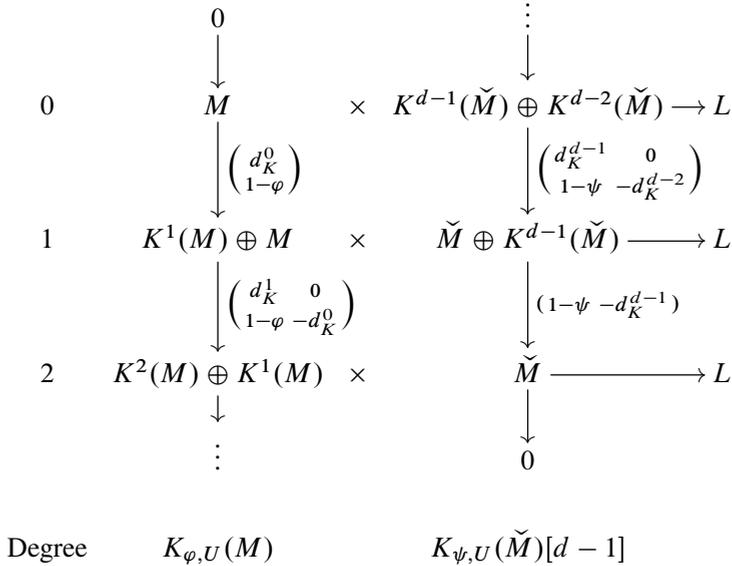
Figure 5.3. Duality for generalized Herr complexes.

Recalling that $W = V^*(1)$ is L-analytic, set $M = D_{\mathrm{rig}}^{\dagger}(W)$ and $\check{M} = D_{\mathrm{rig}}^{\dagger}(V(\tau^{-1}))$
$= D_{\mathrm{rig}}^{\dagger}(W^*(\chi_{\mathrm{LT}}))$. We obtain a Fontaine-style, explicit map

$$\mathrm{pr}_U : D_{\mathrm{rig}}^{\dagger}\big(V(\tau^{-1})\big)^{\psi=1} \to h^1\big(K_{\psi,U'}(\check{M}^{\Delta})[d-1]\big), \quad m \mapsto [(\bar{m},0)], \tag{5.25}$$

where $\bar{m} = \frac{1}{\#\Delta}\sum_{\delta\in\Delta}\delta m$ denotes the image of m under the map $\check{M} \twoheadrightarrow \check{M}_{\Delta} \cong \check{M}^{\Delta}$.

Remark 5.2.17. Let $U_1 \subseteq U$ be an open subgroup with torsion subgroups Δ_1 and Δ, respectively. Assume that the torsion-free parts U_1' and U' are generated by $\gamma_1^{p^n}, \ldots,$ $\gamma_d^{p^n}$ and $\gamma_1, \ldots, \gamma_d$, respectively. Then, for M any complete locally convex L-vector space with continuous U-action, the restriction and corestriction maps of Koszul complexes from Section 5.2.2 extend by functoriality to the mapping fiber

$$\mathrm{cor}_U^{U_1} := \mathrm{cor}_{U'}^{U'} \circ K_{\varphi, U_1'}(N_{\Delta/\Delta_1}) : K_{\varphi, U_1'}(M^{\Delta_1}) \to K_{\varphi, U'}(M^\Delta),$$

$$\mathrm{res}_{U_1}^U := K_{\varphi, U_1'}(\iota) \circ \mathrm{res}_{U_1'}^{U'} : K_{\varphi, U'}(M^\Delta) \to K_{\varphi, U_1'}(M^{\Delta_1}).$$

Here $N_{\Delta/\Delta_1} : M^{\Delta_1} \to M^\Delta$ denotes the norm/trace map sending m to $\sum_{\delta \in \Delta/\Delta_1} \delta m$, while $\iota : M^\Delta \to M^{\Delta_1}$ is the inclusion. Taking duals as in (5.23), we also obtain

$$\mathrm{cor}_U^{U_1} := (\mathrm{res}_{U_1}^U)^*[1-d] : K_{\psi, U_1'}(M^{\Delta_1}) \to K_{\psi, U'}(M^\Delta),$$

$$\mathrm{res}_{U_1}^U := (\mathrm{cor}_U^{U_1})^*[1-d] : K_{\psi, U'}(M^\Delta) \to K_{\psi, U_1'}(M^{\Delta_1})$$

(co)restriction maps for the ψ-Herr complexes.

Since inflation is compatible with restriction and corestriction, one checks that the above maps are compatible under the isomorphism (5.15) with the usual maps in Galois cohomology. Moreover, they define such maps on $H_{\dagger}^1$ and $H_{/\dagger}^1$ via (5.17) and

$$h^1\big(K_{\psi, U'}\big(D_{\mathrm{rig}}^\dagger\big(W^*(\chi_{\mathrm{LT}})\big)^\Delta[d-1]\big)\big) \cong H_{/\dagger}^1\big(L', W^*(1)\big).$$

By the discussion at the end of Section 5.2.2 the restriction map

$$K_{\varphi, U'}(M^\Delta) \xrightarrow{\mathrm{res}_{U_1}^U} K_{\varphi, U_1'}(M^{\Delta_1})$$

and corestriction map $K_{\varphi, U_1'}(M^{\Delta_1}) \xrightarrow{\mathrm{cor}_U^{U_1}} K_{\varphi, U'}(M^\Delta)$ in degree 0 are given as inclusion $M^\Delta \hookrightarrow M^{\Delta_1}$ and norm $M^{\Delta_1} \xrightarrow{N_{U', U_1'} \circ N_{\Delta/\Delta_1}} M^\Delta$, respectively, where

$$N_{U', U_1'} := \prod_{i=1}^{d} \sum_{k=0}^{p^n-1} \gamma_i^k \in \Lambda(U').$$

Hence, by duality the restriction map

$$K_{\psi, U}(\check{M}^\Delta)[d-1]^2 \xrightarrow{\mathrm{res}_{U_1}^U} K_{\psi, U_1}(\check{M}^{\Delta_1})[d-1]^2$$

and corestriction map $K_{\psi, U_1}(\check{M}^{\Delta_1})[d-1]^2 \xrightarrow{\mathrm{cor}_U^{U_1}} K_{\psi, U}(\check{M}^\Delta)[d-1]^2$ are given by

the norm $\check{M}^\Delta \xrightarrow{(\Delta:\Delta_1)\iota(N_{U',U'_1})} \check{M}^{\Delta_1}$ and projection map $\check{M}^{\Delta_1} \xrightarrow{\frac{1}{(\Delta:\Delta_1)}N_{\Delta/\Delta_1}} \check{M}^\Delta$, respectively. Here ι denotes the involution of $\Lambda(U)$ sending u to u^{-1}. Note that the latter two descriptions also hold for the first components of

$$K_{\psi,U}(\check{M}^\Delta)[d-1]^1 \xrightarrow{\mathrm{res}_{U_1}^U} K_{\psi,U_1}(\check{M}^{\Delta_1})[d-1]^1,$$

$$K_{\psi,U_1}(\check{M}^{\Delta_1})[d-1]^1 \xrightarrow{\mathrm{cor}_U^{U_1}} K_{\psi,U}(\check{M}^\Delta)[d-1]^1,$$

respectively. Hence, we obtain

$$\mathrm{cor}_U^{U_1} \circ \mathrm{pr}_{U_1} = \mathrm{pr}_U \quad \text{and} \quad \mathrm{res}_{U_1}^U \circ \mathrm{pr}_U = \mathrm{pr}_{U_1} \circ N_{\Delta/\Delta_1} \circ \iota(N_{U',U'_1}).$$

Berger and Fourquaux by contrast define a different Fontaine-style map[8] in [7, Thm. 2.5.8] for an L-analytic representation Z and $N = D_{\mathrm{rig}}^\dagger(Z)$

$$\begin{aligned} h_{L_\infty^U,Z}^1 : D_{\mathrm{rig}}^\dagger(Z)^{\psi_L=\frac{q}{\pi_L}} &\to H_{\mathrm{an}}^1\big(U, D_{\mathrm{rig}}^\dagger(Z)^{\psi_L=\frac{q}{\pi_L}}\big) \\ &\to h^1\big(\mathcal{T}_{\varphi_L,U}(N)\big) \xrightarrow{\cong} h^1\big(K_{\varphi_L,U}(N^\Delta)\big) \qquad (5.26) \\ y &\mapsto [c_b(y)] \mapsto \big[(c_b(y),-m_c)\big] \mapsto \big[(\widetilde{c_b}(y),-\widetilde{m_c})\big], \end{aligned}$$

in which the cocycle $h_{L_\infty^U,Z}^1(y)$ is given in terms of the pair $(c_b(y),-m_c)$ in the notation of loc. cit.: m_c is the unique element in $D_{\mathrm{rig}}^\dagger(Z)^{\psi_L=0}$ such that

$$(\varphi_L - 1)c_b(y)(\gamma) = (\gamma - 1)m_c \qquad (5.27)$$

for all $\gamma \in U$ and this pair defines the extension class in the sense of Lemma 5.2.11. Here, the first map is implicity given by [7, Prop. 2.5.1], the second one is the composite from maps arising in [7, Cor. 2.2.3, Thm. 2.2.4] with the natural map from analytic to continuous cohomology

$$\begin{aligned} H_{\mathrm{an}}^1\big(U, D_{\mathrm{rig}}^\dagger(Z)^{\psi_L=\frac{q}{\pi_L}}\big) &\to H_{\mathrm{an}}^1\big(U \times \Psi, D_{\mathrm{rig}}^\dagger(Z)\big) \\ &\cong H_{\mathrm{an}}^1\big(U \times \Phi, D_{\mathrm{rig}}^\dagger(Z)\big) \to H^1\big(U \times \Phi, D_{\mathrm{rig}}^\dagger(Z)\big) \end{aligned}$$

---

[8]We do not know whether this map coincides with the following composite we had used in older versions and which uses the shuffle maps from Proposition 5.2.9:

$$\begin{aligned} h_{L_\infty^U,Z}^1 : D_{\mathrm{rig}}^\dagger(Z)^{\psi_L=\frac{q}{\pi_L}} &\to H_{\mathrm{an}}^1\big(U, D_{\mathrm{rig}}^\dagger(Z)^{\psi_L=\frac{q}{\pi_L}}\big) \\ &\to h^1\big(\mathcal{T}_{\frac{\pi}{q}\psi_L,U}^{\mathrm{an}}(N)\big) \cong h^1\big(\mathcal{T}_{\varphi_L,U}^{\mathrm{an}}(M)\big) \\ &\to h^1\big(\mathcal{T}_{\varphi_L,U}(N)\big) \cong H_\dagger^1(L_\infty^U, Z). \end{aligned}$$

Here the second map is stemming from the spectral sequence (5.14), the third from Proposition 5.2.9, the fourth is the natural map (5.13), and the last one is (5.17).

combined with the interpretation of extension classes (see [7, §1.4] and Lemma 5.2.11), and the last one is (5.22) (the concrete image $(\widetilde{c_b}(y), -\widetilde{m_c})$ will be of interest for us only in the situation where Δ is trivial, when $\widetilde{m_c} = m_c$).

According to [7, Prop. 2.5.6, Rem. 2.5.7], this map also satisfies

$$\mathrm{cor}_{U'}^{U} \circ h_{L_\infty^{U'}, Z}^1 = h_{L_\infty^U, Z}^1. \tag{5.28}$$

Since $D_{\mathrm{rig}}^\dagger(V(\tau^{-1}))^\vee \cong D_{\mathrm{rig}}^\dagger(V^*(1))$ by (5.24), concerning the Iwasawa pairing, we have the following.

Proposition 5.2.18. *For a G_L-representation V such that $V^*(1)$ is L-analytic the following diagram consisting of $D(\Gamma_L, K)$-ι_*-sesquilinear pairings (in the sense of (4.87)) is commutative:*

$$h^1\big(K_{\varphi,U'}\big(D_{\mathrm{rig}}^\dagger\big(V^*(1)(\chi_{\mathrm{LT}}^j)\big)^\Delta\big)\big) \times h^1\big(K_{\psi,U'}\big(D_{\mathrm{rig}}^\dagger\big(V(\chi_{\mathrm{LT}}^{-j})\big)^\Delta\big)[d-1]\big) \xrightarrow{\cup_{K,\psi}} L \subseteq \mathbb{C}_p$$

$$\Big\uparrow{\scriptstyle h^1_{L,V^*(1)(\chi_{\mathrm{LT}}^j)}\,{}^{\mathrm{otw}}\,\chi_{\mathrm{LT}}^j} \qquad\qquad \Big\uparrow{\scriptstyle \mathrm{pr}_U\,{}^{\mathrm{otw}}\,\chi_{\mathrm{LT}}^{-j}} \qquad\qquad \Big\uparrow{\scriptstyle \mathrm{ev}_{\chi_{\mathrm{LT}}^{-j}}}$$

$$D_{\mathrm{rig}}^\dagger\big(V^*(1)\big)^{\psi_L=\frac{q}{\pi_L}} \qquad \times \qquad D_{\mathrm{rig}}^\dagger\big(V(\tau^{-1})\big)^{\psi_L=1} \xrightarrow{\frac{q-1}{q}\{\,,\,\}_{\mathrm{Iw},\Gamma_L}} D(\Gamma_L, \mathbb{C}_p)$$

taking $U' \times \Delta = U = \Gamma_L$.

Proof. By Lemma 4.5.22, it suffices to show the case $j = 0$, i.e., the trivial character χ_{triv}. Furthermore, it suffices to show the following statement for any subgroup of the form Γ_n without any p-torsion:

$$q^{-n}\,\mathrm{ev}_{L_n, \chi_{\mathrm{triv}|\Gamma_n}} \circ \{x, y\}_{\mathrm{Iw}, \Gamma_n} = h_{L_n, V^*(1)}^1(x) \cup_{K,\psi} \mathrm{pr}_{\Gamma_n}(y) \tag{5.29}$$

for $x \in D_{\mathrm{rig}}^\dagger(V^*(1))^{\psi_L=\frac{q}{\pi_L}}$, $y \in D_{\mathrm{rig}}^\dagger(V(\tau^{-1}))^{\psi_L=1}$.

Indeed, by Remark 5.2.17, for every such n, we have the commutative diagram

$$h^1\big(K_{\varphi,\Gamma_n}\big(D_{\mathrm{rig}}^\dagger(V^*(1))\big)\big) \qquad \times \qquad h^1\big(K_{\psi,\Gamma_n}\big(D_{\mathrm{rig}}^\dagger(V)\big)[d-1]\big) \xrightarrow{\cup_{K,\psi}} L$$

$$\Big\downarrow{\scriptstyle \mathrm{cor}} \qquad\qquad\qquad\qquad\qquad \Big\uparrow{\scriptstyle \mathrm{res}} \qquad\qquad\qquad \Big\Vert$$

$$h^1\big(K_{\varphi,U'}\big(D_{\mathrm{rig}}^\dagger(V^*(1))^\Delta\big)\big) \qquad \times \qquad h^1\big(K_{\psi,U'}\big(D_{\mathrm{rig}}^\dagger(V)^\Delta\big)[d-1]\big) \xrightarrow{\cup_{K,\psi}} L.$$

Hence, we obtain using (5.28)

$$h_{L',V^*(1)}^1(x) \cup_{K,\psi} \mathrm{pr}_U(y) = \big(\mathrm{cor} \circ h_{L_n,V^*(1)}^1(x)\big) \cup_{K,\psi} \mathrm{pr}_U(y)$$

$$= h_{L_n,V^*(1)}^1(x) \cup_{K,\psi} \big(\mathrm{res} \circ \mathrm{pr}_U(y)\big)$$

$$= h_{L_n,V^*(1)}^1(x) \cup_{K,\psi} \big(\mathrm{pr}_{\Gamma_n}\big(N_\Delta \circ \iota(N_{U',\Gamma_n})y\big)\big),$$

where we use Remark 5.2.17 for the last equality. On the other hand, one easily checks that[9]

$$\mathrm{ev}_{L_n, \chi_{\mathrm{triv}|\Gamma_n}} \circ \mathrm{pr}_{U, \Gamma_n} \circ N_\Delta \circ \iota(N_{U', \Gamma_n}) = \mathrm{ev}_{L, \chi_{\mathrm{triv}}} : D(U, \mathbb{C}_p) \to \mathbb{C}_p,$$

whence

$$
\begin{aligned}
\mathrm{ev}_{L, \chi_{\mathrm{triv}}} \circ \tfrac{q-1}{q}\{x, y\}_{\mathrm{Iw}, U} &= \tfrac{q-1}{q}\, \mathrm{ev}_{L_n, \chi_{\mathrm{triv}|\Gamma_n}} \circ \mathrm{pr}_{U, \Gamma_n} \left(N_\Delta \circ \iota(N_{U', \Gamma_n})\{x, y\}_{\mathrm{Iw}, U} \right) \\
&= \tfrac{q-1}{q}\, \mathrm{ev}_{L_n, \chi_{\mathrm{triv}|\Gamma_n}} \circ \mathrm{pr}_{U, \Gamma_n} \left(\{x, N_\Delta \circ \iota(N_{U', \Gamma_n}) y\}_{\mathrm{Iw}, U} \right) \\
&= \tfrac{q-1}{q} [U : \Gamma_n]^{-1}\, \mathrm{ev}_{L_n, \chi_{\mathrm{triv}|\Gamma_n}} \circ \{x, N_\Delta \circ \iota(N_{U', \Gamma_n}) y\}_{\mathrm{Iw}, \Gamma_n} \\
&= q^{-n}\, \mathrm{ev}_{L_n, \chi_{\mathrm{triv}|\Gamma_n}} \circ \{x, N_\Delta \circ \iota(N_{U', \Gamma_n}) y\}_{\mathrm{Iw}, \Gamma_n},
\end{aligned}
$$

where we have used Remark 4.5.21 for the last equation.

In order to prove (5.29), choose $n = n_0$ (see Section 4.3.4). As recalled in (5.26), the map

$$h^1_{L_{n_0}, V^*(1)} : D^\dagger_{\mathrm{rig}}\big(V^*(1)\big)^{\psi_L = \frac{q}{\pi_L}} \to h^1\big(K_{\varphi_L, \Gamma_{n_0}}\big(D^\dagger_{\mathrm{rig}}\big(V^*(1)\big)\big)\big)$$

is given by the cocycle $h^1_{L_{n_0}, V^*(1)}(x)$ in terms of the pair $(\widetilde{c_b}(x), -m_c)$. Note that we have

$$m_c = \widehat{\Xi}_b(\varphi_L - 1)x.$$

Indeed, by [7, Thm. 2.5.8] we have $c_b(x)(b_j^k) = (b_j^k - 1)\widehat{\Xi}_b x$ for all $j, k \geq 0$, which together with (5.27) and the uniqueness of m_c (cf. loc. cit.) implies the claim. On the other hand, we have the map (5.25)

$$\mathrm{pr}_{\Gamma_{n_0}} : D^\dagger_{\mathrm{rig}}\big(V(\tau^{-1})\big)^{\psi_L = 1} \to h^1\big(K_{\psi, \Gamma_{n_0}}\big(D^\dagger_{\mathrm{rig}}\big(V(\tau^{-1})\big)\big)[d-1]\big)$$

$$y \mapsto \text{class of } (y, 0).$$

Thus, the pairing $\cup_{K, \psi}$ sends by construction (see diagram (5.3)) the above classes to

$$
\begin{aligned}
h^1_{L_n, V^*(1)}(x) \cup_{K, \psi} \mathrm{pr}_{\Gamma_n}(y) &= 0\big(\widetilde{c_b}(x)\big) + \big\{ -\widehat{\Xi}_b(\varphi_L - 1)x, y \big\} \\
&= \big\{ \widehat{\Xi}_b(\varphi_L - 1)x, \big(\tfrac{\pi_L}{q}\varphi_L - 1\big)y \big\} \\
&= \big\langle \widehat{\Xi}_b, \{x, y\}_{\mathrm{Iw}, \Gamma_n} \big\rangle_{\Gamma_n} \\
&= q^{-n}\mathrm{aug}\big(\{x, y\}_{\mathrm{Iw}, \Gamma_n}\big).
\end{aligned}
$$

Here the second equality holds due to Lemma 4.5.1 because the left-hand side belongs to $D^\dagger_{\mathrm{rig}}(V^*(1))^{\psi_L = 0}$, the third one is (4.89), and the last one comes from (4.81). ∎

---

[9]This is obvious if you decompose $D(U, \mathbb{C}_p) = \bigoplus_{g \in U/\Gamma_n} D(\Gamma_n, \mathbb{C}_p)g$ with respect to the inverses of the representatives used in the definition of $N_\Delta \circ \iota(N_{U', \Gamma_n})$.

Proposition 5.2.19. *For W an L-analytic representation we have a canonical commutative diagram*

$$\cup_{K,\psi} : \; h^1\big(K_{\varphi,U'}(D_{\mathrm{rig}}^\dagger(W)^\Delta)\big) \times h^1\big(K_{\psi,U'}\big(D_{\mathrm{rig}}^\dagger\big(W^*(\chi_{\mathrm{LT}})\big)^\Delta\big)[d-1]\big) \to L$$

$$b \Big\downarrow \cong \qquad\qquad\qquad a \Big\downarrow \cong \qquad\qquad \Big\|$$

$$\langle\,,\,\rangle_{\mathrm{Tate},L,\dagger} : \qquad H_\dagger^1(L',W) \qquad \times \qquad H_{/\dagger}^1(L',W^*(1)) \to H^2(L',L(1)) \cong L$$

$$\Big\downarrow \qquad\qquad\qquad\qquad \mathrm{pr}\Big\Uparrow \qquad\qquad \Big\|$$

$$\langle\,,\,\rangle_{\mathrm{Tate},L'} : \qquad H^1(L',W) \qquad \times \qquad H^1\big(L',W^*(1)\big) \to H^2\big(L',L(1)\big) \cong L.$$

Moreover, the isomorphism a is compatible with the middle map of the diagram (A.5).

Proof. The lower square of pairings comes from Tate duality as in Proposition 5.2.14 and (5.19). Its commutativity holds by definition. In the upper square of pairings the left upper vertical isomorphism b arises from (5.17) combined with (5.22), while the middle vertical isomorphism a is uniquely determined as adjoint of the latter because both pairings are non-degenerate: the middle one by definition of $H_{/\dagger}^1$ while the upper one due to Corollary A.4 (ii) with $W = V^*(1)$. Therefore, one immediately checks that $a^{-1} \circ \mathrm{pr}$ is induced by the cohomology of the middle map (going down) in diagram (A.5) (being the same as the middle map (going from right to left) of diagram (A.6) upon identifying $h^1(K_{\psi,U'}^\bullet(D(V(\tau^{-1})^\Delta)[d-1])$ and $H^1(L',V)$ by the isomorphism described there). $\blacksquare$

Combining the last two propositions, we get the following result.

Corollary 5.2.20. *For a G_L-representation V such that $V^*(1)$ is L-analytic the following diagram is commutative:*

$$H_\dagger^1\big(L,V^*(1)(\chi_{\mathrm{LT}}^j)\big) \quad\times\quad H_{/\dagger}^1\big(L,V(\chi_{\mathrm{LT}}^{-j})\big) \xrightarrow{\;\langle\,,\,\rangle_{\mathrm{Tate},L,\dagger}\;} H^2\big(L,L(1)\big) \cong L \subseteq \mathbb{C}_p$$

$$\Big\uparrow h_{L,V^*(1)}^1 {\scriptstyle\mathrm{otw}}\chi_{\mathrm{LT}}^j \qquad\qquad \Big\uparrow \mathrm{pr}_L {\scriptstyle\mathrm{otw}}\chi_{\mathrm{LT}}^{-j} \qquad\qquad\qquad \Big\uparrow {\scriptstyle\mathrm{ev}}\chi_{\mathrm{LT}}^{-j}$$

$$D_{\mathrm{rig}}^\dagger\big(V^*(1)\big)^{\psi_L=\frac{q}{\pi_L}} \quad\times\quad D_{\mathrm{rig}}^\dagger\big(V(\tau^{-1})\big)^{\psi_L=1} \xrightarrow{\;\frac{q-1}{q}\{\,,\,\}_{\mathrm{Iw}}\;} D(\Gamma_L,\mathbb{C}_p).$$

Remark 5.2.21. For applications it might be useful to renormalize $\cup_{K,\psi}$ by the factor $\frac{q}{q-1}$, i.e., setting $\cup_{K,\psi}' := \frac{q}{q-1}\cup_{K,\psi}$. Then we would get rid of the factor $\frac{q-1}{q}$ in front of the Iwasawa pairing in the above results. Moreover and more importantly, the new normalization would be compatible with the cyclotomic situation taking $L = \mathbb{Q}_p$, $\pi_L = p = q$; i.e., the upper pairing in Proposition 5.2.18 would coincide – at least up to a sign – with the cup product pairing of Galois cohomology

$$H^1\big(\mathbb{Q}_p,V^*(j+1)\big) \quad\times\quad H^1\big(\mathbb{Q}_p,V(-j)\big) \longrightarrow H^2\big(\mathbb{Q}_p,\mathbb{Q}_p(1)\big) \xrightarrow[\cong]{\;\mathrm{inv}\;} \mathbb{Q}_p,$$

using Tate's trace map

$$\mathrm{inv} : H^2\big(\mathbb{Q}_p,\mathbb{Q}_p(1)\big) \cong \mathbb{Q}_p.$$

given by class field theory, if one chooses $Z = \gamma - 1$ for a topological generator γ of $\Gamma_{\mathbb{Q}_p}$ satisfying $\log \chi_{\mathrm{cyc}}(\gamma) = 1$. This follows from [3, Prop. 1.3.4, Thm. 2.2.6], [38, Thm. 5.2, Rem. 5.3], and [42, Rem. 2.3.11/12]: they claim that $-\frac{p-1}{p}\mathrm{inv}$ corresponds to the trace map from the second cohomology group of the φ-Herr complex induced by sending $f \otimes \eta$ to $\frac{1}{\log \chi_{\mathrm{cyc}}(\gamma)}\mathrm{res}_{\omega_{\mathrm{cyc}}}(f \frac{d\omega_{\mathrm{cyc}}}{1+\omega_{\mathrm{cyc}}})$.

With respect to evaluating at a character, we have the following analogue of Corollary 5.2.20.

Proposition 5.2.22. *For a G_L-representation V such that $V^*(1)$ is L-analytic the following diagram is commutative:*

$$
\begin{array}{ccc}
\mathbb{C}_p \otimes_L D_{\mathrm{cris},L}\big(V^*(1)(\chi_{\mathrm{LT}}^j)\big) \;\times\; \mathbb{C}_p \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})(\chi_{\mathrm{LT}}^{-j})\big) & \xrightarrow{[\,,\,]_{\mathrm{cris}}} & \mathbb{C}_p \otimes_L D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}})\big) \\
{\scriptstyle \mathrm{ev}_{\chi_{\mathrm{LT}}^{-j}} \otimes t_{\mathrm{LT}}^{-j}\eta^{\otimes j}} \Big\uparrow & {\scriptstyle \mathrm{ev}_{\chi_{\mathrm{LT}}^{j}} \otimes t_{\mathrm{LT}}^{j}\eta^{\otimes -j}} \Big\uparrow & {\scriptstyle \mathrm{ev}_{\chi_{\mathrm{LT}}^{-j}}}\Big\uparrow \\
D(\Gamma_L,\mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big) \times D(\Gamma_L,\mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) & \xrightarrow{[\,,\,]^0} & D(\Gamma_L,\mathbb{C}_p),
\end{array}
$$

where in the right upper corner we identify $\mathbb{C}_p \otimes_L D_{\mathrm{cris},L}(L(\chi_{\mathrm{LT}})) \cong \mathbb{C}_p$ by choosing $t_{\mathrm{LT}}^{-1} \otimes \eta$ as a basis.

Proof. Using Lemma 5.2.23 below, the statement is reduced to $j = 0$. Evaluation of (5.9) implies the claim in this case. ∎

Lemma 5.2.23. *There is a commutative diagram*

$$
\begin{array}{ccc}
D(\Gamma_L,\mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}\big(V^*(1)(\chi_{\mathrm{LT}}^j)\big) \times D(\Gamma_L,\mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})(\chi_{\mathrm{LT}}^{-j})\big) & \xrightarrow{[\,,\,]^0} & D(\Gamma_L,\mathbb{C}_p) \\
{\scriptstyle \mathrm{Tw}_{\chi_{\mathrm{LT}}^{-j}} \otimes t_{\mathrm{LT}}^{-j}\eta^{\otimes j}} \Big\uparrow & {\scriptstyle \mathrm{Tw}_{\chi_{\mathrm{LT}}^{j}} \otimes t_{\mathrm{LT}}^{j}\eta^{\otimes -j}} \Big\uparrow & {\scriptstyle \mathrm{Tw}_{\chi_{\mathrm{LT}}^{-j}}}\Big\uparrow \\
D(\Gamma_L,\mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big) \quad\times\quad D(\Gamma_L,\mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) & \xrightarrow{[\,,\,]^0} & D(\Gamma_L,\mathbb{C}_p).
\end{array}
$$

Proof. The claim follows immediately from (5.9), the compatibility of the usual D_{cris}-pairing with twists, and the fact that $\mathrm{Tw}_{\chi_{\mathrm{LT}}^{j}}(\lambda\iota_*(\mu)) = \mathrm{Tw}_{\chi_{\mathrm{LT}}^{j}}(\lambda)\iota_*(\mathrm{Tw}_{\chi_{\mathrm{LT}}^{-j}}(\mu))$ holds. ∎

5.2.4 The interpolation formula for the regulator map

In this subsection, we are going to prove the interpolation property for $\mathcal{L}_V$. First, recall that we introduced in Section 3.1 the notation $D_{\mathrm{dR},L'}(V) := (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_{L'}}$. Since B_{dR} contains the algebraic closure $\overline{L}$ of L, we have the isomorphism

$$
B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L) \otimes_L V \xrightarrow{\cong} \prod_{\sigma \in G_{\mathbb{Q}_p}/G_L} B_{\mathrm{dR}} \otimes_{\sigma,L} V
$$

which sends $b \otimes v$ to $(b \otimes v)_\sigma$. The tensor product in the factor $B_{\mathrm{dR}} \otimes_{\sigma,L} V$ is formed with respect to L acting on B_{dR} through σ. With respect to the G_L-action, the right-hand side decomposes according to the double cosets in $G_L \backslash G_{\mathbb{Q}_p}/G_L$. It follows, in

particular, that $D_{dR}^{id}(V):=(B_{dR}\otimes_L V)^{G_L}$ is a direct summand of $D_{dR,L}(V)$ and we denote by pr^{id} the corresponding projection. Similarly, $\tan_{L,id}(V):=(B_{dR}/B_{dR}^+\otimes_L V)^{G_L}$ is a direct summand of $\tan_L(V):=(B_{dR}\otimes_L V)^{G_L}$. More generally, also the filtration $D_{dR,L}^i(V)$ decomposes into direct summands.

According to [80, Appx. A], the dual Bloch–Kato exponential map is uniquely determined by the commutativity of the following diagram, in which all pairings are perfect:

$$
\begin{array}{ccccccc}
H^1(L',W) & \times & H^1(L',W^*(1)) & \xrightarrow{\langle\,,\,\rangle_{\mathrm{Tate},L'}} & & & L \\
\downarrow{\scriptstyle\exp_{L',W}^*} & & \uparrow{\scriptstyle\exp_{L',W^*(1)}} & & & & \downarrow \\
D_{dR,L'}^0(W) & \times & \tan_{L'}(W^*(1)) & \longrightarrow & D_{dR,L'}(\mathbb{Q}_p(1)) & \xrightarrow{\cong} & L' \\
\downarrow & & \uparrow{\scriptstyle pr} & & \| & & \| \\
D_{dR,L'}(W) & \times & D_{dR,L'}(W^*(1)) & \longrightarrow & D_{dR,L'}(\mathbb{Q}_p(1)) & \xrightarrow{\cong} & L'.
\end{array}
$$

Note that we have the compatibility of the following pairings:

$$
\begin{array}{ccccccc}
D_{cris,L}(V^*(1)) & \times & D_{cris,L}(V(\tau^{-1})) & \xrightarrow{[\,,\,]_{cris}} & D_{cris,L}(L(\chi_{LT})) & \xrightarrow{\cong} & L \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \| \\
D_{dR}^{id}(V^*(1)) & \times & D_{dR}^{id}(V(\tau^{-1})) & \xrightarrow{[\,,\,]_{dR}^{id}} & D_{dR}^{id}(L(\chi_{LT})) & \xrightarrow{\cong} & L \\
\downarrow{\scriptstyle\text{injective}} & & {\scriptstyle pr^{id}}\downarrow{\scriptstyle\text{surjective}} & & & & \\
D_{dR}(V^*(1)) & \times & D_{dR}(V(\tau^{-1})) \longrightarrow D_{dR}(L(\chi_{LT})) & \xrightarrow{\cong} & L\otimes_{\mathbb{Q}_p}L & & \\
\|{\scriptstyle=} & & \uparrow{\scriptstyle id\otimes t_{\mathbb{Q}_p}t_{LT}^{-1}\otimes\eta\otimes(\eta_{cyc})^{-1}} & & & pr^{id} & \\
D_{dR}(V^*(1)) & \times & D_{dR}(V) & \xrightarrow{[\,,\,]_{dR}} & D_{dR}(\mathbb{Q}_p(1)) & \xrightarrow{\cong} & L
\end{array}
$$

(cf. [80, (57) in the appendix]). Therefore, in the Lubin–Tate setting we can also consider the dual of the identity component $\exp_{L',W^*(1),id}$ of $\exp_{L',W^*(1)}$:

$$
\begin{array}{ccccccc}
H^1(L',W) & \times & H^1(L',W^*(1)) & \xrightarrow{\langle\,,\,\rangle_{\mathrm{Tate},L'}} & & & L \\
\downarrow{\scriptstyle\widetilde{\exp}_{L',W,id}^*} & & \uparrow{\scriptstyle\exp_{L',W^*(1),id}} & & & & \downarrow \\
D_{dR,L'}^{id,0}(W(\tau^{-1})) & \times & \tan_{L',id}(W^*(1)) & \longrightarrow & D_{dR,L'}^{id}(L(\chi_{LT})) & \xrightarrow{\cong} & L' \qquad(5.30) \\
\downarrow & & \uparrow{\scriptstyle pr} & & \| & & \| \\
D_{dR,L'}^{id}(W(\tau^{-1})) & \times & D_{dR,L'}^{id}(W^*(1)) & \longrightarrow & D_{dR,L'}^{id}(L(\chi_{LT})) & \xrightarrow{\cong} & L'.
\end{array}
$$

Upon noting that under the identifications

$$D_{\mathrm{dR},L'}\big(\mathbb{Q}_p(1)\big) \cong L' \quad \text{and} \quad D^{\mathrm{id}}_{\mathrm{dR},L'}\big(\mathbb{Q}_p(1)\big) \cong L'$$

the elements $t_{\mathbb{Q}_p} \otimes \eta_{\mathrm{cyc}}$ and $t_{\mathrm{LT}} \otimes \eta$ are sent to 1, one easily checks that if $W^*(1)$ is L-analytic, whence the inclusion $\tan_{L',\mathrm{id}}(W^*(1)) \subseteq \tan_{L'}(W^*(1))$ is an equality and $\exp_{L',W^*(1),\mathrm{id}} = \exp_{L',W^*(1)}$, then it holds that

$$\mathbb{T}_{\tau^{-1}} \circ \exp^*_{L',W} = \widetilde{\exp}^*_{L',W,\mathrm{id}},$$

where

$$\mathbb{T}_{\tau^{-1}} : D^0_{\mathrm{dR},L'}(W) \to D^{\mathrm{id},0}_{\mathrm{dR},L'}\big(W(\tau^{-1})\big)$$

is the isomorphism which sends $b \otimes v$ to $b\frac{t_{\mathbb{Q}_p}}{t_{\mathrm{LT}}} \otimes v \otimes \eta \otimes (\eta_{\mathrm{cyc}})^{\otimes -1}$; note that $\frac{t_{\mathbb{Q}_p}}{t_{\mathrm{LT}}} \in (B^+_{\mathrm{dR}})^{\times}$, whence the filtration is preserved.

Now let W be an L-analytic, crystalline L-linear representation of G_L. Recall that $\eta = (\eta_n)_n$ denotes a fixed generator of T_π and that the map

$$\mathrm{tw}_{\chi^j_{\mathrm{LT}}} : D^{\dagger}_{\mathrm{rig}}(W) \to D^{\dagger}_{\mathrm{rig}}\big(W(\chi^j_{\mathrm{LT}})\big)$$

has been defined before Lemma 4.5.22. For D_{cris} the twisting map

$$D_{\mathrm{cris},L}(W) \xrightarrow{-\otimes e_j} D_{\mathrm{cris},L}\big(W(\chi^j_{\mathrm{LT}})\big)$$

sends d to $d \otimes e_j$ with the element e_j given as $e_j = t_{\mathrm{LT}}^{-j} \otimes \eta^{\otimes j} \in D_{\mathrm{cris},L}(L(\chi^j_{\mathrm{LT}}))$.

If we assume, in addition, that

(i) W has Hodge–Tate weights ≤ 0, whence $W^*(1)$ has Hodge–Tate weights ≥ 1 and $D^0_{\mathrm{dR},L}(W^*(1)) = 0$, and

(ii) $D_{\mathrm{cris},L}(W^*(\chi_{\mathrm{LT}}))^{\varphi_L = \frac{q}{\pi_L}} = 0$,

then by our assumption $\exp_{L,W^*(1)} : D_{\mathrm{dR},L}(W^*(1)) \hookrightarrow H^1(L,W^*(1))$ is injective with image $H^1_e(L,W^*(1)) = H^1_f(L,W^*(1))$ (see [12, Cor. 3.8.4]). We denote its inverse by

$$\log_{L,W^*(1)} : H^1_f\big(L,W^*(1)\big) \to D_{\mathrm{dR},L}\big(W^*(1)\big)$$

and define

$$\widetilde{\log}_{L,W^*(1)} : H^1_f\big(L,W^*(1)\big) \to D_{\mathrm{dR},L}\big(W^*(1)\big) \xrightarrow{\mathbb{T}_{\tau^{-1}}} D_{\mathrm{cris},L}\big(W^*(\chi_{\mathrm{LT}})\big),$$

where (by abuse of notation) we also write

$$\mathbb{T}_{\tau^{-1}} : D_{\mathrm{dR},L}\big(W^*(1)\big) \to D^{\mathrm{id}}_{\mathrm{dR},L}\big(W^*(\chi_{\mathrm{LT}})\big) = D_{\mathrm{cris},L}\big(W^*(\chi_{\mathrm{LT}})\big)$$

for the isomorphism, which sends $b \otimes v$ to $b \frac{t_{\mathbb{Q}_p}}{t_{\mathrm{LT}}} \otimes v \otimes \eta \otimes (\eta_{\mathrm{cyc}})^{\otimes -1}$. We obtain the following commutative diagram, which defines the dual map $\log^*_{L,W}$ being inverse to $\exp^*_{L,W}$ (up to factorization over $H^1(L,W)/H^1_f(L,W)$):

$$
\begin{array}{ccccc}
H^1(L,W)/H^1_f(L,W) & \times & H^1_f\big(L,W^*(1)\big) & \xrightarrow{\ \langle\,,\,\rangle_{\mathrm{Tate},L}\ } & L \\[2pt]
\log^*_{L,W}\big\uparrow & & \log_{L,W^*(1)}\big\downarrow & & \big\| \\[2pt]
D_{\mathrm{dR},L}(W) & \times & D_{\mathrm{dR},L}\big(W^*(1)\big) \longrightarrow D_{\mathrm{dR},L}\big(\mathbb{Q}_p(1)\big) & \xrightarrow{\ \cong\ } & L.
\end{array}
$$

Similarly as above, we obtain a commutative diagram more convenient for the Lubin–Tate setting:

$$
\begin{array}{ccccc}
H^1(L,W)/H^1_f(L,W) & \times & H^1_f\big(L,W^*(1)\big) & \xrightarrow{\ \langle\,,\,\rangle_{\mathrm{Tate},L}\ } & L \\[2pt]
\log^*_{L,W,\mathrm{id}}\big\uparrow & & \widetilde{\log}_{L,W^*(1)}\big\downarrow & & \big\| \\[2pt]
D^{\mathrm{id}}_{\mathrm{dR},L}(W) & \times & D^{\mathrm{id}}_{\mathrm{dR},L}\big(W^*(\chi_{\mathrm{LT}})\big) \longrightarrow D^{\mathrm{id}}_{\mathrm{dR},L}\big(L(\chi_{\mathrm{LT}})\big) & \xrightarrow{\ \cong\ } & L.
\end{array}
$$

We write $\mathrm{Ev}_{W,n} : \mathcal{O}_L \otimes_L D_{\mathrm{cris},L}(W) \to L_n \otimes_L D_{\mathrm{cris},L}(W)$ for the composite map $\partial_{D_{\mathrm{cris},L}(W)} \circ \varphi_q^{-n}$ from the introduction of [7], which actually sends $f(Z) \otimes d$ to $f(\eta_n) \otimes \varphi_L^{-n}(d)$. By abuse of notation we also use $\mathrm{Ev}_{W,0}$ for the analogous map

$$
\mathcal{O}_K \otimes_L D_{\mathrm{cris},L}(W) \to K \otimes_L D_{\mathrm{cris},L}(W).
$$

For $x \in D(\Gamma_L, K) \otimes_L D_{\mathrm{cris},L}(W)$ we denote by $x(\chi^j_{\mathrm{LT}})$ the image under the map

$$
D(\Gamma_L, K) \otimes_L D_{\mathrm{cris},L}(W) \to K \otimes_L D_{\mathrm{cris},L}(W), \quad \lambda \otimes d \mapsto \lambda(\chi^j_{\mathrm{LT}}) \otimes d.
$$

Lemma 5.2.24. *Assume that Ω is contained in K. Then there are commutative diagrams*

$$
\begin{array}{ccccc}
D(\Gamma_L,K) \otimes_L D_{\mathrm{cris},L}(W) & \xrightarrow{\mathfrak{M}\otimes\mathrm{id}} & \mathcal{O}_K \otimes_L D_{\mathrm{cris},L}(W) & \xleftarrow{1-\varphi_L\otimes\varphi_L} & \mathcal{O}_K \otimes_L D_{\mathrm{cris},L}(W) \\[2pt]
\mathrm{ev}_{\mathrm{triv}}\big\downarrow & & \mathrm{Ev}_{W,0}\big\downarrow & & \mathrm{Ev}_{W,0}\big\downarrow \\[2pt]
K \otimes_L D_{\mathrm{cris},L}(W) & = & K \otimes_L D_{\mathrm{cris},L}(W) & \xleftarrow{1-\mathrm{id}\otimes\varphi_L} & K \otimes_L D_{\mathrm{cris},L}(W)
\end{array}
$$

and

$$
\begin{array}{ccccc}
D(\Gamma_L,K) \otimes_L D_{\mathrm{cris},L}(W) & \xrightarrow{\mathfrak{M}\otimes\mathrm{id}} & \mathcal{O}_K \otimes_L D_{\mathrm{cris},L}(W) & \xleftarrow{1-\varphi_L\otimes\varphi_L} & \mathcal{O}_K \otimes_L D_{\mathrm{cris},L}(W) \\[2pt]
\mathrm{Tw}_{\chi^{-j}_{\mathrm{LT}}}\otimes e_j\big\downarrow & & (\frac{\partial}{\partial\Omega})^{-j}\otimes e_j\big\downarrow & & (\frac{\partial}{\partial\Omega})^{-j}\otimes e_j\big\downarrow \\[2pt]
D(\Gamma_L,K) \otimes_L D_{\mathrm{cris},L}\big(W(\chi^j_{\mathrm{LT}})\big) & \xrightarrow{\mathfrak{M}\otimes\mathrm{id}} & \mathcal{O}_K \otimes_L D_{\mathrm{cris},L}\big(W(\chi^j_{\mathrm{LT}})\big) & \xleftarrow{1-\varphi_L\otimes\varphi_L} & \mathcal{O}_K \otimes_L D_{\mathrm{cris},L}\big(W(\chi^j_{\mathrm{LT}})\big).
\end{array}
$$

In the latter we follow the (for $j > 0$) abusive notation ∂^{-j} from [7, Rem. 3.5.5].

Proof. For the upper diagram note that $\eta_0 = 0$ and $(\delta_g \cdot \eta(1, Z))_{|Z=0} = 1$, from which the claim follows for Dirac distributions, whence in general. For the right square we observe that $\varphi_L(g(Z))_{|Z=0} = g(0)$. Regarding the lower diagram, we use Lemma 4.3.25 and the relation $\partial_{\mathrm{inv}} \circ \varphi_L = \pi_L \varphi_L \circ \partial$. ∎

With this notation, Berger's and Fourquaux's interpolation property reads as follows.

Theorem 5.2.25 (Berger–Fourquaux [7, Thm. 3.5.3]). *Let W be L-analytic and $h \geq 1$ such that $\mathrm{Fil}^{-h} D_{\mathrm{cris},L}(W) = D_{\mathrm{cris},L}(W)$. For any $f \in (\mathcal{O}^{\psi=0} \otimes_L D_{\mathrm{cris},L}(W))^{\Delta=0}$ and $y \in (\mathcal{O} \otimes_L D_{\mathrm{cris},L}(W))^{\psi=\frac{q}{\pi_L}}$ with $f = (1 - \varphi_L)y$, we have: If $h + j \geq 1$, then*

$$h^1_{L_n, W(\chi^j_{\mathrm{LT}})}\big(\mathrm{tw}_{\chi^j_{\mathrm{LT}}}\big(\Omega_{W,h}(f)\big)\big)$$
$$= (-1)^{h+j-1}(h + j - 1)!$$
$$\times \begin{cases} \exp_{L_n, W(\chi^j_{\mathrm{LT}})}\big(q^{-n}\mathrm{Ev}_{W(\chi^j_{\mathrm{LT}}),n}(\partial^{-j}_{\mathrm{inv}} y \otimes e_j)\big) & \text{if } n \geq 1; \\ \exp_{L, W(\chi^j_{\mathrm{LT}})}\big((1 - q^{-1}\varphi_L^{-1})\mathrm{Ev}_{W(\chi^j_{\mathrm{LT}}),0}(\partial^{-j}_{\mathrm{inv}} y \otimes e_j)\big) & \text{if } n = 0. \end{cases}$$

If $h + j \leq 0$, then[10]

$$- \exp^*_{L_n, W(\chi^j_{\mathrm{LT}})}\big(h^1_{L_n, W(\chi^j_{\mathrm{LT}})}\big(\mathrm{tw}_{\chi^j_{\mathrm{LT}}}\big(\Omega_{W,h}(f)\big)\big)\big)$$
$$= \frac{1}{(-h - j)!} \begin{cases} q^{-n}\mathrm{Ev}_{W(\chi^j_{\mathrm{LT}}),n}(\partial^{-j}_{\mathrm{inv}} y \otimes e_j) & \text{if } n \geq 1; \\ (1 - q^{-1}\varphi_L^{-1})\mathrm{Ev}_{W(\chi^j_{\mathrm{LT}}),0}(\partial^{-j}_{\mathrm{inv}} y \otimes e_j) & \text{if } n = 0. \end{cases}$$

By abuse of notation, we shall denote the base change $K \otimes_L -$ of the (dual) Bloch–Kato exponential map by the same expression. Using Lemma 5.2.24, we deduce the following interpolation property for the modified big exponential map with $x \in D(\Gamma_L, K) \otimes_L D_{\mathrm{cris},L}(W)$: If $j \geq 0$, then

$$h^1_{L, W(\chi^j_{\mathrm{LT}})}\big(\mathrm{tw}_{\chi^j_{\mathrm{LT}}}\big(\Omega_{W,1}(x)\big)\big)$$
$$= (-1)^j j! \Omega^{-j-1} \exp_{L, W(\chi^j_{\mathrm{LT}})}\big((1 - q^{-1}\varphi_L^{-1})(1 - \varphi_L)^{-1}\big(x(\chi^{-j}_{\mathrm{LT}}) \otimes e_j\big)\big); \tag{5.31}$$

if $j < 0$, then

$$h^1_{L, W(\chi^j_{\mathrm{LT}})}\big(\mathrm{tw}_{\chi^j_{\mathrm{LT}}}\big(\Omega_{W,1}(f)\big)\big)$$
$$= \frac{-\Omega^{-j-1}}{(-1 - j)!} \log^*_{L, W(\chi^j_{\mathrm{LT}})}\big((1 - q^{-1}\varphi_L^{-1})(1 - \varphi_L)^{-1}\big(x(\chi^{-j}_{\mathrm{LT}}) \otimes e_j\big)\big),$$

---

[10]Note that the definition of the dual exponential map in [7] coincides with (5.30) only up to reversing the order of the cohomology groups in degree one, which induces a sign due to the skew symmetry of the cup product.

assuming in both cases that the operator $1 - \varphi_L$ is invertible on $D_{\mathrm{cris},L}(W(\chi_{\mathrm{LT}}^j))$ and for $j < 0$ also that the operator $1 - q^{-1}\varphi_L^{-1}$ is invertible on $D_{\mathrm{cris},L}(W(\chi_{\mathrm{LT}}^j))$ (in order to grant the existence of $\log_{L,W(\chi_{\mathrm{LT}}^j)}$).

Recall that the generalized Iwasawa cohomology of $T \in \mathrm{Rep}_{o_L}(G_L)$ is defined by

$$H_{\mathrm{Iw}}^*(L_\infty/L, T) := \varprojlim_{L'} H^*(L', T),$$

where L' runs through the finite Galois extensions of L contained in L_∞ and the transition maps in the projective system are the cohomological corestriction maps. For $V := T \otimes_{o_L} L \in \mathrm{Rep}_L(G_L)$ we define

$$H_{\mathrm{Iw}}^*(L_\infty/L, V) := H_{\mathrm{Iw}}^*(L_\infty/L, T) \otimes_{o_L} L,$$

which is independent of the choice of T. As usual, we denote by

$$\mathrm{cor} : H_{\mathrm{Iw}}^*(L_\infty/L, T) \to H^*(L', T)$$

the projection map and analogously for rational coefficients. In the same way as in (5.25), we have a map

$$\mathrm{pr}_U : D\big(V(\tau^{-1})\big)^{\psi=1} \to h^1\big(K_{\psi,U'}\big(D\big(V(\tau^{-1})\big)^\Delta\big)[d-1]\big) \cong H^1(L', V)$$
$$m \mapsto \big[(\bar{m}, 0)\big],$$

where $\bar{m} = \frac{1}{\#\Delta} \sum_{\delta \in \Delta} \delta m$ denotes the image of m under the map $\check{M} \twoheadrightarrow \check{M}_\Delta \cong \check{M}^\Delta$. Note that under the assumptions of Lemma 3.3.6 for $V(\tau^{-1})$, there is a commutative diagram

$$\begin{array}{ccccc}
H_{\mathrm{Iw}}^1(L_\infty/L, T) & \overset{\cong}{\longrightarrow} & D_{\mathrm{LT}}\big(T(\tau^{-1})\big)^{\psi=1} & \hookrightarrow & D_{\mathrm{rig}}^\dagger\big(V(\tau^{-1})\big)^{\psi=1} \\
{\scriptstyle \mathrm{cor}}\downarrow & & {\scriptstyle \mathrm{pr}_U}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_U} \\
H^1(L', V) & =\!\!=\!\!= & H^1(L', V) & \longrightarrow\!\!\!\!\!\rightarrow & H_{/\dagger}^1(L', V),
\end{array} \qquad (5.32)$$

where the right vertical map is induced by (5.25). Indeed, for the commutativity of the left rectangle and the right rectangle we refer the reader to (B.5) and (A.6), respectively. Let $y_{\chi_{\mathrm{LT}}^{-j}}$ denote the image of y under the map

$$H_{\mathrm{Iw}}^1(L_\infty/L, T) \xrightarrow{\cdot \otimes \eta^{\otimes -j}} H_{\mathrm{Iw}}^1\big(L_\infty/L, T(\chi_{\mathrm{LT}}^{-j})\big) \xrightarrow{\mathrm{cor}} H^1\big(L, T(\chi_{\mathrm{LT}}^{-j})\big)$$
$$\to H^1\big(L, V(\chi_{\mathrm{LT}}^{-j})\big).$$

The following result generalizes [51, Thm. A.2.3] and [52, Thm. B.5] from the cyclotomic case.

Theorem 5.2.26. *Assume that $V^*(1)$ is L-analytic and fulfills*

$$\mathrm{Fil}^{-1} D_{\mathrm{cris},L}(V^*(1)) = D_{\mathrm{cris},L}(V^*(1)),$$

$$D_{\mathrm{cris},L}(V^*(1))^{\varphi_L = \pi_L^{-1}} = D_{\mathrm{cris},L}(V^*(1))^{\varphi_L = 1} = 0.$$

Then, for all $y \in H^1_{\mathrm{Iw}}(L_\infty/L, T)$, it holds that for $j \geq 0$

$$\Omega^j \mathbf{L}_V(y)(\chi_{\mathrm{LT}}^j)$$

$$= -j!\big((1 - \pi_L^{-1}\varphi_L^{-1})^{-1}(1 - \tfrac{\pi_L}{q}\varphi_L)\widetilde{\exp}^*_{L,V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-j}})\big) \otimes e_j$$

$$= -j!(1 - \pi_L^{-1-j}\varphi_L^{-1})^{-1}(1 - \tfrac{\pi_L^{j+1}}{q}\varphi_L)\big(\widetilde{\exp}^*_{L,V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-j}}) \otimes e_j\big)$$

and for $j \leq -1$

$$\Omega^j \mathbf{L}_V(y)(\chi_{\mathrm{LT}}^j)$$

$$= \frac{(-1)^{j+1}}{(-1-j)!}\big((1 - \pi_L^{-1}\varphi_L^{-1})^{-1}(1 - \tfrac{\pi_L}{q}\varphi_L)\widetilde{\log}_{L,V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-j}})\big) \otimes e_j \qquad (5.33)$$

$$= \frac{(-1)^{j+1}}{(-1-j)!}(1 - \pi_L^{-1-j}\varphi_L^{-1})^{-1}(1 - \tfrac{\pi_L^{j+1}}{q}\varphi_L)\big(\widetilde{\log}_{L,V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-j}}) \otimes e_j\big)$$

if the operators $1 - \pi_L^{-1-j}\varphi_L^{-1}$, $1 - \tfrac{\pi_L^{j+1}}{q}\varphi_L$ or equivalently $1 - \pi_L^{-1}\varphi_L^{-1}$, $1 - \tfrac{\pi_L}{q}\varphi_L$ are invertible on $D_{\mathrm{cris},L}(V(\tau^{-1}))$ and $D_{\mathrm{cris},L}(V(\tau^{-1}\chi_{\mathrm{LT}}^j))$, respectively.

Proof. From the reciprocity formula in Corollaries 5.2.2 and 5.2.20 and Proposition 5.2.22 we obtain for $x \in D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}(V^*(1))$, $y \in D(V(\tau^{-1}))^{\psi_L = 1}$ and $j \geq 0$ using (5.32)

$$\big[x(\chi_{\mathrm{LT}}^{-j}) \otimes e_j, (-1)^j \mathbf{L}_V(y)(\chi_{\mathrm{LT}}^j) \otimes e_{-j}\big]_{\mathrm{cris}}$$

$$= \Omega\big[x, \tfrac{\sigma_{-1}\mathbf{L}_V(y)}{\Omega}\big]^0(\chi_{\mathrm{LT}}^{-j})$$

$$= \Omega\tfrac{q-1}{q}\{\mathbf{\Omega}_{V^*(1),1}(x), y\}_{\mathrm{Iw}}(\chi_{\mathrm{LT}}^{-j})$$

$$= \Omega\langle h^1_L \circ \mathrm{tw}_{\chi_{\mathrm{LT}}^j}(\mathbf{\Omega}_{V^*(1),1}(x)), y_{\chi_{\mathrm{LT}}^{-j}}\rangle_{\mathrm{Tate}}$$

$$= \Omega\langle(-1)^j j!\Omega^{-j-1}\exp_{L,V^*(1)(\chi_{\mathrm{LT}}^j)}\big((1 - q^{-1}\varphi_L^{-1})(1 - \varphi_L)^{-1}(x(\chi_{\mathrm{LT}}^{-j}) \otimes e_j)\big), y_{\chi_{\mathrm{LT}}^{-j}}\rangle_{\mathrm{Tate}}$$

$$= -(-1)^j \Omega^{-j} j!\big[(1 - q^{-1}\varphi_L^{-1})(1 - \varphi_L)^{-1}(x(\chi_{\mathrm{LT}}^{-j}) \otimes e_j), \widetilde{\exp}^*_{L,V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-j}})\big]_{\mathrm{cris}}$$

$$= \big[x(\chi_{\mathrm{LT}}^{-j}) \otimes e_j, -(-1)^j \Omega^{-j} j!(1 - \pi_L^{-1}\varphi_L^{-1})^{-1}(1 - \tfrac{\pi_L}{q}\varphi_L)\widetilde{\exp}^*_{L,V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-j}})\big]_{\mathrm{cris}}.$$

Here we used (5.31) in the fourth equation for the interpolation property of $\mathbf{\Omega}_{V^*(1),1}$. The fifth equation is the defining equation for the dual exponential map resulting from (5.30) upon taking the skew-symmetry of the cup product into account. Furthermore, for the last equality we use that $\pi_L^{-1}\varphi_L^{-1}$ is adjoint to φ_L under the lower pairing. The claim follows since the evaluation map is surjective and $[\,,\,]_{\mathrm{cris}}$ is non-degenerate.

Now assume that $j < 0$:

$$
\begin{aligned}
&\big[x(\chi_{\mathrm{LT}}^{-j}) \otimes e_j, (-1)^j \mathbf{L}_V(y)(\chi_{\mathrm{LT}}^j) \otimes e_{-j}\big]_{\mathrm{cris}}\\
&= \Omega\big[x, \tfrac{\sigma_{-1}\mathbf{L}_V(y)}{\Omega}\big]^0 (\chi_{\mathrm{LT}}^{-j})\\
&= \Omega\tfrac{q-1}{q}\big\{\boldsymbol{\Omega}_{V^*(1),1}(x), y\big\}_{\mathrm{Iw}}(\chi_{\mathrm{LT}}^{-j})\\
&= \Omega\big\langle h_L^1 \circ \mathrm{tw}_{\chi_{\mathrm{LT}}^j}(\boldsymbol{\Omega}_{V^*(1),1}(x)), y_{\chi_{\mathrm{LT}}^{-j}}\big\rangle_{\mathrm{Tate}}\\
&= \Omega\big\langle \tfrac{-\Omega^{-j-1}}{(-1-j)!}\log^*_{L,W(\chi_{\mathrm{LT}}^j)}\big((1-q^{-1}\varphi_L^{-1})(1-\varphi_L)^{-1}\big(x(\chi_{\mathrm{LT}}^{-j})\otimes e_j\big)\big), y_{\chi_{\mathrm{LT}}^{-j}}\big\rangle_{\mathrm{Tate}}\\
&= \tfrac{-\Omega^{-j}}{(-1-j)!}\big[(1-q^{-1}\varphi_L^{-1})(1-\varphi_L)^{-1}\big(x(\chi_{\mathrm{LT}}^{-j})\otimes e_j\big), \widetilde{\log}_{L,V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-j}})\big]_{\mathrm{cris}}\\
&= \big[x(\chi_{\mathrm{LT}}^{-j})\otimes e_j, \tfrac{-\Omega^{-j}}{(-1-j)!}(1-\pi_L^{-1}\varphi_L^{-1})^{-1}\big(1-\tfrac{\pi_L}{q}\varphi_L\big)\widetilde{\log}_{L,V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-j}})\big]_{\mathrm{cris}}. \quad\blacksquare
\end{aligned}
$$

Now consider $V = L(\tau\chi_{\mathrm{LT}})$ and $W = V(\chi_{\mathrm{LT}})$. Then the latter satisfies the condition of Theorem 5.2.26 and using Proposition 5.1.1 and Lemma 5.1.2 one easily derives the following interpolation property concerning the former for $y = \kappa(u)$, $u \in U$ for all $r \geq 1$:

$$
\begin{aligned}
\mathbf{L}_V(y)(\chi_{\mathrm{LT}}^r) &= -\tfrac{r!}{\Omega^r}\big((1-\pi_L^{-1}\varphi_L^{-1})^{-1}\big(1-\tfrac{\pi_L}{q}\varphi_L\big)\widetilde{\exp}^*_{L,V(\chi_{\mathrm{LT}}^{-r}),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-r}})\big) \otimes e_r\\
&= -\tfrac{r!}{\Omega^r}(1-\pi_L^{-r})^{-1}\big(1-\tfrac{\pi_L^r}{q}\big)\widetilde{\exp}^*_{L,V(\chi_{\mathrm{LT}}^{-r}),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-r}}) \otimes e_r.
\end{aligned}
$$

On the other hand, we have $\mathbf{L}_V(y) \otimes \mathbf{d}_1 = \mathscr{L}_V(y)$ and hence by the claim concerning (5.4)

$$
\mathbf{L}_V(y)(\chi_{\mathrm{LT}}^r) \otimes \mathbf{d}_1 \otimes \eta^{-1} \otimes t_{\mathrm{LT}} = \mathscr{L}_V(y)(\chi_{\mathrm{LT}}^r) \otimes \eta^{-1} \otimes t_{\mathrm{LT}} = \mathscr{L}(u \otimes \eta^{-1})(\chi_{\mathrm{LT}}^r),
$$

whence

$$
\mathscr{L}(-u \otimes \eta^{-1})(\chi_{\mathrm{LT}}^r) \otimes e_{1-r} = \tfrac{r!}{\Omega^r}(1-\pi_L^{-r})^{-1}\big(1-\tfrac{\pi_L^r}{q}\big)\exp^*_{L,V(\chi_{\mathrm{LT}}^{-r}),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-r}}).
$$

This is (5.2); i.e., together with (5.3) we have just obtained a new proof of Kato's reciprocity law (Theorem 5.1.3) and we may consider Theorem 5.2.26 as a vast generalization of it. In [66], the general version of Kato's reciprocity will be derived from a refined version of Theorem 5.2.26 describing also the interpolation at Artin characters.

Appendix A

Cup products and local Tate duality

The aim of this subsection is to discuss cup products and to prove Proposition 5.2.19. We fix some open subgroup $U \subseteq \Gamma_L$ and let $L' = L_\infty^U$. Note that we obtain a decomposition $U \cong \Delta \times U'$ with a subgroup $U' \cong \mathbb{Z}_p^d$ of U and Δ the torsion subgroup of U.

Lemma A.1. *Let M_0 be a complete linearly topologized o_L-module with continuous U-action and with a continuous U-equivariant endomorphism f. Then there is a canonical quasi-isomorphism*

$$\mathcal{T}_{f,U}(M_0)\left[\tfrac{1}{\pi_L}\right] \simeq K^\bullet_{f,U'}\left(M_0\left[\tfrac{1}{\pi_L}\right]^\Delta\right).$$

If M_0 is an L-vector space, the inversion of π_L can be omitted on both sides.

Proof. Let $\mathcal{C}^\bullet_n(U, M_0) \subseteq \mathcal{C}^\bullet(U, M_0)$ denote the subcomplex of normalized cochains. Since Δ is finite, [87, Thm. 3.7.6] gives a canonical quasi-isomorphism:

$$\mathcal{C}^\bullet_n(U, M_0) = \mathcal{C}^\bullet_n(\Delta \times U', M_0) \xrightarrow{\simeq} \mathcal{C}^\bullet_n\big(\Delta, \mathcal{C}^\bullet_n(U', M_0)\big).$$

Here we understand the above objects in the sense of hypercohomology as total complexes of the obvious double complexes involved. After inverting π_L, we may compute the right-hand side further as

$$
\begin{array}{ccc}
\mathcal{C}^\bullet_n\big(\Delta, \mathcal{C}^\bullet_n(U', M_0)\big)\left[\tfrac{1}{\pi_L}\right] & & \mathcal{C}^\bullet_n\big(U', M_0^\Delta\big)\left[\tfrac{1}{\pi_L}\right] \\
\| & & \| \\
\mathcal{C}^\bullet_n\big(\Delta, \mathcal{C}^\bullet_n(U', M_0)\left[\tfrac{1}{\pi_L}\right]\big) & \xleftarrow{\;\simeq\;} & \mathcal{C}^\bullet_n\big(U', M_0\big)\left[\tfrac{1}{\pi_L}\right]^\Delta.
\end{array}
$$

Here the middle quasi-isomorphism comes from the fact that a finite group has no cohomology in characteristic zero. The right-hand equality is due to the fact that Δ acts on the cochains through its action on M_0. Altogether we obtain a natural quasi-isomorphism

$$\mathcal{C}^\bullet_n(U, M_0)\left[\tfrac{1}{\pi_L}\right] \cong \mathcal{C}^\bullet_n\big(U', M_0^\Delta\big)\left[\tfrac{1}{\pi_L}\right].$$

By using [87, Prop. 3.3.3] we may replace the normalized cochains again by general cochains obtaining the left-hand quasi-isomorphism in

$$\mathcal{C}^\bullet(U, M_0)\left[\tfrac{1}{\pi_L}\right] \cong \mathcal{C}^\bullet\big(U', M_0^\Delta\big)\left[\tfrac{1}{\pi_L}\right] \cong K^\bullet_{U'}\big(M_0^\Delta\big)\left[\tfrac{1}{\pi_L}\right] = K^\bullet_{U'}\big(M_0\left[\tfrac{1}{\pi_L}\right]^\Delta\big).$$

The middle quasi-isomorphism is (5.20). The claim follows by taking mapping fibers of the attached map $f - 1$ of complexes. ∎

Proposition A.2. *Let M be a φ_L-module over $\mathcal{R} = \mathcal{R}_K$, Definition 4.3.3, and $c \in K^\times$. Then $M/(\psi - c)(M)$ is finite dimensional over K.*

Proof. (The proof follows closely the proof of [42, Prop. 3.3.2] in the cyclotomic situation.) We set $\psi_c := c^{-1}\psi$ and show that $M/(\psi_c - 1)(M)$ is finite dimensional over K.

Choose a model $M^{[r_0,1)}$ of M with $1 > r_0 > p^{\frac{-1}{(q-1)e}}$ and put $r = r_0^{\frac{1}{q^2}}$. Recall that for all $1 > s \geq r$ we have maps $M^{[s,1)} \xrightarrow{\psi_c - 1} M^{[s,1)}$ (where strictly speaking we mean ψ_c followed by the corresponding restriction). We first show that it suffices to prove that $\operatorname{coker}(M^{[r,1)} \xrightarrow{\psi_c - 1} M^{[r,1)})$ has finite dimension over K. Indeed, given any $m \in M$ we have $m \in M^{[s,1)}$ for some $1 > s \geq r$. Then there exists $k \geq 0$ such that $r \geq s^{q^k} \geq r_0$, whence $\psi_c^k(m)$ belongs to $M^{[r,1)}$ and represents the same class in $M/(\psi_c - 1)(M)$ as m.

Choose a basis $\mathbf{e}'_1, \ldots, \mathbf{e}'_n$ of $M^{[r_0,1)}$ and take $\mathbf{e}_i := \varphi(\mathbf{e}'_i) \in M^{[r^q,1)}$; by the φ-module property the latter elements also form a basis of $M^{[r^q,1)}$. Note that by base change these two bases also give rise to bases in $M^{[s,1)}$ for $1 > s \geq r^q$. Thus, we find a matrix F' with entries in $\mathcal{R}^{[r^q,1)}$ such that $\mathbf{e}_j = \sum_i F'_{ij}\mathbf{e}'_i$ and we put $F = \varphi(F')$ with entries in $\mathcal{R}^{[r,1)}$, i.e., $\varphi(\mathbf{e}_j) = \sum F_{ij}\mathbf{e}_i$. Similarly, let G be a matrix with values in $\mathcal{R}^{[r^q,1)} \subseteq \mathcal{R}^{[r,1)}$ such that $\mathbf{e}'_j = \sum_i G_{ij}\mathbf{e}_i$ and hence $\mathbf{e}_j = \varphi(\sum_i G_{ij}\mathbf{e}_i)$.

We identify $M^{[r,1)}$ with $(\mathcal{R}^{[r,1)})^n$ by sending $(\lambda_i)_i$ to $\sum_i \lambda_i\mathbf{e}_i$ and endow it for each $r \leq s < 1$ with the norm given by $\max_i |\lambda_i|_s$. Note that then the "semilinear" map ψ_c (followed by the corresponding restriction) on $(\mathcal{R}^{[r,1)})^n$ is given by the matrix G as follows from the projection formula (4.40):

$$\psi_c\Big(\sum_j \lambda_j\mathbf{e}_j\Big) = \sum_j \psi_c\Big(\lambda_j\varphi\Big(\sum_i G_{ij}\mathbf{e}_i\Big)\Big) = \sum_{i,j} \psi_c(\lambda_j)G_{ij}\mathbf{e}_i.$$

Moreover, the restriction of $\varphi : M^{[r,1)} \to M^{[r^{\frac{1}{q}},1)}$ to $\sum_I \mathcal{O}_K(\mathbf{B})\mathbf{e}_i$ becomes the semilinear map $(\mathcal{O}_K(\mathbf{B}))^n \to (\mathcal{R}^{[r,1)})^n$ given by the matrix F.

Consider, for I any subset of the reals $\mathbf{R}$, the K-linear map $P_I : \mathcal{R}^{[r,1)} \to \mathcal{R}^{[r,1)}$, $\sum a_i Z^i \mapsto \sum_{i \in \mathbf{Z} \cap I} a_i Z^i$. We then introduce K-linear operators P_I and Q_k, $k \geq 0$, on $M^{[r,1)}$ by

$$P_I\big((\lambda_i)_i\big) := \big(P_I(\lambda_i)\big)_i, \quad Q_k := P_{(-\infty,-k)} - \tfrac{c\pi_L}{q}\varphi \circ P_{(k,\infty)},$$

i.e.,

$$Q_k\big((\lambda_i)_i\big) = \big(P_{(-\infty,-k)}(\lambda_i)\big)_i - \tfrac{c\pi_L}{q}F \cdot \big(\varphi\big(P_{(k,\infty)}(\lambda_i)\big)\big)_i,$$

because $P_{(k,\infty)}$ factorizes through $\mathcal{O}_K(\mathbf{B})$. Note then that the K-linear operator $\Psi_k := \mathrm{id} - P_{[-k,k]} + (\psi_c - 1)Q_k$ of $M^{[r,1)}$ satisfies

$$\Psi_k = \psi_c \circ P_{(-\infty,-k)} + \tfrac{c\pi_L}{q}\varphi \circ P_{(k,\infty)},$$

i.e.,

$$\Psi_k\big((\lambda_i)_i\big) = G \cdot \big(\psi_c\big(P_{(-\infty,-k)}(\lambda_i)\big)\big)_i + F \cdot \big(\tfrac{c\pi_L}{q}\varphi\big(P_{(k,\infty)}(\lambda_i)\big)\big)_i,$$

whence its operator norm satisfies

$$\|\Psi_k\|_s \leq \max\left\{\|G\|_s\|\psi_c \circ P_{(-\infty,-k)}\|_s, \frac{c\pi_L}{q}\|F\|_s\|\varphi \circ P_{(k,\infty)}\|_s\right\}.$$

It is easy to check that, for $1 > s > q^{\frac{-1}{q-1}}$, we have $\|\varphi \circ P_{(k,\infty)}\|_s \leq |Z|_s^{(q-1)k} = s^{(q-1)k}$ (using the norm relation after (4.28)) and $\|\psi_c \circ P_{(-\infty,-k)}\|_s \leq C_s s^{k(1-q^{-1})}$ for some constant $C_s > 0$. For example, for the latter we have for $\lambda = \sum_i a_i Z^i \in \mathcal{R}^{[r,1)}$

$$
\begin{aligned}
\left|\sum_{i<-k} \psi_c(a_i Z^i)\right|_s &\leq \sup_{i<-k} |a_i| \left|\psi_c(Z^i)\right|_s \\
&\leq \sup_{i<-k} |a_i| C_s s^{\frac{i}{q}} \\
&\leq C_s \sup_{i<-k} |a_i| |Z|_s^i s^{i(q^{-1}-1)} \\
&\leq C_s |\lambda|_s s^{-k(q^{-1}-1)},
\end{aligned}
$$

where we use that by continuity of ψ_c there exists C_s such that

$$\left|\psi_c(Z^i)\right|_s \leq C_s |Z^i|_{s^{\frac{1}{q}}} = C_s s^{\frac{i}{q}}.$$

Thus, we may and do choose k sufficiently big such that $\|\Psi_k\|_r \leq \frac{1}{2}$. Given $m_0 \in M^{[r,1)}$ we define inductively $m_{i+1} := \Psi_k(m_i)$. This sequence obviously converges to zero with respect to the r-Gauss norm. We shall show below that also for all $s \in (r^{\frac{1}{q}}, 1)$ the series $(m_i)_i$ tends to zero with respect to the Gauss norm $| \ |_s$; i.e., by cofinality the sum $m := \sum_{i\geq 0} m_i$ converges in $M^{[r,1)}$ for the Fréchet topology and satisfies

$$m - m_0 = m - P_{[-k,k]}(m) + (\psi_c - 1)Q_k(m);$$

i.e., $P_{[-k,k]}(m)$ represents the same class as m_0 in $M^{[r,1)}/(\psi_c - 1)(M)$. Since the image of $P_{[-k,k]}$ has finite dimension, the proposition follows, once we have shown the following.

Claim. For all $s \in (r^{\frac{1}{q}}, 1)$ we have

$$\left|\Psi_k(m)\right|_s \leq \max\left\{\frac{1}{2}|m|_s, C_s\|G\|_s\left(\frac{s^{\frac{1}{q}}}{r}\right)^{-k}|m|_r, \left|\frac{c\pi_L}{q}\right|\|F\|_s\left(\frac{s^q}{r}\right)^{k'}|m|_r\right\}.$$

Indeed, we fix such s and may choose $k' \geq k$ such that $\|\Psi_{k'}\|_s \leq \frac{1}{2}$. Then $\Psi_k = \Psi_{k'} - \psi_c \circ P_{[-k',-k)} - \frac{c\pi_L}{q}\varphi \circ P_{(k,k']}$, whence the claim as for $\lambda \in \mathcal{R}^{[r,1)}$

$$\left|\psi_c \circ P_{[-k',-k)}(\lambda)\right|_s \leq C_s\left(\frac{s^{\frac{1}{q}}}{r}\right)^{-k}|\lambda|_r$$

$$\left|\varphi \circ P_{(k,k']}(\lambda)\right|_s \leq \left(\frac{s^q}{r}\right)^{k'}|\lambda|_r$$

by similar estimations as above.   ∎

Remark A.3. This result answers the expectation from [7, Rem. 2.3.7] positively.

Corollary A.4. *Let $V^*(1)$ be L-analytic and $M := D^\dagger_{\mathrm{rig}}(V^*(1))$.*

(i)   *The cohomology group $h^2(K^\bullet_{\psi,U'}((M)^\Delta)[d-1])$ is finite dimensional over L.*

(ii) *We have isomorphisms*

$$h^1\big(K^\bullet_{\psi,U'}\big(D^\dagger_{\mathrm{rig}}\big(V(\tau^{-1})\big)^\Delta\big)[d-1]\big)^* \cong h^1\big(K^\bullet_{\varphi,U'}(M^\Delta)\big) \cong H^1_\dagger\big(L', V^*(1)\big),$$

$$h^2\big(K^\bullet_{\psi,U'}\big(D^\dagger_{\mathrm{rig}}\big(V(\tau^{-1})\big)^\Delta\big)[d-1]\big)^* \cong h^0\big(K^\bullet_{\varphi,U'}(M^\Delta)\big) = \big(V^*(1)\big)^{G_{L'}}.$$

Proof. (i) Since $h^2(K^\bullet_{\psi,U'}((M)^\Delta)[d-1])$ is a quotient of $(M/(\psi-1)(M))^\Delta$ by (5.3), this follows from the proposition. (ii) We are in the situation of Remark 5.2.6 (i) with regard to $\mathcal{C} = K^\bullet_{\psi,U'}(D^\dagger_{\mathrm{rig}}(V(\tau^{-1}))^\Delta)[d-1]$ and $i = 2, 3$ in the notation of the remark. Indeed, regarding $i = 3$ we have $h^3 = \mathcal{C}^4 = 0$ by construction; regarding $i = 2$ we have $\mathcal{C}^3 = 0$ by construction and h^2 is finite by (i). Hence, the first isomorphism follows in both cases from (5.23) using the reflexivity of M. The second isomorphisms arise by Lemma A.1 together with (5.17) and (5.16), respectively. ∎

We quickly discuss the analogues of some results in [19, §I.6]. First, we remind the reader of some definitions: $\widetilde{\mathbf{A}} := W(\mathbb{C}_p^\flat)_L$,

$$\widetilde{\mathbf{A}}^\dagger := \Big\{ x = \sum_{n \geq 0} \pi_L^n [x_n] \in \widetilde{\mathbf{A}} : |\pi_L^n| |x_n|_\flat^r \xrightarrow{n \to \infty} 0 \text{ for some } r > 0 \Big\},$$

$\mathbf{A}^\dagger := \widetilde{\mathbf{A}}^\dagger \cap \mathbf{A}$ and $\mathbf{A}^\dagger_L := (\mathbf{A}^\dagger)^{H_L}$.[1]

Remark A.5. There is also the following more concrete description for $A^\dagger_L$ in terms of Laurent series in ω_{LT}:

$$A^\dagger_L = \big\{ F(\omega_{\mathrm{LT}}) \in \mathbf{A}_L \mid F(Z) \text{ converges on } \rho \leq |Z| < 1 \text{ for some } \rho \in (0, 1) \big\} \subseteq \mathbf{A}_L.$$

Indeed, this follows from the analogue of [18, Lem. II.2.2] upon noting that the latter holds with and without the integrality condition: "$rv_p(a_n) + n \geq 0$ for all $n \in \mathbb{Z}$"

---

[1]In the literature, one also finds the subring $\widetilde{\mathbf{A}}^\dagger_{\leq 1} := \bigcup_{r > 0} W^r_{\leq 1}(\mathbb{C}_p^\flat)_L$ of $\widetilde{\mathbf{A}}^\dagger$, where $W^r_{\leq 1}(\mathbb{C}_p^\flat)_L = \{x \in W^r(\mathbb{C}_p^\flat)_L \mid |x|_r \leq 1\}$ consists of those $x \in \widetilde{\mathbf{A}}$ such that $|\pi_L^n| |x_n|_\flat^r \xrightarrow{n \to \infty} 0$ and $|\pi_L^n| |x_n|_\flat^r \leq 1$ for all n. Denoting by $\widetilde{\mathbf{A}}^{\dagger,s}_{\mathrm{St}}$ the ring defined in [83, Def. 3.4], we have the equality $W^r_{\leq 1}(\mathbb{C}_p^\flat)_L = \widetilde{\mathbf{A}}^{\dagger,\frac{q-1}{qr}}_{\mathrm{St}}$ corresponding to $\widetilde{\mathbf{A}}^\dagger_{(\frac{q-1}{qr})^-}$ in the notation of [18, §II.1] for $q = p$. For these relations use the following normalizations compatible with [83]: $|\pi_L| = \frac{1}{q}$, $v_{\mathbb{C}_p^\flat}(\omega) = \frac{q}{q-1}$, $v_{\pi_L}(\pi_L) = 1$, $|x|_\flat = q^{-v_{\mathbb{C}_p^\flat}}$, $|\omega|_\flat = q^{-\frac{q}{q-1}} = |\pi_L|^{\frac{q}{q-1}}$, where $\omega = \omega_{\mathrm{LT}}$ mod π_L as in Section 3.1. Furthermore, $|x|_r = |\pi_L|^{V(x, \frac{q-1}{rq})}$ and $|\omega_{\mathrm{LT}}|_r = |\pi_L|^{\frac{rq}{q-1}} = |\omega|_\flat^r$, where $V(x, r) := \inf_k (v_{\mathbb{C}_p^\flat}(x_k) \frac{q-1}{rq} + k)$ for $x = \sum_{k \geq 0} \pi_L^k [x_k] \in \widetilde{\mathbf{A}}$. For $x \in \widetilde{\mathbf{A}}^\dagger$ we have $V(x, r) = \frac{q-1}{rq} V_{\mathrm{St}}(x, r)$, where $V_{\mathrm{St}}(x, r)$ uses the notation in [83]. Note also that $\omega_{\mathrm{LT}}^{-1}$ is contained in $W^{\frac{q-1}{q}}_{\leq 1}(\widehat{L_\infty^\flat})_L$ by [83, Lem. 3.10] (in analogy with [18, Cor. II.1.5]).

(for $r \in \bar{\mathbf{R}} \setminus \mathbf{R}$) in the notation of that article.[2] In particular, we obtain canonical embeddings $\mathbf{A}_L^\dagger \subseteq \mathbf{B}_L^\dagger \hookrightarrow \mathcal{R}_L$ of rings.

Now consider the subring

$$A = \mathbf{A}_L^+ \left[\!\left[\tfrac{\pi_L}{Z^{q-1}} \right]\!\right] = \left\{ x = \sum_k a_k Z^k \in \mathbf{A}_L \mid v_{\pi_L}(a_k) \geq -\tfrac{k}{q-1} \right\} \subseteq \mathbf{A}_L.$$

For $x \in \mathbf{A}_L$ and each inter $n \geq 0$, we define $w_n(x)$ to be the smallest integer $k \geq 0$ such that $x \in Z^{-k} A + \pi_L^{n+1} \mathbf{A}_L$. It satisfies $w_n(x+y) \leq \max\{w_n(x), w_n(y)\}$ and $w_n(xy) \leq w_n(x) + w_n(y)$ (since A is a ring) and $w_n(\varphi(x)) \leq q w_n(x)$ (use that $\tfrac{\varphi(Z)}{Z^q} \in A^\times$, whence $\varphi(Z^{-k})A = Z^{-qk}A$). Set for $n \geq 2$, $m \geq 0$ the integers $r(n) :=$ $(q-1)q^{n-1}$, $l(m,n) = m(q-1)(q^{n-1}-1) = m(r(n)-(q-1))$ and define $\mathbf{A}_L^{\dagger,n} =$ $\{x = \sum_k a_k Z^k \in \mathbf{A}_L | v_{\pi_L}(a_k) + \tfrac{k}{r(n)} \to \infty$ for $k \to -\infty\}$. Then, by Remark A.5 and Footnote 2, we obtain that $\mathbf{A}_L^\dagger = \bigcup_{n \geq 2} \mathbf{A}_L^{\dagger,n}$.

Lemma A.6. *Let* $x = \sum_k a_k Z^k \in \mathbf{A}_L$ *and* $l \geq 0$, $n \geq 2$. *Then*

(i)   *we have*

$$w_m(x) \leq l \Leftrightarrow v_{\pi_L}(a_k) \geq \min\left\{m+1, -\tfrac{k+l}{q-1}\right\} \quad \text{for } k < -l.$$

(ii) $x \in \mathbf{A}_L^{\dagger,n}$ *if and only if* $w_m(x) - l(m,n)$ *goes to* $-\infty$ *when* m *runs to* ∞.

Item (ii) of the lemma is an analogue of [19, Prop. III 2.1 (ii)] for $\mathbf{A}_L^{\dagger,n}$ instead of

$$\mathbf{A}_{L,\leq 1}^{\dagger,n} = \left\{ x = \sum_k a_k Z^k \in \mathbf{A}_L^{\dagger,n} \mid v_{\pi_L}(a_k) + \tfrac{k}{r(n)} \geq 0 \text{ for all } k \leq 0 \right\}.$$

Proof. (i) follows from the fact that $x \in Z^{-l} A$ if and only if $v_{\pi_L}(a_k) \geq -\tfrac{k+l}{q-1}$ for $k < -l$. (ii) Let $M, N = M(q-1) \gg 0$ be arbitrary huge integers and assume first that $x \in \mathbf{A}_L^{\dagger,n}$. Then

$$w_m(x) - l(m,n) \leq -N \tag{A.1}$$

is equivalent to

$$v_{\pi_L}(a_k) \geq \min\left\{m+1, -\tfrac{k+l(m,n)-N}{q-1}\right\} \quad \text{for } k < -l(m,n) + N \tag{A.2}$$

[2]This description does not require any completeness property! A similar result holds for $\mathbf{A}_{\leq 1,L}^\dagger$ when requiring for the Laurent series in ω_{LT} in addition that $F(Z)$ takes values on $\rho \leq |Z| < 1$ of norm at most 1. More precisely, for $r < 1$ (or equivalently $s(r) := \tfrac{q-1}{rq} > \tfrac{q-1}{q}$) $W^r(\mathbf{C}_p^\flat)_L$ and $W_{\leq 1}^r(\mathbf{C}_p^\flat)_L$ correspond to $\{F(\omega_{\mathrm{LT}}) \in \mathbf{A}_L \mid F(Z)$ converges on $|\omega|^r \leq |Z| < 1\}$ and $\{F(\omega_{\mathrm{LT}}) \in \mathbf{A}_L \mid F(Z)$ converges on $|\omega|^r = q^{-\frac{1}{s(r)}} \leq |Z| < 1$ with values $|F(z)| \leq 1\}$, respectively. The latter condition on the values can also be rephrased as $s(r)v_{\pi_L}(a_m) + m \geq 0$ for all $m \in \mathbb{Z}$ corresponding to $V(x, s(r)) \geq 0$ on the Witt vector side if $F(Z) = \sum_m a_m Z^m \in \mathbf{A}_L$.

by (i). To verify this relation for m sufficiently huge, we choose a $k_0 \in \mathbb{Z}$ such that $v_{\pi_L}(a_k) + \frac{k}{r(n)} \geq N \geq 0$ for all $k \leq k_0$. Now choose m_0 with $-l(m_0, n) < k_0$ and fix $m \geq m_0$. For $\frac{-k}{r(n)} > m$ we obtain $v_{\pi_L}(a_k) \geq m + 1$ because $k < k_0$ holds. For k with

$$k \geq -r(n)m \Leftrightarrow \frac{k}{r(n)} - \frac{k + l(m, n)}{q - 1} \leq 0 \tag{A.3}$$

we obtain $v_{\pi_L}(a_k) \geq -\frac{k + l(m,n) - N}{q-1}$. Thus, the above relation holds true.

Vice versa choose m_0 such that (A.1) holds for all $m \geq m_0$, and fix

$$k \leq k_0 := -r(n) \max\{Mq^{n-1}, m_0\}.$$

Let m_1 be the unique integer satisfying $r(n)M - k \geq r(n)m_1 \geq r(n)M - k - r(n)$. In particular, we have $m_1 + 1 + \frac{k}{r(n)} \geq M$ and $k \geq -r(n)m_1$, which implies

$$-\frac{k + l(m_1, n) - N}{q - 1} + \frac{k}{r(n)} \geq M$$

by (A.3). Moreover, it holds that $m_1 \geq m_0$ and $k < -l(m_1, n) + N$ (using $k \leq r(n)M - r(n)m_1 = -l(m_1, n) + q^{n-1}N - (q-1)m_1$ and $m_1 > (q^{n-1} - 1)M$ by our assumption on k). Hence, we can apply (A.2) to conclude $v_{\pi_L}(a_k) + \frac{k}{r(n)} \geq M$ as desired. ∎

The analogue of [19, Lem. I.6.2] holds by the discussion following [80, Rem. 2.1]. This can be used to show the analogue of [19, Cor. I.6.3], i.e., $w_n(\psi(x)) \leq 1 + \frac{w_n(x)}{q}$. Now fix a basis $(e_1, \ldots, e_d)$ of $D(T)$ over $\mathbf{A}_L$ and denote by $\Phi = (a_{ij})$ the matrix defined by $e_j = \sum_{i=1}^{d} a_{ij}\varphi(e_i)$. The proof of [19, Lem. I.6.4] then carries over to show that for $x = \psi(y) - y$ with $x, y \in D(T)$ we have

$$w_n(y) \leq \max\left\{w_n(x), \frac{q}{q-1}(w_n(\Phi) + 1)\right\}, \tag{A.4}$$

where $w_n(\Phi) = \max_{ij} w_n(a_{ij})$ and $w_n(a) = \max_i w_n(a_i)$ for $a = \sum_{i=1}^{d} a_i \varphi(e_i)$ with $a_i \in \mathbf{A}_L$.

Lemma A.7. *Let $T \in \mathrm{Rep}_{o_L}(G_L)$ such that T is free over o_L and $V = T \otimes_{o_L} L$ is overconvergent. Then the canonical map $D^\dagger(T) \to D(T)$ induces an isomorphism $D^\dagger(T)/(\psi - 1)(D^\dagger(T)) \cong D(T)(\psi - 1)(D(T))$.*

Proof. We follow closely the proof of [50, Lem. 2.6], but note that he claims the statement for $D^\dagger_{\leq 1}(T)$.

Choose a basis $e_1, \ldots, e_d$ of the $\mathbf{A}_L^\dagger$-module $D^\dagger(T)$, which is free because $\mathbf{A}_L^\dagger$ is a henselian discrete valuation ring with respect to the uniformizer π_L (cf. [40, Def. 2.1.4]). Since V is overconvergent, it is also a basis of $D(T)$. Due to étaleness and since $(\mathbf{A}_L^\dagger) \cap \mathbf{A}_L^\times = (\mathbf{A}_L^\dagger)^\times$, also $\varphi(e_1), \ldots, \varphi(e_d)$ is a basis of all these modules. Given $x = \psi(y) - y$ with $x \in D^\dagger(T)$ and $y \in D(T)$, there is an $m > 0$ such that all x_i, a_{ij} lie in $\mathbf{A}_L^{\dagger, m}$ for some m. Since $q \geq 2$, it follows from the criterion in Lemma A.6 (ii) combined with (A.4) that all y_i belong to $\mathbf{A}_L^{\dagger, m+1}$, whence $y \in D^\dagger(T)$. This shows injectivity.

In order to show surjectivity, we apply Nakayama's lemma with regard to the ring o_L upon recalling that $D(T)/(\psi - 1)$ is of finite type over it. Indeed, by left exactness of $D^\dagger$ we obtain $D^\dagger(T)/\pi_L D^\dagger(T) \subseteq D^\dagger(T/\pi_{LT}) = D(T/\pi_{LT})$. Since these are vector spaces over $\mathbf{E}_L$ of the same dimension, they are equal, whence

$$\left(D^\dagger(T)/(\psi - 1)\right)/(\pi_L) = \left(D^\dagger(T)/(\pi_L)\right)/(\psi - 1) = \left(D(T)/(\pi_L)\right)/(\psi - 1)$$
$$= \left(D(T)/(\psi - 1)\right)/(\pi_L). \qquad \blacksquare$$

Corollary A.8. *Under the assumption of Lemma 3.3.6 for $V(\tau^{-1})$, the inclusion of complexes*

$$K_{\psi,U'}^\bullet\left(D^\dagger\left(V(\tau^{-1})\right)^\Delta\right) \subseteq K_{\psi,U'}^\bullet\left(D\left(V(\tau^{-1})\right)^\Delta\right)$$

is a quasi-isomorphism.

Proof. Forming Koszul complexes with regard to U', we obtain a diagram of (double) complexes with exact columns as illustrated in Figure A.1 in which the bottom line is an isomorphism of complexes because under the assumptions $\psi - 1$ induces an automorphism of $D(V(\tau^{-1}))/D^\dagger(V(\tau^{-1}))$ and as the action of Δ commutes with ψ. Hence, going over to total complexes gives an exact sequence

$$0 \to K_{\psi,U'}^\bullet\left(D^\dagger\left(V(\tau^{-1})\right)^\Delta\right) \to K_{\psi,U'}^\bullet\left(D\left(V(\tau^{-1})\right)^\Delta\right)$$
$$\to K_{\psi,U'}^\bullet\left(\left(D\left(V(\tau^{-1})\right)/D^\dagger\left(V(\tau^{-1})\right)\right)^\Delta\right) \to 0,$$

in which $K_{\psi,U'}^\bullet((D(V(\tau^{-1}))/D^\dagger(V(\tau^{-1})))^\Delta)$ is acyclic, whence the statement follows.

Figure A.1. Comparison of Herr complexes of D and $D^\dagger$. $\blacksquare$

Remark A.9. Instead of using Lemma 3.3.6 (for crystalline, analytic representations) one can probably show by the same techniques as in [19, Prop. III.3.2 (ii)] that for any overconvergent representation V we have

$$D^\dagger(V)^{\psi=1} = D(V)^{\psi=1}.$$

Recall from [58, (5.2.1)] the quasi-isomorphism

$$L[-2] \xrightarrow{\iota} \tau_{\geq 2} \mathcal{C}^\bullet(G_{L'}, L(1))$$

which allows us to define a trace map

$$\mathrm{tr}_{\mathcal{C}} : \mathcal{C}^\bullet(G_{L'}, L(1)) \to L[-2]$$

in the derived category of L-vector spaces as

$$\mathcal{C}^\bullet(G_{L'}, L(1)) \to \tau_{\geq 2} \mathcal{C}^\bullet(G_{L'}, L(1)) \xleftarrow{\iota} L[-2].$$

Then local Tate duality is induced by the following pairing on cocycles:

$$\mathcal{C}^\bullet(G_{L'}, V^*(1)) \times \mathcal{C}^\bullet(G_{L'}, V) \xrightarrow{\cup_{G_{L'}}} \mathcal{C}^\bullet(G_{L'}, L(1)) \xrightarrow{\mathrm{tr}_{\mathcal{C}}} L[-2].$$

The interest in the following diagram, the commutativity of which is shown before Lemma B.5, stems from the discrepancy that the reciprocity law has been formulated and proved in the setting of $K^\bullet_{\psi,U'}(D^\dagger_{\mathrm{rig}}(V(\tau^{-1}))^\Delta)[d-1]$ while the regulator map originally lives in the setting of $K^\bullet_{\psi,U'}(D(V(\tau^{-1}))^\Delta)[d-1]$:

$$
\begin{array}{cccccc}
\mathcal{C}^\bullet(G_{L'},V^*(1))\times & \mathcal{C}^\bullet(G_{L'},V) & \xrightarrow{\cup_{G_{L'}}} & \mathcal{C}^\bullet(L',L(1)) & \xrightarrow{\mathrm{tr}_{\mathcal{C}}} & L[-2] \\
\downarrow \simeq & e \downarrow \simeq & & & & \| \\
K^\bullet_{\varphi,U'}(M^\Delta) \times & K^\bullet_{\psi,U'}(D(V(\tau^{-1}))^\Delta)[d-1] & \xrightarrow{\cup_{K,\psi}} & & & L[-2] \\
\uparrow & \uparrow & & & & \| \\
K^\bullet_{\varphi,U'}((M^\dagger)^\Delta) \times & K^\bullet_{\psi,U'}(D^\dagger(V(\tau^{-1}))^\Delta)[d-1] & \xrightarrow{\cup_{K,\psi}} & & & L[-2] \\
\downarrow \simeq & \downarrow & & & & \| \\
K^\bullet_{\varphi,U'}((M^\dagger_{\mathrm{rig}})^\Delta) \times & K^\bullet_{\psi,U'}(D^\dagger_{\mathrm{rig}}(V(\tau^{-1}))^\Delta)[d-1] & \xrightarrow{\cup_{K,\psi}} & & & L[-2],
\end{array}
\tag{A.5}
$$

which in turn induces the commutativity of the lower rectangle in the following diagram (the upper quadrangles commute obviously):

$$
\begin{array}{ccc}
D_{\mathrm{rig}}^{\dagger}\big(V(\tau^{-1})\big)^{\psi=1} & & D\big(V(\tau^{-1})\big)^{\psi=1} \\
& D^{\dagger}\big(V(\tau^{-1})\big)^{\psi=1} & \\
\mathrm{pr}_U \downarrow & \mathrm{pr}_U \downarrow & \mathrm{pr}_U \downarrow \\
h^1\big(K_{\psi,U'}^{\bullet}\big(D_{\mathrm{rig}}^{\dagger}\big(V(\tau^{-1})\big)^{\Delta}\big)[d-1]\big) & h^1\big(K_{\psi,U'}^{\bullet}\big(D^{\dagger}\big(V(\tau^{-1})\big)^{\Delta}\big)[d-1]\big) & h^1\big(K_{\psi,U'}^{\bullet}\big(D\big(V(\tau^{-1})\big)^{\Delta}\big)[d-1]\big) \\
\cong\, a & & c\,\cong \\
H_{/\dagger}^1(L',V) & \xleftarrow{\quad \mathrm{pr} \quad} & H^1(L',V).
\end{array}
$$

$$\tag{A.6}$$

Here the vertical maps pr_U are defined as in (5.25), a and pr are taken from Proposition 5.2.19, and the isomorphism c stems from (B.6). The map a is bijective under the assumption of Lemma 3.3.6, which extends to the map b by Corollary A.8.

Appendix B

Iwasawa cohomology and descent

In this appendix, we recall a crucial observation from [45, 46], which is based on [58] and generalizes [80, Thm. 5.13]. As before, let U be an open subgroup of Γ_L. We set $\mathbb{T} := \Lambda(U) \otimes_{o_L} T$ with actions by $\Lambda(U) := o_L[\![U]\!]$ via left multiplication on the left factor and by $g \in G_{L'}$ given as $\lambda \otimes t \mapsto \lambda \bar{g}^{-1} \otimes g(t)$, where $\bar{g}$ denotes the image of g in U. We write $R\Gamma_{\mathrm{Iw}}(L_\infty/L', T)$ for the continuous cochain complex $C^\bullet(U, \mathbb{T})$ and recall that its cohomology identifies with $H^\bullet_{\mathrm{Iw}}(L_\infty/L', T)$ by [80, Lem. 5.8]. For any continuous endomorphism f of M, we set $\mathcal{T}_f(M) := [M \xrightarrow{f-1} M]$, a complex concentrated in degree 0 and 1.

The map $p : \mathbb{T} \to o_L \otimes_{\Lambda(U)} \mathbb{T} \cong T, t \mapsto 1 \otimes t$, and its dual $i : T^\vee(1) \to \mathbb{T}^\vee(1)$ induce on cohomology the corestriction and restriction map, respectively, and they are linked by the following commutative diagram:

$$
\begin{array}{ccccccc}
\mathcal{C}^\bullet(G_{L'}, \mathbb{T}) & \times & \mathcal{C}^\bullet(G_{L'}, \mathbb{T}^\vee(1)) & \xrightarrow{\cup_{G_{L'}}} & \mathcal{C}^\bullet(L', L/o_L(1)) & \xrightarrow{\mathrm{tr}_{\mathcal{C}}} & L/o_L[-2] \\
\downarrow{\scriptstyle p_*} & & \uparrow{\scriptstyle i_*} & & & & \| \\
\mathcal{C}^\bullet(G_{L'}, T) & \times & \mathcal{C}^\bullet(G_{L'}, T^\vee(1)) & \xrightarrow{\cup_{G_{L'}}} & \mathcal{C}^\bullet(L', L/o_L(1)) & \xrightarrow{\mathrm{tr}_{\mathcal{C}}} & L/o_L[-2].
\end{array}
\tag{B.1}
$$

By [34, Prop. 1.6.5 (3)] (see also [58, (8.4.8.1)]) we have a canonical isomorphism

$$
o_L \otimes^{\mathbb{L}}_{\Lambda(U)} R\Gamma(L', \mathbb{T}) \cong R\Gamma(L', o_L \otimes_{\Lambda(U)} \mathbb{T}) \cong R\Gamma(L', T),
\tag{B.2}
$$

where we denote by $R\Gamma(L', -)$ the complex $\mathcal{C}^\bullet(G_{L'}, -)$ regarded as an object of the derived category. Dually, by a version of Hochschild–Serre, there is a canonical isomorphism

$$
R\mathrm{Hom}_\Lambda(o_L, R\Gamma(L', \mathbb{T}^\vee(1))) \cong R\Gamma(L', T^\vee(1)).
\tag{B.3}
$$

It follows that the isomorphism

$$
R\Gamma_{\mathrm{Iw}}(L_\infty/L', T) \cong R\mathrm{Hom}_{o_L}(R\Gamma(L', \mathbb{T}^\vee(1)), L/o_L)[-2]
$$

induced by the upper line of (B.1) induces an isomorphism

$$
\begin{aligned}
& o_L \otimes^{\mathbb{L}}_{\Lambda(U)} R\Gamma_{\mathrm{Iw}}(L_\infty/L', T) \\
& \quad \cong R\mathrm{Hom}_{o_L}(R\mathrm{Hom}_\Lambda(o_L, R\Gamma(L', \mathbb{T}^\vee(1))), L/o_L)[-2],
\end{aligned}
\tag{B.4}
$$

which is compatible with the lower cup product pairing in (B.1) via the canonical identifications (B.2) and (B.3).

Lemma B.1. *There is a canonical isomorphism*

$$R\Gamma\big(L', \mathbb{T}^\vee(1)\big) \cong \mathcal{T}_\varphi\big(D\big(T^\vee(1)\big)\big)$$

in the derived category.

Proof. See [45, Thm. 5.1.11]. ∎

For the rest of this section we assume that $U \subseteq \Gamma_L$ is an open *torsion-free* subgroup.

Lemma B.2. *Let T be in $\mathrm{Rep}_{o_L}(G_L)$ of finite length. Set $\Lambda := \Lambda(U)$ and let $\gamma_1, \ldots, \gamma_d$ be topological generators of U. Then we have an up to signs canonical isomorphism of complexes*

$$\mathrm{Hom}^\bullet_\Lambda \big(K_\bullet(\gamma), \mathcal{T}_\varphi\big(D\big(T^\vee(1)\big)\big)\big)^\vee[-2]$$
$$\cong \mathrm{tot}\big(\mathcal{T}_\psi\big(D\big(T(\tau^{-1})\big)\big)\big)[-1] \otimes_\Lambda K_\bullet(\gamma^{-1})(\Lambda)^\bullet\big),$$

where $-^\vee$ denotes forming the Pontrjagin dual.

Proof. Upon noting that

$$\mathcal{T}_\varphi\big(D\big(T^\vee(1)\big)\big)^\vee[-2] \cong \mathcal{T}_\psi\big(D\big(T(\tau^{-1})\big)\big)[-1]$$

(canonically up to a sign!), this is easily reduced to the following statement:

$$\mathrm{Hom}^\bullet_\Lambda \big(K_\bullet(\gamma), M\big)^\vee \cong M^\vee \otimes_\Lambda K_\bullet(\gamma^{-1})(\Lambda)^\bullet,$$

which can be proved in the same formal way as (5.12), together with a consideration of signs. ∎

Remark B.3. For every $M \in \mathfrak{M}(\mathbf{A}_L)$, we have a canonical isomorphism

$$\mathrm{Hom}^\bullet_\Lambda \big(K_\bullet^U, \mathcal{T}_\varphi(M)\big) \cong K_{\varphi, U}(M)$$

up to the sign $(-1)^n$ in degree n and a non-canonical isomorphism

$$\mathrm{tot}\big(\mathcal{T}_\psi(M)[-1] \otimes_\Lambda K_\bullet^U(\Lambda)^\bullet\big) \cong K_{\psi, U}(M)[d-1]$$

(involving the self-duality of the Koszul complex). Here, the right-hand sides are formed with respect to the same sequence of topological generators as the left-hand sides.

Proof. By our conventions in Section 5.2.1 $K_{\varphi, U}(M)$ is the total complex of the double complex

$$\mathrm{Hom}^\bullet \big(K_\bullet(\Lambda)^\bullet, M\big) \xrightarrow{1-\varphi_*} \mathrm{Hom}^\bullet \big(K_\bullet(\Lambda)^\bullet, M\big).$$

A comparison with the total Hom-complex (with the same sign rules as in Section 5.2.1) shows the first claim. For the second statement we have

$$
\begin{aligned}
\mathrm{tot}\big(\mathcal{T}_\psi(M)[-1] \otimes_\Lambda K_\bullet(\Lambda)^\bullet\big) &\cong \mathrm{tot}\big(\mathcal{T}_\psi(M) \otimes_\Lambda K_\bullet(\Lambda)^\bullet\big)[-1] \\
&= \mathrm{tot}\big(\mathcal{T}_\psi\big(M \otimes_\Lambda K_\bullet(\Lambda)^\bullet\big)\big)[-1] \\
&\cong \mathrm{tot}\big(\mathcal{T}_\psi\big(M \otimes_\Lambda K^\bullet(\Lambda)[d]\big)\big)[-1] \\
&= \mathrm{tot}\big(\mathcal{T}_\psi\big(K^\bullet(M)[d]\big)\big)[-1] \\
&= \mathrm{cone}\big(K_U^\bullet(M)[d] \xrightarrow{1-\psi} K_U^\bullet(M)[d]\big)[-2] \\
&\cong K_{\psi,U}(M)[d-1].
\end{aligned}
$$

The first isomorphism involves a sign on $\mathcal{T}_\psi^1(M)$. The third isomorphism stems from (5.11), while the last isomorphism again involves signs. ∎

Theorem B.4. *There are canonical isomorphisms*

$$
R\Gamma_{\mathrm{Iw}}(L_\infty/L, T) \cong \mathcal{T}_\psi\big(D\big(T(\tau^{-1})\big)\big)[-1], \tag{B.5}
$$

$$
K_{\psi,U}\big(D\big(T(\tau^{-1})\big)\big)[d-1] \xrightarrow{\simeq} R\Gamma(L', T) \tag{B.6}
$$

in the derived category $D_{\mathrm{perf}}(\Lambda_{o_L}(\Gamma_L))$ of perfect complexes and in the derived category $D^+(o_L)$ of bounded below cochain complexes of o_L-modules, respectively.

Proof. The first isomorphism is [45, Thm. 5.2.54], while the second one follows from this and (B.2) as

$$
\begin{aligned}
R\Gamma_{\mathrm{Iw}}(L_\infty/L, T) \otimes^{\mathbb{L}}_{\Lambda_{o_L}(U)} o_L &\cong \mathcal{T}_\psi\big(D\big(T(\tau^{-1})\big)\big)[-1] \otimes^{\mathbb{L}}_\Lambda K_\bullet(\Lambda)^\bullet \\
&= \mathrm{tot}\big(\mathcal{T}_\psi\big(D\big(T(\tau^{-1})\big)\big)[-1] \otimes_\Lambda K_\bullet(\Lambda)^\bullet\big) \\
&= K_{\psi,U}\big(D\big(T(\tau^{-1})\big)\big)[d-1]
\end{aligned}
$$

by Remark B.3. ∎

By Lemma B.2 and Remark B.3 we see that, for T be in $\mathrm{Rep}_{o_L}(G_L)$ of finite length,

$$
K_{\varphi,U}\big(D\big(T^\vee(1)\big)\big) = R\mathrm{Hom}_\Lambda\big(o_L, \mathcal{T}_\varphi\big(D\big(T^\vee(1)\big)\big)\big)[2]
$$

is dual to

$$
K_{\psi,U}\big(D\big(T(\tau^{-1})\big)\big) = o_L \otimes^{\mathbb{L}}_{\Lambda(U)} \mathcal{T}_\psi\big(D\big(T(\tau^{-1})\big)\big)[-1],
$$

such that the upper rectangle in the diagram (A.5) commutes by (B.4), taking inverse limits and inverting π_L.

Lemma B.5. *Let T be in $\mathrm{Rep}_{o_L}(G_L)$. Then the left rectangle in (5.32) is commutative.*

Sketch of proof. By an obvious analogue of Remark 5.2.17, it suffices to show the statement for $U = \Gamma_n \cong \mathbb{Z}_p^d$. In this situation, we have a homological spectral sequence

$$H_{i,\mathrm{cont}}\big(U, H_{\mathrm{Iw}}^{-j}(L_\infty/L, T)\big) \Longrightarrow H_{\mathrm{cont}}^{-i-j}(L', T)$$

which is induced by (B.2), see [58, (8.4.8.1)] for the statement and missing notation. We may and do assume that T is of finite length. Then, on the one hand, the map $H_{\mathrm{Iw}}^1(L_\infty/L, T) \xrightarrow{\mathrm{cor}} H^1(L', T)$ is dual to $H^1(L', T^\vee(1)) \xrightarrow{\mathrm{res}} H^1(L_\infty, T^\vee(1))$, which sits in the five-term exact sequence of lower degrees associated with the Hochschild–Serre spectral sequence. As explained just before this lemma, the above homological spectral sequence arises by dualizing from the latter. Hence, cor shows up in the five-term exact sequence of lower degrees associated with this homological spectral sequence. On the other hand, via the isomorphisms (B.2) and (B.6), the latter spectral sequence is isomorphic to

$$H_{i,\mathrm{cont}}\big(U, h^{-j}\big(\mathcal{T}_\psi\big(D\big(T(\tau^{-1})\big)\big)\big)[-1]\big) \Longrightarrow h^{-i-j}\big(K_{\psi,U}\big(D\big(T(\tau^{-1})\big)\big)[d-1]\big)$$

and one checks by inspection that cor corresponds to pr_U.   ∎

References

[1] T. Abe and C. Lazda, Proper pushforwards on analytic adic spaces. *Q. J. Math.* **76** (2025), no. 1, 147–183

[2] C. Bănică and O. Stănăşilă, *Algebraic methods in the global theory of complex spaces*. John Wiley & Sons, London, 1976

[3] D. Benois, On Iwasawa theory of crystalline representations. *Duke Math. J.* **104** (2000), no. 2, 211–267

[4] L. Berger, Bloch and Kato's exponential map: three explicit formulas. *Doc. Math.* **Extra Vol.** (2003), 99–129

[5] L. Berger, Limites de représentations cristallines. *Compos. Math.* **140** (2004), no. 6, 1473–1498

[6] L. Berger, Multivariable (φ, Γ)-modules and locally analytic vectors. *Duke Math. J.* **165** (2016), no. 18, 3567–3595

[7] L. Berger and L. Fourquaux, Iwasawa theory and F-analytic Lubin-Tate (φ, Γ)-modules. *Doc. Math.* **22** (2017), 999–1030

[8] L. Berger, P. Schneider, and B. Xie, Rigid character groups, Lubin-Tate theory, and (φ, Γ)-modules. *Mem. Amer. Math. Soc.* **263** (2020), no. 1275, v+79

[9] V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*. Math. Surveys Monogr. 33, American Mathematical Society, Providence, RI, 1990

[10] P. Beyer, On Serre-duality for coherent sheaves on rigid-analytic spaces. *Manuscripta Math.* **93** (1997), no. 2, 219–245

[11] P. Beyer, *Zur Serre-Dualität für kohärente Garben auf rigid-analytischen Räumen*. Schriftenreihe Math. Inst. Univ. Münster 3. Ser. 20, Univ. Münster, Math. Inst., Münster, 1997

[12] S. Bloch and K. Kato, *L*-functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, pp. 333–400, Progr. Math. 86, Birkhäuser, Boston, MA, 1990

[13] A. Borel and N. R. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*. Ann. of Math. Stud. 94, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1980

[14] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean analysis. A systematic approach to rigid analytic geometry*. Grundlehren Math. Wiss. 261, Springer, Berlin, 1984

[15] N. Bourbaki, *Topological vector spaces. Chapters 1–5*. Elem. Math., Springer, Berlin, 1987

[16] N. Bourbaki, *Commutative algebra. Chapters 1–7*. Elem. Math., Springer, Berlin, 1989

[17] N. Bourbaki, *Éléments de mathématique. Topologie générale. Chapitres 5 à 10*. Springer, Berlin, 2007

[18] F. Cherbonnier and P. Colmez, Représentations p-adiques surconvergentes. *Invent. Math.* **133** (1998), no. 3, 581–611

[19] F. Cherbonnier and P. Colmez, Théorie d'Iwasawa des représentations p-adiques d'un corps local. *J. Amer. Math. Soc.* **12** (1999), no. 1, 241–268

[20] B. Chiarellotto, Duality in rigid analysis. In *p-adic analysis (Trento, 1989)*, pp. 142–172, Lecture Notes in Math. 1454, Springer, Berlin, 1990

[21] R. F. Coleman, Division values in local fields. *Invent. Math.* **53** (1979), no. 2, 91–116

[22] P. Colmez, Théorie d'Iwasawa des représentations de de Rham d'un corps local. *Ann. of Math. (2)* **148** (1998), no. 2, 485–571

[23] P. Colmez, Représentations cristallines et représentations de hauteur finie. *J. Reine Angew. Math.* **514** (1999), 119–143

[24] P. Colmez, Espaces de Banach de dimension finie. *J. Inst. Math. Jussieu* **1** (2002), no. 3, 331–439

[25] P. Colmez, Représentations localement analytiques de $\mathbf{GL}_2(\mathbf{Q}_p)$ et (φ, Γ)-modules. *Represent. Theory* **20** (2016), 187–248

[26] P. Colmez and W. Nizioł, Syntomic complexes and p-adic nearby cycles. *Invent. Math.* **208** (2017), no. 1, 1–108

[27] E. de Shalit, *Iwasawa theory of elliptic curves with complex multiplication. p-adic L functions*. Perspect. Math. 3, Academic Press, Boston, MA, 1987

[28] D. Eisenbud, *Commutative algebra*. Grad. Texts in Math. 150, Springer, New York, 1995

[29] M. Emerton, Locally analytic vectors in representations of locally p-adic analytic groups. *Mem. Amer. Math. Soc.* **248** (2017), no. 1175, iv+158

[30] L. Fargues and J.-M. Fontaine, Courbes et fibrés vectoriels en théorie de Hodge p-adique. *Astérisque* **406** (2018), xiii+382

[31] J.-M. Fontaine, Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate. In *Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. III*, pp. 3–80, Astérisque 65, Soc. Math. France, Paris, 1979

[32] L. Fourquaux, *Logarithme de Perrin-Riou pour des extensions associées à un groupe de Lubin-Tate*. Ph.D. thesis, Université Paris 6, 2005, https://math.lionel.fourquaux.org/publications/these.pdf visited on 4 August 2025

[33] L. Fourquaux and B. Xie, Triangulable $\mathcal{O}_F$-analytic (φ_q, Γ)-modules of rank 2. *Algebra Number Theory* **7** (2013), no. 10, 2545–2592

[34] T. Fukaya and K. Kato, A formulation of conjectures on p-adic zeta functions in non-commutative Iwasawa theory. In *Proceedings of the St. Petersburg Mathematical Society. Vol. XII*, pp. 1–85, Amer. Math. Soc. Transl. Ser. 2 219, American Mathematical Society, Providence, RI, 2006

[35] A. Grothendieck and J. A. Dieudonné, *Éléments de géométrie algébrique. I*. Grundlehren Math. Wiss. 166, Springer, Berlin, 1971

[36] L. Gruson, Théorie de Fredholm p-adique. *Bull. Soc. Math. France* **94** (1966), 67–95

[37] M. Hazewinkel, *Formal groups and applications*. Pure Appl. Math. 78, Academic Press, New York, 1978

[38] L. Herr, Une approche nouvelle de la dualité locale de Tate. *Math. Ann.* **320** (2001), no. 2, 307–337

[39] K. S. Kedlaya, Slope filtrations for relative Frobenius. *Astérisque* **319** (2008), 259–301

[40] K. S. Kedlaya, New methods for (Γ, φ)-modules. *Res. Math. Sci.* **2** (2015), article no. 20

[41] K. S. Kedlaya, Some slope theory for multivariate Robba rings. [v1] 2013, [v3] 2020, arXiv:1311.7468v3

[42] K. S. Kedlaya, J. Pottharst, and L. Xiao, Cohomology of arithmetic families of (φ, Γ)-modules. *J. Amer. Math. Soc.* **27** (2014), no. 4, 1043–1115

[43] M. Kisin, Crystalline representations and F-crystals. In *Algebraic geometry and number theory*, pp. 459–496, Progr. Math. 253, Birkhäuser, Boston, MA, 2006

[44] M. Kisin and W. Ren, Galois representations and Lubin-Tate groups. *Doc. Math.* **14** (2009), 441–461

[45] B. Kupferer and O. Venjakob, Herr-complexes in the Lubin-Tate setting. *Mathematika* **68** (2022), no. 1, 74–147

[46] B. J. Kupferer, *Two ways to compute Galois cohomology using Lubin-Tate (φ, Γ)-modules, a reciprocity law and a regulator map*. Ph.D. thesis, Ruprecht-Karls-Universität Heidelberg, 2020, https://archiv.ub.uni-heidelberg.de/volltextserver/28383/ visited on 4 August 2025

[47] S. Lang, *Cyclotomic fields*. Grad. Texts in Math. 59, Springer, New York, 1978

[48] M. Lazard, Les zéros des fonctions analytiques d'une variable sur un corps valué complet. *Publ. Math. Inst. Hautes Études Sci.* **14** (1962), 47–75

[49] M. Lazard, Groupes analytiques p-adiques. *Publ. Math. Inst. Hautes Études Sci.* **26** (1965), 389–603

[50] R. Liu, Cohomology and duality for (ϕ, Γ)-modules over the Robba ring. *Int. Math. Res. Not. IMRN* **2008** (2008), no. 3, article no. rnm150

[51] D. Loeffler, O. Venjakob, and S. L. Zerbes, Local epsilon isomorphisms. *Kyoto J. Math.* **55** (2015), no. 1, 63–127

[52] D. Loeffler and S. L. Zerbes, Iwasawa theory and p-adic L-functions over $\mathbb{Z}_p^2$-extensions. *Int. J. Number Theory* **10** (2014), no. 8, 2045–2095

[53] S. Lubkin, *Cohomology of completions*. North-Holland Math. Stud. 42, North-Holland Publishing, Amsterdam, 1980

[54] M. Malčić, Trace maps on rigid Stein spaces. *Manuscripta Math.* **175** (2024), no. 3–4, 1119–1160

[55] M. Malcic, R. Steingart, O. Venjakob, and M. Witzelsperger, ε-isomorphisms for rank one (φ, Γ)-modules over Lubin-Tate Robba rings. *J. Inst. Math. Jussieu* **24** (2025), no. 5, 1895–1994

[56] M. Manji, *Selmer complexes and Iwasawa theory for* GU(2, 1). Ph.D. thesis, University of Warwick, 2024

[57] H. Matsumura, *Commutative ring theory*. Cambridge Stud. Adv. Math. 8, Cambridge University Press, Cambridge, 1986

[58] J. Nekovář, Selmer complexes. *Astérisque* **310** (2006), viii+559

[59] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*. 2nd edn., Grundlehren Math. Wiss. 323, Springer, Berlin, 2008

[60] W. Nizioł, Cohomology of crystalline representations. *Duke Math. J.* **71** (1993), no. 3, 747–791

[61] C. Perez-Garcia and W. H. Schikhof, *Locally convex spaces over non-Archimedean valued fields*. Cambridge Stud. Adv. Math. 119, Cambridge University Press, Cambridge, 2010

[62] B. Perrin-Riou, Théorie d'Iwasawa des représentations p-adiques sur un corps local. *Invent. Math.* **115** (1994), no. 1, 81–161

[63] F. Prosmans, Derived projective limits of topological abelian groups. *J. Funct. Anal.* **162** (1999), no. 1, 135–177

[64] P. Robba and G. Christol, *Équations différentielles p-adiques*. Actualités Math., Hermann, Paris, 1994

[65] N. Saavedra Rivano, *Catégories Tannakiennes*. Lecture Notes in Mathematics 265, Springer, Berlin, 1972

[66] T. Sano and O. Venjakob, On Lubin-Tate regulator maps and Kato's explicit reciprocity law. 2024, arXiv:2412.12429v1

[67] W. H. Schikhof and Y. Morita, Duality of projective limit spaces and inductive limit spaces over a nonspherically complete non-Archimedean field. *Tohoku Math. J. (2)* **38** (1986), no. 3, 387–397

[68] T. Schmidt, Auslander regularity of p-adic distribution algebras. *Represent. Theory* **12** (2008), 37–57

[69] T. Schmidt, On locally analytic Beilinson–Bernstein localization and the canonical dimension. *Math. Z.* **275** (2013), no. 3–4, 793–833

[70] P. Schneider, Points of rigid analytic varieties. *J. Reine Angew. Math.* **434** (1993), 127–157

[71] P. Schneider, *Nonarchimedean functional analysis*. Springer Monogr. Math., Springer, Berlin, 2002

[72] P. Schneider, *p-adic Lie groups*. Grundlehren Math. Wiss. 344, Springer, Heidelberg, 2011

[73] P. Schneider, *Galois representations and (φ, Γ)-modules*. Cambridge Stud. Adv. Math. 164, Cambridge University Press, Cambridge, 2017

[74] P. Schneider and J. Teitelbaum, *p-adic Fourier theory*. *Doc. Math.* **6** (2001), 447–481

[75] P. Schneider and J. Teitelbaum, Locally analytic distributions and p-adic representation theory, with applications to GL_2. *J. Amer. Math. Soc.* **15** (2002), no. 2, 443–468

[76] P. Schneider and J. Teitelbaum, Algebras of p-adic distributions and admissible representations. *Invent. Math.* **153** (2003), no. 1, 145–196

[77] P. Schneider and J. Teitelbaum, Duality for admissible locally analytic representations. *Represent. Theory* **9** (2005), 297–326

[78] P. Schneider and J. Teitelbaum, Banach-Hecke algebras and p-adic Galois representations. *Doc. Math.* **Extra Vol.** (2006), 631–684

[79] P. Schneider and O. Venjakob, A splitting for K_1 of completed group rings. *Comment. Math. Helv.* **88** (2013), no. 3, 613–642

[80] P. Schneider and O. Venjakob, Coates–Wiles homomorphisms and Iwasawa cohomology for Lubin-Tate extensions. In *Elliptic curves, modular forms and Iwasawa theory*, pp. 401–468, Springer Proc. Math. Stat. 188, Springer, Cham, 2016

[81] J.-P. Serre, *Abelian l-adic representations and elliptic curves*. W. A. Benjamin, New York, 1968

[82] The Stacks project, https://stacks.math.columbia.edu visited on 4 August 2025

[83] R. Steingart, *Frobeniusregularisierung und Limites L-kristalliner Darstellungen*. Master's thesis, Heidelberg University, 2019

[84] R. Steingart, *Analytic cohomology of families of L-analytic Lubin-Tate (φ_L, Γ_L)-modules*. Ph.D. thesis, Heidelberg University, 2022, https://archiv.ub.uni-heidelberg.de/volltextserver/32192/ visited on 11 September 2025

[85] R. Steingart, Finiteness of analytic cohomology of Lubin-Tate (φ_L, Γ_L)-modules. *J. Number Theory* **263** (2024), 24–78

[86] R. Steingart, Iwasawa cohomology of analytic (φ_L, Γ_L)-modules. *Acta Arith.* **216** (2024), no. 2, 123–176

[87] O. Thomas, *On analytic and Iwasawa cohomology*. Ph.D. thesis, Heidelberg University, 2019, https://archiv.ub.uni-heidelberg.de/volltextserver/27443/ visited on 11 September 2025

[88] O. Thomas, Cohomology of topologised monoids. *Math. Z.* **301** (2022), no. 3, 2793–2840

[89] O. Thomas, Duality for K-analytic group cohomology of p-adic Lie groups. *C. R. Math. Acad. Sci. Paris* **360** (2022), 1213–1226

[90] M. van der Put, Serre duality for rigid analytic spaces. *Indag. Math. (N.S.)* **3** (1992), no. 2, 219–235

[91] A. C. M. van Rooij, *Non-Archimedean functional analysis*. Monogr. Textbooks Pure Appl. Math. 51, Marcel Dekker, New York, 1978

[92] O. Venjakob, Explicit reciprocity laws in Iwasawa theory: a survey with some focus on the Lubin–Tate setting. In *Arithmetic of L-functions*, pp. 201–251, EMS Ser. Congr. Rep., EMS Press, Berlin, 2025

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Reciprocity Laws for (φ_L, Γ_L)-Modules over Lubin–Tate Extensions

In the Lubin–Tate setting, we study pairings for analytic (φ_L, Γ_L)-modules and prove an abstract reciprocity law which then implies a relation between the analogue of Perrin-Riou's big exponential map as developed by Berger and Fourquaux and a p-adic regulator map whose construction relies on the theory of Kisin–Ren modules generalizing the concept of Wach modules to the Lubin–Tate situation.