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Bellman Functions on Simple Non-Convex Domains in the Plane

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Abstract

The present memoir provides a generalization of the authors' previous work on Bellman functions for integral functionals on BMO. Those Bellman functions are the minimal locally concave functions on parabolic strips in the plane. We describe an algorithm for constructing minimal locally concave functions on a planar domain that is a difference of two unbounded convex domains. This leads to many sharp estimates for functions in classes such as BMO, A_p , or the Gehring classes.

Keywords. Bellman function, extremal problems, sharp inequalities, bounded mean oscillation, Muckenhoupt classes, Gehring classes

Mathematics Subject Classification (2020): 42B35 (primary); 42A05, 26D07, 49K20 (secondary)

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Chapter 1

Introduction

The Bellman functions and their probabilistic counterparts, the Burkholder functions, play an important role in modern harmonic analysis and probability theory. In fact, these are two names for the same object. The foundations were laid in the groundbreaking papers [4] and [29] (see [28] as well); we refer the reader to the monographs [33] and [58] for a general description of the field; see shorter introductions in [30] and [43].

The field may be, roughly speaking, divided into two parts. The first part uses Bellman functions as tools for proving inequalities without concern for their sharpness. Here, one usually has to come up with a sufficiently good *Bellman supersolution*. The second part aims to provide sharp solutions to Bellman optimization problems. This significantly narrows the area of application since the exact Bellman function is often extremely difficult to describe. However, the result usually comes not only with the sharp constant for the inequality in question, but also with a good understanding of optimal functions.

The present memoir provides the proofs of the results announced in [15]. We will concentrate on a special class of optimization problems that are designed to study summability properties of functions in classes like BMO or A_p . We refer the reader to [15] and to Sections 1.1.1 and 1.1.2 below for a long list of specific problems already solved within the scope of our considerations. In a sense, our main target is to provide a theory for solving optimization problems of a very special kind; this covers only a tiny part of inequalities amenable to the Bellman function method. However, the constructions and ideas arising in this small subfield are universal and they are used in other more involved Bellman problems; see Sections 1.1.3, 1.1.4, and 1.1.5 below. In the forthcoming section, we briefly motivate our study, describe the field, and outline the main examples. We also show how many parts of the present work serve as useful tools for related problems and explain the relationship with the classical moment method and themes in differential geometry. Section 1.2 contains the plan of the memoir. In Section 1.3, we discuss some related questions that fall outside the scope of this work.

1.1 Motivation, examples, and links to classical topics

Let Ξ_0 be a strictly convex unbounded open proper subset of \mathbb{R}^2 . Let Ξ_1 be another strictly convex unbounded open set whose closure lies in Ξ_0 entirely; we set $\Omega = \text{clos } \Xi_0 \setminus \Xi_1$. Let I be an interval in \mathbb{R} . Consider the class $\mathcal{A}_\Omega(I)$ of \mathbb{R}^2 -valued

summable functions $\varphi: I \rightarrow \partial\Xi_0$ such that the point $\langle\varphi\rangle_J$ does not belong to Ξ_1 for any subinterval $J \subset I$; here and in what follows, the symbol $\langle\psi\rangle_E$ denotes the average of a locally summable function ψ over a set E of nonzero measure:

$$\langle\psi\rangle_E = \frac{1}{|E|} \int_E \psi(t) dt, \quad |E| \text{ stands for the Lebesgue measure of } E \subset \mathbb{R}.$$

We assume that both boundaries of Ξ_0 and Ξ_1 may be parametrized by the first coordinate. In other words, they are graphs of convex functions in the standard coordinate system.

Let $f: \partial\Xi_0 \rightarrow \mathbb{R}$ be a function. Assume f is measurable and does not grow too fast at infinity. We address the question of finding sharp estimates for the quantity $\langle f(\varphi)\rangle_I$ under the conditions $\varphi \in A_\Omega(I)$ and $\langle\varphi\rangle_I = x$. This setting naturally leads to the introduction of the Bellman function

$$\mathbf{B}_\Omega(x; f) = \sup\{\langle f(\varphi)\rangle_I \mid \langle\varphi\rangle_I = x, \varphi \in A_\Omega(I)\}, \quad x \in \Omega. \quad (1.1.1)$$

Using dilation invariance of the average, one may prove that the definition of \mathbf{B} does not depend on the particular choice of I . There are two main classes of examples: those related to BMO spaces and those to the Muckenhoupt classes A_p .

1.1.1 BMO case

The case where

$$\Omega = \{x \in \mathbb{R}^2 \mid x_1^2 \leq x_2 \leq x_1^2 + 1\},$$

corresponds to optimizing an integral functional $\langle f(\varphi)\rangle_I$ on the ball of unit radius of the BMO space. Here we have chosen

$$\Xi_0 = \{x \in \mathbb{R}^2 \mid x_1^2 < x_2\} \quad \text{and} \quad \Xi_1 = \{x \in \mathbb{R}^2 \mid x_1^2 + 1 < x_2\}.$$

Note that if $\varphi: I \rightarrow \partial\Xi_0$ belongs to $A_\Omega(I)$, then its first coordinate, $\varphi_1: I \rightarrow \mathbb{R}$, satisfies the inequality

$$\langle\varphi_1^2\rangle_J - (\langle\varphi_1\rangle_J)^2 \leq 1 \quad \text{for any subinterval } J \subset I.$$

This may be rewritten as

$$\langle|\varphi_1 - \langle\varphi_1\rangle_J|^2\rangle_J \leq 1 \quad \text{for any subinterval } J \subset I.$$

This is equivalent to the inequality $\|\varphi_1\|_{\text{BMO}} \leq 1$, provided that we equip the space $\text{BMO}(I)$ with the quadratic seminorm. The general BMO_p seminorm is

$$\|\psi\|_{\text{BMO}_p(I)} = \sup\{(\langle|\psi - \langle\psi\rangle_J|^p\rangle_J)^{1/p} \mid J \text{ is a subinterval of } I\}.$$

For our considerations, the main norm is BMO_2 , so we omit the subscript 2 in this case. By the John–Nirenberg inequality, these norms define the same Banach space of functions. We refer the reader to [46] for general BMO theory.

The BMO setting suggests it might be useful to consider the intermediate domains

$$\Omega_\varepsilon = \{x \in \mathbb{R}^2 \mid x_1^2 \leq x_2 \leq x_1^2 + \varepsilon^2\}, \quad (1.1.2)$$

which are given by the set-theoretic difference between Ξ_0 and

$$\Xi_\varepsilon = \{x \in \mathbb{R}^2 \mid x_1^2 + \varepsilon^2 < x_2\}.$$

The class $\mathcal{A}_{\Omega_\varepsilon}(I)$ generated by Ω_ε corresponds to the ε -ball of BMO. From now on, we fix some $\varepsilon > 0$.

The choice $f(t, t^2) = e^t$ and the analytic expression

$$\mathbf{B}_{\Omega_\varepsilon}(x; e^t) = \frac{1 - \sqrt{x_1^2 - x_2 + \varepsilon^2}}{1 - \varepsilon} e^{x_1 - \varepsilon + \sqrt{x_1^2 - x_2 + \varepsilon^2}}, \quad x \in \Omega_\varepsilon, \varepsilon < 1, \quad (1.1.3)$$

for the Bellman function (1.1.1), found in [41], lead to sharp integral forms of the John–Nirenberg inequality. This formula explains what we mean by the phrase “compute the Bellman function”. While \mathbf{B} is completely defined by (1.1.1), one can extract almost no information from this definition. By computation of a Bellman function we mean finding a good analytic expression for it. Usually our expressions are not as brief as (1.1.3). They are composed of integrations, differentiations, and also include specific functions. What is more, usually we define functions by cases, and the splitting into cases is defined in terms of inequalities for solutions to certain analytic equations. Though sometimes such a description is implicit, we always prove that the solutions exist and are unique.

The choice $f(t, t^2) = |t|^p$ leads to sharp inequalities between BMO_p norms; this case was considered in [42]. The sharp constants in the classical forms of the John–Nirenberg inequalities were found in [57]. The paper [31] derives sharp embeddings of BMO into Lorentz spaces from the Bellman functions in [57] (see [18] for similar estimates for BMO martingales) and [32] uses the case where $f(t) = f(t, t^2)$ is piecewise affine to derive the sharp embedding of BMO into the weak L_∞ -space. The paper [35] employs Bellman functions of the described type to find sharp constants in the embeddings of BMO into weighted BMO. The papers [47] and [59] find the sharp constants in the multiplicative inequalities $\|\varphi\|_{L_q} \leq c_{p,q} \|\varphi\|_{L_p}^{p/q} \|\varphi\|_{\text{BMO}}^{1-p/q}$. The reasoning relies on a three-dimensional Bellman function. In the first paper, it is almost immediately constructed from two-dimensional functions from [42]. The function of the second paper is quite complicated. For the sharp symmetric form of the John–Nirenberg inequality, see the recent paper [8]. The theory for the case of a generic f was created in [17] (see the short report [14], a simpler case study in [16], and an example of application of that theory in [55]).

We note that the Bellman function technique described above is perfectly suited for working with BMO_2 , however, it meets significant difficulties when applied to BMO_p norms with general p . All the papers cited above work with BMO_2 (the only

application of the Bellman function technique to BMO_p is given in [40] and [56]). For results about sharp constants in John–Nirenberg inequalities on BMO_1 , see [23, 27, 40] (the first two papers do not use Bellman function techniques).

The functions $\mathbf{B}_{\Omega_\varepsilon}(\cdot; f)$ for different values of ε are related to each other. The paper [17] suggested describing them all simultaneously. To be more precise, first, the function \mathbf{B} is computed for sufficiently small ε . Then one continuously increases ε and tracks the evolution of \mathbf{B} . This provides an algorithm for constructing \mathbf{B} for any ε . In this memoir, we will follow a similar route. This justifies the need for a continuous family of convex domains Ξ_ε that connect Ξ_0 with Ξ_1 .

1.1.2 The A_{p_1, p_2} classes

Let p_1 and p_2 , $p_1 > p_2$, be real numbers and let $Q > 1$. Consider the domain

$$\Omega_{p_1, p_2, Q} = \{x \in \mathbb{R}^2 \mid x_1, x_2 > 0 \text{ and } x_2^{1/p_2} \leq x_1^{1/p_1} \leq Qx_2^{1/p_2}\}. \quad (1.1.4)$$

This domain corresponds to the so-called A_{p_1, p_2} classes introduced in [54]. A weight $w: I \rightarrow \mathbb{R}_+$ belongs to A_{p_1, p_2} if the quantity

$$[w]_{p_1, p_2} = \sup\{\langle w^{p_1} \rangle_J^{1/p_1} \langle w^{p_2} \rangle_J^{-1/p_2} \mid J \text{ is a subinterval of } I\}$$

is finite. One may see that $[w]_{p_1, p_2} \leq Q$ if and only if $\varphi \in A_{\Omega_{p_1, p_2, Q}}$, where $\varphi = (w^{p_1}, w^{p_2})$.

The paper [54] generalizes [53], where the first sharp Bellman functions for Muckenhoupt weights were found; those Bellman functions lead to sharp constants in the reverse Hölder inequality. The scale A_{p_1, p_2} includes the classical Muckenhoupt classes A_p . More precisely, $A_{p, -1/(p-1)}$ coincides with A_p when $p \in (1, \infty)$. The case $p_1 = 1$ corresponds to the so-called Gehring classes. The paper [7] is devoted to this particular case. The sharp constants for the embedding of a Gehring class into A_p are found. That paper uses the Bellman function generated by an f of the form $f(x) = x_1^q$ for some q . The corresponding sharp weak-type reverse Hölder inequalities for A_{p_1, p_2} -weights were established in [37]; the latter paper uses the Bellman function for $f(x) = \chi_{[1, \infty)}(x_1)$. See [36] for the limiting case of the Muckenhoupt class A_1 .

We note that the definition (1.1.4) extends naturally to the limiting cases where one of the parameters p_1 and p_2 is equal to 1 and the other ∞ . The case of A_∞ equipped with the so-called Hruščev’s “norm”

$$[w]_\infty = \sup\{\langle w \rangle_J \exp(-\langle \log w \rangle_J) \mid J \text{ is a subinterval of } I\},$$

introduced in [12], is of particular importance. This corresponds to the domain

$$\Omega_{1, \infty, Q} = \{x \in \mathbb{R}^2 \mid e^{x_1} \leq x_2 \leq Qe^{x_1}\}, \quad (1.1.5)$$

where $Q > 1$, meaning $[w]_\infty \leq Q$ whenever $\varphi = (\log w, w)$ belongs to $A_{\Omega, \infty, Q}$. This domain and the study of the corresponding Bellman functions go back to [53]. See [1] for the study of this particular case and for applications of the obtained inequalities. The paper [34] delivers sharp forms of the principle that the logarithm of an A_∞ Muckenhoupt weight belongs to BMO; the reasoning relies on a Bellman function on the domain (1.1.5) with $f(x) = x_1^2$. In [40] and [56], the Bellman functions for $f(x) = |x_1|^p$ are computed and used to establish sharp forms of the John–Nirenberg inequalities for BMO_p , with $p \neq 2$.

1.1.3 Monotonic rearrangements and geometry of the Bellman function

One might wonder why a complicated optimization problem (1.1.1) has a transparent solution. The heuristic reason might be described in several ways. First, the notion of a monotonic rearrangement appears useful in this context. For a function $\xi: I \rightarrow \mathbb{R}$, define its non-increasing rearrangement $\xi^*: [0, |I|] \rightarrow \mathbb{R}$ by the formula

$$\xi^*(t) = \inf\{\lambda \mid |\{s \in I \mid \xi(s) > \lambda\}| \leq t\}.$$

Note that $\langle f(\xi) \rangle_I = \langle f(\xi^*) \rangle_{[0, |I|]}$ for any reasonable function f . It was proved in [21, 22] that the monotonic rearrangement operator does not increase the BMO_p norm for all $p \in [1, \infty)$. A similar principle also holds for the A_p constant: $[w^*]_p \leq [w]_p$ (see [2] for the case $p = 1$ and [24] for arbitrary p). Recall that we have assumed that $\partial\Xi_0$ may be parametrized with the first coordinate. A function $\varphi: I \rightarrow \partial\Xi_0$ is called non-increasing when its first coordinate φ_1 does not increase. The notion of monotonic rearrangement of a function $\varphi: I \rightarrow \partial\Xi_0$ is defined accordingly: this is a non-increasing function $\varphi^*: [0, |I|] \rightarrow \partial\Xi_0$ that has the same distribution. The paper [52] suggests a proof of a similar principle in the generality of A_Ω , that is, $\varphi^* \in A_\Omega([0, |I|])$, provided $\varphi \in A_\Omega(I)$. The reasoning is based on Bellman ideas. See [51] for dyadic classes and [45] for generalization to Campanato-type norms on VMO. Thus, since the monotonic rearrangement of a function in A_Ω also belongs to A_Ω , we may restrict the class of functions φ in (1.1.1) to non-increasing ones. This often simplifies the investigation of the function \mathbf{B} and gives hope for a closed formula for \mathbf{B} .

Another reason for a good formula for \mathbf{B} is its geometric description proved in [52] (see [62] for a generalization): this function is the pointwise minimal among all locally concave functions on Ω that satisfy the boundary condition $\mathbf{B}(x) = f(x)$, $x \in \partial\Xi_0$. The main idea of [52] is that the Bellman function (1.1.1) has a probabilistic representation in terms of certain discrete martingales; it resembles the representation formula for harmonic functions in terms of the Brownian motion. See, e.g., [26], for much more general representation formulas for solutions of a Hamilton–Jacobi–Bellman equation; the difference between [52] and [26] is that the latter paper uses continuous time martingales. The development of these ideas allowed us to transfer

all the results about Bellman problems for functions on the interval to the line and the circle, see [49]. In particular, the sharp constants in various forms of the John–Nirenberg inequalities and reverse Hölder inequalities for Muckenhoupt weights are the same for functions on the interval, the circle, and the line. We note that the question of finding any reasonable sharp forms of the said inequalities for functions of several variables is widely open (see [6] for questions about dimensional dependence of constants). Some dimensional estimates are obtained with the help of monotone rearrangement estimates (see [3]) and with the semigroup approach (see [44]).

1.1.4 Miscellaneous direct applications

We will describe two recent applications. The first one deals with sharp estimates of the distributions of martingales whose square functions are bounded. Namely, it turns out that the Bellman function (1.1.1) for the BMO case (i.e., defined on (1.1.2) with arbitrary f) yields good bounds for the following optimization problem: maximize $\mathbb{E}f(\psi)$ when ψ is the limit value of a martingale with the square function uniformly bounded by 1 and prescribed mean. Indeed, one may view ψ as a function whose $\text{BMO}([0, 1])$ norm is bounded by one. The paper [48] (see also the short report [50]) describes the class of f for which this principle leads to the exact solution of the initial problem; for other f , the said principle provides a fine supersolution.

The second application in [44] says that the Bellman function (1.1.1) delivers dimension-free bounds for the classes of functions such as $\text{BMO}(\mathbb{R}^d)$ or $A_p(\mathbb{R}^d)$, provided that one replaces averages over balls in the definitions of these spaces and Bellman functions with averages over specific semigroup kernels and uses the associated Garsia-type norms. The main feature of these semigroup averages is that they are of martingale nature, which allows us to apply martingale representations from [52]. It is unclear whether the resulting estimates are sharp.

1.1.5 Relationship with the moment method and differential geometry

Assume for a while that Ξ_1 is an empty set. Let $(t, g(t))$ be a parametrization of $\partial\Xi_0$. Then we arrive at the following optimization problem:

$$\int_0^1 \varphi_1 = x_1, \quad \int_0^1 g(\varphi_1) = x_2, \quad \int_0^1 f(\varphi_1) \rightarrow \max.$$

Here we have set $I = [0, 1]$. This is a simple example of the moment problem. We refer the reader to [19, 20, 25] for the description of the moment method. The solution to the problem is described geometrically. Let H be the convex hull of the three-dimensional curve $\gamma(t) = (t, g(t), f(t))$. Then the desired maximum equals

$$\sup\{s \in \mathbb{R} \mid (x_1, x_2, s) \in H\}.$$

Similar to (1.1.1), we will denote this quantity by $\mathbf{b}(x)$. Assume for a while that Ξ_0 is bounded. Using the Carathéodory theorem on convex hulls, we see that there exists an optimal φ_1 that attains at most 3 values. One may establish a similar principle for unbounded domains. As a result of these considerations, we see that the graph of \mathbf{b} is somehow flat: each point on this graph is a convex combination of at most 3 points of the boundary curve γ . The most common case is that $(x, \mathbf{b}(x))$ is a convex combination of two points of the boundary curve, A and B . Then we call the segment $[A, B]$ (as well as its projection onto the x_1x_2 -plane) that connects them a *chord*. In differential geometry, the tangent plane to the graph of \mathbf{b} at $(x, \mathbf{b}(x))$ is called a *bitangent plane*, because it is tangent to the boundary curve at both points A and B . For us, the condition on the points A and B for existence of the bitangent plane is expressed in the *cup equation* (see equation (3.4.2) below). The cup equation described in the present memoir has already been used by the authors in several more complicated Bellman problems, see [11, 13, 47, 59, 61].

A solution to the classical moment method as described in the books above works in the case where the boundary curve is somehow regular (say, it has positive torsion); this is related to the notion of a Chebyshev system. According to [25, Chapters III and IV], the boundary of the convex hull of a space curve $(\gamma_1(t), \dots, \gamma_{n+1}(t))$ given on an interval $[a, b]$, where $\gamma_j(t)$ are continuous functions, can be described in terms of *upper and lower principal representations*, provided that $(1, \gamma_1(t), \dots, \gamma_n(t))$ and $(1, \gamma_1(t), \dots, \gamma_{n+1}(t))$ are T_+ -systems on $[a, b]$. Recall that the system of continuous functions $(\gamma_0(t), \dots, \gamma_m(t))$ is called a T_+ -system if

$$\det(\{\gamma_i(t_j)\}_{i,j=0}^m) > 0 \quad \text{on } \Sigma = \{a \leq t_0 < \dots < t_m \leq b\}. \quad (1.1.6)$$

Verifying condition (1.1.6) on the set Σ may turn out to be cumbersome. If the map $t \mapsto (\gamma_0(t), \dots, \gamma_m(t))$ is of class $C^m((a, b))$, then (see [19, Chapter VIII]) the following easier condition:

$$\det(\{\gamma_i^{(j)}(t)\}_{i,j=0}^k) > 0 \quad \text{on } (a, b), \quad (1.1.7)$$

for all $k = 1, \dots, m$, implies (1.1.6). The system $(\gamma_0(t), \dots, \gamma_m(t))$ satisfying (1.1.7) is called an M_+ -system. Clearly, if the system $(\gamma_0(t), \dots, \gamma_m(t))$ is an M_+ -system, then any system $(\gamma_0(t), \dots, \gamma_k(t))$ is also an M_+ -system for any k , $0 \leq k \leq m$. Notice that the system $(1, t, g(t), f(t))$ is an M_+ -system if and only if $g''(t) > 0$ and $g''(t)f'''(t) - f''(t)g'''(t) > 0$ on (a, b) .

If γ does not satisfy any regularity assumptions of type (1.1.6), the combinatorial structure of the boundary of the convex hull can be extremely complicated. We refer the reader to [38] and [60] for descriptions of singularities of convex hulls of generic curves. See also [9] and [39] for relationship with the four vertex theorems.

The minimal concave and locally concave functions may be treated as concave (and locally concave) solutions of the degenerate homogeneous Monge–Ampère equation (see Section 2.2.1 below). See [5] and [10] for regularity results. Note that

the latter paper studies minimal locally concave functions on non-convex domains from a different point of view. In that paper, the boundary conditions are imposed on the whole boundary. We impose the boundary conditions on the convex (or *fixed*, see [52]) part of the boundary. This comes naturally from the Bellman function setting. The difference between the two settings is huge.

1.2 Description of exposition

The main body of the memoir is divided into four chapters. The general line of reasoning follows [17].

Chapter 2 contains a more formal introduction. In Section 2.1, we state the problem, list the requirements on the domains and the function f , define the Bellman function, and describe its simple properties. By simple properties we mean those that follow from the definition directly. We also provide some examples that explain the meaning of various conditions imposed on Ω and f . Section 2.2 surveys the duality theory from [52], introduces the strategy of proof, in particular, the notion of optimizers, and ends with a formal description of our results. We suggest consulting Chapter 2 for definitions while reading the further plan of the memoir or return to this section after reading Chapter 2.

We describe the possible local foliations in Chapter 3. In Section 3.1, we present a general description of what a foliation may look like. Section 3.2 introduces a *fence* – a foliation that consists of line segments attached to the fixed boundary. A fence may consist of chords, or tangents, or both of them. We study the conditions for the function constructed from this foliation to be locally concave. Section 3.3 contains the description of tangent domains and the corresponding standard candidates. In Section 3.4, we characterize chordal domains. Though the reasoning here follows the lines of [17], some non-trivial modifications are needed to make calculations simpler. Section 3.5 is devoted to the construction of a cup. Here the construction significantly differs from the one in [17], being quite lengthy and involved; Section 3.5 is split into several subsections. We introduce *forces* in Section 3.6. The forces are certain quantities that express the concavity of the constructed candidate. The main feature of these quantities is that they are negative and decrease as ε increases. The construction of the Bellman function relies heavily on these monotonicity formulas for the forces. We state and prove them in Section 3.6. We also introduce the notion of the *tail* of a force. We warn the reader that we have slightly changed the point of view concerning forces: now the domain of a force coincides with its tails by definition. In [17], the forces were defined on rays or the entire line and could attain both positive and negative values. In the present memoir, they are always negative. Section 3.7 contains the classification of linearity domains and Section 3.8 describes combinatorial properties of foliations. The material of the latter two sections is similar to the material of the corresponding sections in [17].

Chapter 4 describes the evolution of Bellman candidates and, in particular, contains the proof that a Bellman candidate exists for any admissible f and ε . In Section 4.1, we describe what we will refer to as the simple picture case, when the foliation consists of tangents, angles, cups, and, possibly, simple multicups. We also prove that, given an admissible f , there exists ε_1 such that for any $\varepsilon < \varepsilon_1$, there exists a simple Bellman candidate for f and ε . Section 4.2 contains the local monotonicity formulas for forces, the proofs of the existence of roots of balance equations, and other lemmas that will be used in further proofs. In Section 4.3, we describe the local evolution scenarios. Roughly speaking, the cups and full multicups grow, the trolleybuses shrink, the multibirdies and multitrolleybuses disintegrate. We state and prove a rigorous proposition for each of these principles. Section 4.4 contains the combinatorial reasoning that glues together local evolution scenarios. Since this combinatorial reasoning is identical to the one in [17], we omit it, providing a detailed citation.

Chapter 5 provides the theory of optimizers. In particular, it is proved that all the Bellman candidates constructed in Chapter 4 coincide with the corresponding Bellman functions. The opening Section 5.1 contains a geometric description of optimizers: it is useful to represent them as special curves in Ω , called delivery curves (they deliver an optimizer to a given point in Ω). Section 5.2 considers all local foliations and constructs optimizers for each of them. Then these “local” optimizers are glued together in Section 5.3. The concluding Section 5.4 considers various situations when the summability conditions on f are violated. We study the question of finiteness of the Bellman function in this setting and provide simple necessary and sufficient conditions on f , assuming this function satisfies the regularity condition that the torsion of γ changes sign a finite number of times. Sometimes, the Bellman function is finite when the summability assumptions are violated. In this case, the optimizers do not exist. However, we provide an optimizing sequence for each point.

1.3 Related questions

1.3.1 Case of bounded Ω

We may consider the case where Ω_0 and Ω_1 are both bounded domains. Say, the case where

$$\Omega_0 = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}, \quad \Omega_1 = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1 - \delta\}$$

naturally corresponds to the δ -“ball” of the class of BMO mappings of $[0, 1]$ into the unit circle S^1 . See the third example in [62, Section 2]. While it seems that all the constructions of the present memoir are applicable in this situation, there are two subtleties that distinguish the case where Ω_0 is bounded.

First, the boundaries of Ω_0 and Ω_1 are not graphs of functions anymore. Therefore, to adjust the methods of the current memoir to this situation, we need to find coordinate-free geometric versions of our reasoning. For some of them, this is not difficult. Say, a torsion is a geometric notion that does not depend on the choice of the coordinate system. Similarly, forces can be expressed geometrically as exterior products of certain vectors. However, we were not able to find a geometric interpretation for most of the material of Section 3.5.

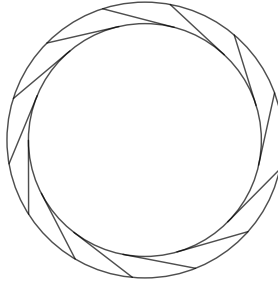


Figure 1.1. Impossible foliation.

Second, some topological effects related to the four vertex theorem (see [9]) come into play. That theorem says that a regular curve that bounds a convex surface changes the sign of its torsion at least four times. Based on the experience of the present memoir, it is natural to expect that the points where the torsion changes sign from $+$ to $-$ are origins of cups (at least for the case where δ is sufficiently small) and there are vertices of angles in neighborhoods of the points where the torsion changes sign from $-$ to $+$. In particular, the foliation of the type presented in Figure 1.1 is impossible.

1.3.2 Higher dimensional cases

One may consider the case where Ω_0 and Ω_1 are subsets of \mathbb{R}^d and state a similar Bellman function problem for them. For example, such a setting appears if one wishes to work with vector-valued BMO functions (see the second example in [62]). As it was proved in [62], the assertion that the Bellman function is the minimal locally concave function holds in this generality. Its geometry for arbitrary boundary conditions might be involved and difficult to describe. However, if one assumes that the domains Ω_0 and Ω_1 possess rotational symmetry, and f is also rotationally symmetric, then, maybe, the computation of the Bellman function may be reduced to the computation of certain functions of the types described in the present memoir. In particular, the problems for vector-valued BMO functions fit into this framework. This has an analogue in the probabilistic setting: many Burkholder problems in [33] allow generalizations to martingales attaining values in a Hilbert space.

Chapter 2

Setting and sketch of proof

2.1 Setting

2.1.1 Domains

Suppose that we have a family of C^3 -smooth curves $\mathbf{g}(\cdot; \varepsilon): \mathbb{R} \rightarrow \mathbb{R}^2$, $\varepsilon \in [0, \varepsilon_{\max})$, and let $\mathbf{g}(t; \varepsilon) = (\mathbf{g}_1(t; \varepsilon), \mathbf{g}_2(t; \varepsilon))$, $t \in \mathbb{R}$. In particular, we assume $\frac{\partial}{\partial t} \mathbf{g}(t; \varepsilon) \neq 0$. We assume that $\mathbf{g}(t, \varepsilon)$ is $C^{3,1}$ -smooth with respect to (t, ε) meaning the mixed derivative $\frac{\partial^4}{\partial t^3 \partial \varepsilon} \mathbf{g}$ is continuous. For any ε fixed, we will write $\mathbf{g}(t) = (\mathbf{g}_1(t), \mathbf{g}_2(t))$ omitting ε and considering it as a function of one variable. By $g(t)$, we will denote $\mathbf{g}(t; 0)$. Suppose that

$$\mathbf{g}'_1 > 0, \quad (2.1.1)$$

$$\mathbf{g}''_2 \mathbf{g}'_1 - \mathbf{g}'_2 \mathbf{g}''_1 > 0, \quad (2.1.2)$$

in particular, $\mathbf{g}(\mathbb{R})$ is a graph of a strictly convex function.

Let Ξ_ε be the strict epigraph of $\mathbf{g}(\cdot; \varepsilon)$:

$$\Xi_\varepsilon = \{(\mathbf{g}_1(t; \varepsilon), x_2) \in \mathbb{R}^2 \mid t \in \mathbb{R}, x_2 > \mathbf{g}_2(t; \varepsilon)\}.$$

We assume that $\Xi_{\varepsilon_1} \supset \Xi_{\varepsilon_2}$ if $\varepsilon_1 < \varepsilon_2$. Moreover, we assume $\mathbf{g}(t; \varepsilon_1) \notin \text{clos } \Xi_{\varepsilon_2}$ for any $t \in \mathbb{R}$ and $\varepsilon_1 < \varepsilon_2$.

For any $\varepsilon \in (0, \varepsilon_{\max})$, we assume that for any $t \in \mathbb{R}$, there are two tangent lines from the point $g(t)$ to Ξ_ε , and that each of these lines intersects with the curve $g(\cdot)$ twice. We note that this assumption slightly differs from the cone condition used in [52] and [62]. The cone condition imposed on the domain by those two papers is weaker than the present one. Let $w_L(t; \varepsilon) = \mathbf{g}(s_L; \varepsilon)$ and $w_R(t; \varepsilon) = \mathbf{g}(s_R; \varepsilon)$ be the tangency points with the curve $\mathbf{g}(\cdot; \varepsilon)$, here s_L and s_R are some functions of t and ε ; we choose $s_R < s_L$. Let $S_L(t; \varepsilon) = [g(t), w_L(t; \varepsilon)]$ and $S_R(t; \varepsilon) = [g(t), w_R(t; \varepsilon)]$ be the tangent segments. Here L and R mean left and right with respect to the tangency point. We will often omit the indices L and R and write simply $S(t; \varepsilon)$ and $w(t; \varepsilon)$ or even $S(t)$ and $w(t)$.

Remark 2.1.1. For ε fixed the functions $t \mapsto s_L$ and $t \mapsto s_R$ are C^2 -smooth and have positive derivatives: $s'_L > 0, s'_R > 0$.

Proof. It follows from (2.1.1) that the functions s_L and s_R are increasing.

Let either $s = s_L$ or $s = s_R$. This number is defined by the tangency equation

$$(\mathbf{g}_2(t; 0) - \mathbf{g}_2(s; \varepsilon)) \mathbf{g}'_1(s; \varepsilon) - (\mathbf{g}_1(t; 0) - \mathbf{g}_1(s; \varepsilon)) \mathbf{g}'_2(s; \varepsilon) = 0. \quad (2.1.3)$$

The partial derivative of the left-hand side with respect to t is

$$\mathbf{g}'_2(t; 0)\mathbf{g}'_1(s; \varepsilon) - \mathbf{g}'_1(t; 0)\mathbf{g}'_2(s; \varepsilon),$$

which is nonzero, because the segment connecting $\mathbf{g}(t; 0)$ and $\mathbf{g}(s; \varepsilon)$ is transversal to the curve $\mathbf{g}(\cdot; 0)$. The partial derivative of the left-hand side of (2.1.3) with respect to s is equal to

$$(\mathbf{g}_2(t; 0) - \mathbf{g}_2(s; \varepsilon))\mathbf{g}_1''(s; \varepsilon) - (\mathbf{g}_1(t; 0) - \mathbf{g}_1(s; \varepsilon))\mathbf{g}_2''(s; \varepsilon).$$

This is nonzero because $\mathbf{g}_1(t; 0) - \mathbf{g}_1(s; \varepsilon) \neq 0$ and due to (2.1.3) and (2.1.2).

The implicit function theorem guarantees that s and t are C^2 -smooth functions of each other and that $s'(t)$ is finite and nonzero. Therefore, $s'(t) > 0$. ■

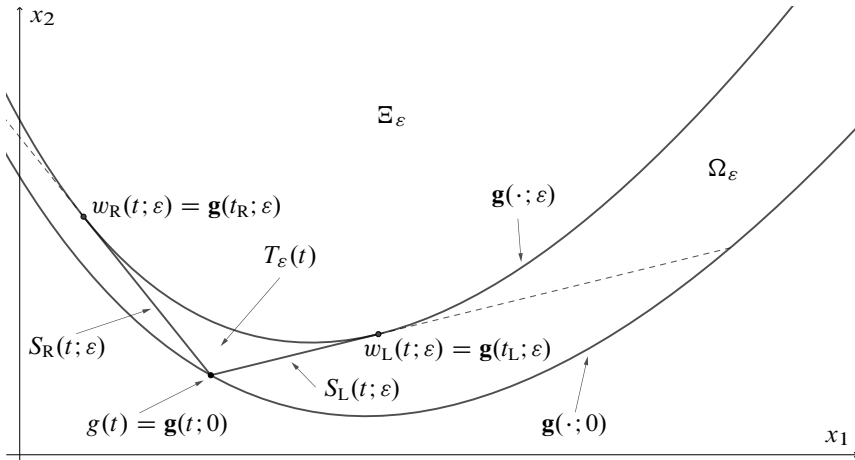


Figure 2.1. Triangle $T_\varepsilon(t)$.

Let $T_\varepsilon(t)$ be the closed curvilinear triangle with the vertex $g(t)$ whose sides are $S_L(t; \varepsilon)$, $S_R(t; \varepsilon)$, and the part of the curve $\mathbf{g}(\cdot; \varepsilon)$ between the two tangency points, see Figure 2.1. We define the domain

$$\Omega_\varepsilon = \bigcup_{t \in \mathbb{R}} T_\varepsilon(t). \quad (2.1.4)$$

One may see that $\Omega_\varepsilon = \text{clos } \Xi_0 \setminus \Xi_\varepsilon$. We provide several examples of admissible domains Ω_ε in Figure 2.2, see Section 5.4.7 for more information concerning these examples. The domain on the last picture of Figure 2.2 formally does not satisfy (2.1.2), but after application of a suitable isometry of the plane, it becomes admissible for the theory. The parametrization by t running from $-\infty$ to $+\infty$ is convenient

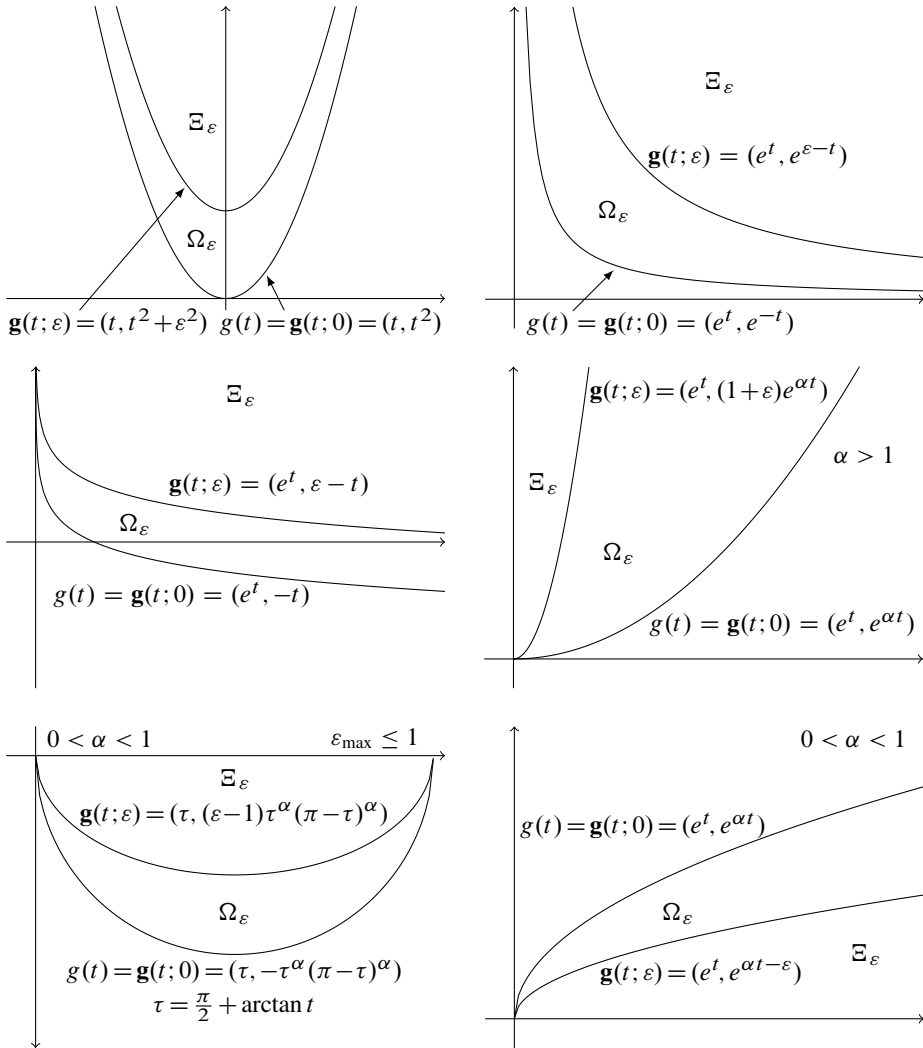


Figure 2.2. Examples of the domains Ω_ε .

for theoretical construction, but sometimes there are more natural parametrizations, as we can see in Figure 2.2.

We also make the following technical assumption. Let $\kappa_L(t; \varepsilon)$ and $\kappa_R(t; \varepsilon)$ be the slopes of the sides S_L and S_R of the triangle $T_\varepsilon(t)$, i.e.,

$$\kappa = \frac{w_2 - g_2}{w_1 - g_1}. \tag{2.1.5}$$

It follows from the implicit function theorem that the functions $s_L, s_R, w_L, w_R,$ and κ are $C^{2,1}$ -smooth with respect to (t, ε) in the same sense as before. The domains Ξ_ε

decrease by inclusion, therefore, the function κ is monotone with respect to ε : κ_L increases while κ_R decreases. We assume a bit more: $\frac{\partial}{\partial \varepsilon} \kappa(t, \varepsilon) \neq 0$ for any t and ε .

The curve $g(\mathbb{R})$ is called the *fixed boundary* of the domain Ω_ε and is denoted by $\partial_{\text{fixed}} \Omega_\varepsilon$. The curve $\mathbf{g}(\mathbb{R}; \varepsilon)$ is called the *free boundary* of Ω_ε and is denoted by $\partial_{\text{free}} \Omega_\varepsilon$.

Remark 2.1.2. We will use the sign \preceq for two points on $\partial_{\text{fixed}} \Omega$ to indicate their disposition: $g(t) \preceq g(s)$ if $t \leq s$.

Since the segment $S(t)$ is tangent to the curve $\tau \mapsto \mathbf{g}(\tau; \varepsilon)$, there exists a scalar function λ satisfying the following vector-valued identity:

$$\lambda(t)(g(t) - w(t)) = w'(t). \quad (2.1.6)$$

There are two cases: $w = w_R$ defines $\lambda = \lambda_R(t; \varepsilon)$ and $w = w_L$ defines $\lambda = \lambda_L(t; \varepsilon)$, respectively. Let us note that $\lambda_R > 0$ and $\lambda_L < 0$ pointwise, the strict inequalities follow from $w'(t) = \mathbf{g}'(s; \varepsilon)s'(t) \neq 0$ that is guaranteed by Remark 2.1.1. We formulate important assumptions on the domain Ω in terms of λ and w : for all $\varepsilon \in (0, \varepsilon_{\max})$ and $t_0 \in \mathbb{R}$,

$$(1 + |w_R(t; \varepsilon) - g(t)|) \cdot \exp\left(-\int_t^{t_0} \lambda_R(\tau; \varepsilon) d\tau\right) \rightarrow 0, \quad t \rightarrow -\infty, \quad (2.1.7)$$

$$(1 + |w_L(t; \varepsilon) - g(t)|) \cdot \exp\left(\int_{t_0}^t \lambda_L(\tau; \varepsilon) d\tau\right) \rightarrow 0, \quad t \rightarrow +\infty. \quad (2.1.8)$$

These conditions are technical. If some of them fail, the general theory works, but some explanations need minor changes. The details are given in Section 5.4.

Recall the following definition from [15].

Definition 2.1.3. Let $\Omega = \Omega_\varepsilon$, let $I \subset \mathbb{R}$ be an interval and let $\varphi: I \rightarrow \partial_{\text{fixed}} \Omega$ be a summable function. We say that the function φ belongs to the class $\mathbf{A}_\Omega(I)$ if $\langle \varphi \rangle_J \in \Omega$ for every subinterval $J \subset I$.

2.1.2 Extremal problem and the Bellman function

Definition 2.1.4. Let $f: \partial_{\text{fixed}} \Omega_\varepsilon \rightarrow \mathbb{R}$ be some measurable function. We define the Bellman function

$$\mathbf{B}_\varepsilon(x; f) \stackrel{\text{def}}{=} \sup\{\langle f(\varphi) \rangle_I \mid \langle \varphi \rangle_I = x, \varphi \in \mathbf{A}_{\Omega_\varepsilon}(I)\}. \quad (2.1.9)$$

In this subsection, we summarize some simple properties of the Bellman function defined by formula (2.1.9) and try to explain the choice of the Bellman function in view of the extremal problem we study. We will often use the function $f: \mathbb{R} \rightarrow \mathbb{R}$ associated with f given above as follows:

$$f(t) = f(g(t)), \quad t \in \mathbb{R}.$$

We will also use the symbols $\mathbf{B}_\varepsilon(\cdot)$ and $\mathbf{B}_\varepsilon(\cdot; f)$ for the Bellman function (2.1.9) depending on what we would like to emphasize. When ε and f are fixed, we will omit them and write simply $\mathbf{B}(x)$. Now, we formulate the easiest properties of the Bellman function that do not need any conditions on f .

Remark 2.1.5. The Bellman function \mathbf{B}_ε does not depend on the interval I where the class $\mathbf{A}_{\Omega_\varepsilon}$ is defined.

Indeed, using a linear change of variables, one can transform a function $\varphi \in \mathbf{A}_{\Omega_\varepsilon}(I)$ into another function $\tilde{\varphi} \in \mathbf{A}_{\Omega_\varepsilon}(\tilde{I})$ so that all the averages in formula (2.1.9) do not change. Thus, the supremum defined by formula (2.1.9) is taken over the same subset of the real numbers.

The next remark allows us to estimate integral functionals from below. One can consider another Bellman function,

$$\mathbf{B}_\varepsilon^{\min}(x; f) \stackrel{\text{def}}{=} \inf\{\langle f(\varphi) \rangle_I \mid \langle \varphi \rangle_I = x, \varphi \in \mathbf{A}_{\Omega_\varepsilon}(I)\}.$$

Of course, the minimal Bellman function can be easily expressed in terms of the maximal one.

Remark 2.1.6. We have $\mathbf{B}_\varepsilon^{\min}(x; f) = -\mathbf{B}_\varepsilon(x; -f)$.

Now we begin to study the domain of \mathbf{B}_ε . The following definition seems to be useful for all further reasoning.

The Bellman function (2.1.9) is defined everywhere on \mathbb{R}^2 . However, this function is equal to $-\infty$ if the supremum is taken over the empty set. We drop such points from the domain of the Bellman function.

Definition 2.1.7. By the *Bellman domain* of the function \mathbf{B}_ε , we mean the set of points $x \in \mathbb{R}^2$ for which there exists a function $\varphi \in \mathbf{A}_{\Omega_\varepsilon}$ satisfying $\langle \varphi \rangle_I = x$.

Proposition 2.1.8. *The domain Ω_ε defined in (2.1.4) coincides with the Bellman domain of \mathbf{B}_ε .*

Proof. First, by the definition of $\mathbf{A}_{\Omega_\varepsilon}$, for any $\varphi \in \mathbf{A}_{\Omega_\varepsilon}$ we have $\langle \varphi \rangle_I \in \Omega_\varepsilon$. Second, for any given point $x \in \Omega_\varepsilon$, there exists some $t \in \mathbb{R}$ such that $x \in T_\varepsilon(t)$. We draw the tangent to $\partial_{\text{free}} \Omega_\varepsilon$ passing through x . It follows from conditions on the domain that this line intersects $\partial_{\text{fixed}} \Omega_\varepsilon$ twice, say, at $g(a)$ and $g(b)$. It is important that $x \in [g(a), g(b)] \subset \Omega_\varepsilon$. Let $\alpha \in [0, 1]$ be such that $x = \alpha g(a) + (1 - \alpha)g(b)$. Define the function φ as follows:

$$\varphi = g(a)\chi_{[0, \alpha]} + g(b)\chi_{[\alpha, 1]}.$$

It is clear that $\langle \varphi \rangle_{[0, 1]} = x$ and $\langle \varphi \rangle_J \in [g(a), g(b)] \subset \Omega_\varepsilon$ for any $J \subset [0, 1]$. Therefore, we have $\varphi \in \mathbf{A}_{\Omega_\varepsilon}$. ■

This proposition does not say whether the Bellman function is finite on Ω_ε , it only cuts off those points of the plane where it is *a priori* infinite.

Proposition 2.1.9. *We have $\mathbf{B}_\varepsilon(g(t); f) = f(t)$ for all $t \in \mathbb{R}$.*

Proof. The function φ whose average $\langle \varphi \rangle_I$ lies on the fixed boundary is constant almost everywhere, because the curve $\partial_{\text{fixed}} \Omega_\varepsilon$ is a graph of a strictly convex function. This constant coincides with the average, thus $\varphi = g(t)$. Consequently, the set we are taking the supremum over consists of a single number $f(t)$, therefore $\mathbf{B}_\varepsilon(g(t); f) = f(t)$. ■

2.1.3 Conditions on f

The function f will be subject to some restrictions. These restrictions are of two types. The first type comes from quantitative aspects. We are interested in finite Bellman functions, and we want the function $f(\varphi)$ to be integrable for any function $\varphi \in \mathcal{A}_{\Omega_\varepsilon}$, $\varepsilon < \varepsilon_{\max}$, in order to have well-defined averages in (2.1.9). These conditions are expressed in terms of summability properties of f . The second type of conditions corresponds to regularity properties of f . These conditions make the structure of the Bellman function less complicated and thus describable.

We begin with conditions of the second type. We require $f \in C^2(\mathbb{R})$. Consider the curve

$$\gamma(t) = (g(t), f(t)), \quad t \in \mathbb{R},$$

which is the graph of the boundary condition, see Proposition 2.1.9. We introduce the following object:

$$\mathbf{T}(t) = \det \begin{pmatrix} \gamma'(t) \\ \gamma''(t) \\ \gamma'''(t) \end{pmatrix} = \det \begin{pmatrix} g_1'(t) & g_2'(t) & f'(t) \\ g_1''(t) & g_2''(t) & f''(t) \\ g_1'''(t) & g_2'''(t) & f'''(t) \end{pmatrix}, \quad t \in \mathbb{R}. \quad (2.1.10)$$

We will impose such conditions on f that \mathbf{T} will be a signed measure, see Condition 2.1.11. The sign of \mathbf{T} coincides with the sign of the torsion of the curve γ . Since we assume γ to be C^2 , the measure \mathbf{T} does not have atoms.

Definition 2.1.10. Let μ be a signed measure on the line. The complement of its support is an open subset of the line, so it is a union of several intervals (finite or countable number of them). We call the closure of each such interval a *solid root* of μ . If μ is neither positive nor negative in every neighborhood of its solid root, then such a solid root is called *essential*.

Let $\mathbf{T} = \mathbf{T}_+ + \mathbf{T}_-$ be the Hahn decomposition of the measure \mathbf{T} . Note that the set $\text{supp } \mathbf{T}_+ \cap \text{supp } \mathbf{T}_-$ is the set of points where \mathbf{T} “changes its sign”. This set is closed. The points of $\text{supp } \mathbf{T}_+ \cap \text{supp } \mathbf{T}_-$ are also called essential roots. Therefore,

an essential root is a maximal by inclusion connected subset of the line such that \mathbf{T} vanishes on it but is neither negative nor positive in any neighborhood of it. The regularity condition we will impose on \mathbf{T} is that it has only a finite number of essential roots. If γ were C^3 -smooth, this condition would be the same as if the function \mathbf{T} had only a finite number of changes of sign.

Let us introduce the following functions:

$$\kappa_2 = \frac{g'_2}{g'_1}, \quad \kappa_3 = \frac{f'}{g'_1}, \quad \mathfrak{K} = \frac{\kappa'_3}{\kappa'_2}. \quad (2.1.11)$$

Note that κ_2 and κ_3 are the slopes of the projections of γ' onto (x_1x_2) - and (x_1x_3) -coordinate planes.

Since $g'_1 > 0$, $\kappa'_2 \neq 0$ (see (2.1.1) and (2.1.2)), and

$$\mathbf{T} = \mathfrak{K}' \cdot (g'_1)^3 \cdot (\kappa'_2)^2, \quad (2.1.12)$$

the signs of \mathbf{T} and \mathfrak{K}' coincide. We formulate the condition discussed above in terms of \mathfrak{K} .

Condition 2.1.11. *The function f is twice continuously differentiable, \mathfrak{K} is piecewise monotone and has only a finite number of monotonicity intervals.*

Remark 2.1.12. Condition 2.1.11 implies that the functions f and κ_3 are piecewise monotone and change their sign only a finite number of times.

Proof. The function \mathfrak{K} changes its sign only a finite number of times, $\kappa'_2 > 0$, therefore the same is true for $\kappa'_3 = \kappa'_2 \mathfrak{K}'$, and hence for κ_3 . But $g'_1 > 0$ and $f' = \kappa_3 g'_1$, thus f' has only a finite number of changes of sign, therefore the same is true for f . ■

The essential roots of \mathbf{T} will play a significant role in what follows, therefore we fix the notation for them.

Definition 2.1.13. The essential roots of \mathbf{T} (or \mathfrak{K}') are closed intervals (which can be single points or rays) c_0, c_1, \dots, c_n and v_1, v_2, \dots, v_n such that $c_0 < v_1 < c_1 < v_2 < \dots < v_n < c_n$. The measure \mathbf{T} “changes sign” from “−” to “+” at v_i , from “+” to “−” at c_i .

We make an agreement that if in a neighborhood of $-\infty$ we have $\mathbf{T} < 0$, then $c_0 = -\infty$. Similarly, if in a neighborhood of $+\infty$ we have $\mathbf{T} > 0$, then $c_n = +\infty$. What is more, v_i or c_i is an interval (not a point) if and only if it is an essential solid root in the sense of Definition 2.1.10.

In light of our definition, sometimes we will have to treat the intervals as if they were points. We write $\text{dist}(x, y)$ for the usual distance between subsets x and y of the real line. We will need it only to denote the distance between either two intervals or an interval and a point. Moreover, sometimes we will write, for example, $a_n \rightarrow w$,

where w is a root, e.g., it can be an interval. In such situations we mean that for every neighborhood of w , all but finite number of members of $\{a_n\}_n$ lie in it. What is more, the set of intervals has an essential ordering, that is, $[a, b]$ is less than $[c, d]$ if and only if $b < c$. We have already used this ordering in Definition 2.1.13. We will also often use the notation α^r and α^l for the right and left endpoints of the interval α .

Let us turn to the conditions of the first type (summability conditions at infinities).

Condition 2.1.14. For any $\varepsilon < \varepsilon_{\max}$, we have

$$\int_{-\infty}^v f'(\tau) \exp\left(-\int_{\tau}^v \lambda_R(s; \varepsilon) ds\right) d\tau > -\infty, \quad (2.1.13)$$

$$\int_v^{+\infty} f'(\tau) \exp\left(\int_v^{\tau} \lambda_L(s; \varepsilon) ds\right) d\tau < +\infty \quad (2.1.14)$$

for any $v \in \mathbb{R}$. Recall that λ_L and λ_R are defined in (2.1.6).

These conditions look cumbersome, but it will be clear that they appear quite naturally, see Chapter 5, Propositions 5.2.6 and 5.2.8. If (2.1.13) fails, but (2.1.7) holds, then the function B is infinite because one can construct a special test function φ (that is, an optimizer in Proposition 5.2.5, see also Definition 2.2.6) such that $\langle f(\varphi) \rangle = +\infty$. Similarly, if (2.1.14) fails, but (2.1.8) holds, then the function B is also infinite. See Section 5.4.6 for the details.

2.1.4 Smoothness conditions

Here we collect all the smoothness conditions we impose on the functions we discussed in this section.

We suppose that \mathbf{g} is $C^{3,1}$ -smooth and then the functions $s_L, s_R, w_L, w_R, \kappa_L, \kappa_R$ are $C^{2,1}$ -smooth with respect to (t, ε) and κ_2 is C^2 -smooth. We also have $\frac{\partial}{\partial t} \mathbf{g}(t, \varepsilon) > 0$, $\frac{\partial}{\partial \varepsilon} \kappa(t, \varepsilon) \neq 0$. The functions λ_R, λ_L are $C^{1,1}$. The function f is C^2 with f'' being piecewise monotone with a finite number of monotonicity intervals, therefore $\kappa_3 \in C^1$ and \mathfrak{K} is a piecewise monotone continuous function with a finite number of monotonicity intervals, and \mathfrak{R}' (as well as \mathbf{T}) is a signed continuous measure.

2.2 Locally concave functions

We will use abstract results from [52] concerning the relation between the Bellman function and locally concave functions. We start with the definition of local concavity.

Definition 2.2.1. The function $G: \Omega \mapsto \mathbb{R}$ is said to be *locally concave* on Ω if it is concave on every convex subdomain of Ω . We introduce the set of locally concave functions on Ω majorising f on $\partial_{\text{fixed}} \Omega$:

$$\Lambda_{\Omega, f} \stackrel{\text{def}}{=} \{G \mid G \text{ is locally concave on } \Omega, G|_{\partial_{\text{fixed}} \Omega} \geq f\}.$$

Define the function $\mathfrak{B}_{\Omega, f}$ to be the pointwise infimum of the functions from $\Lambda_{\Omega, f}$:

$$\mathfrak{B}_{\Omega, f}(x) = \inf_{G \in \Lambda_{\Omega, f}} G(x), \quad x \in \Omega.$$

It is not difficult to see that a function is locally concave if and only if it is concave on every segment that belongs to Ω entirely.

The proposition below resembles [52, Proposition 6.1]. It follows from [52, Proposition 6.1] in the case where the limits $\lim_{t \rightarrow \pm\infty} g(t)$ are infinite. If one of the limits is finite, one should use the projective transformation trick to reduce it to the case of infinite limits, see [52, Appendix A].

Proposition 2.2.2. *Let the function f be locally bounded from below, and let the function $\mathfrak{B}_{\Omega_\varepsilon, \max(f, 0)}$ be finite. Then the integral $\langle f(\varphi) \rangle_I$ is well defined for all $\varphi \in A_{\Omega_\varepsilon}$, and $B_\varepsilon(\cdot; f) = \mathfrak{B}_{\Omega_\varepsilon, f}$.*

Proposition 2.2.2 will serve as one of the main ingredients in the proof of Theorem 2.2.7.

2.2.1 Monge–Ampère equation and ruled surfaces

Using simple convex geometry arguments, one can see that $\mathfrak{B}_{\Omega_\varepsilon, f}$ is affine in some directions. The precise statement is as follows.

Theorem 2.2.3. *Let f satisfy Conditions 2.1.11 and 2.1.14. Then the function $\mathfrak{B} = \mathfrak{B}_{\Omega_\varepsilon, f}$ satisfies the following conditions:*

- (1) *For every point $x \in \text{int } \Omega_\varepsilon$, there is a nonzero vector $\Theta(x)$ such that \mathfrak{B} is affine along the line $\ell(x) = x + \mathbb{R}\Theta(x)$ in a neighborhood of x . If there are at least two noncollinear vectors Θ_1 and Θ_2 with this property, then \mathfrak{B} is affine in a neighborhood of x . If this vector is unique, we call the maximal (by inclusion) segment of $\ell(x)$ containing x on which \mathfrak{B} is affine the extremal segment.*
- (2) *The function \mathfrak{B} is differentiable and its gradient is constant on each extremal segment.*
- (3) *The extremal segments cannot intersect the free boundary transversally, but only tangentially.*

This theorem provides a partition of Ω_ε into sets of two types. The sets of the first type are extremal segments, along which \mathfrak{B} is affine. We note (and this can be easily proved) that the extremal segments cannot “stop” inside Ω_ε . One of the endpoints of an extremal segment belongs to the fixed boundary, the second one could lie either on the fixed boundary as well, or it is the point of tangency with the free boundary. The sets of the second type are the two-dimensional domains where \mathfrak{B} is affine; we call them *domains of linearity*.

We should make a remark on the term “domain”. We call a domain every open connected set united with some part (or none) of its boundary.

We will not prove Theorem 2.2.3, because, from a formal point of view, we do not need it (however, it will follow from our general considerations, e.g., Theorem 5.3.2 far below). It only helps us to guess the Bellman function. It leads us to the notion of a *Bellman candidate*.

Definition 2.2.4. Let B be a continuous locally concave function on a subdomain ω of Ω_ε satisfying the boundary condition on $\omega \cap \partial_{\text{fixed}} \Omega_\varepsilon$. The function B is called a *Bellman candidate* on ω if there exists a *foliation* on ω . This means that ω can be represented as a union of several domains, $\omega = \bigcup_i \omega^i$, such that $B \in C^1(\omega^i)$ and B is either affine in ω^i (thus, ω^i is a domain of linearity) or ω^i is foliated by straight line segments along which the differential of B is constant (they are the extremal segments).

If B is twice differentiable at some inner point $x \in \Omega_\varepsilon$, then $\frac{d^2 B}{dx^2} \leq 0$, i.e., the second differential of B is negative semi-definite, because B is locally concave. This matrix has the vector $\Theta(x)$ in the kernel, thus its determinant is zero. This remark clarifies the name of the subsection, since the resulting equation is called the *homogeneous Monge–Ampère equation*:

$$B_{x_1 x_1} B_{x_2 x_2} - B_{x_1 x_2}^2 = 0.$$

The homogeneous Monge–Ampère equation must hold almost everywhere for the function \mathfrak{B} , because a locally concave function is almost everywhere twice differentiable. However, it does not have to hold everywhere, because the Bellman function does not have to be C^2 -smooth even for very smooth boundary values f . For example, see the function $N_{\varepsilon, p}$ from [42, Lemma 6.1] with $p > 2$; it is not C^2 -smooth, however, its boundary data is C^∞ for p being an even integer greater than 2.

2.2.2 Optimizers

Let B be a Bellman candidate on the whole domain Ω_ε , i.e., let it satisfy Definition 2.2.4 with $\omega = \Omega_\varepsilon$. This subsection provides a method of verification that the candidate B coincides with the Bellman function. By Proposition 2.2.2, the inequality $\mathbf{B}_\varepsilon \leq B$ holds. To prove the reverse inequality, $B(x) \leq \mathbf{B}_\varepsilon(x)$, for a point $x, x \in \Omega_\varepsilon$, it is sufficient to find a function $\varphi \in A_{\Omega_\varepsilon}$, with $\langle \varphi \rangle_I = x$, such that $B(x) \leq \langle f(\varphi) \rangle_I$. Indeed, by the definition of the Bellman function, formula (2.1.9), $\langle f(\varphi) \rangle_I \leq \mathbf{B}_\varepsilon(x)$, consequently, $B(x) \leq \mathbf{B}_\varepsilon(x)$. We introduce some notions.

Definition 2.2.5. Let $x \in \Omega_\varepsilon$. We call a function $\varphi \in A_{\Omega_\varepsilon}$ a *test function* for x if

$$\langle \varphi \rangle_I = x.$$

Definition 2.2.6. Let $x \in \Omega_\varepsilon$ and let B be a Bellman candidate. We call a measurable function φ an *optimizer* for B at x if it satisfies two conditions:

- φ is a test function for x ,
- $B(x) = \langle f(\varphi) \rangle_I$.

So, in order to prove that a candidate B coincides with the Bellman function, it suffices to provide, for each point x in Ω_ε , at least one optimizer for B at x . What is the way to do this? We may consider only monotone optimizers (with respect to the ordering \leq defined in Section 2.1). This follows from the fact that the class $\mathcal{A}_{\Omega_\varepsilon}$ is invariant with respect to monotonic rearrangement. This fact is proved in [52, Corollary 3.12].

The following property of monotonic rearrangement is useful: for the monotone rearrangement φ^* of φ , we have $\langle f(\varphi) \rangle_I = \langle f(\varphi^*) \rangle_I$. Therefore, φ^* is an optimizer provided φ is. We do not need this consideration formally, but it helps us to guess the optimizers. In light of this, we consider only monotone optimizers. A more detailed discussion concerning optimizers is postponed until Chapter 5.

2.2.3 General principles and description of results

One of our main aims is to prove the following theorem.

Theorem 2.2.7. *Let f satisfy Conditions 2.1.11 and 2.1.14. Then*

$$\mathbf{B}_\varepsilon(x; f) = \mathfrak{B}_{\Omega_\varepsilon, f}.$$

In fact, in the whole memoir we study how to construct the minimal locally concave function $\mathfrak{B}_{\Omega_\varepsilon, f}$. In Chapters 3 and 4, for every f satisfying Conditions 2.1.11 and 2.1.14, we construct specific locally concave functions G on Ω_ε with the boundary conditions $G|_{\partial_{\text{fixed}} \Omega_\varepsilon} = f$. These functions are called Bellman candidates, see Definition 2.2.4. Due to Remark 2.1.12, the function f does not change its sign outside a compact set, therefore it is easy to find a function \tilde{f} satisfying Conditions 2.1.11 and 2.1.14, which coincides with $\max(f, 0)$ outside a compact set. This would imply that the function $\mathfrak{B}_{\Omega_\varepsilon, \tilde{f}}$ is finite, whence $\mathfrak{B}_{\Omega_\varepsilon, \max(f, 0)}$ is finite as well. Proposition 2.2.2 now gives the statement of Theorem 2.2.7. In particular, the function G constructed in our considerations satisfies the inequality $\mathbf{B}_\varepsilon(\cdot; f) \leq G$ on Ω_ε . We use optimizers to prove that G coincides with $\mathbf{B}_\varepsilon(\cdot; f)$; this will be done in Chapter 5, see Theorem 5.3.1.

Main results.

1. We provide an algorithm to calculate the Bellman function \mathbf{B}_ε , for f satisfying Conditions 2.1.11 and 2.1.14, and describe the evolution of this function with respect to ε ; as a corollary, we obtain Theorem 2.2.7.

2. We show that for any C^1 function f not satisfying at least one of the two conditions in Condition 2.1.14, $\mathbf{B}_\varepsilon(\cdot; f) = +\infty$ everywhere except the fixed boundary.
3. We investigate the case when the technical conditions (2.1.7) and (2.1.8) for the domain do not hold. The theory still works with minor modifications: it could happen that there exists no optimizer for some points $x \in \Omega_\varepsilon$; in such a case, instead of one optimizer, we construct an optimizing sequence of functions $\varphi_n \in \mathbf{A}_{\Omega_\varepsilon}$ such that $\langle \varphi_n \rangle_I = x$ and $\langle f(\varphi_n) \rangle_I \rightarrow \mathbf{B}_\varepsilon(x; f)$. We also give criteria for the function \mathbf{B}_ε to be finite when conditions (2.1.7) and (2.1.8) fail.

Some explanation is needed. By building the Bellman function we mean mainly the construction of the corresponding foliation. This foliation evolves continuously and obeys certain monotonicity rules that are also described. In the first point we intend to provide some expression for \mathbf{B}_ε that contains integrals, differentiation, and solution of some implicit equations. We always prove that those equations are well solvable, i.e., they do not have infinitely many solutions.

Chapter 3

Patterns for Bellman candidates

3.1 Preliminaries

The purpose of this section is to construct Bellman candidates (see Definition 2.2.4) on various domains. The global foliation for the Bellman function may be rather complicated, but its local structure is easy to describe. We give some heuristics to classify local Bellman candidates.

Consider a minimal locally concave function and its foliation provided by Theorem 2.2.3. We recall that this foliation consists of extremal segments and linearity domains.

The extremal segments are of two types: those that connect two points on the fixed boundary, called *chords*, and those that connect a point on the fixed boundary with a point on the free boundary, called *tangents*. We note that a chord can be tangent to the free boundary. Such a chord is called a *long* one.

It is convenient to classify linearity domains by the number of their points on the fixed boundary. We distinguish the linearity domains that have one point on the fixed boundary, the ones that have two points on the fixed boundary, and all the others. A more detailed classification will be provided later in Section 3.7.

A global foliation is glued from local ones. We explain the informal meaning of the word “glue” we use. Consider two subdomains Ω^1 and Ω^2 of Ω_ε . Let B_1 be a Bellman candidate on Ω^1 and let B_2 be a Bellman candidate on Ω^2 . Suppose that $B_1 = B_2$ on $\Omega^1 \cap \Omega^2$. Consider the function B defined on the union domain $\Omega = \Omega^1 \cup \Omega^2$ as a concatenation of B_1 and B_2 (i.e., $B = B_1$ on Ω^1 and $B = B_2$ on Ω^2). Suppose that this function B is C^1 -smooth. In such a case, it is locally concave, provided the functions B_1 and B_2 are, see Proposition 3.1.2 below. Thus, it is a Bellman candidate on Ω . Its foliation coincides with the foliation for B_1 on Ω^1 and with the foliation for B_2 on Ω^2 . We say that the foliation for B is glued from the foliations for B_1 and B_2 .

We have used the following fact in the explanation: a C^1 -concatenation of two locally concave functions is locally concave. To formulate this claim rigorously, we need several new notions.

Definition 3.1.1. Suppose that Ω is a subdomain of Ω_ε . We call Ω an *induced convex set* if for every segment $l \subset \Omega_\varepsilon$, the set $\Omega \cap l$ is convex. As usual, for any set $\omega \subset \Omega_\varepsilon$, we define its *induced convex hull* $\text{ind conv}(\omega)$ as the minimal induced convex set which contains ω .

All the domains we use for building Bellman candidates are induced convex.

Proposition 3.1.2. *Suppose that the domains Ω^1 and Ω^2 are induced convex in Ω_ε . Suppose that a C^1 -smooth function B is locally concave on each of the domains Ω^i , $i = 1, 2$. Then it is locally concave on $\Omega^1 \cup \Omega^2$.*

Proof. To prove the claim we establish that the restriction of B to every segment $l \subset \Omega^1 \cup \Omega^2$ is concave. We have $l = (l \cap \Omega^1) \cup (l \cap \Omega^2)$. Each of the sets $l \cap \Omega^1$ and $l \cap \Omega^2$ is convex, i.e., they are either segments or empty sets (the latter case is trivial). By the hypothesis, B is concave on each of these segments. Using C^1 -smoothness of B at a common point of these segments, we get that $B|_l$ is concave. ■

Now we can state that a C^1 -smooth concatenation of two Bellman candidates is a Bellman candidate provided their domains are induced convex.

We turn to building Bellman candidates. Usually, we will give only sufficient conditions for a foliation and a function f that generate a Bellman candidate. However, to be ready to construct the Bellman function, we have to examine all possible local Bellman candidates. So, the conditions we provide are usually also necessary. To make the story shorter, sometimes we will not prove this necessity, because we do not need it.

To describe combinatorial properties of foliations, we associate a special oriented graph with each foliation. Generally, its vertices correspond to the linearity domains, whereas its edges correspond to the domains of extremal segments. A vertex is incident to an edge if the corresponding two domains are adjacent. We postpone a more detailed description of the graph to Section 3.8.2.

3.2 Fence

We start by investigating the properties of a Bellman candidate defined on a family of extremal segments with an endpoint on the fixed boundary.

Recall that $g = (g_1, g_2): \mathbb{R} \rightarrow \partial_{\text{fixed}} \Omega$ is the parametrization of the fixed boundary of Ω such that $g'_1 > 0$ and $\kappa'_2 = (g'_2/g'_1)' > 0$. Let I be some interval in \mathbb{R} . Suppose that there exists a family of segments $S(t) = [g(t), w(t)]$, $t \in I$, which foliate a subdomain $\Omega(I) \subset \Omega$. We assume that the foliation is sufficiently smooth: $w \in C^1(I)$. We can consider t as a function from $\Omega(I)$ to I such that $x \in S(t(x))$ for $x \in \Omega(I)$. We assume the function t to be C^1 -smooth as well, and also $\frac{\partial}{\partial x_2} t \neq 0$ (which implies that the segments $S(t)$ are not vertical, i.e., $w_1(t) \neq g_1(t)$). We denote the slope of the segment $S(t)$ by $\kappa(t)$:

$$\kappa(t) \stackrel{\text{def}}{=} \frac{w_2(t) - g_2(t)}{w_1(t) - g_1(t)}. \quad (3.2.1)$$

Proposition 3.2.1. *Suppose that B is a C^1 -smooth function on $\Omega(I)$ that is affine on each segment $S(t)$, and its gradient $\beta = (\beta_1, \beta_2) = \nabla B$ is constant on $S(t)$, $t \in I$.*

Then B is C^2 -smooth. Moreover, there is a representation

$$B(x) = f(t) + (x_1 - g_1(t))[\kappa_3(t) + (\kappa(t) - \kappa_2(t))\beta_2(t)], \quad t = t(x), \quad (3.2.2)$$

where β_2 is given by

$$\begin{aligned} \beta_2(t) = & \exp\left(-\int_{t_0}^t \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \\ & \times \left(\int_{t_0}^t \exp\left(\int_{t_0}^{\tau} \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \frac{\kappa'_3(\tau)}{\kappa_2(\tau) - \kappa(\tau)} d\tau + \beta_2(t_0)\right), \quad t, t_0 \in I, \end{aligned} \quad (3.2.3)$$

and κ_2, κ_3 are defined in (2.1.11); here $\beta_2(t_0)$ is an arbitrary number.

Proof. We have

$$B(x) = f(t) + \beta_1(t)(x_1 - g_1(t)) + \beta_2(t)(x_2 - g_2(t)), \quad (3.2.4)$$

since the function B is affine on $S(t)$. Now, by differentiating the boundary equality $B(g(t)) = f(t)$, we obtain

$$\langle \beta(t), g'(t) \rangle = f'(t), \quad (3.2.5)$$

which is equivalent to (divide by g'_1)

$$\beta_1 + \kappa_2\beta_2 = \kappa_3. \quad (3.2.6)$$

Plugging this into (3.2.4), we get

$$B(x) = f(t) + (\kappa_3(t) - \kappa_2(t)\beta_2(t))(x_1 - g_1(t)) + \beta_2(t)(x_2 - g_2(t)). \quad (3.2.7)$$

We use the formula

$$\kappa(t) = \frac{x_2 - g_2(t)}{x_1 - g_1(t)}, \quad x \in S(t), \quad (3.2.8)$$

to rewrite (3.2.7) in the form (3.2.2).

It remains to prove (3.2.3). Let us note that (3.2.2) at the point $x = w(t)$ implies C^1 -smoothness of β_2 , which in its turn implies C^2 -smoothness of B because β_1 is C^1 -smooth as well due to (3.2.6). Differentiating (3.2.7) with respect to x_2 and using the relations $\kappa_2 g'_1 = g'_2, \kappa_3 g'_1 = f'$, we get

$$\left((\kappa'_3 - \kappa'_2\beta_2 - \kappa_2\beta'_2)(x_1 - g_1(t)) + \beta'_2(x_2 - g_2(t))\right) \cdot \frac{\partial}{\partial x_2} t = 0.$$

We use the fact that $\frac{\partial}{\partial x_2} t \neq 0$ and (3.2.8) to obtain

$$(\kappa_2 - \kappa)\beta'_2 + \kappa'_2\beta_2 = \kappa'_3. \quad (3.2.9)$$

The solution y of the equation $y'(t) + K_1(t)y(t) = K_2(t)$ is given by the formula

$$y(t) = \exp\left(-\int_{t_0}^t K_1(\tau) d\tau\right) \left(\int_{t_0}^t \exp\left(\int_{t_0}^{\tau} K_1(s) ds\right) K_2(\tau) d\tau + y(t_0)\right), \quad (3.2.10)$$

where $y(t_0)$ is an arbitrary parameter. Applying this to (3.2.9) with $y = \beta_2$, we obtain formula (3.2.3). ■

In what follows we will need a slightly different representation for β_2 that can be obtained using integration by parts.

Corollary 3.2.2. *Under the conditions of Proposition 3.2.1, for $t, t_0 \in I$, we have*

$$\begin{aligned} \beta_2(t) &= \mathfrak{R}(t) - \exp\left(-\int_{t_0}^t \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \\ &\quad \times \left(\int_{t_0}^t \exp\left(\int_{t_0}^{\tau} \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \mathfrak{R}'(\tau) d\tau + \text{Const}\right), \end{aligned} \quad (3.2.11)$$

$$\begin{aligned} \beta'_2(t) &= \frac{\kappa'_2(t)}{\kappa_2(t) - \kappa(t)} \exp\left(-\int_{t_0}^t \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \\ &\quad \times \left(\int_{t_0}^t \exp\left(\int_{t_0}^{\tau} \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \mathfrak{R}'(\tau) d\tau + \text{Const}\right), \end{aligned} \quad (3.2.12)$$

where $\text{Const} = \beta'_2(t_0) \frac{\kappa_2(t_0) - \kappa(t_0)}{\kappa'_2(t_0)} = \mathfrak{R}(t_0) - \beta_2(t_0)$.

Proof. Let us integrate by parts in the right-hand side of (3.2.3), using the fact that

$$\exp\left(\int_{t_0}^{\tau} \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \frac{\kappa'_3(\tau)}{\kappa_2(\tau) - \kappa(\tau)} = \left(\exp\left(\int_{t_0}^{\tau} \frac{\kappa'_2}{\kappa_2 - \kappa}\right)\right)' \cdot \mathfrak{R}(\tau).$$

Note that \mathfrak{R}' is a signed measure due to Condition 2.1.11, and the integrals in (3.2.11) and (3.2.12) are considered as the integrals with respect to this measure. Now formula (3.2.12) immediately follows from (3.2.11) and (3.2.9), this also proves that the constants in (3.2.12) and (3.2.11) are the same. ■

We formulate the converse of Proposition 3.2.1.

Proposition 3.2.3. *Suppose that the function β_2 is given by (3.2.3) and the function B is defined by (3.2.2) on the domain $\Omega(I)$. Then the function B is C^2 -smooth on $\Omega(I)$ and affine on each segment $S(t)$. Its gradient coincides with $(\beta_1(t), \beta_2(t))$ on $S(t)$; here β_1 is given by (3.2.6).*

Proof. If we prove $\nabla B = \beta$, then we automatically get that B is C^2 -smooth, since both β and t are C^1 -smooth.

Using (3.2.6) and (3.2.8) we rewrite (3.2.2) in the form (3.2.4). Differentiating formula (3.2.4) with respect to x , we obtain, for $t \in I$, $x \in S(t)$,

$$\nabla B = f'(t)\nabla t + \beta(t) + (\langle \beta'(t), x - g(t) \rangle - \langle \beta(t), g'(t) \rangle)\nabla t. \quad (3.2.13)$$

We recall that (3.2.6) is equivalent to (3.2.5), and therefore $\langle \beta(t), g'(t) \rangle = f'(t)$. Thus, to prove that $\nabla B = \beta$, we only need to show that $\langle \beta'(t), x - g(t) \rangle = 0$. The function β_2 given by (3.2.3) satisfies (3.2.9). Again, using (3.2.6) and (3.2.8), we obtain

$$(\kappa'_3 - \kappa'_2 \beta_2 - \kappa_2 \beta'_2)(x_1 - g_1(t)) + \beta'_2(x_2 - g_2(t)) = 0,$$

which is equivalent to

$$\langle \beta'(t), x - g(t) \rangle = 0, \quad x \in S(t). \quad (3.2.14)$$

This completes the proof. ■

Remark 3.2.4. Dividing (3.2.14) by $x_1 - g_1(t)$, we obtain the following relation:

$$\beta'_1 + \kappa \beta'_2 = 0. \quad (3.2.15)$$

Concluding this subsection, we name the domains $\Omega(I)$ as defined above by the term *fences*. We will need two types of fences: the points $w(t)$ are either on $\partial_{\text{free}} \Omega$ or on $\partial_{\text{fixed}} \Omega$. In the first case, we have a family of tangents to $\partial_{\text{free}} \Omega$, and $\Omega(I)$ is called a tangent domain, and in the second case, we have a family of chords, and $\Omega(I)$ is called a chordal domain. We will use all the relations from this subsection for both types of fences in what follows. Now we consider these two cases separately and show how to find the vector-valued function β and determine the candidate B .

3.3 Tangent domains

As mentioned in Section 3.1, the extremal segments are of two types: the chords and the tangents. This section provides a study of *tangent domains*, i.e., fences that consist of the segments $S(t)$ tangent to $\partial_{\text{free}} \Omega$, see Figure 3.1.

Definition 3.3.1. A fence $\Omega(I)$ with the foliation $S(t) = [g(t), w(t)]$, $t \in I$, is called a tangent domain if $w(t) \in \partial_{\text{free}} \Omega$ and the segment $S(t)$ is tangent to $\partial_{\text{free}} \Omega$ for $t \in I$.

We note that in general there are two possibilities: either $S(t) = S_{\text{R}}(t)$ (right tangents) or $S(t) = S_{\text{L}}(t)$ (left ones), see Section 2.1.1. If we consider one of these two cases, we use the notation $\Omega_{\text{R}}(I)$ and $\Omega_{\text{L}}(I)$, respectively. The segment $S(t)$ is tangent to $\partial_{\text{free}} \Omega$ if and only if the function κ defined in (3.2.1) satisfies the following equation:

$$\kappa(t) = \frac{w'_2(t)}{w'_1(t)}. \quad (3.3.1)$$

Differentiating (3.2.1) and using (3.3.1), one can easily check that

$$\kappa' = \frac{g'_1 \kappa - g'_2}{w_1 - g_1}. \quad (3.3.2)$$

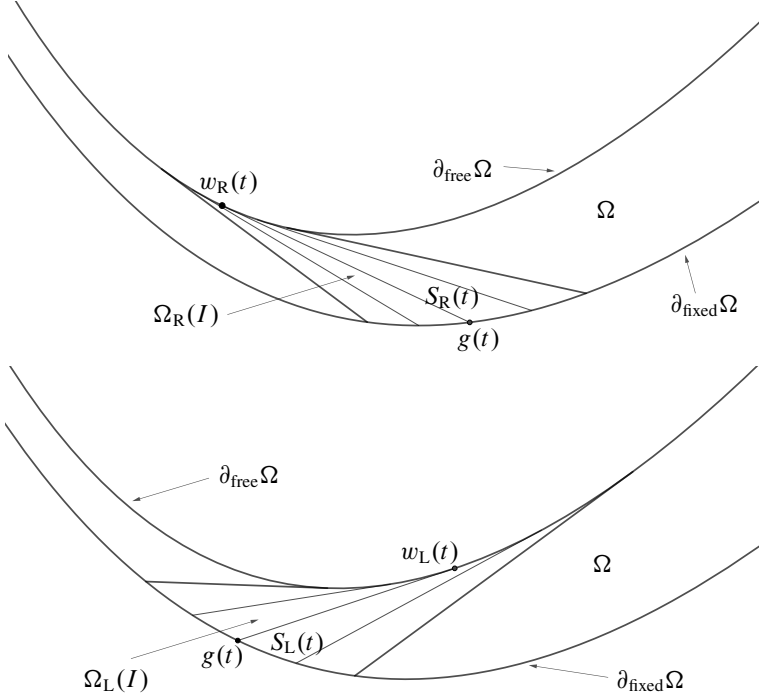


Figure 3.1. Domains Ω_R and Ω_L with right and left tangents.

Differentiating (3.2.8) with respect to x_2 , using (3.3.2) and (3.2.1) once more, we find the partial derivative of the function $t = t(x)$:

$$\begin{aligned} t_{x_2} &\stackrel{\text{def}}{=} \frac{\partial t}{\partial x_2} = \frac{1}{g'_2 + (x_1 - g_1)\kappa' - g'_1\kappa} \\ &= \frac{w_1 - g_1}{(\kappa g'_1 - g'_2)(x_1 - w_1)} = \frac{w_1 - g_1}{g'_1(\kappa - \kappa_2)(x_1 - w_1)}. \end{aligned}$$

Remark 3.3.2. Note that x_1 lies between g_1 and w_1 , $\kappa > \kappa_2$ for $\Omega_L(I)$ and $\kappa < \kappa_2$ for $\Omega_R(I)$. Therefore, $t_{x_2} > 0$ for $\Omega_R(I)$ and $t_{x_2} < 0$ for $\Omega_L(I)$. Recall that B is defined by (3.2.2) and its Hessian is degenerate, therefore the concavity of B follows from $B_{x_2x_2} < 0$. The sign of $B_{x_2x_2} = \beta'_2(t)t_{x_2}$ is determined by the sign of β'_2 , and we obtain the following propositions.

Proposition 3.3.3. *Consider a tangent domain $\Omega_R(I)$ foliated by the family of segments $S_R(t) = [g(t), w(t)]$ such that $w(t) \in \partial_{\text{free}} \Omega$ and $S_R(t)$ is a right tangent to $\partial_{\text{free}} \Omega$ for $t \in I$. Suppose that the function β_2 given by (3.2.3) satisfies the inequality $\beta'_2(t) \leq 0$ for $t \in I$. Then the function B given by (3.2.2) is a Bellman candidate on $\Omega_R(I)$.*

Proof. The functions β_2 given by (3.2.3) and β_1 defined by (3.2.6) are the components of the gradient ∇B by Proposition 3.2.3. Thus, we only need to verify the concavity of B . If $\beta_2'(t) = 0$, then $\beta_1'(t) = 0$ by (3.2.15), and the Hessian of B is the zero matrix. If $\beta_2'(t) < 0$, then $B_{x_2 x_2} = \beta_2'(t)t_{x_2} < 0$, and the Hessian is non-positive definite by Sylvester's criterion. Thus, in both cases the proposition is proved. \blacksquare

Remark 3.3.4. If $I = [t_-, t_+]$, $\beta_2'(t_-) \leq 0$, and the function \mathbf{T} corresponding to γ (defined in (2.1.10)) is non-positive on I , then $\beta_2' \leq 0$ on the entire interval I because of (3.2.12) and (2.1.12).

We state the following symmetrical proposition.

Proposition 3.3.5. Consider a tangent domain $\Omega_L(I)$ foliated by the family of segments $S_L(t) = [g(t), w(t)]$ such that $w(t) \in \partial_{\text{free}} \Omega$ and $S_L(t)$ is a left tangent to $\partial_{\text{free}} \Omega$ for $t \in I$. Suppose that the function β_2 given by (3.2.3) satisfies the inequality $\beta_2'(t) \geq 0$ for $t \in I$. Then the function B given by (3.2.2) is a Bellman candidate on $\Omega_L(I)$.

In the final part of this subsection we consider the case of infinite tangent domains.

Proposition 3.3.6. Let $I = (-\infty, t_2)$, $t_2 \in \mathbb{R} \cup \{+\infty\}$. Consider a tangent domain $\Omega_R(I)$ foliated by the family of segments $S_R(t) = [g(t), w(t)]$ such that $w(t) \in \partial_{\text{free}} \Omega$ and $S_R(t)$ is a right tangent to $\partial_{\text{free}} \Omega$ for $t \in I$. Suppose that the function β_2 , given by (3.2.11) with $\text{Const} = 0$ and $t_0 = -\infty$, i.e.,

$$\beta_2(t) = \mathfrak{K}(t) - \int_{-\infty}^t \exp\left(-\int_{\tau}^t \frac{\kappa_2'}{\kappa_2 - \kappa}\right) \mathfrak{K}'(\tau) d\tau, \quad t \in I, \quad (3.3.3)$$

is finite and satisfies the inequality

$$\beta_2'(t) = \frac{\kappa_2'(t)}{\kappa_2(t) - \kappa(t)} \int_{-\infty}^t \exp\left(-\int_{\tau}^t \frac{\kappa_2'}{\kappa_2 - \kappa}\right) \mathfrak{K}'(\tau) d\tau \leq 0, \quad t \in I. \quad (3.3.4)$$

Then the function B given by (3.2.2) is a Bellman candidate on $\Omega_R(I)$.

Proposition 3.3.7. Let $I = (t_1, +\infty)$, $t_1 \in \mathbb{R} \cup \{-\infty\}$. Consider a tangent domain $\Omega_L(I)$ foliated by a family of segments $S_L(t) = [g(t), w(t)]$ such that $w(t) \in \partial_{\text{free}} \Omega$ and $S_L(t)$ is a left tangent to $\partial_{\text{free}} \Omega$ for $t \in I$. Suppose that the function β_2 , given by formula (3.2.11) with $\text{Const} = 0$ and $t_0 = +\infty$, i.e.,

$$\beta_2(t) = \mathfrak{K}(t) + \int_t^{+\infty} \exp\left(\int_t^{\tau} \frac{\kappa_2'}{\kappa_2 - \kappa}\right) \mathfrak{K}'(\tau) d\tau, \quad t \in I, \quad (3.3.5)$$

is finite and satisfies the inequality

$$\beta_2'(t) = -\frac{\kappa_2'(t)}{\kappa_2(t) - \kappa(t)} \int_t^{+\infty} \exp\left(\int_t^{\tau} \frac{\kappa_2'}{\kappa_2 - \kappa}\right) \mathfrak{K}'(\tau) d\tau \geq 0, \quad t \in I. \quad (3.3.6)$$

Then the function B given by formula (3.2.2) is a Bellman candidate on $\Omega_L(I)$.

Definition 3.3.8. The functions B constructed in Propositions 3.3.3, 3.3.5, 3.3.6, and 3.3.7 are called *standard candidates* for the corresponding tangent domains $\Omega_R(I)$ and $\Omega_L(I)$ if the inequalities for β'_2 are strict in the interior of I , i.e., $\beta'_2 < 0$ for $\Omega_R(I)$ and $\beta'_2 > 0$ for $\Omega_L(I)$.

We comment on the convergence of the integrals in (3.3.3) and (3.3.5).

Remark 3.3.9. Convergence of the integral in (3.3.3) is equivalent to convergence of the integral in condition (2.1.13); this will be proved in Lemma 5.2.4 below. Moreover, if the integral in (2.1.13) is equal to $+\infty$, then the inequality in (3.3.4) fails, i.e., there is no standard candidate on $\Omega_R(-\infty, t_2)$ for any $t_2 \in \mathbb{R}$.

Symmetrically, convergence of the integral in (3.3.5) is equivalent to convergence of the integral in condition (2.1.14). Moreover, if the integral in (2.1.14) is equal to $-\infty$, then the inequality in (3.3.6) fails, i.e., there is no standard candidate on $\Omega_L(t_1, +\infty)$ for any $t_1 \in \mathbb{R}$.

3.4 Chordal domains

As mentioned in Section 3.1, the extremal segments are of two types: the chords and the tangents. In this section we study *chordal domains*, i.e., the domains that consist of chords (see Figure 3.2).

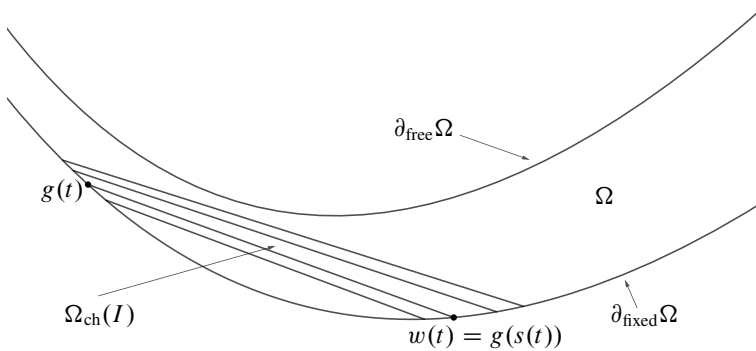


Figure 3.2. Chordal domain $\Omega_{\text{ch}}(I)$.

Definition 3.4.1. A fence $\Omega(I)$ with the foliation $S(t) = [g(t), w(t)]$, $t \in I$, is called a *chordal domain* if $w(t) \in \partial_{\text{fixed}} \Omega$ for $t \in I$. A chordal domain is denoted by $\Omega_{\text{ch}}(I)$.

The function B is uniquely determined in a chordal domain by its boundary values and linearity on segments:

$$B(\alpha g(t) + (1 - \alpha)g(s)) = \alpha f(t) + (1 - \alpha)f(s), \quad \alpha \in [0, 1], \quad (3.4.1)$$

where the function $s: I \rightarrow \mathbb{R}$ is defined by the relation $g(s(t)) = w(t)$ for $t \in I$. Let us note that the function s is decreasing. Our aim in this subsection is to provide conditions for when the function B given by (3.4.1) is a Bellman candidate.

The vectors $\gamma'(t)$, $\gamma'(s)$, and $\gamma(t) - \gamma(s)$ belong to the same tangent plane to the graph of B , therefore

$$\det \begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma(s) - \gamma(t) \end{pmatrix} = \det \begin{pmatrix} g'_1(t) & g'_2(t) & f'(t) \\ g'_1(s) & g'_2(s) & f'(s) \\ g_1(s) - g_1(t) & g_2(s) - g_2(t) & f(s) - f(t) \end{pmatrix} = 0. \quad (3.4.2)$$

We will refer to this equation as the *cup equation*.

Denote by $\theta = \theta(x_1, x_2)$ the vector orthogonal to the graph of B at the point $(x_1, x_2, B(x_1, x_2))$:

$$\theta = \nabla(B(x_1, x_2) - x_3) = (\beta(t(x_1, x_2)), -1). \quad (3.4.3)$$

We can consider θ as a function of t only: $\theta(t) = (\beta(t), -1)$. This vector is orthogonal to the tangent vector of the boundary curve γ , i.e., $\langle \theta(t), \gamma'(t) \rangle = 0$, which is equivalent to

$$\langle \beta(t), g'(t) \rangle = f'(t). \quad (3.4.4)$$

Now we are going to find β' . We differentiate equation (3.4.4) and obtain

$$\langle \beta'(t), g'(t) \rangle = f''(t) - \langle \beta(t), g''(t) \rangle = -\langle \theta(t), \gamma''(t) \rangle, \quad (3.4.5)$$

where θ is defined by (3.4.3).

We know that $\langle \theta(t), \gamma'(t) \rangle = \langle \theta(t), \gamma'(s) \rangle = 0$ and $\langle \theta(t), (0, 0, 1) \rangle = -1$. Therefore, for any $v \in \mathbb{R}^3$, we have $\langle \theta(t), v + \langle \theta(t), v \rangle (0, 0, 1) \rangle = 0$, whence

$$\det \begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ v + \langle \theta(t), v \rangle (0, 0, 1) \end{pmatrix} = 0 \quad \text{or} \quad \langle \theta(t), v \rangle \det \begin{pmatrix} g'(t) \\ g'(s) \\ v \end{pmatrix} = -\det \begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ v \end{pmatrix}.$$

We use it in (3.4.5) and obtain

$$\langle \beta'(t), g'(t) \rangle = -\langle \theta(t), \gamma''(t) \rangle = \frac{\det \begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma''(t) \end{pmatrix}}{\det \begin{pmatrix} g'(t) \\ g'(s) \end{pmatrix}} = \frac{\det \begin{pmatrix} g'_1(t) & g'_2(t) & f'(t) \\ g'_1(s) & g'_2(s) & f'(s) \\ g''_1(t) & g''_2(t) & f''(t) \end{pmatrix}}{\det \begin{pmatrix} g'_1(t) & g'_2(t) \\ g'_1(s) & g'_2(s) \end{pmatrix}}. \quad (3.4.6)$$

We have a system of equations (3.4.6) and (3.2.15) (with $w(t) = g(s)$) for $\beta'(t)$. We can solve it and obtain

$$\beta'_2(t) = \frac{1}{g'_1(t)} \cdot \frac{1}{\kappa_2(t) - \kappa(t)} \cdot \frac{\det \begin{pmatrix} g'_1(t) & g'_2(t) & f'(t) \\ g'_1(s) & g'_2(s) & f'(s) \\ g''_1(t) & g''_2(t) & f''(t) \end{pmatrix}}{\det \begin{pmatrix} g'_1(t) & g'_2(t) \\ g'_1(s) & g'_2(s) \end{pmatrix}}, \quad (3.4.7)$$

where $\kappa(t) = \frac{g_2(s) - g_2(t)}{g_1(s) - g_1(t)}$ was defined by (3.2.1) and $\kappa_2 = g'_2/g'_1$. We recall that $g'_1 > 0$ (see (2.1.1)). It follows from the convexity of $\partial_{\text{fixed}}\Omega$ that for $t < s$, the following inequality holds:

$$\frac{g'_2(t)}{g'_1(t)} = \kappa_2(t) < \kappa(t) < \kappa_2(s) = \frac{g'_2(s)}{g'_1(s)} \quad (3.4.8)$$

(and for $s < t$ both inequalities are opposite), thus, $\text{sign } \beta'_2(t) = -\text{sign } \det \begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma''(t) \end{pmatrix}$.

Note that $\text{sign } \frac{\partial t}{\partial x_2} = \text{sign}(t - s)$, whence

$$\text{sign } \frac{\partial^2 B}{\partial x_2^2} = \text{sign} \left[\beta'_2(t) \frac{\partial t}{\partial x_2} \right] = \text{sign} \left[(s - t) \det \begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma''(t) \end{pmatrix} \right]. \quad (3.4.9)$$

We formulate a proposition that gives sufficient conditions for the function B defined by (3.4.1) to be a Bellman candidate on the chordal domain $\Omega_{\text{ch}}(I)$.

Proposition 3.4.2. *Consider a chordal domain $\Omega_{\text{ch}}(I)$ foliated by the family of segments $S(t) = [g(t), g(s(t))]$, where the function $s: I \rightarrow \mathbb{R}$ satisfies (3.4.2). Let B be defined on $\Omega_{\text{ch}}(I)$ by the boundary conditions $B \circ g = f$ and by linearity on each segment $S(t)$, $t \in I$, i.e., by (3.4.1). Suppose that*

$$(t - s) \det \begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma''(t) \end{pmatrix} \geq 0, \quad t \in I. \quad (3.4.10)$$

Then the function B is a Bellman candidate on $\Omega_{\text{ch}}(I)$.

Proof. We need to check that ∇B is constant on each chord. Using (3.4.2) we find a vector $\beta = \beta(t)$ such that $\theta = (\beta, -1)$ is orthogonal to $\gamma'(t)$, $\gamma'(s)$ and $\gamma(s) - \gamma(t)$. The function β chosen in such a way satisfies relations (3.4.4) and (3.2.14). Identity (3.2.4) holds for $x = g(t)$ and $x = g(s)$, therefore, by linearity it holds for any $x \in S(t)$. Then, by (3.2.13), $\nabla B = \beta(t)$, therefore ∇B is constant on $S(t)$. Under the hypothesis of the proposition, the concavity of B follows from (3.4.9). ■

Definition 3.4.3. The function B constructed in Proposition 3.4.2 is called a *standard candidate* for the chordal domain $\Omega_{\text{ch}}(I)$ if the inequality in (3.4.10) is strict for t in the interior of I .

Definition 3.4.4. Let $I = [t_-, t_+]$ and let either $s(t_-) = t_-$ or $s(t_+) = t_+$ for a function $s: I \rightarrow \mathbb{R}$ corresponding to the chordal domain $\Omega_{\text{ch}}(I)$. Then the chordal domain $\Omega_{\text{ch}}(I)$ is called a *cup* originated at $g(c)$, where $c = s(c)$ coincides with either t_- or t_+ , respectively. The point $g(c)$ as well as the value of the parameter c is called the *origin of the cup*.

The origin $g(c)$ of a cup $\Omega_{\text{ch}}(I)$ is a very special point on the curve g . It should satisfy the following torsion equation:

$$\det \begin{pmatrix} \gamma'(c) \\ \gamma''(c) \\ \gamma'''(c) \end{pmatrix} = 0, \quad (3.4.11)$$

which can be obtained by carefully passing to the limit as $t \rightarrow c$ and $s \rightarrow c$ in (3.4.2).

Geometrically the chordal domains $\Omega_{\text{ch}}(I)$ and $\Omega_{\text{ch}}(s(I))$ are the same. Usually we do not need to distinguish them, so we will use the following common notation. Let $I = [t_-, t_+]$. Denote the numbers t_{\pm} and $s(t_{\pm})$ by a_0, a_1, b_1 and b_0 in the increasing order, i.e., $[g(a_0), g(b_0)]$ is the top chord of the domain and $[g(a_1), g(b_1)]$ is the bottom one. Denote such a chordal domain by $\Omega_{\text{ch}}([a_0, b_0], [a_1, b_1])$. If one of these chords is unimportant, we will use $*$ instead:

$$\Omega_{\text{ch}}([a_0, b_0], *) \quad \text{or} \quad \Omega_{\text{ch}}(*, [a_1, b_1]).$$

3.5 Around the cup

3.5.1 Differentials

Definition 3.5.1. Suppose that a pair (s, t) , $s < t$, satisfies the cup equation (3.4.2). We call the following two expressions the differentials, the left and the right one respectively:

$$D_{\text{L}}(s, t) \stackrel{\text{def}}{=} \frac{\det \begin{pmatrix} \gamma'(s) \\ \gamma'(t) \\ \gamma''(s) \end{pmatrix}}{\det \begin{pmatrix} g'(s) \\ g'(t) \end{pmatrix}}, \quad D_{\text{R}}(s, t) \stackrel{\text{def}}{=} \frac{\det \begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma''(t) \end{pmatrix}}{\det \begin{pmatrix} g'(t) \\ g'(s) \end{pmatrix}}. \quad (3.5.1)$$

It is clear that, formally, $D_{\text{L}}(s, t) = D_{\text{R}}(t, s)$, but it will be more convenient to keep two different symbols for these expressions.

Lemma 3.5.2. Suppose that the pair (s, t) satisfies the cup equation (3.4.2). Then

$$\gamma(t) - \gamma(s) = C_{\text{R}} \gamma'(t) - C_{\text{L}} \gamma'(s), \quad (3.5.2)$$

where

$$C_{\text{L}}(s, t) = \frac{\det \begin{pmatrix} g(s) - g(t) \\ g'(t) \end{pmatrix}}{\det \begin{pmatrix} g'(s) \\ g'(t) \end{pmatrix}}, \quad C_{\text{R}}(s, t) = \frac{\det \begin{pmatrix} g(t) - g(s) \\ g'(s) \end{pmatrix}}{\det \begin{pmatrix} g'(t) \\ g'(s) \end{pmatrix}}.$$

Remark 3.5.3. If $u, v, w \in \mathbb{R}^2$, then

$$u \det \begin{pmatrix} w \\ v \end{pmatrix} + v \det \begin{pmatrix} u \\ w \end{pmatrix} + w \det \begin{pmatrix} v \\ u \end{pmatrix} = 0. \quad (3.5.3)$$

Proof of Lemma 3.5.2. It follows from (3.5.3) that

$$g(t) - g(s) = C_R g'(t) - C_L g'(s). \quad (3.5.4)$$

The vectors $\gamma(t) - \gamma(s)$, $\gamma'(s)$, $\gamma'(t)$ are linearly dependent, and the coefficients of the linear dependence are the same as for their two-dimensional projections $g(t) - g(s)$, $g'(t)$ and $g'(s)$. ■

Remark 3.5.4. For $s \neq t$, we have $\text{sign}(C_R) = -\text{sign}(C_L) = \text{sign}(t - s)$ (see (3.4.8)).

Corollary 3.5.5. Let $\Omega_{\text{ch}}(I)$ be a chordal domain with corresponding function $s = s(t)$, $t \in I$. Then

$$C_L \det \begin{pmatrix} \gamma'(s) \\ \gamma'(t) \\ \gamma''(s) \end{pmatrix} ds + C_R \det \begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma''(t) \end{pmatrix} dt = 0. \quad (3.5.5)$$

Proof. We differentiate (3.4.2) and obtain

$$\begin{aligned} 0 &= \det \begin{pmatrix} \gamma''(t) \\ \gamma'(s) \\ \gamma(t) - \gamma(s) \end{pmatrix} dt + \det \begin{pmatrix} \gamma'(t) \\ \gamma''(s) \\ \gamma(t) - \gamma(s) \end{pmatrix} ds \\ &= C_R \det \begin{pmatrix} \gamma''(t)\gamma'(s) \\ \gamma'(t) \end{pmatrix} dt - C_L \det \begin{pmatrix} \gamma'(t) \\ \gamma''(s) \\ \gamma'(s) \end{pmatrix} ds, \end{aligned}$$

according to (3.5.2). ■

By (3.5.1), the equality (3.5.5) can be rewritten in the following form:

$$C_L D_L ds = C_R D_R dt. \quad (3.5.6)$$

Corollary 3.5.6. We have

$$D_R = \frac{\det \begin{pmatrix} \gamma'(t) \\ \gamma(t) - \gamma(s) \\ \gamma''(t) \end{pmatrix}}{\det \begin{pmatrix} g'(t) \\ g(t) - g(s) \end{pmatrix}}, \quad D_L = \frac{\det \begin{pmatrix} \gamma'(s) \\ \gamma(s) - \gamma(t) \\ \gamma''(s) \end{pmatrix}}{\det \begin{pmatrix} g'(s) \\ g(s) - g(t) \end{pmatrix}}. \quad (3.5.7)$$

Proof. This immediately follows from (3.5.2), (3.5.4), and (3.5.1). ■

Lemma 3.5.7. Let $\Omega_{\text{ch}}(I)$ be a chordal domain with corresponding function $s = s(t)$, $t \in I$. Then

$$dD_R(s, t) = \frac{\det \begin{pmatrix} \gamma'(t) \\ \gamma''(t) \\ \gamma'''(t) \end{pmatrix}}{\det \begin{pmatrix} g'(t) \\ g''(t) \end{pmatrix}} dt + \left(\frac{\det \begin{pmatrix} g'(t) \\ g'''(t) \end{pmatrix}}{\det \begin{pmatrix} g'(t) \\ g''(t) \end{pmatrix}} - \frac{\det \begin{pmatrix} g''(t) \\ g(t) - g(s) \end{pmatrix}}{\det \begin{pmatrix} g'(t) \\ g(t) - g(s) \end{pmatrix}} \right) D_R(s, t) dt. \quad (3.5.8)$$

Proof. Let us differentiate D_R using the representation (3.5.7). First, note that the partial derivative with respect to s is equal to zero:

$$\begin{aligned} & \frac{\det\begin{pmatrix} \gamma'(t) \\ -\gamma'(s) \\ \gamma''(t) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g(t)-g(s) \end{pmatrix}} - \frac{\det\begin{pmatrix} \gamma'(t) \\ \gamma(t)-\gamma(s) \\ \gamma''(t) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g(t)-g(s) \end{pmatrix}^2} \det\begin{pmatrix} g'(t) \\ -g'(s) \end{pmatrix} \\ & \stackrel{(3.5.7)}{=} \stackrel{(3.5.1)}{=} -D_R \frac{\det\begin{pmatrix} g'(t) \\ g'(s) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g(t)-g(s) \end{pmatrix}} + D_R \frac{\det\begin{pmatrix} g'(t) \\ g'(s) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g(t)-g(s) \end{pmatrix}} = 0. \end{aligned}$$

Second, the partial derivative with respect to t is equal to

$$\frac{\det\begin{pmatrix} \gamma'(t) \\ \gamma(t)-\gamma(s) \\ \gamma'''(t) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g(t)-g(s) \end{pmatrix}} - \frac{\det\begin{pmatrix} g''(t) \\ g(t)-g(s) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g(t)-g(s) \end{pmatrix}} D_R. \quad (3.5.9)$$

We use formulas (3.5.2) and (3.5.4) to rewrite the first summand in (3.5.9):

$$\frac{\det\begin{pmatrix} \gamma'(t) \\ \gamma(t)-\gamma(s) \\ \gamma'''(t) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g(t)-g(s) \end{pmatrix}} = \frac{\det\begin{pmatrix} \gamma'(t) \\ \gamma''(t) \\ \gamma'''(t) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g'(s) \end{pmatrix}}. \quad (3.5.10)$$

It follows from (3.5.3), for the vectors $g'(t)$, $g''(t)$ and $g'''(t)$, that

$$\det\begin{pmatrix} g'(t) \\ g''(t) \end{pmatrix} \gamma'''(t) + \det\begin{pmatrix} g'''(t) \\ g'(t) \end{pmatrix} \gamma''(t) + \det\begin{pmatrix} g''(t) \\ g'''(t) \end{pmatrix} \gamma'(t) = \left(0, 0, \det\begin{pmatrix} \gamma'(t) \\ \gamma''(t) \\ \gamma'''(t) \end{pmatrix}\right).$$

We express $\gamma'''(t)$ from this identity and substitute it into (3.5.10) to obtain

$$\frac{\det\begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma'''(t) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g'(s) \end{pmatrix}} = \frac{\det\begin{pmatrix} \gamma'(t) \\ \gamma''(t) \\ \gamma'''(t) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g'(s) \end{pmatrix}} - \frac{\det\begin{pmatrix} \gamma'(t) \\ \gamma''(t) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g'(s) \end{pmatrix}} \frac{\det\begin{pmatrix} g'''(t) \\ g'(t) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g'(s) \end{pmatrix}}.$$

This formula, together with (3.5.10) and (3.5.9), and definition (3.5.1), completes the proof. \blacksquare

Remark 3.5.8. Formula (3.5.8) can be rewritten in terms of the functions κ , κ_2 , κ_3 , and \mathfrak{K} (see (2.1.5) and (2.1.11)):

$$\frac{dD_R(s(t), t)}{dt} = \kappa_2' g_1' \mathfrak{K}' + \left[\frac{(g_1' \kappa_2')'}{g_1' \kappa_2'} - \frac{\kappa_2'}{\kappa_2 - \kappa} \right] D_R(s(t), t). \quad (3.5.11)$$

Proof. The coincidence of the first summands of (3.5.11) and (3.5.8) follows from equation (2.1.12) and the identity

$$\det\begin{pmatrix} g'(t) \\ g''(t) \end{pmatrix} = g_1' g_2'' - g_1'' g_2' = (g_1')^2 \kappa_2'.$$

As for the second summand, we write the following chain of elementary equalities:

$$\begin{aligned} \det \begin{pmatrix} g'(t) \\ g(t) - g(s) \end{pmatrix} &= g'_1(g_2(t) - g_2(s)) - g'_2(g_1(t) - g_1(s)) \\ &= g'_1[g_2(t) - g_2(s) - \kappa_2(g_1(t) - g_1(s))]. \end{aligned}$$

The coefficient in the big parentheses in (3.5.8) is equal to

$$\begin{aligned} \frac{\det \begin{pmatrix} g'(t) \\ g'''(t) \end{pmatrix}}{\det \begin{pmatrix} g'(t) \\ g''(t) \end{pmatrix}} - \frac{\det \begin{pmatrix} g''(t) \\ g(t) - g(s) \end{pmatrix}}{\det \begin{pmatrix} g'(t) \\ g(t) - g(s) \end{pmatrix}} &= \frac{\partial}{\partial t} \log \frac{\det \begin{pmatrix} g'(t) \\ g''(t) \end{pmatrix}}{\det \begin{pmatrix} g'(t) \\ g(t) - g(s) \end{pmatrix}} \\ &= \frac{\partial}{\partial t} \log \frac{\kappa'_2 g'_1}{g_2(t) - g_2(s) - \kappa_2(g_1(t) - g_1(s))} \\ &= \frac{(\kappa'_2 g'_1)'}{\kappa'_2 g'_1} - \frac{g'_2 - \kappa'_2(g_1(t) - g_1(s)) - \kappa_2 g'_1}{(g_1(t) - g_1(s))(\kappa - \kappa_2)} \\ &= \frac{(\kappa'_2 g'_1)'}{\kappa'_2 g'_1} + \frac{\kappa'_2}{\kappa - \kappa_2}. \quad \blacksquare \end{aligned}$$

3.5.2 Cup construction

Proposition 3.5.9. *Suppose that the torsion of γ changes its sign from $+$ to $-$ at the point t_0 , i.e., \mathbf{T} is positive in a left neighborhood of t_0 and is negative in a right one. Then, for $\delta > 0$ sufficiently small, there exists a chordal domain $\Omega_{\text{ch}}(t_0, t_0 + \delta)$ with the standard Bellman candidate on it.*

Proposition 3.5.9 emphasizes the importance of Condition 2.1.11 because cups can originate only at the points $t_0 = c_i$, see Definition 2.1.13.

First, we need to obtain several useful formulas.

Lemma 3.5.10. *For any s, t the following identity holds:*

$$\begin{aligned} \det \begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma(t) - \gamma(s) \end{pmatrix} &= g'_1(t)g'_1(s) \int_s^t \mathfrak{R}'(\sigma) \left(\int_s^\sigma \int_\sigma^t \kappa'_2(u)\kappa'_2(v)(g_1(v) - g_1(u)) dv du \right) d\sigma \quad (3.5.12) \\ &= -g'_1(t)g'_1(s) \int_s^t \mathfrak{R}'(\sigma) \left(\int_s^t \kappa'_2(\sigma)\kappa'_2(v)(g_1(v) - g_1(\sigma)) dv \right) d\sigma. \quad (3.5.13) \end{aligned}$$

Proof. We have

$$\det \begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma(t) - \gamma(s) \end{pmatrix} = \int_s^t \det \begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma'(\tau) \end{pmatrix} d\tau$$

$$\begin{aligned}
 &= \int_s^t g_1'(t)g_1'(s)g_1'(\tau) \det \begin{pmatrix} 1 & \kappa_2(t) & \kappa_3(t) \\ 1 & \kappa_2(s) & \kappa_3(s) \\ 1 & \kappa_2(\tau) & \kappa_3(\tau) \end{pmatrix} d\tau \\
 &= -g_1'(t)g_1'(s) \int_s^t g_1'(\tau) \det \begin{pmatrix} \kappa_2(t) - \kappa_2(\tau) & \kappa_3(t) - \kappa_3(\tau) \\ \kappa_2(\tau) - \kappa_2(s) & \kappa_3(\tau) - \kappa_3(s) \end{pmatrix} d\tau \\
 &= -g_1'(t)g_1'(s) \int_s^t g_1'(\tau) \int_\tau^t \int_s^\tau \det \begin{pmatrix} \kappa_2'(v) & \kappa_3'(v) \\ \kappa_2'(u) & \kappa_3'(u) \end{pmatrix} du dv d\tau \\
 &= g_1'(t)g_1'(s) \int_s^t g_1'(\tau) \int_\tau^t \int_s^\tau \kappa_2'(v)\kappa_2'(u) \int_u^v \mathfrak{R}'(\sigma) d\sigma du dv d\tau \\
 &= g_1'(t)g_1'(s) \int_s^t \mathfrak{R}'(\sigma) \left(\int_s^\sigma \int_\sigma^t \kappa_2'(u)\kappa_2'(v) \int_u^v g_1'(\tau) d\tau dv du \right) d\sigma \\
 &= g_1'(t)g_1'(s) \int_s^t \mathfrak{R}'(\sigma) \left(\int_s^\sigma \int_\sigma^t \kappa_2'(u)\kappa_2'(v)(g_1(v) - g_1(u)) dv du \right) d\sigma \\
 &= -g_1'(t)g_1'(s) \int_s^t \mathfrak{R}(\sigma) \left(\int_\sigma^t \kappa_2'(\sigma)\kappa_2'(v)(g_1(v) - g_1(\sigma)) dv \right. \\
 &\quad \left. - \int_s^\sigma \kappa_2'(u)\kappa_2'(\sigma)(g_1(\sigma) - g_1(u)) du \right) d\sigma \\
 &= -g_1'(t)g_1'(s) \int_s^t \mathfrak{R}(\sigma) \left(\int_s^t \kappa_2'(\sigma)\kappa_2'(v)(g_1(v) - g_1(\sigma)) dv \right) d\sigma,
 \end{aligned}$$

where, in the penultimate line, we performed integration by parts. \blacksquare

Corollary 3.5.11. *For any s, t , the following identity holds true:*

$$\begin{aligned}
 \det \begin{pmatrix} \gamma''(t) \\ \gamma'(s) \\ \gamma(t) - \gamma(s) \end{pmatrix} &= \frac{g_1''(t)}{g_1'(t)} \det \begin{pmatrix} \gamma'(t) \\ \gamma'(s) \\ \gamma(t) - \gamma(s) \end{pmatrix} - g_1'(t)g_1'(s)\kappa_2'(t) \\
 &\quad \times \int_s^t (\mathfrak{R}(u) - \mathfrak{R}(t))\kappa_2'(u)(g_1(t) - g_1(u)) du. \quad (3.5.14)
 \end{aligned}$$

Proof. The result is obtained by the direct differentiation of (3.5.13): the first summand is obtained from the differentiation of the factor $g_1'(t)$, the second one is the result of the differentiation of the integral. \blacksquare

Lemma 3.5.12. *Let $s, t, c \in \mathbb{R}$ and $s < c < t$. Suppose that a function Φ is continuous on $[s, t]$, non-decreasing on (s, c) , non-increasing on (c, t) , and not constant on $[s, t]$. Let the functions ϕ and ψ on $[s, t]$ be such that $\phi > 0$ and $\psi' > 0$. Suppose that*

$$\int_s^t \Phi(\sigma) \left(\int_s^t \phi(\sigma)\phi(v)(\psi(v) - \psi(\sigma)) dv \right) d\sigma = 0. \quad (3.5.15)$$

Then

$$\int_s^t (\Phi(u) - \Phi(t))\phi(u)(\psi(t) - \psi(u)) du > 0. \quad (3.5.16)$$

Proof. The inequality (3.5.16) and the equality (3.5.15) do not change if we add a constant to the function ψ . Therefore, we may assume that $\psi(t) = 0$. Under this assumption, $\psi < 0$ on (s, t) .

We rewrite (3.5.15) in the following way:

$$\int_s^t \Phi(\sigma)\phi(\sigma) d\sigma \int_s^t \phi(v)\psi(v) dv = \int_s^t \Phi(\sigma)\phi(\sigma)\psi(\sigma) d\sigma \int_s^t \phi(v) dv. \quad (3.5.17)$$

We see that the problem is not sensitive to adding a constant to Φ . The integral $\int \phi\psi$ is strictly negative, so we can assume $\int_s^t \Phi\phi\psi = 0$ by adding a constant to Φ , if necessary. It now follows from (3.5.17) that $\int_s^t \Phi\phi = 0$. Under our assumptions, (3.5.16) takes the following form:

$$\Phi(t) \int \phi\psi > 0,$$

which is equivalent to $\Phi(t) < 0$. If this is not true, then $\Phi(t) \geq 0$ and $\Phi \geq 0$ on $[c, t]$. We can find a point $\sigma_0 \in (s, c)$ such that $\Phi \geq 0$ on (σ_0, t) and $\Phi \leq 0$ on (s, σ_0) . However,

$$\int_s^t \Phi(\sigma)\phi(\sigma)(\psi(\sigma) - \psi(\sigma_0)) d\sigma = 0,$$

while the integrand is non-negative. Thus, the integrand is identically zero, therefore Φ vanishes as well. This contradicts our requirement that Φ is not constant. ■

Corollary 3.5.13. *Let $s, t, t_0 \in \mathbb{R}$ and $s < t_0 < t$. Suppose that \mathfrak{K} is non-decreasing on (s, t_0) , non-increasing on (t_0, t) , and not constant on $[s, t]$. Suppose that the pair (s, t) satisfies the cup equation (3.4.2). Then $D_{\mathbb{R}}(s, t) < 0$ and $D_{\mathbb{L}}(s, t) < 0$.*

Proof. We will prove only $D_{\mathbb{R}}(s, t) < 0$; the remaining inequality is symmetric.

We want to use Lemma 3.5.12 with $\Phi = \mathfrak{K}$, $\psi = g_1$, $\phi = \kappa'_2$, and $c = t_0$. Then (3.5.15) holds due to (3.5.13). Thus, we obtain (3.5.16), which is equivalent to

$$\det \begin{pmatrix} \gamma''(t) \\ \gamma'(s) \\ \gamma(t) - \gamma(s) \end{pmatrix} < 0,$$

by (3.5.14). We use (3.5.2) to rewrite this in terms of $D_{\mathbb{R}}$:

$$0 > \det \begin{pmatrix} \gamma''(t) \\ \gamma'(s) \\ \gamma(t) - \gamma(s) \end{pmatrix} = C_{\mathbb{R}} \det \begin{pmatrix} \gamma''(t) \\ \gamma'(s) \\ \gamma'(t) \end{pmatrix} = -C_{\mathbb{R}} D_{\mathbb{R}} \det \begin{pmatrix} g'(t) \\ g'(s) \end{pmatrix}.$$

We recall that $C_{\mathbb{R}} > 0$ and $\det \begin{pmatrix} g'(t) \\ g'(s) \end{pmatrix} < 0$ when $t > s$. Thus, we obtain the claimed inequality $D_{\mathbb{R}} < 0$. ■

Proof of Proposition 3.5.9. Take $\theta_0 > 0$ to be a small number such that the torsion \mathbf{T} of γ is positive on $(t_0 - \theta_0, t_0)$ and negative on $(t_0, t_0 + \theta_0)$. Recall that the sign of \mathbf{T} coincides with the sign of \mathfrak{K}' , see (2.1.12).

Consider the following function h :

$$h(t, \theta) = \det \begin{pmatrix} \gamma'(t) \\ \gamma'(t - \theta) \\ \gamma(t) - \gamma(t - \theta) \end{pmatrix}, \quad \theta \in (0, \theta_0], t \in [t_0, t_0 + \theta].$$

This function is continuous and differentiable on its domain. It follows from the conditions on the curve g that $\kappa_2'(u)\kappa_2'(v)(g_1(v) - g_1(u)) > 0$ for any u, v such that $u < v$. For any $\theta \in (0, \theta_0]$, the measure \mathfrak{K}' is positive on $(t_0 - \theta, t_0)$ and negative on $(t_0, t_0 + \theta)$, therefore formula (3.5.12) implies that $h(t_0, \theta) > 0 > h(t_0 + \theta, \theta)$ for any $\theta \in (0, \theta_0]$. So, for any such θ , there exists $t_\theta \in (t_0, t_0 + \theta)$ such that $h(t_\theta, \theta) = 0$. We will show that such t_θ is unique for any $\theta \in (0, \theta_0]$.

Corollary 3.5.13 implies that $D_R = D_R(t_\theta - \theta, t_\theta)$ and $D_L = D_L(t_\theta - \theta, t_\theta)$ are negative. We use formula (3.5.2) to obtain

$$\begin{aligned} \frac{\partial h}{\partial t}(t_\theta, \theta) &= \det \begin{pmatrix} \gamma''(t_\theta) \\ \gamma'(t_\theta - \theta) \\ \gamma(t_\theta) - \gamma(t_\theta - \theta) \end{pmatrix} + \det \begin{pmatrix} \gamma'(t_\theta) \\ \gamma''(t_\theta - \theta) \\ \gamma(t_\theta) - \gamma(t_\theta - \theta) \end{pmatrix} \\ &= \det \begin{pmatrix} g'(t_\theta) \\ g'(t_\theta - \theta) \end{pmatrix} (C_L D_L - C_R D_R) < 0, \end{aligned}$$

because $\det \begin{pmatrix} g'(t_\theta) \\ g'(t_\theta - \theta) \end{pmatrix} < 0$, $C_R > 0 > C_L$, $D_R < 0$, and $D_L < 0$. It follows that the function $h(\cdot, \theta)$ changes its sign from plus to minus at any root on $(t_0, t_0 + \theta)$. Therefore, it has only one root t_θ there.

We now calculate

$$\frac{\partial h}{\partial \theta}(t_\theta, \theta) = -\det \begin{pmatrix} \gamma'(t_\theta) \\ \gamma''(t_\theta - \theta) \\ \gamma(t_\theta) - \gamma(t_\theta - \theta) \end{pmatrix} = -\det \begin{pmatrix} g'(t_\theta) \\ g'(t_\theta - \theta) \end{pmatrix} C_L D_L > 0.$$

It now follows from the implicit function theorem that t_θ is a differentiable function of θ and

$$\frac{\partial t_\theta}{\partial \theta} = \frac{C_L D_L}{C_L D_L - C_R D_R} \in (0, 1)$$

for $\theta \in (0, \theta_0)$. Thus, $t_\theta - \theta$ decreases. Moreover, $t_\theta \in (t_0, t_0 + \theta)$ for any θ , therefore, $\lim_{\theta \rightarrow 0^+} t_\theta = t_0$.

Let $\delta = t_{\theta_0} - t_0$, $I = [t_0, t_{\theta_0}]$. We define the function $s: I \rightarrow [t_{\theta_0} - \theta_0, t_0]$ by the rule $s(t_\theta) = t_\theta - \theta$ for $\theta \in (0, \theta_0]$, and $s(t_0) = t_0$. This function decreases, and the corresponding family of segments $S(t) = [g(t), g(s(t))]$, $t \in I$, forms a chordal domain. By the definition of t_θ , the pair $(s(t), t)$ satisfies the cup equation (3.4.2) for any $t \in I$, and moreover $D_R(s, t) < 0$ and $D_L(s, t) < 0$. Therefore, this chordal domain satisfies the conditions of Proposition 3.4.2 and the proof is complete. ■

3.5.3 How to grow a chordal domain from a single chord

Now we turn to a more general situation. In this subsection we work under the following assumption.

Condition 3.5.14. *There is a pair of numbers a_0, b_0 , $a_0 < b_0$, satisfying the cup equation (3.4.2), that is,*

$$\det \begin{pmatrix} \gamma'(a_0) \\ \gamma'(b_0) \\ \gamma(b_0) - \gamma(a_0) \end{pmatrix} = 0,$$

such that for some $\theta_0 > 0$, the determinant

$$L(t) = \det \begin{pmatrix} \gamma'(t) \\ \gamma'(a_0) \\ \gamma'(b_0) \end{pmatrix}$$

is negative for $t \in (b_0, b_0 + \theta_0)$ and positive for $t \in (a_0 - \theta_0, a_0)$.

Remark 3.5.15. If (a_0, b_0) satisfies the cup equation (3.4.2) and the inequalities $D_R(a_0, b_0) < 0$ and $D_L(a_0, b_0) < 0$, then Condition 3.5.14 holds.

Proof. It follows immediately from the fact that $L(a_0) = L(b_0) = 0$ and

$$\frac{L'(a_0)}{D_L(a_0, b_0)} = \frac{L'(b_0)}{D_R(a_0, b_0)} = \det \begin{pmatrix} g'(a_0) \\ g'(b_0) \end{pmatrix} > 0. \quad \blacksquare$$

We are ready to present the main result of this subsection.

Proposition 3.5.16. *Under Condition 3.5.14, there exist $\delta > 0$ and two differentiable functions $a, b: [0, \delta] \rightarrow \mathbb{R}$ such that:*

- (1) $a' < 0 < b'$, $a(0) = a_0$, $b(0) = b_0$,
- (2) $b(\theta) - a(\theta) = b_0 - a_0 + \theta$ and the pair $(a(\theta), b(\theta))$ satisfies the cup equation (3.4.2) for any $\theta \in [0, \delta]$,
- (3) $D_L(a(\theta), b(\theta)) < 0$ and $D_R(a(\theta), b(\theta)) < 0$ for any $\theta \in (0, \delta]$.

Proof. We can always assume that $f'(a_0) = f'(b_0) = 0$. If this is not the case, then we can replace the function f by $f + c_1 g_1 + c_2 g_2$ with appropriate $c_1, c_2 \in \mathbb{R}$. In such a case, the determinants in the cup equation and in the differentials do not depend on the choice of c_1 and c_2 . Since $\det \begin{pmatrix} g'(a_0) \\ g'(b_0) \end{pmatrix} \neq 0$, we can choose the constants c_1, c_2 in such a way that $f'(a_0) = f'(b_0) = 0$ for the modified function. In this situation the sign of $L(t)$ coincides with the sign of $f'(t)$, therefore Condition 3.5.14 implies that f' is negative on the right neighborhood of b_0 . From the cup equation (3.4.2), we obtain $f(a_0) = f(b_0)$.

Let us prove that κ'_3 is non-positive in some right neighborhood of b_0 . The function \mathfrak{K} is piecewise monotone by Condition 2.1.11, therefore we can find a right neighborhood of b_0 where it is monotone and does not change its sign. Then $\kappa'_3 = \mathfrak{K}\kappa'_2$ does not change its sign in the same right neighborhood of b_0 as well. We know that $\kappa_3(b_0) = 0$ and

$$\kappa_3 = \frac{f'}{g'_1} < 0 \quad \text{on the right of } b_0, \quad (3.5.18)$$

therefore $\kappa'_3 \leq 0$ there. Without loss of generality, in what follows we assume that

$$\kappa'_3 \leq 0 \quad \text{on } (b_0, b_0 + \theta_0). \quad (3.5.19)$$

We write the following inequality for $t \in (b_0, b_0 + \theta_0)$:

$$\begin{aligned} f(t) - f(a_0) &= f(t) - f(b_0) \\ &= \int_{b_0}^t f'(\tau) d\tau = \int_{b_0}^t g'_1(\tau)\kappa_3(\tau) d\tau \geq (g_1(t) - g_1(b_0))\kappa_3(t) \end{aligned}$$

because $g'_1 > 0$ and $\kappa'_3 \leq 0$. We use this inequality to estimate the determinant:

$$\begin{aligned} \det \begin{pmatrix} \gamma'(t) \\ \gamma'(a_0) \\ \gamma(t) - \gamma(a_0) \end{pmatrix} &\leq \det \begin{pmatrix} g'_1(t) & g'_2(t) & f'(t) \\ g'_1(a_0) & g'_2(a_0) & 0 \\ g_1(t) - g_1(a_0) & g_2(t) - g_2(a_0) & (g_1(t) - g_1(b_0))\kappa_3(t) \end{pmatrix} \\ &= f'(t)g'_1(a_0) \det \begin{pmatrix} 1 & \kappa_2(t) & 1 \\ 1 & \kappa_2(a_0) & 0 \\ g_1(t) - g_1(a_0) & g_2(t) - g_2(a_0) & g_1(t) - g_1(b_0) \end{pmatrix}. \end{aligned}$$

When t approaches b_0 from the right, the latter determinant tends to

$$g_2(b_0) - g_2(a_0) - \kappa_2(a_0)(g_1(b_0) - g_1(a_0)).$$

This expression is strictly positive due to convexity of the curve g . Since $f'(t) < 0$, this leads to the inequality

$$\det \begin{pmatrix} \gamma'(t) \\ \gamma'(a_0) \\ \gamma(t) - \gamma(a_0) \end{pmatrix} < 0 \quad (3.5.20)$$

for $t > b_0$ sufficiently close to b_0 . Reducing θ_0 if needed, we can assume that (3.5.19) and (3.5.20) hold for all $t \in (b_0, b_0 + \theta_0)$.

Similarly, we can prove that for all t in a left neighborhood of a_0 , we have that $\kappa'_3(t) \geq 0$ and

$$\det \begin{pmatrix} \gamma'(b_0) \\ \gamma'(t) \\ \gamma(b_0) - \gamma(t) \end{pmatrix} > 0. \quad (3.5.21)$$

We can reduce θ_0 once more, and in what follows, we assume that (3.5.21) holds for all $t \in (a_0 - \theta_0, a_0)$. Consider the function

$$h(t, \theta) = \det \begin{pmatrix} & \gamma'(t) \\ \gamma'(t - (b_0 - a_0) - \theta) & \end{pmatrix}, \quad \theta \in (0, \theta_0], t \in [b_0, b_0 + \theta].$$

This function is continuous and differentiable on its domain. It follows from (3.5.20) and (3.5.21) that

$$h(b_0, \theta) > 0 > h(b_0 + \theta, \theta), \quad \theta \in (0, \theta_0).$$

Thus, for any such θ , there exists $b_\theta \in (b_0, b_0 + \theta)$ such that $h(b_\theta, \theta) = 0$. We will show that such b_θ is unique. Let $a_\theta = b_\theta - (b_0 - a_0) - \theta$. We would like to verify that the functions $a(\theta) = a_\theta$ and $b(\theta) = b_\theta$ are the function whose existence is stated in the proposition.

The remaining part of the proof is based on the following lemma.

Lemma 3.5.17. *If $\theta \in (0, \theta_0)$ and θ is sufficiently small, then $D_R(a_\theta, b_\theta) < 0$ and $D_L(a_\theta, b_\theta) < 0$.*

The proof of this lemma is rather technical, and we present it after completing the proof of the proposition.

The remaining part of the proof literally repeats the proof of Proposition 3.5.9. We can assume that θ_0 is so small that the statement of Lemma 3.5.17 holds for all θ in the interval $(0, \theta_0)$. We calculate the derivative of h :

$$\frac{\partial h}{\partial t}(b_\theta, \theta) = \det \begin{pmatrix} g'(b_\theta) \\ g'(a_\theta) \end{pmatrix} \cdot [C_L D_L - C_R D_R](a_\theta, b_\theta) < 0. \quad (3.5.22)$$

Therefore, the function $h(\cdot, \theta)$ has unique root on $(b_0, b_0 + \theta)$. We save the notation b_θ for this unique root. One can calculate the partial derivative with respect to θ :

$$\frac{\partial h}{\partial \theta}(b_\theta, \theta) = -\det \begin{pmatrix} g'(b_\theta) \\ g'(a_\theta) \end{pmatrix} \cdot [C_L D_L](a_\theta, b_\theta) > 0. \quad (3.5.23)$$

The implicit function theorem implies that b_θ is a differentiable function of θ and

$$\frac{\partial b_\theta}{\partial \theta} = \frac{C_L D_L}{C_L D_L - C_R D_R} \in (0, 1)$$

for $\theta \in (0, \theta_0)$. It follows that $a_\theta = b_\theta - \theta - (b_0 - a_0)$ decreases and

$$\lim_{\theta \rightarrow 0^+} b_\theta = b_0, \quad \lim_{\theta \rightarrow 0^+} a_\theta = a_0.$$

Proposition 3.5.16 is proved. ■

Proof of Lemma 3.5.17. We will deal with D_R only, the other inequality is symmetric. Since $f'(b_0) = 0$ and $f'(t) < 0$ for $t \in (b_0, b_0 + \theta_0)$, we have $f''(b_0) \leq 0$. Since also $f'(a_0) = 0$, this yields $D_R(a_0, b_0) \leq 0$. If $D_R(a_0, b_0) < 0$, then $D_R(a_\theta, b_\theta) < 0$ for small θ , by continuity. The same reasoning shows that $D_L(a_0, b_0) \leq 0$, and if $D_L(a_0, b_0) < 0$, then $D_L(a_\theta, b_\theta) < 0$ for θ sufficiently small.

Let us consider the case when one of $D_L(a_0, b_0)$ and $D_R(a_0, b_0)$ is strictly negative while the other one is zero. The case where both differentials vanish will be considered later. Without loss of generality assume $D_L(a_0, b_0) < 0$ and $D_R(a_0, b_0) = 0$. Then $f''(b_0) = 0$, and therefore

$$\kappa'_3(b_0) = \frac{f''(b_0)g'_1(b_0) - f'(b_0)g''_1(b_0)}{(g'_1(b_0))^2} = 0, \quad \mathfrak{K}(b_0) = \frac{\kappa'_3(b_0)}{\kappa'_2(b_0)} = 0.$$

Condition 2.1.11 implies that \mathfrak{K}' does not change its sign in a right neighborhood of b_0 . From (3.5.19) we know that \mathfrak{K} is non-positive on $(b_0, b_0 + \theta_0)$, therefore $\mathfrak{K}' \leq 0$ in a right neighborhood of b_0 as well. Let us note that

$$\mathfrak{K}' \text{ is not identically zero in any right neighborhood of } b_0. \quad (3.5.24)$$

Assume on the contrary that \mathfrak{K}' is identically zero in a right neighborhood of b_0 . Then the function \mathfrak{K} is constant there and coincides with $\mathfrak{K}(b_0) = 0$. Therefore, $\kappa'_3 = 0$ in the same neighborhood. Then κ_3 is constant there and equal to $\kappa_3(b_0) = 0$. This contradicts (3.5.18).

Inequalities (3.5.22) and (3.5.23) hold for $\theta = 0$, therefore, by continuity, they also hold for θ sufficiently small. Thus,

$$\frac{\partial b_\theta}{\partial \theta} = \frac{C_L D_L}{C_L D_L - C_R D_R} > 0.$$

Let $F(\theta) = D_R(a_\theta, b_\theta)$. From (3.5.11), we obtain

$$F'(\theta) = \left(\kappa'_2(b_\theta)g'_1(b_\theta)\mathfrak{K}'(b_\theta) + \left[\frac{(g'_1\kappa'_2)'}{g'_1\kappa'_2}(b_\theta) - \frac{\kappa'_2}{\kappa_2 - \kappa}(b_\theta) \right] F(\theta) \right) \frac{\partial b_\theta}{\partial \theta}.$$

We use the integrating factor

$$M(\theta) = \exp\left(-\int_0^\theta \left[\frac{(g'_1\kappa'_2)'}{g'_1\kappa'_2}(b_\eta) - \frac{\kappa'_2}{\kappa_2 - \kappa}(b_\eta) \right] \frac{\partial b_\eta}{\partial \eta} d\eta\right), \quad \theta > 0,$$

to conclude

$$\frac{d}{d\theta}(F(\theta)M(\theta)) = M(\theta)\kappa'_2(b_\theta)g'_1(b_\theta)\mathfrak{K}'(b_\theta) \frac{\partial b_\theta}{\partial \theta} \leq 0. \quad (3.5.25)$$

We know that $F(0) = 0$, therefore $F(\eta) \leq 0$ for $\eta > 0$. If $F(\eta) = 0$ for some $\eta > 0$, then the expression in (3.5.25) vanishes on $(0, \eta)$. Therefore, $\mathfrak{K}'(b_\theta) = 0$ on $(0, \eta)$ as

well, which contradicts (3.5.24). Hence, we have $D_R(a_\theta, b_\theta) = F(\theta) < 0$ for sufficiently small $\theta > 0$. The inequality $D_L(a_\theta, b_\theta) < 0$ follows from continuity and our assumption $D_L(a_0, b_0) < 0$.

It remains to prove the statement for the case $D_L(a_0, b_0) = D_R(a_0, b_0) = 0$. In this case, $f''(a_0) = f''(b_0) = 0$, $\mathfrak{K}' \leq 0$ in a right neighborhood of b_0 , and $\mathfrak{K}' \geq 0$ in a left neighborhood of a_0 . Instead of f consider the function

$$\tilde{f} = (1 - \chi_{[a_0, b_0]})f + \chi_{[a_0, b_0]}f(a_0),$$

and the corresponding function $\tilde{\mathfrak{K}}$. The function \tilde{f} is C^2 -smooth. For any θ , the pair (a_θ, b_θ) satisfies the cup equation (3.4.2) with \tilde{f} instead of f . Also, outside (a_0, b_0) , we have $\tilde{\mathfrak{K}} = \mathfrak{K}$. The function $\tilde{\mathfrak{K}} = \mathfrak{K}$ is not constant on (b_0, b_θ) by (3.5.24). For any $t_0 \in (a_0, b_0)$, we obtain that $\tilde{\mathfrak{K}}$ satisfies the assumptions of Corollary 3.5.13, which yields $D_R(a_\theta, b_\theta) < 0$ and $D_L(a_\theta, b_\theta) < 0$. ■

3.6 Forces

Definition 3.6.1. Let B be a Bellman candidate on a fence $\Omega(I)$. The function

$$\mathcal{F}_I = \frac{\kappa_2 - \kappa}{\kappa'_2} \beta'_2 \tag{3.6.1}$$

on I is called its *force function*.

Remark 3.6.2. Suppose that $\Omega_{\text{ch}}(I)$ is a chordal domain with corresponding function $s: I \rightarrow \mathbb{R}$, and that B is the standard Bellman candidate on $\Omega_{\text{ch}}(I)$. Then

$$\mathcal{F}_I(t) = \frac{D}{g'_1(t)\kappa'_2(t)}, \quad t \in I,$$

where $D = D_R(s, t)$ if $t > s(t)$ and $D = D_L(t, s)$ if $t < s(t)$.

Proof. This is a direct consequence of formulas (3.4.7), (3.5.1), and (3.6.1). ■

Remark 3.6.3. The force function \mathcal{F} is non-positive on the domain of its definition.

Proof. If $\Omega(I)$ is a right tangent domain, then we have $\beta'_2 \leq 0$ (see Remark 3.3.2) and $\kappa_2 - \kappa > 0$ (this follows from geometric properties of Ω). For the case of left tangents, both inequalities are opposite. Therefore, $\mathcal{F} \leq 0$.

For the case of a chordal domain, both differentials D_L and D_R are negative, hence $\mathcal{F} \leq 0$. ■

3.6.1 Gluing fences

In this part we present necessary and sufficient conditions for gluing two Bellman candidates on fences with a common point. By this we mean the following situation.

Proposition 3.6.4. Let $t_-, t_0, t_+ \in \mathbb{R}$, $t_- \leq t_0 \leq t_+$, $I_- = [t_-, t_0]$ and $I_+ = [t_0, t_+]$. Suppose that $\Omega(I_\pm)$ are two fences with Bellman candidates B_\pm on them with corresponding functions w_\pm, S_\pm, β_\pm . Let $\text{int } \Omega(I_-) \cap \text{int } \Omega(I_+) = \emptyset$. Suppose that

$$\beta_+(t_0) = \beta_-(t_0) \stackrel{\text{def}}{=} \beta_0. \quad (3.6.2)$$

Then the function B defined by the formula

$$B(x) = \begin{cases} B_-(x), & x \in \Omega(I_-), \\ f(t_0) + \langle \beta_0, x - g(t_0) \rangle, & x \in \text{ind conv}(S_-(t_0) \cup S_+(t_0)), \\ B_+(x), & x \in \Omega(I_+). \end{cases} \quad (3.6.3)$$

is a C^1 -smooth Bellman candidate on its domain (see Definition 3.1.1 for the induced convex hull).

Proof. The C^1 -smoothness of B follows from (3.6.2) and the C^1 -smoothness of B_\pm . The concavity of B is then implied by Proposition 3.1.2. ■

Corollary 3.6.5. The conclusion of Proposition 3.6.4 holds if we replace (3.6.2) by the forces equality $\mathcal{F}_{I_+}(t_0) = \mathcal{F}_{I_-}(t_0)$.

Proof. Since κ_2 and κ_3 do not depend on the fences, it follows from (3.2.6) and (3.2.9) applied to t_0 that $\beta_+(t_0) = \beta_-(t_0)$. ■

Now we consider specific cases of gluing fences we have investigated in Sections 3.3 and 3.4.

3.6.2 Forces and tails

In Definition 3.6.1, forces were defined on fences. For a given chordal domain, we have a force function defined inside this domain. We wish to extend the forces outside for the purpose of continuation of a Bellman candidate from a chordal domain via tangents.

Let $I = [t_-, t_+]$ be an interval, and let $\Omega_{\text{ch}}(I)$ be a chordal domain with corresponding function $s: I \rightarrow \mathbb{R}$ such that $s(t) \leq t$ for $t \in I$. We define a right force of the chordal domain $\Omega_{\text{ch}}(I)$ in the following way. We define t^{R} to be the supremum of the numbers τ , $\tau \geq t_+$, such that there exists a standard Bellman candidate on the right tangent domain $\Omega_{\text{R}}([t_+, \tau])$ satisfying

$$\mathcal{F}_{[t_-, t_+]}(t_+) = \mathcal{F}_{[t_+, \tau]}(t_+).$$

Note that t^{R} can be equal to $+\infty$. The concatenation of the functions \mathcal{F}_I and $\mathcal{F}_{[t_+, t^{\text{R}}]}$ will be called the right force function of the chordal domain $\Omega_{\text{ch}}(I)$.

Definition 3.6.6. The function

$$F_R(t, \Omega_{\text{ch}}(I)) = \begin{cases} \mathcal{F}_I(t), & t \in I, \\ \mathcal{F}_{[t_+, t^R]}(t), & t \in [t_+, t^R], \end{cases}$$

is called the *right force function* of the chordal domain $\Omega_{\text{ch}}(I)$. The set (t_-, t^R) is called the *right tail* of the chordal domain $\Omega_{\text{ch}}(I)$.

Remark 3.6.7. If $t^R < +\infty$, then $\mathcal{F}_{[t_+, t^R]}(t^R) = 0$.

Proof. This immediately follows from the maximality of t^R , and the continuity and non-positivity of the force. ■

Remark 3.6.8. If $t_+ < t^R < +\infty$, then there exists $\bar{t}, \bar{t} < t^R$, such that the torsion of the curve γ is strictly positive on (\bar{t}, t^R) . If $t^R > t_1 \geq t_+$ and the torsion of the curve γ is non-positive on an interval (t_1, t_2) , then $t^R > t_2$.

Proof. Due to Condition 2.1.11, we can find $\bar{t}, \bar{t} < t^R$, sufficiently close to t^R such that either $\mathfrak{K}' > 0$ on (\bar{t}, t^R) or $\mathfrak{K}' \leq 0$ on (\bar{t}, t^R) . If $\mathfrak{K}' \leq 0$, then we use (3.2.12) for $t_0 = \bar{t}, t = t^R$, and obtain that $\mathcal{F}_{[t_+, t^R]}(t^R) < 0$, which contradicts Remark 3.6.7. Thus, $\mathfrak{K}' > 0$ on (\bar{t}, t^R) . The second claim follows immediately from the first one. ■

Similarly, we define the left force function and the left tail of the chordal domain. Let $I = [t_-, t_+]$ be an interval, and let $\Omega_{\text{ch}}(I)$ be a chordal domain with corresponding function $s: I \rightarrow \mathbb{R}$ such that $s(t) \geq t$ for $t \in I$. We define t^L to be the infimum of the numbers $\tau, \tau \leq t_-$, such that there exists a standard Bellman candidate on the left tangent domain $\Omega_L([\tau, t_-])$ satisfying

$$\mathcal{F}_{[\tau, t_-]}(t_-) = \mathcal{F}_{[t_-, t_+]}(t_-).$$

Note that t^L can be equal to $-\infty$.

Definition 3.6.9. The function

$$F_L(t, \Omega_{\text{ch}}(I)) = \begin{cases} \mathcal{F}_{[t^L, t_-]}(t), & t \in [t^L, t_-], \\ \mathcal{F}_I(t), & t \in I, \end{cases}$$

is called the *left force function* of the chordal domain $\Omega_{\text{ch}}(I)$. The set (t^L, t_+) is called the *left tail* of the chordal domain $\Omega_{\text{ch}}(I)$.

The following remark is the “left” analog of Remarks 3.6.7 and 3.6.8.

Remark 3.6.10. If $t^L > -\infty$, then $\mathcal{F}_{[t^L, t_-]}(t^L) = 0$. If $-\infty < t^L < t_-$, then there exists $\bar{t}, t^L < \bar{t}$, such that the torsion of the curve γ is strictly negative on (t^L, \bar{t}) . If $t^L < t_2 \leq t_-$ and the torsion of the curve γ is non-negative on an interval (t_1, t_2) , then $t^L < t_1$.

Remark 3.6.11. Suppose that $\Omega_{\text{ch}}(I)$ is a chordal domain, and that its upper chord is $[g(a), g(b)]$. Since the force function outside the chordal domain does not depend on the foliation inside it, we will use the notation $F_{\text{R}}(t; a, b)$ instead of $F_{\text{R}}(t, \Omega_{\text{ch}}(I))$, and $F_{\text{L}}(t; a, b)$ instead of $F_{\text{L}}(t, \Omega_{\text{ch}}(I))$ for t outside the chordal domain. The forces satisfy the following identities:

$$F_{\text{R}}(t; a, b) = \exp\left(-\int_b^t \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \left(\int_b^t \exp\left(\int_b^\tau \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \mathfrak{R}'(\tau) d\tau + \frac{D_{\text{R}}(a, b)}{g'_1(b)\kappa'_2(b)} \right),$$

$$t \in [b, t^{\text{R}}], \quad (3.6.4)$$

$$F_{\text{L}}(t; a, b) = \exp\left(\int_t^a \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \left(-\int_t^a \exp\left(-\int_\tau^a \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \mathfrak{R}'(\tau) d\tau + \frac{D_{\text{L}}(a, b)}{g'_1(a)\kappa'_2(a)} \right),$$

$$t \in [t^{\text{L}}, a]. \quad (3.6.5)$$

Analogously, we define the forces and tails from the infinities. Suppose $t \in \mathbb{R}$ and that there exists the standard Bellman candidate on the right tangent domain $\Omega_{\text{R}}(-\infty, t)$ (see Proposition 3.3.6). Let $t^{\text{R}} \leq +\infty$ be the supremum of such t .

Definition 3.6.12. The function

$$F_{\text{R}}(t, -\infty) = \mathfrak{F}_{(-\infty, t^{\text{R}})}(t), \quad t \in (-\infty, t^{\text{R}}),$$

is called the *right force function of $-\infty$* . The ray $(-\infty, t^{\text{R}})$ is called the *right tail of $-\infty$* .

Similarly, we define the left force function and the left tail of $+\infty$. Let $t^{\text{L}} \geq -\infty$ be the infimum of such $t \in \mathbb{R}$ that there exists the standard Bellman candidate on the left tangent domain $\Omega_{\text{L}}(t^{\text{L}}, +\infty)$ (see Proposition 3.3.7).

Definition 3.6.13. The function

$$F_{\text{L}}(t, +\infty) = \mathfrak{F}_{(t^{\text{L}}, +\infty)}(t), \quad t \in (t^{\text{L}}, +\infty),$$

is called the *left force function of $+\infty$* . The ray $(t^{\text{L}}, +\infty)$ is called the *left tail of $+\infty$* .

3.6.3 Properties of forces

Though the expressions for the forces are well defined for arbitrary pairs of points (a_0, b_0) , when we write a force concerning such a pair, we always assume that the pair (a_0, b_0) satisfies the cup equation (3.4.2). We study differential properties of forces.

Proposition 3.6.14. Let $\Omega_{\text{ch}}([a, b], *)$ be a chordal domain. Then its forces satisfy the following differential equation on the corresponding tails:

$$F' = \mathfrak{R}' - \beta'_2 = \mathfrak{R}' - \frac{\kappa'_2}{\kappa_2 - \kappa} F, \quad \text{where } F \text{ is } F_{\text{R}} \text{ or } F_{\text{L}}. \quad (3.6.6)$$

Proof. Taking into account the definition (3.6.1), this statement may be obtained via division of (3.2.9) by κ'_2 and differentiation. ■

Remark 3.6.15. The functions $F_R - \mathfrak{R}$ and $F_L - \mathfrak{R}$ are strictly increasing and strictly decreasing on their domains, respectively.

Proof. According to (3.6.6), $F' - \mathfrak{R}' = -\frac{\kappa'_2}{\kappa_2 - \kappa} F$. We always have $F < 0$ and $\kappa'_2 > 0$. It remains to notice that $\kappa_2 - \kappa > 0$ in the case of the right force, and $\kappa_2 - \kappa < 0$ in the case of the left force. ■

Lemma 3.6.16. Suppose that $\Omega_{\text{ch}}(I)$ is a chordal domain, and that its upper chord is $[g(a), g(b)]$. For a fixed t , $t > b$, consider the right force function $F_R(t, a(b), b)$ as a function of b . Then its derivative with respect to b satisfies the following equality:

$$\begin{aligned} \frac{\partial F_R(t, a(b), b)}{\partial b} &= \frac{D_R(a, b)}{g'_1(b)} \cdot \left[\frac{1}{\kappa_2(b) - \kappa_R(b)} - \frac{1}{\kappa_2(b) - \kappa_{\text{chord}}(b)} \right] \\ &\quad \times \exp\left(-\int_b^t \frac{\kappa'_2(\tau)}{\kappa_2(\tau) - \kappa_R(\tau)} d\tau\right), \end{aligned} \quad (3.6.7)$$

where κ_R is the slope of the right tangent (see (2.1.5)), and $\kappa_{\text{chord}}(b)$ is the slope of the upper chord of the chordal domain, i.e.,

$$\kappa_{\text{chord}}(b) = \frac{g_2(b) - g_2(a)}{g_1(b) - g_1(a)}. \quad (3.6.8)$$

For a fixed t , $t < a$, we get a symmetric formula:

$$\begin{aligned} \frac{\partial F_L(t, a, b(a))}{\partial a} &= \frac{D_L(a, b)}{g'_1(a)} \cdot \left[\frac{1}{\kappa_2(a) - \kappa_L(a)} - \frac{1}{\kappa_2(a) - \kappa_{\text{chord}}(a)} \right] \\ &\quad \times \exp\left(\int_t^a \frac{\kappa'_2(\tau)}{\kappa_2(\tau) - \kappa_L(\tau)} d\tau\right), \end{aligned} \quad (3.6.9)$$

where κ_L is the slope of the left tangent (see (2.1.5)), and $\kappa_{\text{chord}}(a) = \kappa_{\text{chord}}(b)$ is given by (3.6.8).

Proof. We differentiate $F_R(t, a, b)$ (see (3.6.4)) with respect to b , regarding $a = a(b)$:

$$\begin{aligned} \frac{\partial F_R(t, a(b), b)}{\partial b} &= \frac{\kappa'_2(b)}{\kappa_2(b) - \kappa(b)} F_R(t, a, b) \\ &\quad + \left\{ -\mathfrak{R}'(b) - \frac{\kappa'_2(b)}{\kappa_2(b) - \kappa(b)} \int_b^t \exp\left(\int_b^\tau \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \mathfrak{R}'(\tau) d\tau \right. \\ &\quad \left. + \frac{1}{g'_1(b)\kappa'_2(b)} \frac{dD_R(a(b), b)}{db} - \frac{(g'_1(b)\kappa'_2(b))'}{(g'_1(b)\kappa'_2(b))^2} D_R(a, b) \right\} \exp\left(-\int_b^t \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \\ &= \left\{ -\mathfrak{R}' + \frac{\kappa'_2}{\kappa_2 - \kappa} \cdot \frac{D_R}{g'_1\kappa'_2} + \frac{1}{g'_1\kappa'_2} \cdot \frac{dD_R}{db} - \frac{(g'_1\kappa'_2)'}{(g'_1\kappa'_2)^2} D_R \right\} \exp\left(-\int_b^t \frac{\kappa'_2}{\kappa_2 - \kappa}\right). \end{aligned} \quad (3.6.10)$$

We use formula (3.5.11) to simplify the expression in the braces. We note that κ in (3.6.10) differs from the one in (3.5.11). We write κ_R for that which appears in (3.6.10) and κ_{chord} for the one in (3.5.11):

$$\begin{aligned}
& \left\{ -\mathfrak{R}' + \frac{\kappa_2'}{\kappa_2 - \kappa_R} \cdot \frac{D_R}{g_1' \kappa_2'} + \frac{1}{g_1' \kappa_2'} \cdot \frac{dD_R}{db} - \frac{(g_1' \kappa_2')'}{(g_1' \kappa_2')^2} D_R \right\} \\
&= -\mathfrak{R}' + \frac{\kappa_2'}{\kappa_2 - \kappa_R} \cdot \frac{D_R}{g_1' \kappa_2'} \\
&+ \frac{1}{g_1' \kappa_2'} \cdot \left[\kappa_2' g_1' \mathfrak{R}' + \left(\frac{(g_1' \kappa_2')'}{g_1' \kappa_2'} - \frac{\kappa_2'}{\kappa_2 - \kappa_{\text{chord}}} \right) D_R \right] - \frac{(g_1' \kappa_2')'}{(g_1' \kappa_2')^2} D_R \\
&= \frac{D_R}{g_1'} \left(\frac{1}{\kappa_2 - \kappa_R} - \frac{1}{\kappa_2 - \kappa_{\text{chord}}} \right). \quad \blacksquare
\end{aligned}$$

3.7 Linearity domains

As stated in Section 3.1, we classify linearity domains by the number of points on the fixed boundary. For each linearity domain \mathfrak{L} , we will define the corresponding force function on $\mathfrak{L} \cap \partial_{\text{fixed}} \Omega$.

3.7.1 Angle

The first linearity domain we study, an *angle*, has only one point $g(u)$ on the fixed boundary. Recall that $S_L(u)$ and $S_R(u)$ are the left and the right tangent segments to the free boundary of Ω starting at the point $g(u)$, and $T(u)$ is the closed curvilinear triangle with the vertex $g(u)$ whose sides are $S_L(u)$, $S_R(u)$, and the part of $\partial_{\text{free}} \Omega$ between the two tangency points. We will use the symbol $\Omega_{\text{ang}}(u)$ for this domain of linearity; note that geometrically $\Omega_{\text{ang}}(u) = T(u)$.

Definition 3.7.1. An affine function B on an angle $\Omega_{\text{ang}}(u)$ is called a *standard candidate* on $\Omega_{\text{ang}}(u)$ if the vector $\gamma'(u)$ is parallel to the graph of B on $\Omega_{\text{ang}}(u)$.

An angle is the linearity domain that appeared in Proposition 3.6.4 in the case where $\Omega(I_-)$ is a right tangent domain and $\Omega(I_+)$ is a left one, see Figure 3.3. The function B defined by (3.6.3) is a standard candidate on the angle. See the graphical representation of this situation in Section 3.7.3, Figure 3.9.

3.7.2 Linearity domains with two points on the fixed boundary

Consider a linearity domain \mathfrak{L} that has two points $g(a)$ and $g(b)$ on the fixed boundary $\partial_{\text{fixed}} \Omega$, assuming $a < b$ and $[g(a), g(b)] \subset \Omega$. Surely, the segment $[g(a), g(b)]$ is a part of the boundary of the linearity domain. It is natural to assume that there are

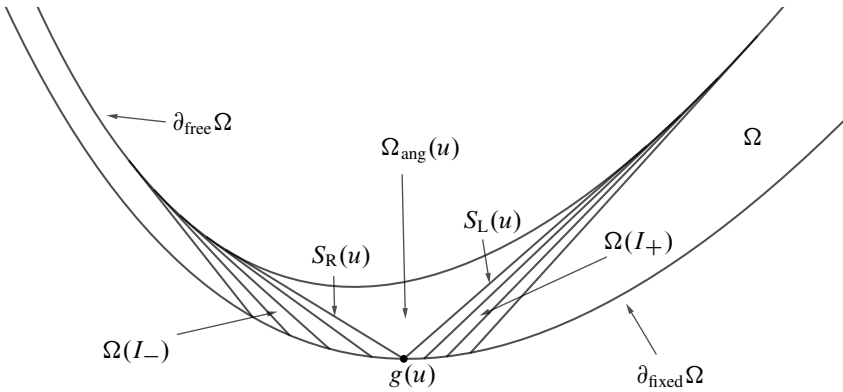


Figure 3.3. An angle $\Omega_{\text{ang}}(u)$ with adjacent domains.

two extremal segments tangent to the free boundary, $S(a)$ and $S(b)$, bounding our linearity domain from the left and right. If they have the same orientation, namely, they are either $S_L(a)$ and $S_L(b)$, or $S_R(a)$ and $S_R(b)$, then the linearity domain is called a *trolleybus*, the *left* one or the *right* one, respectively. These trolleybuses will be denoted by $\Omega_{\text{tr,L}}(a, b)$ and $\Omega_{\text{tr,R}}(a, b)$, see Figure 3.4 and Figure 3.5.

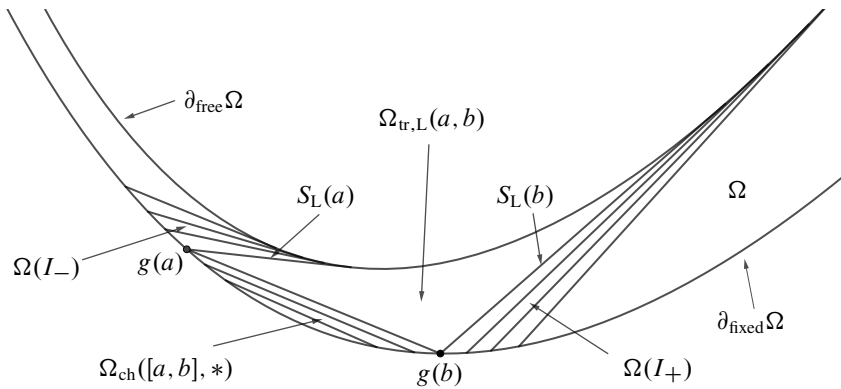


Figure 3.4. Trolleybus $\Omega_{\text{tr,L}}(a, b)$ with adjacent domains.

A linearity domain whose border tangents have different orientation, i.e., $S_R(a)$ and $S_L(b)$, is called a *birdie*, see Figure 3.6; we denote it by $\Omega_{\text{bird}}(a, b)$. It is clear that the opposite situation, $S_L(a)$ and $S_R(b)$, is impossible because these two segments intersect each other.

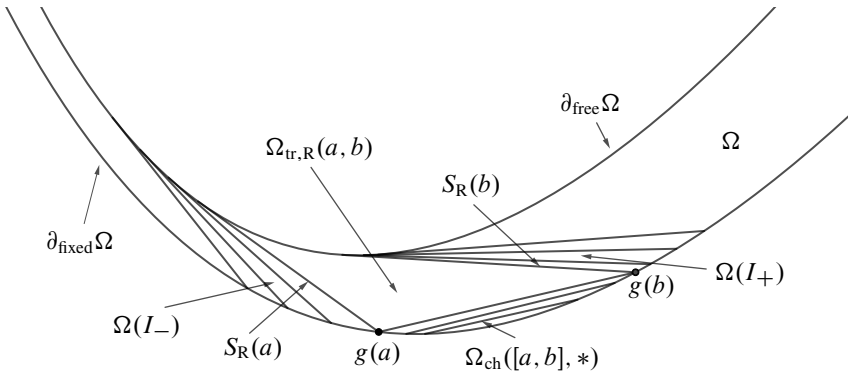


Figure 3.5. Trolleybus $\Omega_{\text{tr},R}(a, b)$ with adjacent domains.

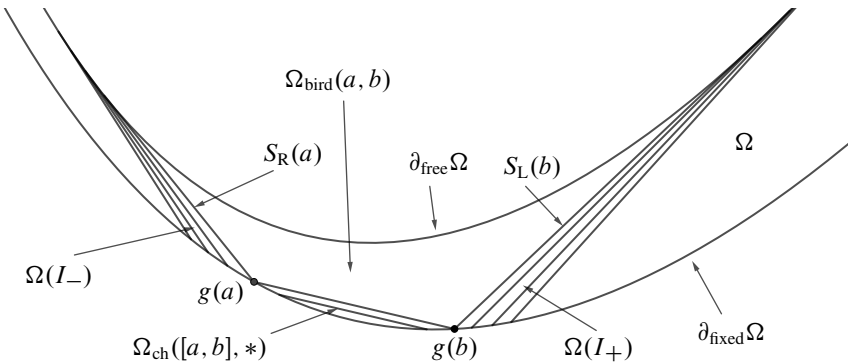


Figure 3.6. A birdie $\Omega_{\text{bird}}(a, b)$ with the adjacent domains.

As in the case of an angle, we suppose that $g(a)$ and $g(b)$ are the points where we glue two different fences. So, there are three fences: a tangent domain $\Omega(I_-)$ adjacent to \mathcal{L} along the tangent $S(a)$, a tangent domain $\Omega(I_+)$ adjacent to \mathcal{L} along $S(b)$, and a chordal domain $\Omega_{\text{ch}}([a, b], *)$.

Proposition 3.7.2. *Let $I_- = [t_-, a]$, $I_+ = [b, t_+]$. Suppose that B_{\pm} are Bellman candidates on $\Omega(I_{\pm})$, and $B_{\Omega_{\text{ch}}([a,b],*)}$ is the Bellman candidate on $\Omega_{\text{ch}}([a, b], *)$. Suppose that the gluing conditions hold for each of two pairs of fences, i.e.,*

$$\mathcal{F}_{I_-}(a) = F_L(a, \Omega_{\text{ch}}([a, b], *)), \quad \mathcal{F}_{I_+}(b) = F_R(b, \Omega_{\text{ch}}([a, b], *)).$$

Let $B_{\mathcal{L}}(x)$ be the affine function defined by the equation

$$\det \begin{pmatrix} g'_1(a) & g'_2(a) & f'(a) \\ g'_1(b) & g'_2(b) & f'(b) \\ x_1 - g_1(a) & x_2 - g_2(a) & B_{\mathcal{L}}(x) - f(a) \end{pmatrix} = 0, \quad x \in \mathcal{L}. \quad (3.7.1)$$

Then the function B defined by the formula

$$B(x) = \begin{cases} B_-(x), & x \in \Omega(I_-), \\ B_{\mathcal{L}}(x), & x \in \mathcal{L}, \\ B_+(x), & x \in \Omega(I_+), \\ B_{\Omega_{\text{ch}}([a,b],*)}, & x \in \Omega_{\text{ch}}([a,b],*), \end{cases}$$

is a C^1 -smooth Bellman candidate on its domain.

Proof. Condition (2.1.2) implies

$$\det \begin{pmatrix} g'_1(a) & g'_2(a) \\ g'_1(b) & g'_2(b) \end{pmatrix} \neq 0,$$

therefore (3.7.1) defines $B_{\mathcal{L}}$ correctly. The cup equation (3.4.2) guarantees that the affine function $B_{\mathcal{L}}$ coincides with the unique affine extension of $B_{\Omega_{\text{ch}}([a,b],*)}$ to \mathcal{L} . Therefore, it also coincides with the affine parts of the functions from (3.6.3) for both pairs of glued fences: for $\Omega(I_-)$ and Ω_{ch} , and for Ω_{ch} and $\Omega(I_+)$. Thus, Proposition 3.6.4 gives that the function B is a C^1 -smooth Bellman candidate. ■

Definition 3.7.3. The function $B_{\mathcal{L}}$ defined by (3.7.1) in the linearity domain \mathcal{L} with two points on the fixed boundary is called a *standard candidate* there.

Remark 3.7.4. By (3.7.1) we have the following formula for the derivative of the affine function $B_{\mathcal{L}}$:

$$\beta_2 = \frac{\partial B_{\mathcal{L}}}{\partial x_2} = \frac{\det \begin{pmatrix} g'_1(a) & f'(a) \\ g'_1(b) & f'(b) \end{pmatrix}}{\det \begin{pmatrix} g'_1(a) & g'_2(a) \\ g'_1(b) & g'_2(b) \end{pmatrix}} = \frac{\kappa_3(b) - \kappa_3(a)}{\kappa_2(b) - \kappa_2(a)}. \quad (3.7.2)$$

Definition 3.7.5. Let \mathcal{L} be a domain of linearity with two points on the fixed boundary: $g(a), g(b) \in \mathcal{L} \cap \partial_{\text{fixed}} \Omega$, with $g(a) \prec g(b)$. Define the force function F on $\{a, b\}$ as follows:

$$F(a) = \frac{D_{\text{L}}(a, b)}{g'_1(a)\kappa'_2(a)}, \quad F(b) = \frac{D_{\text{R}}(a, b)}{g'_1(b)\kappa'_2(b)}.$$

Let us note that a birdie can be regarded as a union of a trolleybus and an angle (there are two symmetric ways):

$$\begin{aligned} \Omega_{\text{bird}}(a, b) &= \Omega_{\text{tr,R}}(a, b) \uplus \Omega_{\text{R}}(b, b) \uplus \Omega_{\text{ang}}(b), \\ \Omega_{\text{bird}}(a, b) &= \Omega_{\text{ang}}(a) \uplus \Omega_{\text{L}}(a, a) \uplus \Omega_{\text{tr,L}}(a, b). \end{aligned} \quad (3.7.3)$$

Note that the right and the left sides of the equalities are equal as planar domains, provided we substitute \cup for \uplus .

Remark 3.7.6. The symbol \uplus in (3.7.3) means the following: if a function B on this domain is continuous and its restriction to each single subdomain of one side of the formula is a standard candidate, then this function B is a standard candidate for each subdomain of the other side of the formula. This easily follows from Proposition 3.6.4 and Proposition 3.7.2. Graphical representation of this equality is presented in Figure 3.14 (see Section 3.7.3 below).

It is convenient to introduce two more “linearity domains” for the purposes of formalization. First, sometimes we will treat a single chord $[g(a), g(b)]$, where (a, b) satisfies the cup equation (3.4.2), as a linearity domain \mathfrak{L} . The standard candidate B inside $[g(a), g(b)]$ is given by linearity. We note that Proposition 3.7.2 is valid for the following two “hidden” subcases. The first one is when the chord $[g(a), g(b)]$ is tangent to the free boundary of Ω , $\Omega(I_-) = \Omega_L(I_-)$ is a left tangent domain, and $\Omega(I_+) = \Omega_R(I_+)$ is a right tangent domain. We obtain the chordal domain with the glued tangent domains (see Figure 3.7).

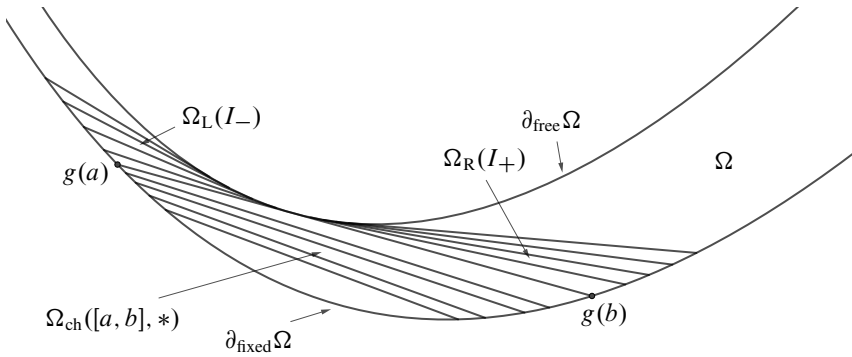


Figure 3.7. A chordal domain $\Omega_{\text{ch}}([a, b], *)$ with tangent domains attached to it.

The second subcase is when the chord $[g(a), g(b)]$ is not tangent to the free boundary and $\Omega(I_{\pm}) = \Omega_{\text{ch}}(*, [a, b])$ are the right and the left parametrization of the same chordal domain, which lies above the chord $[g(a), g(b)]$. In this subcase we obtain two glued chordal domains $\Omega_{\text{ch}}(*, [a, b])$ and $\Omega_{\text{ch}}([a, b], *)$, which will appear when it is impossible to consider their union as one chordal domain (if either $D_R(a, b) = 0$ or $D_L(a, b) = 0$), see Figure 3.8.

Second, sometimes it is useful to treat a single tangent $\Omega_R(u, u)$ or $\Omega_L(u, u)$ as a linearity domain \mathfrak{L} . Moreover, no matter how strange it seems, it is natural to think of it as a domain with two points on the fixed boundary. We treat this single tangent as a trolleybus of zero width, i.e., its base is the chord $[g(u), g(u)]$. This can be considered as a limit of a sequence of trolleybuses on the chord shrinking to

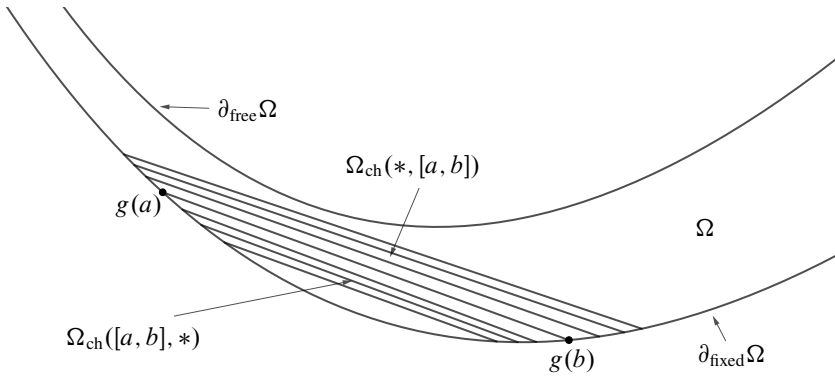


Figure 3.8. Two chordal domains $\Omega_{\text{ch}}([a, b], *)$ and $\Omega_{\text{ch}}(*, [a, b])$ with the chord between them.

the point u . It will appear only when $u = c_i$ for some i , where c_i is a single point root from Definition 2.1.13. The standard candidate B in this domain is obtained by passing to the limit in (3.7.1):

$$\det \begin{pmatrix} g'_1(u) & g'_2(u) & f'(u) \\ g''_1(u) & g''_2(u) & f''(u) \\ x_1 - g_1(u) & x_2 - g_2(u) & B_{\mathfrak{L}}(x) - f(u) \end{pmatrix} = 0, \quad x \in \mathfrak{L}. \quad (3.7.4)$$

This construction will appear between two tangent domains of the same direction when the standard candidates on them cannot be concatenated to be a standard candidate on the union of the domains. The obstacle of such a concatenation is the vanishing of the forces at the point u .

3.7.3 Graphical representation of the elementary domains

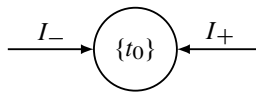


Figure 3.9. Graphical representation of an angle $\Omega_{\text{ang}}(t_0)$ with adjacent tangent domains.

As it was said in Section 3.1, we give a graphical representation for a combinatorial structure of foliations. The material of this subsection essentially repeats analogous constructions in [17]. We start with the representation of the simplest local foliations: fences (tangent domains and chordal domains) and some linearity domains. In Section 3.8.2 we will give a graphical representation describing a foliation of the whole domain Ω_ε .

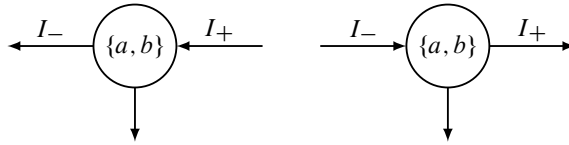


Figure 3.10. Graphical representation of trolleybuses $\Omega_{\text{tr,L}}(a, b)$ and $\Omega_{\text{tr,R}}(a, b)$ with adjacent tangent domains and a chordal domain.



Figure 3.11. Graphical representation of two chordal domains $\Omega_{\text{ch}}([a, b], *)$ and $\Omega_{\text{ch}}(*, [a, b])$ glued along the chord.

Usually vertices will correspond to linearity domains, and oriented edges will always correspond to fences. We draw a vertex of the graph that corresponds to an angle, and two incoming edges that correspond to the neighbor tangent domains, see Figure 3.9. We equip the elements of the graph with numerical parameters corresponding to the points on the fixed boundary.

Now we give a graphical representation of possible variants described in Proposition 3.7.2 (trolleybus and birdie). We draw a vertex for the domain of linearity and edges for the fences – a chordal domain and two tangent domains. The edge corresponding to the chordal domain is outgoing. We will draw the edges corresponding to the tangent domains horizontally, and their directions agree with the directions of the tangents (either left or right). A trolleybus always has one incoming and two outgoing edges, and for a birdie, we have two incoming and one outgoing edge, see Figure 3.10 for the graphs corresponding to trolleybuses and the left drawing in Figure 3.12 for the birdie.

We give a graphical representation of a full chordal domain with two neighbor tangent domains, see the right drawing in Figure 3.12. The vertex corresponds to the “linearity domain”, being the chord $[g(a), g(b)]$, and the edges correspond to the chordal domain and tangent domains. Figure 3.11 gives a graphical representation of two chordal domains $\Omega_{\text{ch}}([a, b], *)$ and $\Omega_{\text{ch}}(*, [a, b])$ glued along the chord (Figure 3.8). The vertex here corresponds to the chord $[g(a), g(b)]$, and the incoming and outgoing edges correspond to $\Omega_{\text{ch}}(*, [a, b])$ and $\Omega_{\text{ch}}([a, b], *)$, respectively.

Figure 3.13 gives a graphical representation of two tangent domains $\Omega_{\text{L}}(t_-, u)$ and $\Omega_{\text{L}}(u, t_+)$ glued along a zero-width “trolleybus” $\Omega_{\text{L}}(u, u)$ (on the left), and of

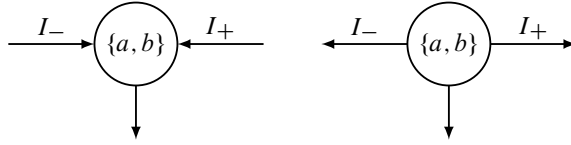


Figure 3.12. Graphical representation of a birdie $\Omega_{\text{bird}}(a, b)$ and of a full chordal domain $\Omega_{\text{ch}}([a, b], *)$ with adjacent domains.

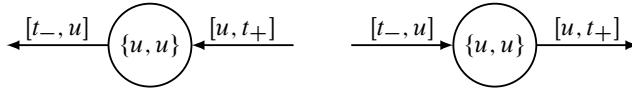


Figure 3.13. Graphical representation of two tangent domains glued along a “trolleybus” of zero width: the left and the right cases.

the symmetric case where $\Omega_{\text{R}}(t_-, u)$ and $\Omega_{\text{R}}(u, t_+)$ are glued along a zero-width “trolleybus” $\Omega_{\text{R}}(u, u)$ (on the right). The vertex here corresponds to the “trolleybus” and the edges correspond to the tangent domains.

Finally, Figure 3.14 demonstrates a graphical representation of equality (3.7.3).

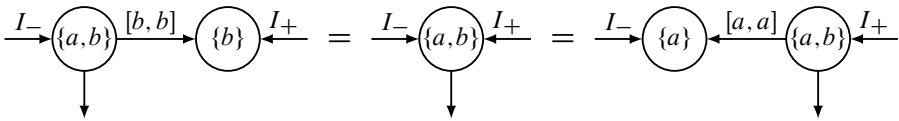


Figure 3.14. Graphical representation of the equality “birdie = angle + trolleybus”.

3.7.4 Multifigures

We begin with a structural agreement. For each linearity domain \mathfrak{L} , we make the following finiteness assumption: the intersection $\mathfrak{L} \cap \partial_{\text{fixed}} \Omega$ is assumed to be a union of a finite number of arcs (one or two of these arcs may be unbounded, i.e., parametrized by a ray),

$$\mathfrak{L} \cap \partial_{\text{fixed}} \Omega = \bigcup_{i=1}^k \{g(t) \mid t \in \alpha_i\},$$

where $\{\alpha_i\}_{i=1}^k$ is a finite set of disjoint closed intervals, which can be single points. The curvilinear arc that corresponds to α_i is $g(\alpha_i)$. We remind the reader the notation introduced in Section 2.1.3: the left endpoint of α_i is α_i^l and the right endpoint is α_i^r . As we will see, due to Condition 2.1.11, all linearity domains under consideration satisfy this finiteness assumption.

Let \mathcal{L} be a linearity domain. We know that the set $\{\gamma(a) \mid g(a) \in \mathcal{L} \cap \partial_{\text{fixed}} \Omega\}$ lies in a two-dimensional plane in \mathbb{R}^3 . Therefore, there exist a function $P_{\mathcal{L}}$, which is a linear combination of g_1, g_2 , and a constant function such that

$$f(a) = P_{\mathcal{L}}(a), \quad g(a) \in \mathcal{L} \cap \partial_{\text{fixed}} \Omega. \quad (3.7.5)$$

Surely, the converse is also true: if there exist some linear combination $P_{\mathcal{L}}$ of g_1, g_2 , and a constant function such that (3.7.5) holds, then there exists an affine function $B_{\mathcal{L}}$ such that $B_{\mathcal{L}}(g(a)) = f(a)$ for all $g(a) \in \mathcal{L} \cap \partial_{\text{fixed}} \Omega$. Namely, if $P_{\mathcal{L}} = \beta_0 + \beta_1 g_1 + \beta_2 g_2$, then

$$B_{\mathcal{L}}(x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2. \quad (3.7.6)$$

This function $B_{\mathcal{L}}$ is a Bellman candidate in \mathcal{L} .

Remark 3.7.7. Similar to the case where the linearity domain has only two points on the fixed boundary, identities (3.7.1) and (3.7.2) hold true for any $g(a), g(b) \in \mathcal{L} \cap \partial_{\text{fixed}}$. All the vectors $\gamma'(a)$ such that $g(a) \in \mathcal{L} \cap \partial_{\text{fixed}} \Omega$ lie in one plane orthogonal to the vector $(\beta_1, \beta_2, -1)$.

Definition 3.7.8. The function B defined by formula (3.7.1) in the linearity domain \mathcal{L} , where $g(a)$ and $g(b)$ are arbitrary points from $\mathcal{L} \cap \partial_{\text{fixed}} \Omega$, is called the *standard candidate* in \mathcal{L} .

As we have verified, the standard candidate in \mathcal{L} does not depend on the choice of a and b in the definition and coincides with the function given by (3.7.6).

In the following lemma, we use Definition 3.5.1.

Lemma 3.7.9. *Let a_1, a_2 , and a_3 be such that $\gamma'(a_i), i = 1, 2, 3$, lie in one plane. If $a_1 \leq a_2 \leq a_3$, then*

$$\begin{aligned} D_L(a_1, a_2) &= D_L(a_1, a_3), \\ D_R(a_1, a_3) &= D_R(a_2, a_3), \\ D_R(a_1, a_2) &= D_L(a_2, a_3). \end{aligned}$$

Proof. Let us prove the first identity; the others are similar. We find coefficients α_1 and α_2 such that $\gamma'(a_3) = \alpha_1 \gamma'(a_1) + \alpha_2 \gamma'(a_2)$. Then $g'(a_3) = \alpha_1 g'(a_1) + \alpha_2 g'(a_2)$. We substitute this to $D_L(a_1, a_3)$ and obtain

$$D_L(a_1, a_3) = \frac{\det \begin{pmatrix} \gamma'(a_1) \\ \gamma'(a_3) \\ \gamma''(a_1) \end{pmatrix}}{\det \begin{pmatrix} g'(a_1) \\ g'(a_3) \end{pmatrix}} = \frac{\alpha_2 \det \begin{pmatrix} \gamma'(a_1) \\ \gamma'(a_2) \\ \gamma''(a_1) \end{pmatrix}}{\alpha_2 \det \begin{pmatrix} g'(a_1) \\ g'(a_2) \end{pmatrix}} = D_L(a_1, a_2). \quad \blacksquare$$

Definition 3.7.10. Let \mathcal{L} be a linearity domain such that the intersection $\mathcal{L} \cap \partial_{\text{fixed}} \Omega$ contains at least two points; let $g(a) \in \mathcal{L} \cap \partial_{\text{fixed}} \Omega$. By analogy with Definition 3.7.5,

we define the value of the force function F at a as

$$F(a) = \frac{D}{g'_1(a)\kappa'_2(a)},$$

where $D = D_L(a, b)$ if $a < b$ and $D = D_R(b, a)$ if $a > b$, and $b \neq a$ is any point such that $g(b) \in \mathcal{L} \cap \partial_{\text{fixed}} \Omega$.

Remark 3.7.11. The force function is well defined (does not depend on the choice of the point b) on the set $\{a \in \mathbb{R} \mid g(a) \in \mathcal{L}\}$ due to Lemma 3.7.9.

Lemma 3.7.12. *Let $a_1, a_2,$ and a_3 be such that $\gamma'(a_i), i = 1, 2, 3,$ lie in one plane. Suppose that the pairs (a_1, a_2) and (a_2, a_3) satisfy the cup equation. Then (a_1, a_3) satisfies the cup equation as well.*

Proof. The cup equations for the pairs (a_1, a_2) and (a_2, a_3) mean that the vectors $\gamma(a_2) - \gamma(a_1)$ and $\gamma(a_3) - \gamma(a_2)$ lie in the same plane as all the $\gamma'(a_i)$, hence $\gamma(a_3) - \gamma(a_1)$ also lies there. Thus, the cup equation holds for the pair (a_1, a_3) . ■

Now we are ready to describe all the remaining linearity domains. We start with the domains that are not separated from the free boundary of Ω . The boundary of such a domain, provided it is compact, consists of the arcs $g(\alpha_i), i = 1, 2, \dots, k,$ the chords $[g(\alpha_i^r), g(\alpha_{i+1}^l)], i = 1, 2, \dots, k - 1,$ two tangents $S(\alpha_1^l)$ and $S(\alpha_k^r),$ and the arc of the free boundary. We classify the multifigures with respect to the orientation of these tangents. Namely, if we have $S_R(\alpha_1^l)$ and $S_R(\alpha_k^r),$ then we get a *right multitrolleybus* denoted by $\Omega_{\text{Mtr,R}}(\{\alpha_i\}_{i=1}^k),$ see Figure 3.15; and if we have $S_L(\alpha_1^l)$ and $S_L(\alpha_k^r),$ then we obtain a *left multitrolleybus* denoted by $\Omega_{\text{Mtr,L}}(\{\alpha_i\}_{i=1}^k).$ If we have $S_L(\alpha_1^l)$ and $S_R(\alpha_k^r),$ then the linearity domain is called a *multicup* and is denoted by $\Omega_{\text{Mcup}}(\{\alpha_i\}_{i=1}^k),$ see Figure 3.16.

We distinguish the case where the two border tangents $S_L(\alpha_1^l)$ and $S_R(\alpha_k^r)$ lie on one line and say that in this case the multicup is *full*. Finally, if we have $S_R(\alpha_1^l)$ and $S_L(\alpha_k^r),$ then the domain of linearity \mathcal{L} is called a *multibirdie* and is denoted by $\Omega_{\text{Mbird}}(\{\alpha_i\}_{i=1}^k),$ see Figure 3.17. Graphical representation for a multifigure \mathcal{L} built over $\{\alpha_i\}_{i=1}^k$ is drawn by the following rule. The domain \mathcal{L} corresponds to a single vertex, and it has $k - 1$ outgoing edges representing the chordal domains $\Omega_{\text{ch}}([\alpha_i^r, \alpha_{i+1}^l], *), i = 1, 2, \dots, k - 1.$ There are two more edges corresponding to two tangent domains surrounding \mathcal{L} . They are both outgoing if \mathcal{L} is a multicup and both incoming in the case where \mathcal{L} is a multibirdie. If \mathcal{L} is a multitrolleybus, then it has one incoming and one outgoing edge. We provide examples of graphs for the multifigures drawn in Figures 3.15, 3.16, and 3.17.

We will also consider three types of unbounded domains of linearity: a multicup $\Omega_{\text{Mcup}}(\{\alpha_i\}_{i=1}^k)$ with at least one of the intervals α_1 or α_k being a ray, a right multitrolleybus $\Omega_{\text{Mtr,R}}(\{\alpha_i\}_{i=1}^k),$ where α_k is a ray that lasts to $+\infty,$ or a left multitrolleybus $\Omega_{\text{Mtr,L}}(\{\alpha_i\}_{i=1}^k),$ where α_1 is a ray that lasts to $-\infty.$ Such domains do not

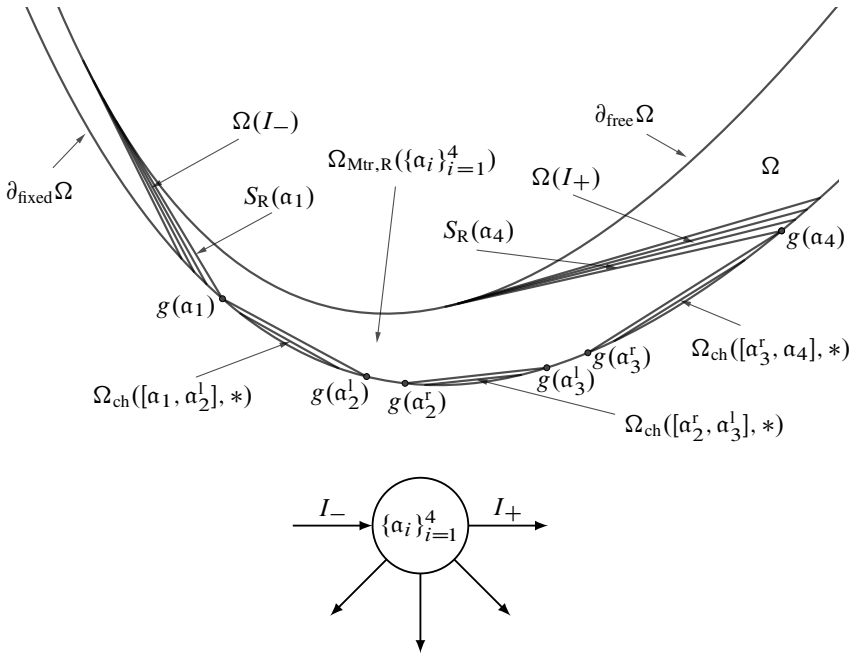


Figure 3.15. A right multitrolleybus for $k = 4$ with adjacent domains and its graphical representation.

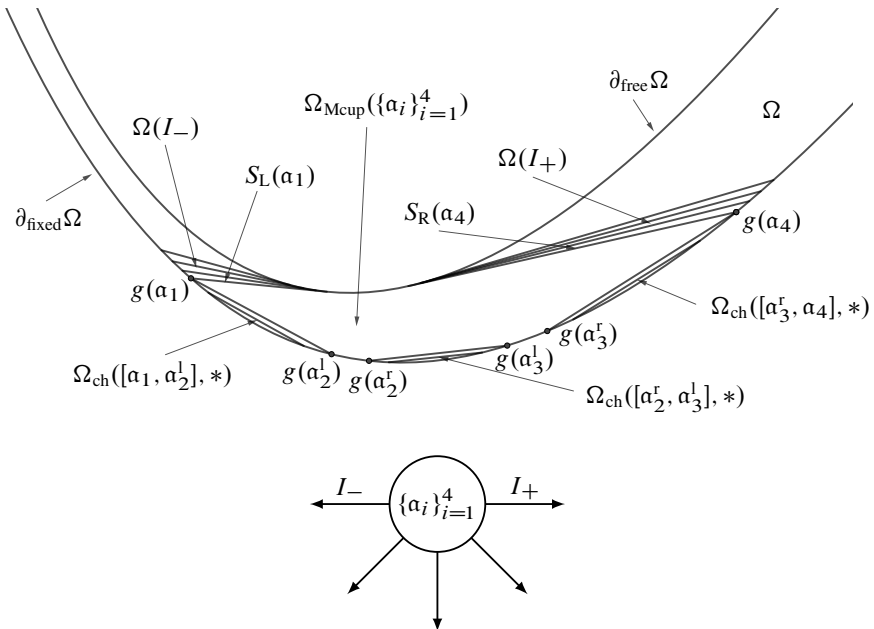


Figure 3.16. A mutlicup for $k = 4$ with adjacent domains and its graphical representation.

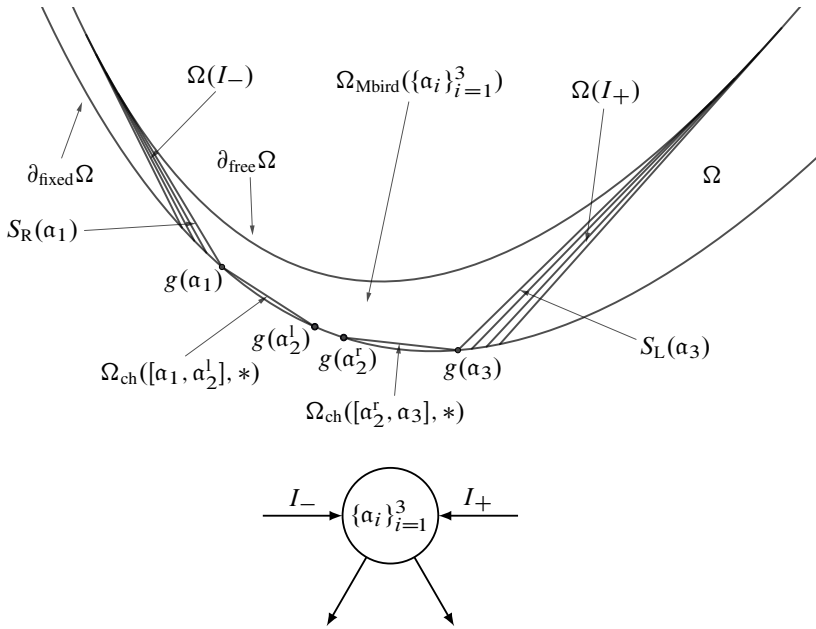


Figure 3.17. A multibirdie for $k = 3$ with adjacent domains and its graphical representation.

have one of border tangents. In such a case, the vertex representing this domain does not have the corresponding edge.

Now, consider the case of a linearity domain \mathcal{L} that is separated from the free boundary. The boundary of \mathcal{L} consists of the arcs $g(\alpha_i)$, $i = 1, 2, \dots, k$, the chords $[g(\alpha_i^r), g(\alpha_{i+1}^l)]$, $i = 1, 2, \dots, k - 1$, and the chord $[g(\alpha_1^l), g(\alpha_k^r)]$. Such a construction is called a *closed multicup*, denoted by $\Omega_{\text{CIMcup}}(\{\alpha_i\}_{i=1}^k)$. It is represented graphically in the following way. It has one incoming edge representing $\Omega_{\text{ch}}(*, [\alpha_1^l, \alpha_k^r])$ and several outgoing edges corresponding to the chordal domains $\Omega_{\text{ch}}([\alpha_i^r, \alpha_{i+1}^l], *)$, $i = 1, 2, \dots, k - 1$. For example, it may look like the one in Figure 3.18.

The following proposition gives sufficient conditions for concatenation of a linearity domain with the surrounding fences (tangent domains and chordal domains). Formally, it is more general than Proposition 3.7.2, but the proof is similar.

Proposition 3.7.13. *Let \mathcal{L} be a domain of linearity surrounded by several fences. Suppose that the function B is a standard candidate on \mathcal{L} and on each surrounding fence. Let*

$$a = \inf\{t \in \mathbb{R} \mid g(t) \in \mathcal{L}\} \quad \text{and} \quad b = \sup\{t \in \mathbb{R} \mid g(t) \in \mathcal{L}\}.$$

If the force function is continuous at a and b , then B is a C^1 -smooth Bellman candidate on its domain.

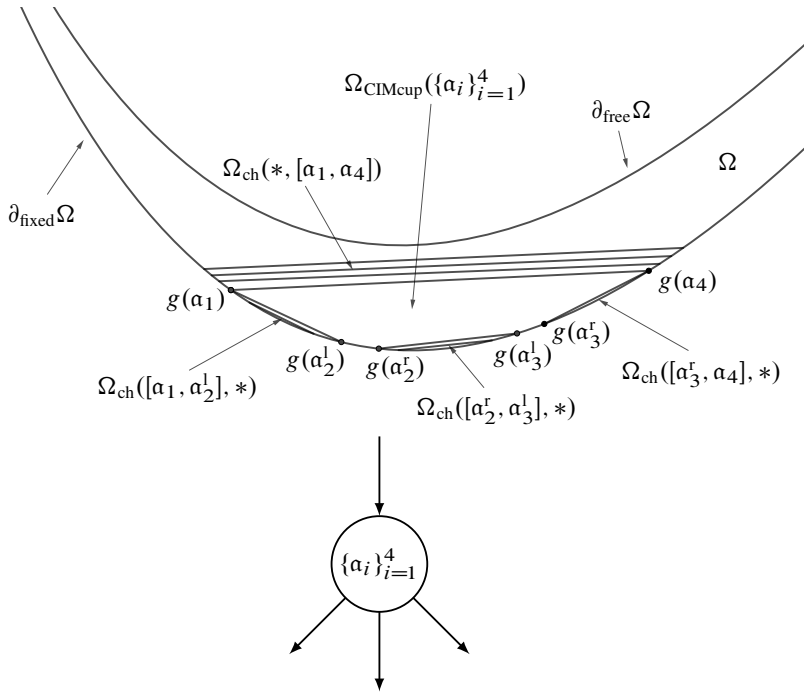


Figure 3.18. An example of the graph for a closed multicup with adjacent chordal domains.

Remark 3.7.14. In the previous proposition, if \mathcal{L} is a closed multicup, then it is surrounded by the chordal domains, and the condition of continuity of the force function always takes place.

Now is the time to define tails of a linearity domain \mathcal{L} that contains at least two points on the lower boundary. As in Proposition 3.7.13, we define $a = \inf\{t \in \mathbb{R} \mid g(t) \in \mathcal{L}\}$ and $b = \sup\{t \in \mathbb{R} \mid g(t) \in \mathcal{L}\}$.

Definition 3.7.15. We define the tails and the forces of a linearity domain \mathcal{L} as the tails and the forces for the chord $[g(a), g(b)]$ (which possibly does not lie in Ω). Namely, see Definitions 3.6.6, 3.6.9 for the tails and formulas (3.6.4), (3.6.5) for the corresponding forces.

3.8 Combinatorial properties of foliations

The material of this section essentially repeats that of [17, Section 3.5]. The reason for this repetition is a slight change of the notation we are forced to do since some natural parameters used in [17] do not exist in our general setting.

3.8.1 Gluing composite figures

In this subsection we present several formulas that allow us to consider a part of the foliation as a union of elementary domains in different ways. An example has already been given in (3.7.3) (see also the description of the symbol \uplus in Remark 3.7.6).

We start with the formula which describes gluing of an angle $\Omega_{\text{ang}}(a)$ with a long chord $[g(a), g(b)]$ (see Figure 3.19). Their union forms a trolleybus $\Omega_{\text{tr}}(a, b)$:

$$\Omega_{\text{ang}}(a) \uplus \Omega_{\text{L}}(a, a) \uplus [g(a), g(b)] = \Omega_{\text{tr},\text{R}}(a, b). \quad (3.8.1)$$

Similarly,

$$[g(a), g(b)] \uplus \Omega_{\text{R}}(b, b) \uplus \Omega_{\text{ang}}(b) = \Omega_{\text{tr},\text{L}}(a, b).$$

Both these formulas can be informally named as “angle + long chord = trolleybus”.

We have already considered an example of the more complicated formula (3.7.3):

$$\begin{aligned} \Omega_{\text{tr},\text{R}}(a, b) \uplus \Omega_{\text{R}}(b, b) \uplus \Omega_{\text{ang}}(b) &= \Omega_{\text{bird}}(a, b), \\ \Omega_{\text{ang}}(a) \uplus \Omega_{\text{L}}(a, a) \uplus \Omega_{\text{tr},\text{L}}(a, b) &= \Omega_{\text{bird}}(a, b), \end{aligned} \quad (3.8.2)$$

which can be informally named as “birdie = angle + trolleybus”.

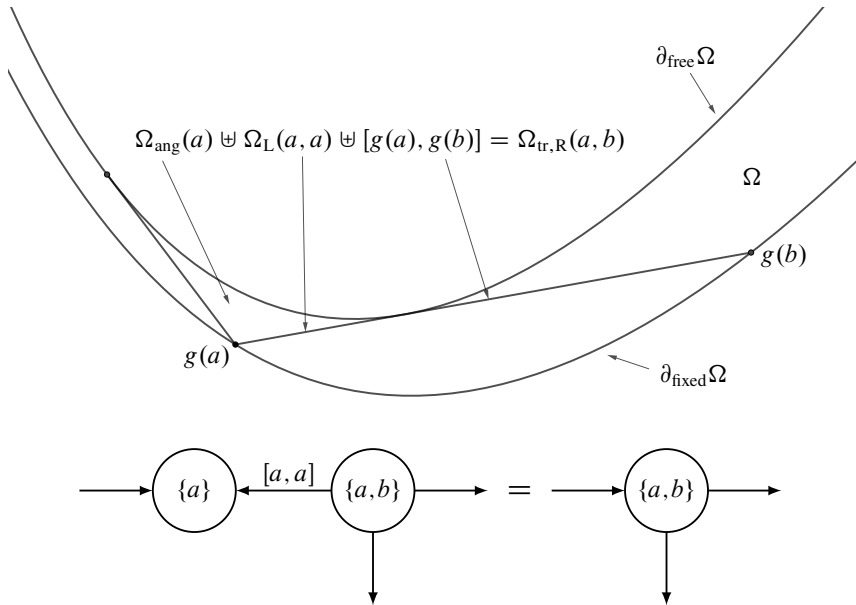


Figure 3.19. A graphical representation of formula (3.8.1).

We provide similar formulas for other domains. We leave their verification to the reader.

Angle + multicup = multitrolleybus. We have

$$\Omega_{\text{ang}}(\alpha_1^l) \uplus \Omega_L(\alpha_1^l, \alpha_1^l) \uplus \Omega_{\text{Mcup}}(\{\alpha_i\}_{i=1}^k) = \Omega_{\text{Mtr,R}}(\{\alpha_i\}_{i=1}^k), \quad (3.8.3)$$

$$\Omega_{\text{Mcup}}(\{\alpha_i\}_{i=1}^k) \uplus \Omega_R(\alpha_k^r, \alpha_k^r) \uplus \Omega_{\text{ang}}(\alpha_k^r) = \Omega_{\text{Mtr,L}}(\{\alpha_i\}_{i=1}^k). \quad (3.8.4)$$

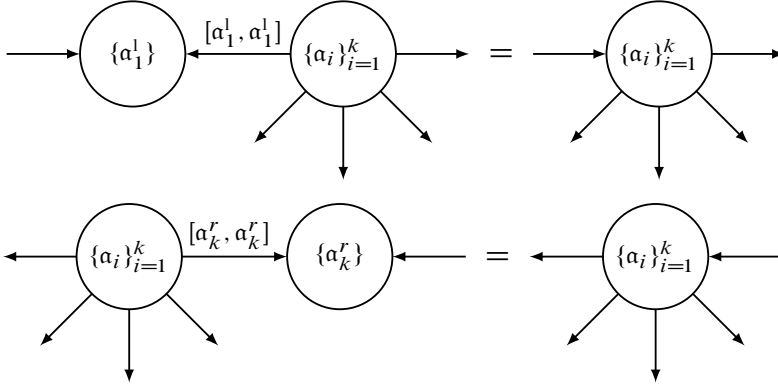


Figure 3.20. The graphs for formulas (3.8.3) and (3.8.4).

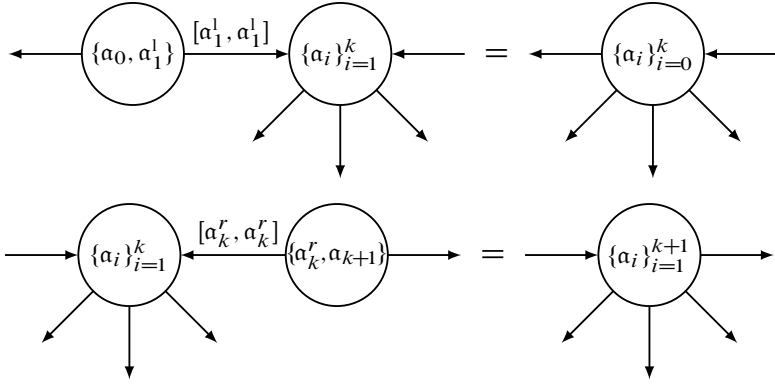


Figure 3.21. The graphs for formulas (3.8.5) and (3.8.6).

Long chord + multibirdie = multitrolleybus. We have

$$[g(a_0), g(\alpha_1^l)] \uplus \Omega_R(\alpha_1^l, \alpha_1^l) \uplus \Omega_{\text{Mbird}}(\{\alpha_i\}_{i=1}^k) = \Omega_{\text{Mtr,L}}(\{a_0\} \cup \{\alpha_i\}_{i=1}^k), \quad (3.8.5)$$

where $[g(a_0), g(\alpha_1^l)]$ is a long chord, and

$$\begin{aligned} & \Omega_{\text{Mbird}}(\{\alpha_i\}_{i=1}^k) \uplus \Omega_L(\alpha_k^r, \alpha_k^r) \uplus [g(\alpha_k^r), g(a_{k+1})] \\ & = \Omega_{\text{Mtr,R}}(\{\alpha_i\}_{i=1}^k \cup \{a_{k+1}\}), \end{aligned} \quad (3.8.6)$$

where $[g(\alpha_k^r), g(a_{k+1})]$ is a long chord.

Angle + multitrolleybus = multibirdie. We have

$$\Omega_{\text{Mtr,R}}(\{\alpha_i\}_{i=1}^k) \uplus \Omega_{\text{R}}(\alpha_k^r, \alpha_k^r) \uplus \Omega_{\text{ang}}(\alpha_k^r) = \Omega_{\text{Mbird}}(\{\alpha_i\}_{i=1}^k), \quad (3.8.7)$$

$$\Omega_{\text{ang}}(\alpha_1^l) \uplus \Omega_{\text{L}}(\alpha_1^l, \alpha_1^l) \uplus \Omega_{\text{Mtr,L}}(\{\alpha_i\}_{i=1}^k) = \Omega_{\text{Mbird}}(\{\alpha_i\}_{i=1}^k). \quad (3.8.8)$$

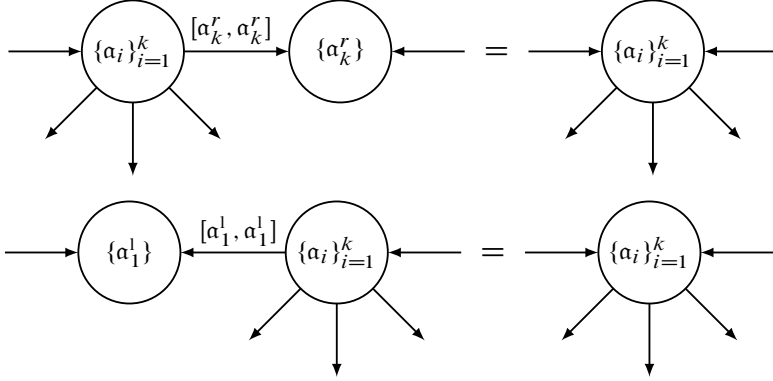


Figure 3.22. The graphs for formulas (3.8.7) and (3.8.8).

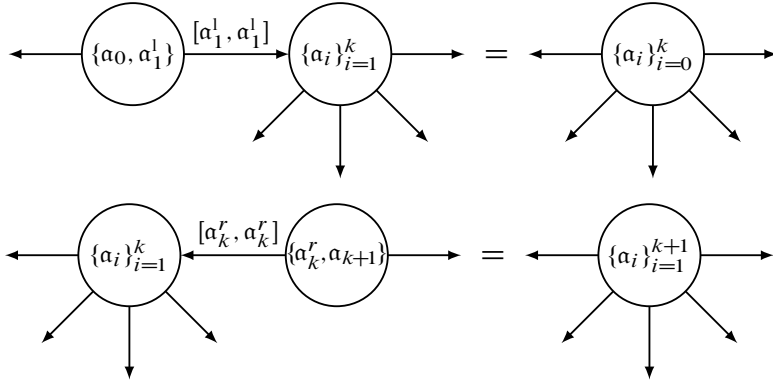


Figure 3.23. The graphs for formulas (3.8.9) and (3.8.10).

Long chord + multitrolleybus = multicup. We have

$$[g(a_0), g(\alpha_1^l)] \uplus \Omega_{\text{R}}(\alpha_1^l, \alpha_1^l) \uplus \Omega_{\text{Mtr,R}}(\{\alpha_i\}_{i=1}^k) = \Omega_{\text{Mcup}}(\{a_0\} \cup \{\alpha_i\}_{i=1}^k), \quad (3.8.9)$$

where $[g(a_0), g(\alpha_1^l)]$ is a long chord, and

$$\begin{aligned} & \Omega_{\text{Mtr,L}}(\{\alpha_i\}_{i=1}^k) \uplus \Omega_{\text{L}}(\alpha_k^r, \alpha_k^r) \uplus [g(\alpha_k^r), g(a_{k+1})] \\ & = \Omega_{\text{Mcup}}(\{\alpha_i\}_{i=1}^k \cup \{a_{k+1}\}), \end{aligned} \quad (3.8.10)$$

where $[g(\alpha_k^r), g(a_{k+1})]$ is a long chord.

Multitrolleybus = trolleybus parade. We have

$$\begin{aligned} \Omega_{\text{Mtr,R}}(\{\alpha_i\}_{i=1}^k) &= \left(\bigoplus_{i=1}^k \Omega_{\text{Mtr,R}}(\{\alpha_i\}) \right) \\ &\quad \uplus \left(\bigoplus_{i=1}^{k-1} (\Omega_{\text{R}}(\alpha_i^r, \alpha_i^r) \uplus \Omega_{\text{tr,R}}(\alpha_i^r, \alpha_{i+1}^1) \uplus \Omega_{\text{R}}(\alpha_{i+1}^1, \alpha_{i+1}^1)) \right) \end{aligned} \quad (3.8.11)$$

$$\begin{aligned} \Omega_{\text{Mtr,L}}(\{\alpha_i\}_{i=1}^k) &= \left(\bigoplus_{i=1}^k \Omega_{\text{Mtr,L}}(\{\alpha_i\}) \right) \\ &\quad \uplus \left(\bigoplus_{i=1}^{k-1} (\Omega_{\text{L}}(\alpha_i^r, \alpha_i^r) \uplus \Omega_{\text{tr,L}}(\alpha_i^r, \alpha_{i+1}^1) \uplus \Omega_{\text{L}}(\alpha_{i+1}^1, \alpha_{i+1}^1)) \right). \end{aligned} \quad (3.8.12)$$

In (3.8.11), if some α_i is a single point, then we omit the degenerate tangent domains $\Omega_{\text{R}}(\alpha_i^1, \alpha_i^1)$ and $\Omega_{\text{R}}(\alpha_i^r, \alpha_i^r)$ and replace the multitrolleybus $\Omega_{\text{Mtr,R}}(\{\alpha_i\})$ by the degenerate tangent domain $\Omega_{\text{R}}(\alpha_i, \alpha_i)$. We perform similar replacements of the left tangent domains and multitrolleybuses in (3.8.12).

Multibirdie = right multitrolleybus + angle + left multitrolleybus. We have

$$\begin{aligned} \Omega_{\text{Mbird}}(\{\alpha_i\}_{i=1}^k) &= \Omega_{\text{Mtr,R}}(\{\alpha_i\}_{i=1}^{j-1} \cup \{\alpha_j^1\}) \uplus \Omega_{\text{R}}(\alpha_j^1, \alpha_j^1) \uplus \Omega_{\text{Mbird}}(\{\alpha_j\}) \\ &\quad \uplus \Omega_{\text{L}}(\alpha_j^r, \alpha_j^r) \uplus \Omega_{\text{Mtr,L}}(\{\alpha_j^r\} \cup \{\alpha_i\}_{i=j+1}^k). \end{aligned} \quad (3.8.13)$$

Here j is an arbitrary number, $j = 1, 2, \dots, k$ (see a graphical example at Figure 3.24). Both multitrolleybuses can be disintegrated according to equations (3.8.11) and (3.8.12). If α_j is a single point, then one should change $\Omega_{\text{Mbird}}(\{\alpha_j\})$ for $\Omega_{\text{ang}}(\alpha_j)$. If α_j is a solid root, then $\Omega_{\text{Mbird}}(\{\alpha_j\})$ can be further disintegrated in two ways according to (3.8.8) and (3.8.7).

Closed multicup + trolleybus = multitrolleybus. We have

$$\begin{aligned} \Omega_{\text{ClMcup}}(\{\alpha_i\}_{i=1}^k) \uplus \Omega_{\text{ch}}([\alpha_1^1, \alpha_k^r], [\alpha_1^1, \alpha_k^r]) \uplus \Omega_{\text{tr,R}}(\alpha_1^1, \alpha_k^r) &= \Omega_{\text{Mtr,R}}(\{\alpha_i\}_{i=1}^k), \\ \Omega_{\text{ClMcup}}(\{\alpha_i\}_{i=1}^k) \uplus \Omega_{\text{ch}}([\alpha_1^1, \alpha_k^r], [\alpha_1^1, \alpha_k^r]) \uplus \Omega_{\text{tr,L}}(\alpha_1^1, \alpha_k^r) &= \Omega_{\text{Mtr,L}}(\{\alpha_i\}_{i=1}^k). \end{aligned}$$

Closed multicup + birdie = multibirdie. We have

$$\Omega_{\text{ClMcup}}(\{\alpha_i\}_{i=1}^k) \uplus \Omega_{\text{ch}}([\alpha_1^1, \alpha_k^r], [\alpha_1^1, \alpha_k^r]) \uplus \Omega_{\text{bird}}(\alpha_1^1, \alpha_k^r) = \Omega_{\text{Mbird}}(\{\alpha_i\}_{i=1}^k).$$

3.8.2 General foliations

It is natural to draw a special graph Γ corresponding to a foliation to describe its combinatorial properties. The vertices correspond to the linearity domains. Two vertices are joined with an edge if there is a fence that is their common neighbor. Such a graph

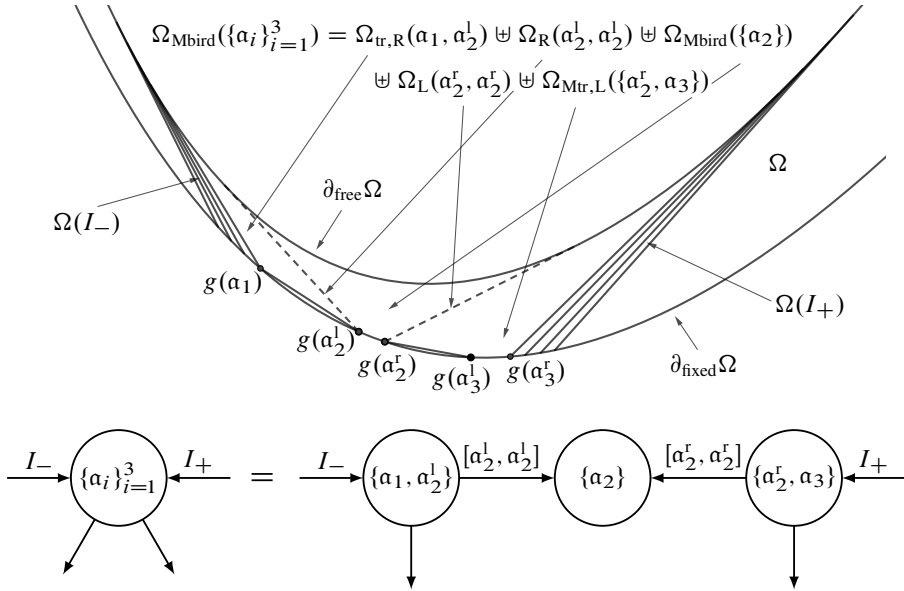


Figure 3.24. An example for formula (3.8.13) and its graphical representation.

is drawn in the plane by the mapping $\nabla B: \Omega_\varepsilon \rightarrow \mathbb{R}^2$. However, we need to clarify some details.

We will use a small amount of graph theory terminology. Since we study very special graphs, the use of the terminology will also be special. Our graphs are oriented trees (i.e., trees whose edges possess orientation). We call a vertex that does not have incoming edges a root, and a vertex that does not have outgoing edges a leaf (a leaf may have several incoming edges). By a path we call an oriented path, i.e., we move from the beginning of the edge to its end while exploring the path. Other terminology is clear.

The vertices of the graph will be denoted by $\{\mathcal{L}_i\}_i$, the edges will be denoted by $\{\mathcal{E}_i\}_i$. Edges and vertices are of different types, moreover, they are also equipped with numerical parameters to be specified later. We begin the description with the edges.

Each edge \mathcal{E} represents either a chordal domain $\Omega_{\text{ch}}([a^{\text{top}}, b^{\text{top}}], [a^{\text{bot}}, b^{\text{bot}}])$ or one of the tangent domains: either $\Omega_{\text{R}}(u_l, u_r)$ or $\Omega_{\text{L}}(u_l, u_r)$. The edge representing a chordal domain $\Omega_{\text{ch}}([a^{\text{top}}, b^{\text{top}}], [a^{\text{bot}}, b^{\text{bot}}])$ is oriented from its upper neighbor to its lower neighbor. We consider the functions a and b associated with a chordal domain as its numeric parameters. The edge of $\Omega_{\text{R}}(u_l, u_r)$ is oriented from the vertex of its left neighbor to the vertex of its right neighbor. The edge representing $\Omega_{\text{L}}(u_l, u_r)$ is oriented symmetrically. The closed interval $[u_l, u_r]$ is the numerical parameter of such an edge.

The vertices correspond to the linearity domains. For angles, trolleybuses, birdies, and multifigures, the graphical representation was given in the subsections where they were introduced (Sections 3.7.3 and 3.7.4). These vertices are of their individual types (i.e., there are several vertices of the type “angle” in the graph, several vertices of the type “birdie”, etc.). Each such vertex is equipped with its numerical characteristics that are the values of the parameters corresponding to its points on the fixed boundary $\partial_{\text{fixed}} \Omega$. For example, a vertex of the type “angle” has one numerical parameter u of the point $g(u)$ the angle is sitting on, whereas the collection of the intervals $\{\alpha_i\}_{i=1}^k$ plays the role of the numerical parameter for a vertex that has the type “multicup”, or “multitrolleybus”, or “multibirdie”.

However, we also need some fictitious vertices, which do not correspond to any linearity domain of nonzero area. For example, in Figure 3.12 the vertex representing the long chord is fictitious. There will be five types of such vertices.

First, there will be some \mathcal{L}_i that correspond to long chords (the chords that are tangent to the free boundary of Ω_ε). Namely, suppose that we have a full chordal domain $\Omega_{\text{ch}}([a^{\text{top}}, b^{\text{top}}], *)$ such that $D_{\text{R}}(a^{\text{top}}, b^{\text{top}}) \neq 0$ and $D_{\text{L}}(a^{\text{top}}, b^{\text{top}}) \neq 0$, and two tangent domains $\Omega_{\text{R}}(b^{\text{top}}, u_2)$ and $\Omega_{\text{L}}(u_1, a^{\text{top}})$. Then the vertex \mathcal{L} corresponding to the chord $[g(a^{\text{top}}), g(b^{\text{top}})]$ has three outgoing edges representing $\Omega_{\text{ch}}([a^{\text{top}}, b^{\text{top}}], *)$, $\Omega_{\text{R}}(b^{\text{top}}, u_2)$, and $\Omega_{\text{L}}(u_1, a^{\text{top}})$. The set $\{a^{\text{top}}, b^{\text{top}}\}$ is the numerical parameter for \mathcal{L} . The example is given in Figure 3.12.

Second, there will be some vertices \mathcal{L}_i that correspond to points of the fixed boundary. Suppose we have a chordal domain $\Omega_{\text{ch}}(*, [a^{\text{bot}}, b^{\text{bot}}])$ with $a^{\text{bot}} = b^{\text{bot}}$ (we recall that such chordal domains are called cups). In our foliations, all such points will coincide with some c_j from Definition 2.1.13, see explanation around (3.4.11). Then the vertex \mathcal{L} corresponding to $a^{\text{bot}} = b^{\text{bot}} = c_j$ has one incoming edge matching $\Omega_{\text{ch}}(*, [a^{\text{bot}}, b^{\text{bot}}])$ and one numerical parameter that equals c_j .

Third, sometimes we will need to paste a chord between two chordal domains (this will be done when one of the differentials vanishes, see Definition 3.5.1). Suppose we have two chordal domains $\Omega_{\text{ch}}([a_1, b_1], [a_2, b_2])$ and $\Omega_{\text{ch}}([a_2, b_2], [a_3, b_3])$. In such a case, we paste a vertex \mathcal{L} that corresponds to the chord $[g(a_2), g(b_2)]$, see Figure 3.11. It has one incoming edge and one outgoing edge and the numerical parameter $\{a_2, b_2\}$. Long chords, one or both differentials of which vanish, are also considered as fictitious vertices of the third type.

Fourth, there might be one or two vertices at infinity. If we have a tangent domain $\Omega(u_1, u_r)$ with $u_1 = -\infty$, then there is a vertex \mathcal{L} that corresponds to $-\infty$. It has the numerical parameter $-\infty$ and one edge representing $\Omega(-\infty, u_r)$ that is outgoing for the case of Ω_{R} and incoming for the case of Ω_{L} . Similarly, if we have $u_r = +\infty$, then we have a vertex \mathcal{L} that corresponds to $+\infty$ with the numerical parameter $+\infty$ and one edge representing $\Omega(u_1, +\infty)$ that is incoming for the case of Ω_{R} and outgoing for the case of Ω_{L} . If $u_1 = -\infty$ and $u_r = +\infty$ simultaneously, then we have both such vertices and one edge between them.

Fifth, there might be a vertex corresponding to a single tangent. Suppose we have a tangent domain $\Omega(u_1, u_2)$ such that $\mathcal{F} < 0$ on $[u_1, u_2]$ except for some point u , where \mathcal{F} equals zero.¹ For the case of right tangent domain, it is useful to decompose $\Omega_R(u_1, u_2)$ as

$$\Omega_R(u_1, u) \uplus \Omega_R(u, u) \uplus \Omega_R(u, u_2)$$

and paste a vertex representing $\Omega_R(u, u)$ (alternatively, one can consider it as a multitrolleybus on a single point). It has one outgoing edge $\Omega_R(u, u_2)$ and one incoming edge $\Omega_R(u_1, u)$, see Figure 3.13. Its numerical parameter is u . The same can be done for the case of left tangents. We note that the fictitious vertices of the fifth type may be right and left (the same as the trolleybuses).

The rules listed above define the graph of the foliation. However, we provide a further description to make its structure more transparent. It is useful to introduce a partial ordering on the set of linearity domains.

Definition 3.8.1. Let \mathfrak{L}_1 and \mathfrak{L}_2 be two linearity domains. We say that \mathfrak{L}_2 is subordinate to \mathfrak{L}_1 and write $\mathfrak{L}_2 < \mathfrak{L}_1$ if \mathfrak{L}_1 separates \mathfrak{L}_2 from the free boundary.

Note that if $\mathfrak{L}_2 < \mathfrak{L}_1$, then \mathfrak{L}_2 is a closed multicup. We can also let \mathfrak{L}_1 and \mathfrak{L}_2 be chords, and let \mathfrak{L}_2 be a point on the fixed boundary. Another point to note is that the numerical parameters of the vertices \mathfrak{L}_1 and \mathfrak{L}_2 are sufficient to define whether the statement $\mathfrak{L}_2 < \mathfrak{L}_1$ is true.

We explain how to construct the graph from a foliation. Our graph is a tree if we disregard the orientation. First, we describe its subgraph Γ^{free} spanned by the edges representing tangent domains. This subgraph describes the trace of the foliation on the free boundary. Formally, we can define Γ^{free} to be the set of vertices that are not subordinated by any other vertex, and the edges between them. If we forget the orientation of edges, Γ^{free} is a path, i.e., a tree whose vertices have degree two, except, possibly for two leaves at infinity. The leaves are usually fictitious vertices of the fourth type, however, if there is a multicup or a multitrolleybus that lasts to infinity, its vertex is a leaf (in such a case there is no fictitious vertex representing the corresponding infinity). The orientation of edges has already been described. We only say that the roots of Γ^{free} are the fictitious vertices of the first and third type (the latter, of course, should belong to Γ^{free} , i.e., represent a long chord), the vertices that correspond to multicups, and possibly, the vertices at infinity. The leaves in Γ^{free} correspond to angles, birdies, multibirdies, and possibly, vertices at infinities. The necessary and sufficient condition for Γ^{free} to be a subgraph spanned by the edges corresponding to tangent domains of some foliation is that the foliation reconstructed from it covers the free boundary without intersections.

¹In such a situation, $u = c_j$ for some root c_j (see Definition 2.1.13).

Second, we describe the graph Γ^{fixed} spanned by the edges that correspond to chordal domains. The graph Γ^{fixed} is a forest (i.e., a finite collection of trees). Each tree of the forest is oriented from its root, being any vertex of Γ^{free} except for fictitious vertices of the fourth or fifth types, and multifigures sitting on single arcs, to its leaves. The leaves of Γ^{fixed} are the fictitious vertices of the second type (corresponding to the origins of cups) and closed multicups sitting on single arcs. All other vertices are closed multicups and fictitious vertices of the third type. We note that this graph is generated by the ordering introduced in Definition 3.8.1: each edge \mathfrak{C} goes from \mathfrak{L}_1 to \mathfrak{L}_2 if and only if $\mathfrak{L}_2 \prec \mathfrak{L}_1$ and there are no vertices \mathfrak{L}_3 such that $\mathfrak{L}_2 \prec \mathfrak{L}_3 \prec \mathfrak{L}_1$. The necessary and sufficient condition for Γ^{fixed} to be a subgraph spanned by the edges corresponding to chordal domains of some foliation is that the linearity domains built from its vertices do not intersect; the edges are generated by the ordering from Definition 3.8.1.

So, the graph of the foliation is a finite oriented tree whose vertices and edges have types (they correspond either to some figures or to fictitious constructions described above) and several numerical characteristics regarding their type. We warn the reader that we do not write down all the numerical parameters when we draw graphs, this makes our illustrations clearer. We would like to underline that the foliation could be restored from the graph and the numerical parameters determined by this foliation.

Chapter 4

Evolution of Bellman candidates

In this chapter we provide an algorithm for building a special Bellman candidate on Ω_ε for each ε , with $0 < \varepsilon < \varepsilon_{\max}$. In Chapter 5 we will prove that this candidate coincides with the Bellman function $\mathbf{B}_\varepsilon(\cdot; f)$ using optimizers.

The algorithm starts with sufficiently small ε . In such a case, the foliation for the Bellman candidate can be composed of cups (multicups), angles, and tangent domains. Then we increase ε in order to construct the Bellman candidates for larger ε . Formally, there will be statements of two kinds (they can be called “induction steps of the first and second kind”). The first ones state that the set of ε , for which there is a Bellman candidate of a given structure, is open. They are of the form: “if for some η there is a Bellman candidate with the graph Γ , then there is some positive δ such that for all ε in $[\eta, \eta + \delta]$, the foliation with the graph Γ and perturbed numerical parameters provides a Bellman candidate for f in Ω_ε ”. The second ones state that the set of those ε , for which there are a graph Γ and a collection of numerical parameters that provide a Bellman candidate for f and ε , is closed. They are of the form: “if for each ε_n there is a Bellman candidate with the graph Γ , $\varepsilon_n \nearrow \varepsilon$, and the numerical parameters converge to some limits as $\varepsilon_n \rightarrow \varepsilon$, then Γ with the limiting parameters provide a foliation for f in Ω_ε ”. We note that the limits of numerical parameters may be degenerate in a sense (for example, a trolleybus may become a fictitious vertex of the fifth type), so Γ changes after passing to the limit. Each such induction step, in its turn, can be reduced to similar local statements, i.e., statements about the evolution behavior of lonely figures, e.g., cups, angles, etc.

The main law that rules the evolution of the foliation is “the forces decrease (grow in absolute value) as ε grows”. As a result, long chords and multicups grow (Propositions 4.3.1 and 4.3.2), trolleybuses decrease (Propositions 4.3.4 and 4.3.5), multitrolleybuses, birdies, and multibirdies disintegrate (Propositions 4.3.10, 4.3.11, and 4.3.13). What is more, single figures can collide; formally this happens in the induction steps of the second kind when one of the edges has “zero length” at the limit. In the case of a collision, we use formulas from Section 3.8.1 to continue the evolution.

4.1 Simple picture

Definition 4.1.1. Let Γ be a foliation graph. We call it (and the foliation itself) *simple* if it has no oriented paths longer than one and no closed multicups.

Simple foliations consist of alternating cups (or multicups on single arcs; by a multicup on a single arc we mean $\Omega_{\text{Mcup}}(\{\alpha\})$, where α is an interval) and angles connected by tangent domains. If Γ is a graph of a simple foliation consisting of N edges, then there are either $\frac{N}{2}$, or $\frac{N-1}{2}$, or $\frac{N-2}{2}$ angles in the foliation. In Γ^{free} , the vertices corresponding to angles alternate the vertices representing multicups and long chords. Each multicup is sitting on an arc whose convex hull is not contained in Ω_ε . Each long chord has a cup below it. See Figure 4.1 for the visualization.

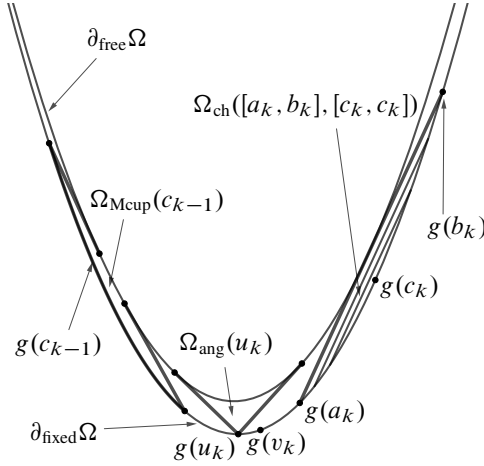


Figure 4.1. An example of simple picture.

For didactic reasons, we explain how a simple graph generates a Bellman candidate B (similar essence for general graphs will be explained in Section 4.4). Suppose Γ is a simple graph. First, we consider its roots, which are long chords, multicups on solid roots, and vertices at infinities. For long chords and multicups, we build the standard candidate by linearity (see (3.4.1)). Second, consider the edges of Γ . For the edges corresponding to chordal domains, we construct standard candidates again by linearity on chords (see (3.4.1)). For each edge corresponding to a tangent domain, we continuously glue a standard candidate in this domain to the already built standard candidate corresponding to the source of the edge. This is done by choosing an appropriate $\beta_2(t_0)$ in (3.2.3). For tangent domains whose source is infinity, we do not have to glue anything, we simply consider the standard candidates on them; such standard candidates are uniquely defined, see Definition 3.3.8. In the angles, we choose the standard candidates by Proposition 3.6.4. The constructed function B is C^1 -smooth, and thus, by Proposition 3.1.2, it is locally concave.

In the theorem below, we use the notation for the essential roots, see Definition 2.1.13.

Theorem 4.1.2. *For any function f satisfying Conditions 2.1.11 and 2.1.14, there exists $\varepsilon_1 > 0$ such that for any $\varepsilon < \varepsilon_1$, there exist a simple graph and a collection of numerical parameters such that the function B constructed from this graph, f , and ε as described above is a C^1 -smooth Bellman candidate. Moreover, its foliation satisfies the following properties: the origins of the cups coincide with those c_i that are single points; the multicups are sitting on those c_i that are intervals; for any $k = 1, 2, \dots, n$, the parameter of the vertex u_k of the k -th angle in Γ^{free} tends to v_k as $\varepsilon \rightarrow 0$.*

The proof of this theorem will be presented at the end of this section, because it requires some preparation.

Consider two neighbor points c_k and v_{k+1} , where the torsion \mathbf{T} of the boundary curve γ changes its sign. If c_k is a solid root, then for sufficiently small ε , we can build a multicup on c_k and define the standard candidate there. If c_k is a single point, for small ε , we use Proposition 3.5.9 to build a full cup on it. Let its upper chord be $[g(a_k), g(b_k)] = [g(a_k(\varepsilon)), g(b_k(\varepsilon))]$ (for a cup, we take its upper chord, whereas for a multicup, we consider the chord connecting its endpoints). Then, by (3.6.4), we have $F_R(u; a_k, b_k; \varepsilon) < 0$ when $u \in (b_k, v_{k+1}]$, because $\text{sign}(\mathbf{T}) = \text{sign}(\mathfrak{R}')$ (see (2.1.12)). Thus, the right tail of the cup or multicup built on c_k always contains v_{k+1} . Similarly, the left tail of the cup or multicup built over c_{k+1} contains v_{k+1} . The following lemma says that the ends of these tails tend to v_{k+1} as $\varepsilon \rightarrow 0$.

Lemma 4.1.3. *Let $(g(a_k(\varepsilon)), g(b_k(\varepsilon)))$ be the upper chord of a cup or of a multicup built over c_k , and let $t_k^R = t_k^R(\varepsilon)$ be the endpoint of its right tail. Then $t_k^R \rightarrow v_{k+1}$ as $\varepsilon \rightarrow 0$. Similarly, the endpoint $t_k^L(\varepsilon)$ of the left tail tends to v_k . A similar convergence statement holds for the forces coming from the infinities.*

Proof. We will deal with the case of the right tail. We will write a for a_k and b for b_k . It suffices to prove that for each point w_+ such that $v_{k+1} < w_+$ (we also assume that w_+ is not far from v_{k+1} ; we want \mathfrak{R} to increase on (v_{k+1}, w_+)), the inequality $t_k^R < w_+$ holds eventually as $\varepsilon \rightarrow 0$. We use (3.6.4) for the force $F_R(t) = F_R(t; a(\varepsilon), b(\varepsilon); \varepsilon)$ outside the chordal domain to obtain

$$F_R(w_+) = \int_b^{w_+} \exp\left(-\int_\tau^{w_+} \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \mathfrak{R}'(\tau) d\tau + \exp\left(-\int_b^{w_+} \frac{\kappa'_2}{\kappa_2 - \kappa}\right) F_R(b). \quad (4.1.1)$$

It suffices to prove $F_R(w_+) > 0$ for ε sufficiently small. We first deal with the first summand, which will be split into two integrals $\int_b^{w_+} = \int_b^{v_{k+1}^r} + \int_{v_{k+1}^r}^{w_+}$ (as usual, by v_{k+1}^r we denote the right endpoint of v_{k+1}). First,

$$\begin{aligned} & \left| \int_b^{v_{k+1}^r} \exp\left(-\int_\tau^{w_+} \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \mathfrak{R}'(\tau) d\tau \right| \\ & \leq \exp\left(-\int_{v_{k+1}^r}^{w_+} \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \cdot \int_b^{v_{k+1}^r} |\mathfrak{R}'(\tau)| d\tau. \end{aligned} \quad (4.1.2)$$

Take $w \in (v_{k+1}^r, w_+)$. Then

$$\begin{aligned} \int_{v_{k+1}^r}^{w_+} \exp\left(-\int_{\tau}^{w_+} \frac{\kappa_2'}{\kappa_2 - \kappa}\right) \mathfrak{R}'(\tau) d\tau &\geq \int_w^{w_+} \exp\left(-\int_{\tau}^{w_+} \frac{\kappa_2'}{\kappa_2 - \kappa}\right) \mathfrak{R}'(\tau) d\tau \\ &\geq \exp\left(-\int_w^{w_+} \frac{\kappa_2'}{\kappa_2 - \kappa}\right) \cdot \int_w^{w_+} \mathfrak{R}'(\tau) d\tau \\ &= \exp\left(-\int_w^{w_+} \frac{\kappa_2'}{\kappa_2 - \kappa}\right) \cdot (\mathfrak{R}(w_+) - \mathfrak{R}(w)). \end{aligned} \quad (4.1.3)$$

We multiply both sides of (4.1.1) by $\exp(\int_w^{w_+} \frac{\kappa_2'}{\kappa_2 - \kappa})$, use (4.1.2) and (4.1.3), and obtain

$$\begin{aligned} \exp\left(\int_w^{w_+} \frac{\kappa_2'}{\kappa_2 - \kappa}\right) F_R(w_+) &\geq \mathfrak{R}(w_+) - \mathfrak{R}(w) - \exp\left(-\int_{v_{k+1}^r}^w \frac{\kappa_2'}{\kappa_2 - \kappa}\right) \\ &\quad \times \int_b^{v_{k+1}^r} |\mathfrak{R}'(\tau)| d\tau + \exp\left(-\int_b^w \frac{\kappa_2'}{\kappa_2 - \kappa}\right) F_R(b). \end{aligned} \quad (4.1.4)$$

When $\varepsilon \rightarrow 0$, κ tends to κ_2 from below pointwise, therefore both exponents on the right-hand side of (4.1.4) tend to zero. These exponents are multiplied by bounded factors. Indeed,

$$\int_b^{v_{k+1}^r} |\mathfrak{R}'(\tau)| d\tau \leq \int_{c_k}^{v_{k+1}^r} |\mathfrak{R}'(\tau)| d\tau.$$

Boundedness of the second factor is more cumbersome. If $(g(a), g(b))$ is the upper chord of a multicup, then $F_R(b) = 0$ and there is nothing to do. Consider the case of a cup. In this case we use (3.6.4):

$$F_R(b) = \frac{D_R(a, b)}{g_1'(b)\kappa_2'(b)}. \quad (4.1.5)$$

Let us rewrite formula (3.5.14). After applying (3.5.2) and (3.5.1), its left-hand side takes the following form:

$$\begin{aligned} \det \begin{pmatrix} \gamma''(b) \\ \gamma'(a) \\ \gamma(b) - \gamma(a) \end{pmatrix} &= \det \begin{pmatrix} \gamma''(b) \\ \gamma'(a) \\ C_R \gamma'(b) - C_L \gamma'(a) \end{pmatrix} \\ &= C_R \det \begin{pmatrix} \gamma''(b) \\ \gamma'(a) \\ \gamma'(b) \end{pmatrix} = -C_R D_R \det \begin{pmatrix} g'(b) \\ g'(a) \end{pmatrix}. \end{aligned}$$

By (3.5.2) again, the non-integral term in the right-hand side of (3.5.14) vanishes. Thus, we obtain

$$C_R D_R \det \begin{pmatrix} g'(b) \\ g'(a) \end{pmatrix} = g_1'(b)g_1'(a)\kappa_2'(b) \int_a^b (\mathfrak{R}(u) - \mathfrak{R}(b))\kappa_2'(u)(g_1(b) - g_1(u)) du.$$

The function $W(u) = \kappa'_2(u)(g_1(b) - g_1(u))$ is non-negative and its integral over $[a, b]$ is equal to

$$\begin{aligned} \int_a^b W(u) du &= \kappa_2(u)(g_1(b) - g_1(u)) \Big|_a^b + \int_a^b \kappa_2(u)g'_1(u) du \\ &= -\kappa_2(a)(g_1(b) - g_1(a)) + g_2(b) - g_2(a) \\ &= -\frac{1}{g'_1(a)} \det \begin{pmatrix} g(b) - g(a) \\ g'(a) \end{pmatrix} = -\frac{1}{g'_1(a)} \det \begin{pmatrix} g'(b) \\ g'(a) \end{pmatrix} C_R. \end{aligned}$$

Therefore, (4.1.5) takes the form

$$F_R(b) = \frac{\int_a^b (\mathfrak{R}(b) - \mathfrak{R}(u))W(u) du}{\int_a^b W(u) du}.$$

By the mean value theorem, there exists $u \in [a, b]$ such that $F_R(b) = \mathfrak{R}(b) - \mathfrak{R}(u)$, which is bounded and, moreover, tends to zero when $\varepsilon \rightarrow 0$. Hence, the right-hand side of (4.1.4) tends to $\mathfrak{R}(w_+) - \mathfrak{R}(w)$ which is positive. Thus, the left-hand side of (4.1.4) is positive for sufficiently small ε . ■

The notation in the following lemma is the same as in the previous one.

Lemma 4.1.4. *The difference of forces, $F_R(u; a_k, b_k) - F_L(u; a_{k+1}, b_{k+1})$, is strictly increasing (as a function of u) on the interval $(t_{k+1}^L, t_k^R) \cap (b_k, a_{k+1})$.*

Proof. We differentiate the function in question with respect to u , use (3.6.6), and obtain

$$(F_R - F_L)' = -\beta'_{2,R} + \beta'_{2,L} > 0,$$

because $\beta'_{2,L} > 0 > \beta'_{2,R}$ on $(t_{k+1}^L, t_k^R) \cap (b_k, a_{k+1})$ by Definition 3.3.8. The lemma is proved. ■

Corollary 4.1.5. *The balance equation*

$$F_R(u; a_k, b_k; \varepsilon) = F_L(u; a_{k+1}, b_{k+1}; \varepsilon) \quad (4.1.6)$$

has a unique root $u = u_{k+1}$ in (t_{k+1}^L, t_k^R) for sufficiently small ε .

Proof. First, by Lemma 4.1.3, we have $[t_{k+1}^L, t_k^R] \subset (b_k, a_{k+1})$ for sufficiently small ε . By the definition of tails and the continuity of forces, the force function is zero at the endpoint of its tail: $F_R(t_k^R; a_k, b_k; \varepsilon) = 0$ and $F_L(t_{k+1}^L; a_{k+1}, b_{k+1}; \varepsilon) = 0$. Therefore,

$$F_R(t_{k+1}^L; a_k, b_k; \varepsilon) - F_L(t_{k+1}^L; a_{k+1}, b_{k+1}; \varepsilon) = F_R(t_{k+1}^L; a_k, b_k; \varepsilon) < 0,$$

because $t_{k+1}^L \in (b_k, t_k^R)$. Similarly,

$$F_R(t_k^R; a_k, b_k; \varepsilon) - F_L(t_k^R; a_{k+1}, b_{k+1}; \varepsilon) = -F_L(t_k^R; a_{k+1}, b_{k+1}; \varepsilon) > 0.$$

By the Bolzano–Weierstrass principle, there exists a root u_{k+1} of the balance equation in (t_{k+1}^L, t_k^R) . By Lemma 4.1.4, this root is unique. ■

Remark 4.1.6. The results of the preceding lemma and the corollary hold true if one of the cups (or both) sit at infinity, i.e., the corresponding c_k is infinite.

Now we have all the ingredients to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. First, we take ε to be a small number such that we can build a full cup around each c_k (or a multicup if c_k is a solid root) with the help of Proposition 3.5.9. Moreover, we take ε to be so small that all these figures have no intersections. This is possible because we have only a finite number of roots by our assumptions. Then, if ε satisfies the assumptions of Corollary 4.1.5 (together with Remark 4.1.6) for each k , one can paste an angle with the vertex at the unique root of the balance equation (4.1.6) between each pair of consecutive cups or multicups, see Corollary 3.6.5. The relation $u_{k+1} \rightarrow v_{k+1}$ is an immediate consequence of Lemma 4.1.3 and the inclusion $u_{k+1} \in (t_{k+1}^L, t_k^R)$. ■

4.2 Preparation to evolution

In this section, we collect technical lemmas that are useful for the evolution. There will be three groups of lemmas. The first group consists of lemmas that describe the places where the fictitious vertices of the third type may occur, the second is about tails and forces, and the third one works with the balance equation.

4.2.1 Structural lemmas for chords

We make a convention on chordal domains: the inequalities $D_L < 0$ and $D_R < 0$ hold true inside the chordal domain. Note that the same inequalities are required to build the standard candidate in a chordal domain.

Lemma 4.2.1. *Let $\Omega_{\text{ch}}([a, b], *)$ be a chordal domain. If $D_L(a, b) = 0$, then \mathfrak{K} is decreasing on the right of a ; if $D_R(a, b) = 0$, then \mathfrak{K} is increasing on the left of b .*

Proof. We treat the case of the right differential only. The remaining case is symmetric. We will use formula (3.5.8). Let

$$U(t) = \frac{\det\begin{pmatrix} g'(t) \\ g'''(t) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g''(t) \end{pmatrix}} - \frac{\det\begin{pmatrix} g''(t) \\ g(t) - g(s) \end{pmatrix}}{\det\begin{pmatrix} g'(t) \\ g(t) - g(s) \end{pmatrix}},$$

where $s = s(t)$, $s < t$, is the corresponding function to the chordal domain defined in Section 3.4. The function U is in fact a measure of bounded variation in a left

neighborhood of b , hence we can find an increasing function A such that $A' + U$ is a positive measure in a left neighborhood of b . Consider the function $V(t) = e^{A(t)} D_{\mathbb{R}}(s, t)$ in the left neighborhood of b . We use equations (3.5.8) and (2.1.10):

$$e^{-A(t)} V'(t) = \frac{\mathbf{T}(t)}{\det \begin{pmatrix} g'(t) \\ g''(t) \end{pmatrix}} + (A'(t) + U(t)) D_{\mathbb{R}}(s, t).$$

If \mathfrak{K} does not increase to the left of b , then $\mathbf{T}(t) < 0$ in a small left neighborhood of b according to (2.1.12) and Condition 2.1.11. Since $A'(t) + U(t) > 0$ and $D_{\mathbb{R}}(s, t) < 0$, we obtain $V'(t) < 0$. Therefore, V decreases, which contradicts the fact $V(t) < 0$ for $t < b$ and $V(b) = 0$. ■

Lemma 4.2.2. *Suppose that a pair (a, b) satisfies the cup equation, and the chord $[g(a), g(b)]$ has nonzero tails (i.e., $t^{\mathbb{L}} < a$ and $b < t^{\mathbb{R}}$). If $D_{\mathbb{L}}(a, b) = 0$, then \mathfrak{K} increases on the left of a ; if $D_{\mathbb{R}}(a, b) = 0$, then \mathfrak{K} decreases on the right of b .*

Proof. We treat the case of the right differential only. The remaining case is symmetric. We will use formula (3.6.4). By Condition 2.1.11 for the function f , the function \mathfrak{K} either increases or decreases in a right neighborhood of b . If it increases, then the force $F_{\mathbb{R}}$ is non-negative, which contradicts the assumption $t^{\mathbb{R}} > b$. Therefore, \mathfrak{K} decreases on the right of b . ■

Combining these two lemmas we obtain the following corollary: during the evolution, the differentials can vanish only in some very special situations (we use the notation from Definition 2.1.13 in the corollary below).

Corollary 4.2.3. *Suppose that the chordal domain $\Omega_{\text{ch}}([a, b], *)$ has nonzero tails. If $D_{\mathbb{R}}(a, b) = 0$, then $b = c_i$ for some i ; if $D_{\mathbb{L}}(a, b) = 0$, then $a = c_i$ for some i .*

4.2.2 Tails growth lemmas

Lemma 4.2.4. *Let $\Omega_{\text{ch}}([a_1, b_1], [a_0, b_0])$ be embedded into $\Omega_{\text{ch}}([a_2, b_2], [a_0, b_0])$, i.e., the foliation of the former chordal domain coincides with some part of the foliation of the latter. Then the forces of $\Omega_{\text{ch}}([a_1, b_1], *)$ do not exceed the corresponding forces of $\Omega_{\text{ch}}([a_2, b_2], *)$. More precisely, they are equal on $[a_1, a_0] \cup [b_0, b_1]$, while outside $[a_1, b_1]$, the inequalities are strict.*

Proof. For the right forces, the statement follows from equation (3.6.7) and the fact that $\kappa_{\mathbb{R}}(b) < \kappa_{\text{chord}}(b) < \kappa_2(b)$. Note that $D_{\mathbb{R}}(a, b) < 0$, therefore $F_{\mathbb{R}}$ increases as the chordal domain enlarges, i.e., b increases. Similar arguments work for the case of the left forces: we use (3.6.9) and the opposite relation between the coefficients: $\kappa_{\mathbb{L}}(a) > \kappa_{\text{chord}}(a) > \kappa_2(a)$. Thus, $F_{\mathbb{L}}$ increases as the chordal domain enlarges, i.e., a decreases. ■

Corollary 4.2.5. *Suppose $\Omega_{\text{ch}}([a_1, b_1], *)$ is embedded into $\Omega_{\text{ch}}([a_2, b_2], *)$ in the sense of Lemma 4.2.4. Then the tails of the former chordal domain strictly contain the tails of the latter.*

Proof. This follows from Lemma 4.2.4 and the definition of tails. ■

The previous statements can be reformulated informally as follows: the less the chordal domain is, the larger the tails are and the less (the larger in absolute value) the forces are inside the tails. The following lemma describes monotonicity of forces of a fixed chordal domain for a family of enlarging domains Ω_ε .

Lemma 4.2.6. *Let $\Omega_{\text{ch}}([a, b], *)$ be a chordal domain and let F be its left or right force. If t belongs to the closure of the corresponding tail of $\Omega_{\text{ch}}([a, b], *)$, then*

$$\frac{\partial F(t; \varepsilon)}{\partial \varepsilon} \leq 0,$$

and the inequality is strict outside $[a, b]$.

Proof. We treat the case of the right force only, the other one is symmetric. First, we note that inside $[a, b]$ the force does not depend on ε , therefore $\frac{\partial F_{\text{R}}(t; \varepsilon)}{\partial \varepsilon} = 0$. Second, we use formula (3.6.6) and see that

$$F'_{\text{R}}(t; \varepsilon) + \frac{\kappa'_2(t)}{\kappa_2(t) - \kappa(t; \varepsilon)} F_{\text{R}}(t; \varepsilon) = \mathfrak{K}'(t)$$

outside this interval. We cautiously differentiate this equation with respect to ε and see that

$$\frac{\partial}{\partial \varepsilon} F'_{\text{R}}(t; \varepsilon) + \frac{\kappa'_2(t)}{\kappa_2(t) - \kappa(t; \varepsilon)} \frac{\partial}{\partial \varepsilon} F_{\text{R}}(t; \varepsilon) = - \frac{\kappa'_2(t) F_{\text{R}}(t; \varepsilon)}{(\kappa_2(t) - \kappa(t; \varepsilon))^2} \frac{\partial}{\partial \varepsilon} \kappa(t; \varepsilon).$$

After interchanging the differentiations with respect to t and ε in the first summand, we see that $y(t) = \frac{\partial}{\partial \varepsilon} F_{\text{R}}(t; \varepsilon)$ is a solution of the first-order differential equation with respect to t :

$$y'(t) + \frac{\kappa'_2(t)}{\kappa_2(t) - \kappa(t; \varepsilon)} y(t) = - \frac{\kappa'_2(t) F_{\text{R}}(t; \varepsilon)}{(\kappa_2(t) - \kappa(t; \varepsilon))^2} \frac{\partial}{\partial \varepsilon} \kappa(t; \varepsilon).$$

We use (3.2.10) to express the solution:

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} F_{\text{R}}(t; \varepsilon) &= \exp\left(- \int_b^t \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \\ &\times \left(\int_b^t \exp\left(\int_b^\tau \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \left[- \frac{\kappa'_2 F_{\text{R}}}{(\kappa_2 - \kappa)^2} \frac{\partial \kappa}{\partial \varepsilon} \right](\tau) d\tau + \text{Const}(\varepsilon) \right), \end{aligned}$$

were the constant is zero, because $\lim_{t \rightarrow b+} \frac{\partial}{\partial \varepsilon} F_{\text{R}}(t; \varepsilon) = 0$ (this limit relation can be verified by a straightforward calculation using formula (3.6.4)). Now the result follows immediately, since $F_{\text{R}} < 0$ inside the tail, $\kappa'_2 > 0$, and $\frac{\partial \kappa(t; \varepsilon)}{\partial \varepsilon} < 0$ for geometrical reasons. ■

Corollary 4.2.7. *Let $\Omega_{\text{ch}}([a, b], *)$ be a chordal domain with nonzero tails. If we increase ε a little, the tails of the chordal domain strictly enlarge.*

Proof. By Lemma 4.2.6, the forces decrease (increase in absolute value) on the corresponding tails, therefore the tails cannot decrease. What is more, if we increase ε a little, the force at the end of the tail becomes negative, and the tail enlarges. ■

Remark 4.2.8. The results of Lemma 4.2.6 and Corollary 4.2.7 hold for the forces coming from the infinities and multicups as well.

For a moment, we let chordal domains fall out of Ω_ε , i.e., their chords may intersect the free boundary. Surely, when we were working with chordal domains, we did not need the upper boundary, therefore, such an assumption does not break all the results concerning chordal domains.

Lemma 4.2.9. *Let $\Omega_{\text{ch}}([a_2, b_2], [a_1, b_1])$ be a chordal domain with the standard Bellman candidate there. Suppose that $\varepsilon_1 < \varepsilon_2$, and that there are two differentiable functions $a: [\varepsilon_1, \varepsilon_2] \rightarrow [a_2, a_1]$ and $b: [\varepsilon_1, \varepsilon_2] \rightarrow [b_1, b_2]$, such that the chord $[g(a(\varepsilon)), g(b(\varepsilon))]$ is tangent to the free boundary of Ω_ε for each $\varepsilon \in [\varepsilon_1, \varepsilon_2]$. Then, for any $\tilde{\varepsilon} \in [\varepsilon_1, \varepsilon_2]$, we have*

$$\frac{\partial}{\partial \varepsilon} [F(t; a(\tilde{\varepsilon}), b(\tilde{\varepsilon}); \varepsilon)] \Big|_{\varepsilon=\tilde{\varepsilon}} = \frac{\partial}{\partial \varepsilon} [F(t; a(\varepsilon), b(\varepsilon); \varepsilon)] \Big|_{\varepsilon=\tilde{\varepsilon}},$$

where F stands for the right (see (3.6.4)) or the left (see (3.6.5)) force function.

Proof. This identity follows from Lemma 3.6.16 because $\kappa_R = \kappa_L = \kappa_{\text{chord}}$ when the chord is tangent to the free boundary. ■

Lemma 4.2.10. *Under the hypothesis of Lemma 4.2.9, the tails of $\Omega_{\text{ch}}([a(\varepsilon), b(\varepsilon)], *)$ strictly enlarge in ε .*

Proof. Consider a point t that belongs to one of the tails of $\Omega_{\text{ch}}([a(\varepsilon), b(\varepsilon)], *)$ for some ε . First, we need to prove that $F(t; a(\varepsilon), b(\varepsilon); \varepsilon)$ decreases in ε . The derivative of this function with respect to ε is non-positive, because, by Lemma 4.2.9, for each ε it equals the corresponding derivative taken as if the chordal domain had fixed upper chord, which is non-positive by Lemma 4.2.6. Thus, the tails do not decrease. Moreover, the derivative of the corresponding force is nonzero at the end of each tail, again by Lemma 4.2.6, therefore the tail grows. ■

Definition 4.2.11. Let $0 \leq \varepsilon_1 < \varepsilon_2 < \varepsilon_{\text{max}}$. Let $\Omega_{\text{ch}}([a_1, b_1], [a_0, b_0])$ be a chordal domain. Suppose that there are two continuous functions $a: [\varepsilon_1, \varepsilon_2] \rightarrow [a_1, a_0]$ and $b: [\varepsilon_1, \varepsilon_2] \rightarrow [b_0, b_1]$ such that $[g(a(\varepsilon)), g(b(\varepsilon))]$ is a chord of this chordal domain for each $\varepsilon \in [\varepsilon_1, \varepsilon_2]$. We call the family

$$\{\Omega_{\text{ch}}([a(\varepsilon), b(\varepsilon)], [a_0, b_0])\}_\varepsilon$$

a flow of chordal domains. This flow generates the corresponding forces:

$$\begin{aligned} F_R(u; \Omega_{\text{ch}}([a(\varepsilon), b(\varepsilon)], [a_0, b_0]); \varepsilon), \quad u \in [b_0, t^R(\varepsilon)], \\ F_L(u; \Omega_{\text{ch}}([a(\varepsilon), b(\varepsilon)], [a_0, b_0]); \varepsilon), \quad u \in (t^L(\varepsilon), a_0], \end{aligned} \quad (4.2.1)$$

where $t^R(\varepsilon)$ and $t^L(\varepsilon)$ are the right and the left endpoints of the tails of the chordal domain $\Omega_{\text{ch}}([a(\varepsilon), b(\varepsilon)], [a_0, b_0])$. We say that a flow is *decreasing* if b is decreasing (a is increasing). We say that a flow is *full* if the chord $[g(a(\varepsilon)), g(b(\varepsilon))]$ is tangent to the free boundary of Ω_ε for all $\varepsilon \in [\varepsilon_1, \varepsilon_2]$.

We gather the statements of the previous lemmas in the following corollary.

Corollary 4.2.12. *Consider a decreasing or a full flow of chordal domains. The corresponding forces (4.2.1) are strictly decreasing functions of ε outside the chordal domain and constant inside the chordal domain. Consequently, the tails enlarge.*

Proof. For the case of a decreasing flow, we use Lemmas 4.2.6 and 4.2.4. For the case of a full flow, we use Lemma 4.2.10. \blacksquare

Definition 4.2.13. Let $0 \leq \varepsilon_1 < \varepsilon_2 < \varepsilon_{\max}$, and let $u_-: [\varepsilon_1, \varepsilon_2] \rightarrow \mathbb{R} \cup \{-\infty\}$ be a non-increasing function and $u_+: [\varepsilon_1, \varepsilon_2] \rightarrow \mathbb{R} \cup \{+\infty\}$ a non-decreasing function such that $u_- \leq u_+$. Suppose that for any $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, there is a fence $\Omega(I_\varepsilon)$, such that $(u_-(\varepsilon), u_+(\varepsilon)) \subset I_\varepsilon$.

A continuous function $\mathfrak{F}_R: \{(u, \varepsilon) \mid u \in [u_-(\varepsilon), u_+(\varepsilon)] \cap \mathbb{R}, \varepsilon \in [\varepsilon_1, \varepsilon_2]\} \rightarrow \mathbb{R}$ is called a *right monotone force flow* if the following hold:

- (1) for any $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, the fence $\Omega(I_\varepsilon)$ is a right tangent domain, the function $\mathfrak{F}_R(\cdot; \varepsilon)$ is the force of a standard candidate on $\Omega_R(I_\varepsilon)$ introduced in Definition 3.3.8,
- (2) $\mathfrak{F}_R(u_+(\varepsilon); \varepsilon) = 0$, provided that $u_+(\varepsilon) < +\infty$,
- (3) for any η_1 and η_2 such that $\varepsilon_1 < \eta_1 < \eta_2 < \varepsilon_2$, we have $\mathfrak{F}_R(v; \eta_1) > \mathfrak{F}_R(v; \eta_2)$ whenever $v \in (u_-(\eta_1), u_+(\eta_1)) \cap \mathbb{R}$.

A continuous function $\mathfrak{F}_L: \{(u, \varepsilon) \mid u \in [u_-(\varepsilon), u_+(\varepsilon)] \cap \mathbb{R}, \varepsilon \in [\varepsilon_1, \varepsilon_2]\} \rightarrow \mathbb{R}$ is called a *left monotone force flow* the following hold:

- (1) for any $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, the fence $\Omega(I_\varepsilon)$ is a left tangent domain, and the function $\mathfrak{F}_L(\cdot; \varepsilon)$ is the force of a standard candidate on $\Omega_L(I_\varepsilon)$,
- (2) $\mathfrak{F}_L(u_-(\varepsilon); \varepsilon) = 0$, provided that $u_-(\varepsilon) > -\infty$,
- (3) for any η_1 and η_2 such that $\varepsilon_1 < \eta_1 < \eta_2 < \varepsilon_2$, we have $\mathfrak{F}_L(v; \eta_1) > \mathfrak{F}_L(v; \eta_2)$ whenever $v \in [u_-(\eta_1), u_+(\eta_1)) \cap \mathbb{R}$.

Monotone force flows may be generated by flows of chordal domains as in (4.2.1). Then we have $u_+ = t^R$ for the right case, and $u_- = t^L$ for the left case, see Section 3.6.2 for the definition of t^R and t^L .

Remark 4.2.14. We underline that for any fixed $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, if $u_-(\varepsilon) < u_+(\varepsilon)$, then the monotone force flows $\mathfrak{F}_R(\cdot; \varepsilon)$ and $\mathfrak{F}_L(\cdot; \varepsilon)$ satisfy the differential equations

$$\mathfrak{F}'_R = \mathfrak{K}' - \frac{\kappa'_2}{\kappa_2 - \kappa_R} \mathfrak{F}_R, \quad \mathfrak{F}'_L = \mathfrak{K}' - \frac{\kappa'_2}{\kappa_2 - \kappa_L} \mathfrak{F}_L. \quad (4.2.2)$$

Therefore, the statements of Remarks 3.6.8 and 3.6.10 hold true for the forces:

- if $u_-(\varepsilon) < u_+(\varepsilon) < \infty$, then $\mathfrak{K}' > 0$ in a left neighborhood of $u_+(\varepsilon)$,
- if $u_+(\varepsilon) > u_-(\varepsilon) > -\infty$, then $\mathfrak{K}' < 0$ in a right neighborhood of $u_-(\varepsilon)$.

Remark 4.2.15. Note that the function u_+ in the definition of the right monotone force flow is strictly increasing while it is finite. Similarly, the function u_- in the definition of the left monotone force flow is strictly decreasing while it is finite.

A simple example of a monotone force flow is given by a force of a chord that does not depend on ε (the monotonicity follows from Lemma 4.2.6).

Remark 4.2.16. Let $\{\Omega_{\text{ch}}([a(\varepsilon), b(\varepsilon)], *)\}$, $\varepsilon \in [\eta_1, \eta_2]$, be either decreasing or full flow of chordal domains. Then

$$\mathfrak{F}_R(u; \varepsilon) = F_R(u; \Omega_{\text{ch}}([a(\varepsilon), b(\varepsilon)], *); \varepsilon), \quad u \in [b(\varepsilon), t^R(\varepsilon)] \cap \mathbb{R}, \quad \varepsilon \in [\eta_1, \eta_2],$$

is a right monotone force flow, and

$$\mathfrak{F}_L(u; \varepsilon) = F_L(u; \Omega_{\text{ch}}([a(\varepsilon), b(\varepsilon)], *); \varepsilon), \quad u \in [t^L(\varepsilon), a(\varepsilon)] \cap \mathbb{R}, \quad \varepsilon \in [\eta_1, \eta_2],$$

is a left monotone force flow.

4.2.3 Balance equation lemma

Definition 4.2.17. Let F_R and F_L be forces of chordal domains, infinities, or multicups (for the definitions, see Section 3.6.2 and Definition 3.7.15) such that their domains intersect. Then the *balance equation* is

$$F_R(u) = F_L(u), \quad (4.2.3)$$

where u belongs to the intersection of the domains of the forces.

We are looking for solutions of balance equations. Lemma 4.1.4 helped us to establish the existence of the solution in Corollary 4.1.5.

Lemma 4.2.18. *Let F_R and F_L be two forces of chordal domains, infinities, multicups, or simply chords such that their tails intersect. Then the function $F_R(u) - F_L(u)$ strictly increases on this intersection.*

Proof. The proof is identical to that of Lemma 4.1.4. ■

4.3 Local evolution theorems

In this section all theorems have the following form: if for some ε we can build a Bellman candidate on a specific domain using specific formulas, then, for a slightly larger ε , we can also build a Bellman candidate on a perturbed domain using similar formulas with perturbed parameters. The statements are rather formal and somewhat bulky, so, before each statement we give a short heuristic explanation. We also recall our convention that the forces are strictly negative inside chordal domains and any tail.

The proposition below says that any full chordal domain (with nonzero differentials of the upper chord) surrounded by tangent domains enlarges as ε increases (in other words, in view of Definition 4.2.11, a full chordal domain generates a full flow of chordal domains starting from it).

Proposition 4.3.1 (Induction step for a chordal domain). *Let $\Omega_{\text{ch}}([a_1, b_1], [a_0, b_0])$ be a full chordal domain, $u_1 < a_1 \leq a_0 \leq b_0 \leq b_1 < u_2$, and $0 < \eta_1 < \varepsilon_{\text{max}}$. Let a continuous function B coincide with the standard candidates on $\Omega_{\text{ch}}([a_1, b_1], [a_0, b_0])$, $\Omega_{\text{L}}(u_1, a_1; \eta_1)$, and $\Omega_{\text{R}}(b_1, u_2; \eta_1)$. If $D_{\text{L}}(a_1, b_1) < 0$ and $D_{\text{R}}(a_1, b_1) < 0$, then there exist $\eta_2, \eta_2 > \eta_1$, and a full flow $\{\Omega_{\text{ch}}([a(\varepsilon), b(\varepsilon)], [a_0, b_0])\}_{\varepsilon \in [\eta_1, \eta_2]}$ of chordal domains such that $a(\eta_1) = a_1, b(\eta_1) = b_1$ and for each $\varepsilon \in [\eta_1, \eta_2]$, there exists a continuous function B_ε that coincides with the standard candidates on $\Omega_{\text{L}}(u_1, a(\varepsilon); \varepsilon)$, $\Omega_{\text{ch}}([a(\varepsilon), b(\varepsilon)], [a_0, b_0])$, and $\Omega_{\text{R}}(b(\varepsilon), u_2; \varepsilon)$.*

Proof. We use Remark 3.5.15 and Proposition 3.5.16 for the pair (a_1, b_1) and find $\delta > 0$ and functions \tilde{a} and \tilde{b} acting from $[0, \delta]$ to \mathbb{R} such that \tilde{a} is decreasing, $\tilde{a}(0) = a_1, \tilde{b}$ is increasing, $\tilde{b}(0) = b_1, \tilde{b}(\tau) - \tilde{a}(\tau) = b_1 - a_1 + \tau$, and the pair $(\tilde{a}(\tau), \tilde{b}(\tau))$ satisfies (3.4.2) as well as the inequalities $D_{\text{L}}(\tilde{a}(\tau), \tilde{b}(\tau)) < 0$ and $D_{\text{R}}(\tilde{a}(\tau), \tilde{b}(\tau)) < 0$ for $\tau \in [0, \delta]$. We take $\eta_2 > \eta_1$ in such a way that the chord $[g(\tilde{a}(\delta)), g(\tilde{b}(\delta))]$ intersects the free boundary of Ω_{η_2} . Then, for any $\varepsilon \in [\eta_1, \eta_2]$, there exists a unique $\tau = \tau(\varepsilon) \in [0, \delta]$ such that the chord $[g(\tilde{a}(\tau)), g(\tilde{b}(\tau))]$ is tangent to the free boundary of Ω_ε . We put $a(\varepsilon) = \tilde{a}(\tau(\varepsilon)), b(\varepsilon) = \tilde{b}(\tau(\varepsilon))$.

By our assumptions, u_2 belongs to the right tail of $\Omega_{\text{ch}}([a_1, b_1], [a_0, b_0])$ and u_1 belongs to its left tail. Due to Lemma 4.2.10, u_1 and u_2 belong to the left and the right tail of $\Omega_{\text{ch}}([a(\varepsilon), b(\varepsilon)], [a_0, b_0])$, $\varepsilon \in [\eta_1, \eta_2]$, respectively. This allows us to define the required B_ε on the union of $\Omega_{\text{ch}}([a(\varepsilon), b(\varepsilon)], [a_0, b_0])$, $\Omega_{\text{L}}(u_1, a(\varepsilon); \varepsilon)$, and $\Omega_{\text{R}}(b(\varepsilon), u_2; \varepsilon)$ for $\varepsilon \in [\eta_1, \eta_2]$. \blacksquare

By Remark 4.2.16, the functions

$$\mathfrak{F}_{\text{R}}(u; \varepsilon) = F_{\text{R}}(u; [a(\varepsilon), b(\varepsilon)]; \varepsilon) \quad \text{and} \quad \mathfrak{F}_{\text{L}}(u; \varepsilon) = F_{\text{L}}(u; [a(\varepsilon), b(\varepsilon)]; \varepsilon)$$

are the right and left monotone force flows on the corresponding domains.

The following proposition describes how a non-full multicup (i.e., a multicup $\Omega_{\text{Mcup}}(\{\alpha_i\}_{i=1}^k; \varepsilon)$ such that the chord $[g(\alpha_1^l), g(\alpha_k^r)]$ does not lie in Ω_ε) evolves in ε .

Proposition 4.3.2 (Induction step for a multicup). *Let $\Omega_{\text{Mcup}}(\{\alpha_i\}_{i=1}^k; \eta_1)$, with $\eta_1 \in (0, \varepsilon_{\max})$, be a multicup such that the chord $[g(\alpha_1^l), g(\alpha_k^r)]$ does not lie in Ω_{η_1} , and let $u_1 < \alpha_1^l$ and $\alpha_k^r < u_2$. Let a continuous function B coincide with the standard candidates on $\Omega_{\text{Mcup}}(\{\alpha_i\}_{i=1}^k; \eta_1)$, $\Omega_L(u_1, \alpha_1^l; \eta_1)$, and $\Omega_R(\alpha_k^r, u_2; \eta_1)$. Then there exists η_2 , $\eta_1 < \eta_2$, such that for each $\varepsilon \in [\eta_1, \eta_2]$, there exists a continuous function B_ε that coincides with the standard candidates on $\Omega_L(u_1, \alpha_1^l; \varepsilon)$, $\Omega_{\text{Mcup}}(\{\alpha_i\}_{i=1}^k; \varepsilon)$, and $\Omega_R(\alpha_k^r, u_2; \varepsilon)$.*

Proof. We may take any $\eta_2 > \eta_1$ such that the chord $[g(\alpha_1^l), g(\alpha_k^r)]$ intersects the free boundary of Ω_{η_2} . Take any ε from the interval prescribed. By Remark 4.2.8, the tails of the multicup enlarge with ε , therefore the points u_1 and u_2 belong to them. Thus, one can build the required function B_ε . ■

Clearly, the functions

$$F_R(u; [\alpha_1^l, \alpha_k^r]; \varepsilon) \quad \text{and} \quad F_L(u; [\alpha_1^l, \alpha_k^r]; \varepsilon)$$

are the right and left monotone force flows on the corresponding domains.

The following proposition says that a long chord with nonzero tails gives rise to a chordal domain. We note that this generalizes Proposition 4.3.1 (in the latter case the differentials are nonzero, and thus the tails of the upper chord are nonzero). However, for didactic reasons, we prefer to separate these two propositions. In a sense, the cases where one or both differentials are zero differ from what is described in Proposition 4.3.1. Indeed, suppose that one of the differentials is zero and we have a chordal domain below a chord $[g(a_0), g(b_0)]$ with the standard candidate on it. By Proposition 4.3.3, after we increase ε , we can build a chordal domain above $[g(a_0), g(b_0)]$ and the standard candidate there. However, we cannot glue these standard candidates into a single one, because the differentials should be strictly negative for a standard candidate. We also note that the proposition below may be applied to the upper chord of a full multicup (i.e., a multicup such that the chord $[g(\alpha_1^l), g(\alpha_k^r)]$ is tangent to the free boundary of Ω_ε).

Proposition 4.3.3 (Induction step for a long chord). *Let $a_0, b_0, u_1, u_2 \in \mathbb{R}$. Suppose that $a_0 < b_0$, the pair (a_0, b_0) satisfies the cup equation (3.4.2), and the chord $[g(a_0), g(b_0)]$ is tangent to the free boundary of Ω_{η_1} , $\eta_1 \in (0, \varepsilon_{\max})$. Assume also that $u_1 < a_0$ and $u_2 > b_0$, and that u_1 belongs to the left tail of the chord $[g(a_0), g(b_0)]$, while u_2 belongs to its right tail. Then there exist η_2 , $\eta_2 > \eta_1$, and a full flow of chordal domains $\{\Omega_{\text{ch}}([a(\varepsilon), b(\varepsilon)], [a_0, b_0])\}_{\varepsilon \in [\eta_1, \eta_2]}$ such that for each $\varepsilon \in [\eta_1, \eta_2]$, there exists a continuous function B_ε that coincides with the standard candidates on $\Omega_{\text{ch}}([a(\varepsilon), b(\varepsilon)], [a_0, b_0])$, $\Omega_L(u_1, a(\varepsilon); \varepsilon)$, and $\Omega_R(b(\varepsilon), u_2; \varepsilon)$.*

Proof. The proof is a repetition of the proof of Proposition 4.3.1; the only difference is that here we need to verify Condition 3.5.14 directly in order to use Proposition 3.5.16.

We need to check that the quantity

$$L(t) = \det \begin{pmatrix} \gamma'(t) \\ \gamma'(a_0) \\ \gamma'(b_0) \end{pmatrix} = g'_1(t) \det \begin{pmatrix} 1 & \kappa_2(t) & \kappa_3(t) \\ g'_1(a_0) & g'_2(a_0) & f'(a_0) \\ g'_1(b_0) & g'_2(b_0) & f'(b_0) \end{pmatrix}$$

is positive for t on the left of a_0 and negative on the right of b_0 . We consider the right case; the left one is analogous. Since the right tail of the chord $[g(a_0), g(b_0)]$ is nonempty, we have $D_{\mathbb{R}}(a_0, b_0) \leq 0$. If $D_{\mathbb{R}}(a_0, b_0) < 0$, then the claim is trivial (see Remark 3.5.15). If $D_{\mathbb{R}}(a_0, b_0) = 0$, then we use Lemma 4.2.2 and conclude that \mathfrak{K} decreases on $(b_0, b_0 + \delta)$ for some $\delta > 0$. Note that $L(b_0) = L'(b_0) = 0$ and for $t \in (b_0, b_0 + \delta)$, we have

$$\begin{aligned} L_1(t) &\stackrel{\text{def}}{=} \frac{\left(\frac{L(t)}{g'_1(t)}\right)'}{\kappa'_2(t)} = \mathfrak{K}(t) \det \begin{pmatrix} g'(a_0) \\ g'(b_0) \end{pmatrix} + \text{Const} \\ &< \mathfrak{K}(b_0) \det \begin{pmatrix} g'(a_0) \\ g'(b_0) \end{pmatrix} + \text{Const} = L_1(b_0) = 0. \end{aligned}$$

Since $\kappa'_2 > 0$, the function $\frac{L}{g'_1}$ strictly decreases on $(b_0, b_0 + \delta)$, so $\frac{L}{g'_1}(t) < \frac{L}{g'_1}(b_0) = 0$ and $L(t) < 0$ for $t \in (b_0, b_0 + \delta)$ because $g'_1 > 0$. ■

Now we turn to trolleybuses. The next two propositions claim that the base of a trolleybus shrinks when ε increases. On a more formal way, there exists a decreasing flow of chordal domains such that for each ε , we can build a trolleybus on the corresponding chordal domain. In what follows, we will use the notation $\overline{\mathbb{R}}$ for the extended real line $\mathbb{R} \cup \{\pm\infty\}$.

Proposition 4.3.4 (Induction step for a right trolleybus). *Let $\eta_1, \eta_3 \in \mathbb{R}$, $0 < \eta_1 < \eta_3 < \varepsilon_{\max}$. Suppose $a_1, b_1, a_2, b_2, u \in \mathbb{R}$, $a_2 < a_1 < b_1 < b_2 \leq u$. Let $\Omega([a_2, a_1])$ be the fence with corresponding function $s: [a_2, a_1] \rightarrow [b_1, b_2]$, $s(a_i) = b_i$, $i = 1, 2$. This fence coincides with $\Omega_{\text{ch}}([a_2, b_2], [a_1, b_1])$. Suppose that $\mathfrak{F}_{\mathbb{R}}$ is a right monotone force flow with corresponding functions $u_{\pm}: [\eta_1, \eta_3] \rightarrow \overline{\mathbb{R}}$ such that $a_2 \in [u_-(\eta_1), u_+(\eta_1)]$. Suppose that there exists a continuous function B that coincides with the standard candidates on $\Omega_{\mathbb{R}}(u_-(\eta_1), a_2; \eta_1)$, $\Omega_{\text{tr}, \mathbb{R}}(a_2, b_2; \eta_1)$, $\Omega_{\mathbb{R}}(b_2, u; \eta_1)$, and $\Omega_{\text{ch}}([a_2, a_1])$. Moreover, the force of B on $\Omega_{\mathbb{R}}(u_-(\eta_1), a_2; \eta_1)$ coincides with $\mathfrak{F}_{\mathbb{R}}(\cdot; \eta_1)$. Then there are $\eta_2 \in (\eta_1, \eta_3)$ and a strictly increasing function $a: [\eta_1, \eta_2] \rightarrow [a_2, a_1]$ such that $a(\eta_1) = a_2$, and for any $\varepsilon \in [\eta_1, \eta_2]$, there exists a continuous function B_{ε} that coincides with the standard candidates on $\Omega_{\mathbb{R}}(u_-(\varepsilon), a(\varepsilon); \varepsilon)$, $\Omega_{\text{ch}}([a(\varepsilon), a_1])$, $\Omega_{\text{tr}, \mathbb{R}}(a(\varepsilon), b(\varepsilon); \varepsilon)$, and $\Omega_{\mathbb{R}}(b(\varepsilon), u; \varepsilon)$, where $b(\varepsilon) = s(a(\varepsilon))$. In addition, the force of B_{ε} on $\Omega_{\mathbb{R}}(u_-(\varepsilon), a(\varepsilon); \varepsilon)$ is $\mathfrak{F}_{\mathbb{R}}(\cdot; \varepsilon)$. The functions*

$$F_{\mathbb{R}}(t; a(\varepsilon), b(\varepsilon); \varepsilon) \quad \text{and} \quad F_{\mathbb{L}}(t; a(\varepsilon), b(\varepsilon); \varepsilon) \quad (4.3.1)$$

are the right and the left monotone force flows on the corresponding domains.

We note that $\{\Omega_{\text{ch}}([a(\varepsilon), a_1])\}_{\varepsilon \in [\eta_1, \eta_2]}$ is a decreasing flow of chordal domains.

Proof. Our first step is to find $\eta_2 \in (\eta_1, \eta_3)$ and a point a^- in a right neighborhood of a_2 such that $a^- < u_+(\varepsilon)$ and

$$\mathfrak{F}_{\text{R}}(a^-; \varepsilon) > F_{\text{L}}(a^-; \Omega_{\text{ch}}([a_2, a_1])), \quad \varepsilon \in (\eta_1, \eta_2). \quad (4.3.2)$$

We consider two cases: $u_+(\eta_1) > a_2$ and $u_+(\eta_1) = a_2$.

First, if $u_+(\eta_1) > a_2$, then, by Lemma 4.2.18, $\mathfrak{F}_{\text{R}}(\cdot; \eta_1) - F_{\text{L}}(\cdot; \Omega_{\text{ch}}([a_2, a_1]))$ is strictly increasing in some right neighborhood of a_2 . Since the point a_2 is the root of the balance equation for these forces, i.e., $\mathfrak{F}_{\text{R}}(a_2; \eta_1) = F_{\text{L}}(a_2; \Omega_{\text{ch}}([a_2, a_1]))$, we may simply take a point $a^- \in (a_2, u_+(\eta_1)) \cap (a_2, a_1)$ and obtain $\mathfrak{F}_{\text{R}}(a^-; \eta_1) > F_{\text{L}}(a^-; \Omega_{\text{ch}}([a_2, a_1]))$. It follows from the continuity of forces that for some $\eta_2 \in (\eta_1, \eta_3)$, inequality (4.3.2) holds.

Now we assume that $u_+(\eta_1) = a_2$. Then $\mathfrak{F}_{\text{R}}(a_2; \eta_1) = F_{\text{L}}(a_2; \Omega_{\text{ch}}([a_2, a_1])) = 0$, and, according to Lemma 4.2.1, the function \mathfrak{R} decreases on an interval (a_2, a^-) , with $a^- < a_1$. It follows from Remark 4.2.15 that $u_+(\varepsilon) > a_2$ for any $\varepsilon \in (\eta_1, \eta_3]$, therefore we have $u_+(\varepsilon) > a^-$ by Remark 4.2.14. Lemma 4.2.18 implies that the function $\mathfrak{F}_{\text{R}}(\cdot; \varepsilon) - F_{\text{L}}(\cdot; \Omega_{\text{ch}}([a_2, a_1]))$ strictly increases on (a_2, a^-) . If there were no required η_2 , then $\mathfrak{F}_{\text{R}}(a^-; \varepsilon) - F_{\text{L}}(a^-; \Omega_{\text{ch}}([a_2, a_1])) \leq 0$ for all $\varepsilon \in (\eta_1, \eta_3)$. However,

$$\mathfrak{F}_{\text{R}}(a_2; \varepsilon) \rightarrow \mathfrak{F}_{\text{R}}(a_2; \eta_1) = F_{\text{L}}(a_2; \Omega_{\text{ch}}([a_2, a_1])), \quad \varepsilon \rightarrow \eta_1+,$$

therefore $\mathfrak{F}_{\text{R}}(\cdot; \varepsilon) - F_{\text{L}}(\cdot; \Omega_{\text{ch}}([a_2, a_1]))$ converges uniformly to zero on $[a_2, a^-]$ when $\varepsilon \rightarrow \eta_1+$, or, equivalently,

$$\mathfrak{F}_{\text{R}}(\cdot; \varepsilon) - \mathfrak{R} \rightarrow F_{\text{L}}(\cdot; \Omega_{\text{ch}}([a_2, a_1])) - \mathfrak{R}, \quad \varepsilon \rightarrow \eta_1+, \quad (4.3.3)$$

uniformly on $[a_2, a^-]$. Due to Remark 3.6.15, the function on the right-hand side of (4.3.3) is strictly decreasing while the function on the left-hand side is strictly increasing. This contradicts the uniform convergence (4.3.3). Therefore, there exists $\eta_2 \in (\eta_1, \eta_3)$ such that (4.3.2) holds.

By the definition of the monotone force flow (see Definition 4.2.13),

$$\mathfrak{F}_{\text{R}}(a_2; \varepsilon) - F_{\text{L}}(a_2; \Omega_{\text{ch}}([a_2, a_1])) < 0, \quad \varepsilon \in (\eta_1, \eta_2),$$

since $F_{\text{L}}(a_2; \Omega_{\text{ch}}([a_2, a_1]))$ does not depend on ε . Therefore, there exists a point $a = a(\varepsilon) \in [a_2, a^-]$ that solves the balance equation $\mathfrak{F}_{\text{R}}(a; \varepsilon) = F_{\text{L}}(a; \Omega_{\text{ch}}([a_2, a_1]))$. We note that the function a is increasing. For the existence of the desired function B_ε , we only need to verify that

$$F_{\text{R}}(t; a(\varepsilon), b(\varepsilon); \varepsilon) < 0, \quad \varepsilon \in [\eta_1, \eta_2], \quad t \in [b(\varepsilon); u],$$

which follows from Corollary 4.2.12, because $\{\Omega_{\text{ch}}([a(\varepsilon), a_1])\}$ is a decreasing flow of chordal domains.

By Remark 4.2.16, the force functions (4.3.1) form monotone force flows. \blacksquare

Proposition 4.3.5 (Induction step for a left trolleybus). *Let $\eta_1, \eta_3 \in \mathbb{R}$, $0 < \eta_1 < \eta_3 < \varepsilon_{\max}$. Suppose that $a_1, b_1, a_2, b_2, u \in \mathbb{R}$, $u \leq a_2 < a_1 < b_1 < b_2$. Let $\Omega_{\text{ch}}([b_1, b_2])$ be the fence with corresponding function $s: [b_1, b_2] \rightarrow [a_2, a_1]$, $s(b_i) = a_i$, $i = 1, 2$. This fence coincides with $\Omega_{\text{ch}}([a_2, b_2], [a_1, b_1])$. Suppose that \mathfrak{F}_L is a left monotone force flow with corresponding functions $u_{\pm}: [\eta_1, \eta_3] \rightarrow \overline{\mathbb{R}}$ such that $b_2 \in [u_-(\eta_1), u_+(\eta_1)]$. Suppose that there exists a continuous function B that coincides with the standard candidates on $\Omega_L(b_2, u_+(\eta_1); \eta_1)$, $\Omega_{\text{tr},L}(a_2, b_2; \eta_1)$, $\Omega_L(u, a_2; \eta_1)$, and $\Omega_{\text{ch}}([b_1, b_2])$. Moreover, the force of B on $\Omega_L(b_2, u_+(\eta_1); \eta_1)$ is $\mathfrak{F}_L(\cdot; \eta_1)$. Then there exist η_2 , with $\eta_1 < \eta_2 < \eta_3$, and a strictly decreasing function $b: [\eta_1, \eta_2] \rightarrow [b_1, b_2]$ such that $b(\eta_1) = b_2$, and for any $\varepsilon \in [\eta_1, \eta_2]$, there exists a continuous function B_{ε} that coincides with the standard candidates on $\Omega_L(b(\varepsilon), u_+(\varepsilon); \varepsilon)$, $\Omega_{\text{tr},L}(a(\varepsilon), b(\varepsilon); \varepsilon)$, $\Omega_L(u, a(\varepsilon); \varepsilon)$, and $\Omega_{\text{ch}}([b_1, b(\varepsilon)])$, where $a(\varepsilon) = s(b(\varepsilon))$. Moreover, the force of B_{ε} on $\Omega_L(b(\varepsilon), u_+(\varepsilon); \varepsilon)$ is $\mathfrak{F}_L(\cdot; \varepsilon)$. The forces (4.3.1) form monotone force flows.*

Remark 4.3.6. It follows from Lemma 4.2.18 that the functions a and b constructed in Propositions 4.3.4 and 4.3.5 are unique (at least when $\eta_2 - \eta_1$ is sufficiently small).

The following four propositions describe the behavior under evolution of multi-trolleybuses. It appears that each multitrolleybus immediately splits into a trolleybus parade (by formulas (3.8.11) and (3.8.12)), and each of the trolleybuses decreases. We consider two simpler cases separately to make the presentation smoother.

Proposition 4.3.7 (Induction step for a right multitrolleybus on a solid root). *Let $\eta_1, \eta_3 \in \mathbb{R}$, $0 < \eta_1 < \eta_3 < \varepsilon_{\max}$. Consider a right multitrolleybus $\Omega_{\text{Mtr},R}(\{\alpha\}; \eta_1)$ on a solid root $\alpha = [\alpha^l, \alpha^r]$ (the case $\alpha^l = \alpha^r$ is not excluded). Let $u \in \mathbb{R}$, $\alpha^r \leq u$. Suppose that \mathfrak{F}_R is a right monotone force flow with corresponding functions $u_{\pm}: [\eta_1, \eta_3] \rightarrow \overline{\mathbb{R}}$ such that $\alpha^l \in [u_-(\eta_1), u_+(\eta_1)]$. Suppose that there exists a continuous function B that coincides with the standard candidates on $\Omega_R(u_-(\eta_1), \alpha^l; \eta_1)$, $\Omega_{\text{Mtr},R}(\{\alpha\}; \eta_1)$, and $\Omega_R(\alpha^r, u; \eta_1)$. Moreover, the force of B on $\Omega_R(u_-(\eta_1), \alpha^l; \eta_1)$ is $\mathfrak{F}_R(\cdot; \eta_1)$. Then for any $\varepsilon \in (\eta_1, \eta_3]$, there exists a continuous function B_{ε} that coincides with the standard candidates on $\Omega_R(u_-(\varepsilon), u; \varepsilon)$, and the force of B_{ε} on $\Omega_R(u_-(\varepsilon), u; \varepsilon)$ is $\mathfrak{F}_R(\cdot; \varepsilon)$.*

Proof. First we note that $\mathfrak{F}_R(\alpha^l; \eta_1) = 0$, therefore $\alpha^l = u_+(\eta_1)$. The only thing we need to prove is that $u_+(\varepsilon) \geq u$ for any $\varepsilon \in (\eta_1, \eta_3]$. By Remark 4.2.15 we know that $u_+(\varepsilon) > u_+(\eta_1) = \alpha^l$. By Remark 4.2.14, $u_+(\varepsilon) \notin \alpha$, therefore $u_+(\varepsilon) > \alpha^r$. From (4.2.2), for any $t \in (\alpha^r, \min(u_+(\varepsilon), u)]$, we have

$$\mathfrak{F}_R(t; \varepsilon) = \int_{\alpha^r}^t \exp\left(-\int_{\tau}^t \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \mathfrak{R}'(\tau) d\tau + \exp\left(-\int_{\alpha^r}^t \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \mathfrak{F}_R(\alpha^r; \varepsilon). \quad (4.3.4)$$

The second summand in (4.3.4) is negative because $\alpha^r \in [u_-(\varepsilon), u_+(\varepsilon)]$. The first summand in (4.3.4) is equal to $F_R(t; \alpha^l, \alpha^r; \varepsilon)$ (see (3.6.4), $D_R(\alpha^l, \alpha^r) = 0$). Further,

the inequality

$$F_R(t; \alpha^l, \alpha^r; \varepsilon) < F_R(t; \alpha^l, \alpha^r; \eta_1) \quad (4.3.5)$$

follows from monotonicity of forces with respect to ε . The right-hand side of (4.3.5) is non-positive because $t \in (\alpha^r, u]$ and $F_R(\cdot; \alpha^l, \alpha^r; \eta_1)$ is the force of the standard candidate on $\Omega_R(\alpha^r, u; \eta_1)$. Thus, in particular, $\mathfrak{F}_R(\min(u_+(\varepsilon), u); \varepsilon) < 0$. Since $\mathfrak{F}_R(u_+(\varepsilon); \varepsilon) = 0$, we have $\min(u_+(\varepsilon), u) \neq u_+(\varepsilon)$, therefore $u_+(\varepsilon) > u$. ■

Remark 4.3.8. Note that the case $\alpha^l = \alpha^r$ in Proposition 4.3.7 means that the multitrolleybus $\Omega_{\text{Mtr},R}(\{\alpha\}; \eta_1)$ is a fictitious vertex $\Omega_R(\alpha, \alpha)$ of the fifth type. A similar statement holds for the left case.

Proposition 4.3.9 (Induction step for a right multitrolleybus with one underlying chordal domain). *Let $\eta_1, \eta_3 \in \mathbb{R}$, $0 < \eta_1 < \eta_3 < \varepsilon_{\max}$. Suppose that $a_1, b_1, a_2, b_2, u \in \mathbb{R}$, $a_2 < a_1 < b_1 < b_2 \leq u$. Let $\alpha = [\alpha^l, \alpha^r]$ be a solid root, $\alpha^l < \alpha^r = a_2$. Consider a right multitrolleybus $\Omega_{\text{Mtr},R}(\{\alpha, b_2\}; \eta_1)$. Let $\Omega_{\text{ch}}([a_2, a_1])$ be the chordal domain with corresponding function $s: [a_2, a_1] \rightarrow [b_1, b_2]$, $s(a_i) = b_i$, $i = 1, 2$. Suppose that \mathfrak{F}_R is a right monotone force flow with corresponding functions $u_{\pm}: [\eta_1, \eta_3] \rightarrow \overline{\mathbb{R}}$ such that $\alpha^l \in [u_-(\eta_1), u_+(\eta_1)]$. Suppose that there exists a continuous function B that coincides with the standard candidates on $\Omega_R(u_-(\eta_1), \alpha^l; \eta_1)$, $\Omega_{\text{Mtr},R}(\{\alpha, b_2\}; \eta_1)$, $\Omega_R(b_2, u; \eta_1)$, and $\Omega_{\text{ch}}([a_2, a_1])$. Moreover, the force of B on $\Omega_R(u_-(\eta_1), \alpha^l; \eta_1)$ is $\mathfrak{F}_R(\cdot; \eta_1)$. Then there exist η_2 , $\eta_1 < \eta_2 < \eta_3$, and a strictly increasing function $a: [\eta_1, \eta_2] \rightarrow [a_2, a_1]$ such that $a(\eta_1) = a_2$, and for any $\varepsilon \in (\eta_1, \eta_2]$, there exists a continuous function B_ε that coincides with the standard candidates on $\Omega_R(u_-(\varepsilon), a(\varepsilon); \varepsilon)$, $\Omega_{\text{tr},R}(a(\varepsilon), b(\varepsilon); \varepsilon)$, $\Omega_{\text{ch}}([a(\varepsilon), a_1])$, and $\Omega_R(b(\varepsilon), u; \varepsilon)$, where $b(\varepsilon) = s(a(\varepsilon))$. Moreover, the force of B_ε on $\Omega_R(u_-(\varepsilon), a(\varepsilon); \varepsilon)$ is $\mathfrak{F}_R(\cdot; \varepsilon)$. The forces (4.3.1) form monotone force flows.*

Proof. The force of B on α is equal to zero, in particular, $\mathfrak{F}_R(\alpha^l; \eta_1) = 0$, therefore $u_+(\eta_1) = \alpha^l$. We cannot directly apply Proposition 4.3.4 in this case, but its proof works, because for any $\varepsilon \in (\eta_1, \eta_3)$, by Proposition 4.3.7, we have $u_+(\varepsilon) > \alpha^r$, and the arguments of the second case in the proof of Proposition 4.3.4 (where $u_+(\eta_1) = a_2$) apply verbatim. ■

Proposition 4.3.10 (Induction step for a general right multitrolleybus). *Suppose that $\eta_1, \eta_3 \in \mathbb{R}$ and $\eta_1 < \eta_3 < \varepsilon_{\max}$. Consider a right multitrolleybus $\Omega_{\text{Mtr},R}(\{\alpha_i\}_{i=1}^k; \eta_1)$. Let $u \in \mathbb{R}$, $\alpha_k^r \leq u$. Suppose that \mathfrak{F}_R is a right monotone force flow with corresponding functions $u_{\pm}: [\eta_1, \eta_3] \rightarrow \overline{\mathbb{R}}$ such that $\alpha_1^l \in [u_-(\eta_1), u_+(\eta_1)]$. We also suppose that for each $i = 1, 2, \dots, k-1$, there are chordal domains $\Omega_{\text{ch}}([\alpha_i^r, \alpha_{i+1}^l], *)$ with corresponding functions s_i , $s_i(\alpha_i^r) = \alpha_{i+1}^l$. Suppose that there exists a continuous function B that coincides with the standard candidates on $\Omega_R(u_-(\eta_1), \alpha_1^l; \eta_1)$, $\Omega_{\text{Mtr},R}(\{\alpha_i\}_{i=1}^k; \eta_1)$, $\Omega_R(\alpha_k^r, u; \eta_1)$, and every $\Omega_{\text{ch}}([\alpha_i^r, \alpha_{i+1}^l], *)$. Additionally, suppose that the force of B on $\Omega_R(u_-(\eta_1), \alpha_1^l; \eta_1)$ is $\mathfrak{F}_R(\cdot; \eta_1)$. Then there exist a*

number η_2 , $\eta_2 > \eta_1$, and a collection of strictly increasing functions $a_i: [\eta_1, \eta_2] \rightarrow \mathbb{R}$, $a_i(\eta_1) = \alpha_i^r$, $i = 1, 2, \dots, k-1$, such that for every $\varepsilon \in (\eta_1, \eta_2]$, there exists a continuous function B_ε defined on the domain

$$\begin{aligned} & \Omega_{\mathbb{R}}(u_-(\varepsilon), a_1(\varepsilon); \varepsilon) \cup \left(\bigcup_{i=1}^{k-2} \Omega_{\mathbb{R}}(b_i(\varepsilon), a_{i+1}(\varepsilon); \varepsilon) \right) \cup \left(\bigcup_{i=1}^{k-1} \Omega_{\text{tr},\mathbb{R}}(a_i(\varepsilon), b_i(\varepsilon); \varepsilon) \right) \\ & \cup \left(\bigcup_{i=1}^{k-1} \Omega_{\text{ch}}([a_i(\varepsilon), b_i(\varepsilon)], *) \right) \cup \Omega_{\mathbb{R}}(b_{k-1}(\varepsilon), u; \varepsilon), \quad b_i(\varepsilon) = s_i(a_i(\varepsilon)), \end{aligned}$$

that coincides with the standard candidate inside each subdomain of the partition. Moreover, the force of B_ε on $\Omega_{\mathbb{R}}(u_-(\varepsilon), a_1(\varepsilon); \varepsilon)$ is $\mathfrak{F}_{\mathbb{R}}(\cdot; \varepsilon)$. The functions

$$F_{\mathbb{R}}(\cdot; a_i(\varepsilon), b_i(\varepsilon); \varepsilon) \quad \text{and} \quad F_{\mathbb{L}}(\cdot; a_i(\varepsilon), b_i(\varepsilon); \varepsilon), \quad 1 \leq i \leq k-1,$$

are the right and the left monotone force flows on the corresponding domains.

Proof. We can represent $\Omega_{\text{Mtr},\mathbb{R}}(\{\alpha_i\}_{i=1}^k; \eta_1)$ as the union of $\Omega_{\text{Mtr},\mathbb{R}}(\{\alpha_i, \alpha_{i+1}^1\}; \eta_1)$, where $i = 1, \dots, k-1$, and possibly $\Omega_{\text{Mtr},\mathbb{R}}(\{\alpha_k\}; \eta_1)$ if $\alpha_k^1 < \alpha_k^r$. We apply Proposition 4.3.9 if $\alpha_i^1 < \alpha_i^r$, and Proposition 4.3.4 if $\alpha_i^1 = \alpha_i^r$ to the multitrolleybuses $\Omega_{\text{Mtr},\mathbb{R}}(\{\alpha_i, \alpha_{i+1}^1\}; \eta_1)$ successively. To conclude the induction, we apply Proposition 4.3.7 to $\Omega_{\text{Mtr},\mathbb{R}}(\{\alpha_k\}; \eta_1)$ if $\alpha_k^1 < \alpha_k^r$. ■

Proposition 4.3.11 (Induction step for a general left multitrolleybus). *Suppose that $\eta_1, \eta_3 \in \mathbb{R}$, $\eta_1 < \eta_3 < \varepsilon_{\max}$. Consider a left multitrolleybus $\Omega_{\text{Mtr},\mathbb{L}}(\{\alpha_i\}_{i=1}^k; \eta_1)$. Let $u \in \mathbb{R}$, $u \leq \alpha_1^l$. Suppose that $\mathfrak{F}_{\mathbb{L}}$ is a left monotone force flow with corresponding functions $u_{\pm}: [\eta_1, \eta_3] \rightarrow \overline{\mathbb{R}}$ such that $\alpha_k^r \in [u_-(\eta_1), u_+(\eta_1)]$. We also suppose that for each $i = 1, 2, \dots, k-1$, there are chordal domains $\Omega_{\text{ch}}([\alpha_i^r, \alpha_{i+1}^1], *)$ with corresponding functions s_i , $s_i(\alpha_{i+1}^1) = \alpha_i^r$. Suppose that there exists a continuous function B that coincides with the standard candidates on the domains $\Omega_{\mathbb{L}}(u, \alpha_1^l; \eta_1)$, $\Omega_{\text{Mtr},\mathbb{L}}(\{\alpha_i\}_{i=1}^k; \eta_1)$, $\Omega_{\mathbb{L}}(\alpha_k^r, u_+(\eta_1); \eta_1)$, and every $\Omega_{\text{ch}}([\alpha_i^r, \alpha_{i+1}^1], *)$. Moreover, the force of B on $\Omega_{\mathbb{L}}(\alpha_k^r, u_+(\eta_1); \eta_1)$ is $\mathfrak{F}_{\mathbb{L}}(\cdot; \eta_1)$. Then there exist a number η_2 , with $\eta_2 > \eta_1$, and a collection of strictly decreasing functions $b_i: [\eta_1, \eta_2] \rightarrow \mathbb{R}$, $b_i(\eta_1) = \alpha_{i+1}^l$, $i = 1, 2, \dots, k-1$, such that for every $\varepsilon \in (\eta_1, \eta_2]$, there exists a continuous function B_ε , defined on the domain*

$$\begin{aligned} & \Omega_{\mathbb{L}}(u, a_1(\varepsilon); \varepsilon) \cup \left(\bigcup_{i=1}^{k-2} \Omega_{\mathbb{L}}(b_i(\varepsilon), a_{i+1}(\varepsilon); \varepsilon) \right) \cup \left(\bigcup_{i=1}^{k-1} \Omega_{\text{tr},\mathbb{L}}(a_i(\varepsilon), b_i(\varepsilon); \varepsilon) \right) \\ & \cup \left(\bigcup_{i=1}^{k-1} \Omega_{\text{ch}}([a_i(\varepsilon), b_i(\varepsilon)], *) \right) \cup \Omega_{\mathbb{L}}(b_{k-1}(\varepsilon), u_+(\varepsilon); \varepsilon), \quad a_i(\varepsilon) = s_i(b_i(\varepsilon)), \end{aligned}$$

that coincides with the standard candidate inside each subdomain of the partition.

Moreover, the force of B_ε on $\Omega_L(b_{k-1}(\varepsilon), u_+(\varepsilon); \varepsilon)$ is $\mathfrak{F}_L(\cdot; \varepsilon)$. The functions

$$F_R(t; a_i(\varepsilon), b_i(\varepsilon); \varepsilon) \quad \text{and} \quad F_L(t; a_i(\varepsilon), b_i(\varepsilon); \varepsilon), \quad 1 \leq i \leq k-1,$$

are the right and the left monotone force flows on the corresponding domains.

In the following proposition, we show that angles move continuously.

Proposition 4.3.12 (Induction step for an angle). *Let $\eta_1, \eta_3 \in \mathbb{R}$, $0 < \eta_1 < \eta_3 < \varepsilon_{\max}$, and let $u_0 \in \mathbb{R}$. Suppose that \mathfrak{F}_L is a left monotone force flow with corresponding functions $u_\pm^l: [\eta_1, \eta_3] \rightarrow \bar{\mathbb{R}}$, and \mathfrak{F}_R is a right monotone force flow with corresponding functions $u_\pm^r: [\eta_1, \eta_3] \rightarrow \bar{\mathbb{R}}$ such that $u_0 \in (u_-^r(\eta_1), u_+^r(\eta_1)) \cap [u_-^l(\eta_1), u_+^l(\eta_1))$. Suppose that there exists a continuous function B that coincides with the standard candidates on $\Omega_R(u_-^r(\eta_1), u_0; \eta_1)$, $\Omega_{\text{ang}}(u_0; \eta_1)$, and $\Omega_L(u_0, u_+^l(\eta_1); \eta_1)$. Moreover, the forces of B on $\Omega_R(u_-^r(\eta_1), u_0; \eta_1)$ and $\Omega_L(u_0, u_+^l(\eta_1); \eta_1)$ are $\mathfrak{F}_R(\cdot; \eta_1)$ and $\mathfrak{F}_L(\cdot; \eta_1)$, respectively. Then there exist η_2 , $\eta_1 < \eta_2 < \eta_3$, and a continuous function $u: [\eta_1, \eta_2] \rightarrow \mathbb{R}$ such that $u(\eta_1) = u_0$ and for any $\varepsilon \in [\eta_1, \eta_2]$, we have $u(\varepsilon) \in [u_-^l(\varepsilon), u_+^l(\varepsilon)] \cap (u_-^r(\varepsilon), u_+^r(\varepsilon))$, and there exists a continuous function B_ε that coincides with the standard candidates on $\Omega_R(u_-^r(\varepsilon), u(\varepsilon); \varepsilon)$, $\Omega_{\text{ang}}(u(\varepsilon); \varepsilon)$, and $\Omega_L(u(\varepsilon), u_+^l(\varepsilon); \varepsilon)$. In addition, the forces of B_ε on $\Omega_R(u_-^r(\varepsilon), u(\varepsilon); \varepsilon)$ and $\Omega_L(u(\varepsilon), u_+^l(\varepsilon); \varepsilon)$ are $\mathfrak{F}_R(\cdot; \varepsilon)$ and $\mathfrak{F}_L(\cdot; \varepsilon)$, respectively.*

Proof. We first note that in order to construct a desired function B_ε , it suffices to find a root $u(\varepsilon)$ of the balance equation (4.2.3) for $\mathfrak{F}_R(\cdot; \varepsilon)$ and $\mathfrak{F}_L(\cdot; \varepsilon)$ (see Corollary 3.6.5).

If $u_+^r(\eta_1) > u_0 > u_-^l(\eta_1)$, then the proof is simple. By Lemma 4.2.18, the function $\mathfrak{F}_R(\cdot; \eta_1) - \mathfrak{F}_L(\cdot; \eta_1)$ strictly increases on $(u_-^l(\eta_1), u_+^l(\eta_1)) \cap (u_-^r(\eta_1), u_+^r(\eta_1))$, therefore it is positive on some right neighborhood of u_0 and negative on a left one. By continuity with respect to ε , the function $\mathfrak{F}_R(\cdot; \varepsilon) - \mathfrak{F}_L(\cdot; \varepsilon)$ has a root $u(\varepsilon)$ in a fixed neighborhood of u_0 for ε sufficiently close to η_1 , $\varepsilon > \eta_1$. Again, by Lemma 4.2.18, this root is unique in the intersection of the tails.

If $u_+^r(\eta_1) = u_0$ or $u_0 = u_-^l(\eta_1)$, then $\mathfrak{F}_R(u_0; \eta_1) = \mathfrak{F}_L(u_0; \eta_1) = 0$, because u_0 is the root of the balance equation (4.2.3) for $\mathfrak{F}_R(\cdot; \eta_1)$ and $\mathfrak{F}_L(\cdot; \eta_1)$. Therefore, $u_+^r(\eta_1) = u_-^l(\eta_1) = u_0$. It follows from Remark 4.2.14 that there exist $u_-, u_+ \in \mathbb{R}$ such that $u_- < u_0 < u_+$, $\mathfrak{K}' > 0$ on (u_-, u_0) , and $\mathfrak{K}' < 0$ on (u_0, u_+) . Thus, $u_0 = c_i$ for some i (see Definition 2.1.13). For any $\varepsilon \in (\eta_1, \eta_3)$, we have $u_-^l(\varepsilon) < u_-$, and $u_+ < u_+^r(\varepsilon)$. The only thing we need to check is that for any $\delta > 0$, the function $\mathfrak{F}_R(\cdot; \varepsilon) - \mathfrak{F}_L(\cdot; \varepsilon)$ has a root $u(\varepsilon) \in (u_0 - \delta, u_0 + \delta)$, provided ε sufficiently close to η_1 .

If this is not the case, then there exist a positive δ and a sequence $\varepsilon_n \rightarrow \eta_1$, with $\varepsilon_n > \eta_1$, such that the functions $\Phi_n(\cdot) = \mathfrak{F}_R(\cdot; \varepsilon_n) - \mathfrak{F}_L(\cdot; \varepsilon_n)$ have no roots on $(u_0 - \delta, u_0 + \delta)$. Without loss of generality, we may assume that Φ_n is negative on this intersection. The function Φ_n is strictly increasing and negative on $[u_0, u_0 + \delta]$,

and $\Phi_n(u_0) \rightarrow 0, n \rightarrow +\infty$. Thus, Φ_n converges to zero uniformly on $[u_0, u_0 + \delta]$. It follows that

$$\lim_{n \rightarrow \infty} \mathfrak{F}_R(t; \varepsilon_n) - \mathfrak{R}(t) = \lim_{n \rightarrow \infty} \mathfrak{F}_L(t; \varepsilon_n) - \mathfrak{R}(t) = \mathfrak{F}_L(t; \eta_1) - \mathfrak{R}(t), \quad t \in [u_0, u_0 + \delta],$$

where the function on the right-hand side is strictly decreasing on $(u_0, u_0 + \delta)$, whereas the functions on the left-hand side are strictly increasing, by Remark 3.6.15. This leads to the contradiction and proves the claim. \blacksquare

Proposition 4.3.13 (Induction step for a multibirdie). *Let $\eta_1, \eta_3 \in \mathbb{R}$, with $\eta_1 < \eta_3 < \varepsilon_{\max}$. Consider a multibirdie $\Omega_{\text{Mbird}}(\{\alpha_i\}_{i=1}^k; \eta_1)$. Suppose that \mathfrak{F}_L and \mathfrak{F}_R are left and right monotone force flows with corresponding functions u_{\pm}^L and u_{\pm}^R acting from $[\eta_1, \eta_3]$ to \mathbb{R} such that $\alpha_1^L \in (u_-^R(\eta_1), u_+^R(\eta_1))$ and $\alpha_k^r \in [u_-^L(\eta_1), u_+^L(\eta_1))$. We also suppose that for each $i, i = 1, 2, \dots, k-1$, there are chordal domains $\Omega_{\text{ch}}([\alpha_i^r, \alpha_{i+1}^L], *)$. Suppose that there exists a continuous function B that coincides with the standard candidates on the domains $\Omega_R(u_-^R(\eta_1), \alpha_1^L; \eta_1)$, $\Omega_{\text{Mbird}}(\{\alpha_i\}_{i=1}^k; \eta_1)$, $\Omega_L(\alpha_k^r, u_+^L(\eta_1); \eta_1)$, and every $\Omega_{\text{ch}}([\alpha_i^r, \alpha_{i+1}^L], *)$. In addition, the forces of B on $\Omega_R(u_-^R(\eta_1), \alpha_1^L; \eta_1)$ and $\Omega_L(\alpha_k^r, u_+^L(\eta_1); \eta_1)$ are $\mathfrak{F}_R(\cdot; \eta_1)$ and $\mathfrak{F}_L(\cdot; \eta_1)$, respectively. Then there exist a number η_2 , with $\eta_1 < \eta_2 < \eta_3$, and a collection of strictly monotone functions a_i and b_i acting from $[\eta_1, \eta_2]$ to \mathbb{R} such that the a_i are increasing and $a_i(\eta_1) = \alpha_i^r$, the b_i are decreasing and $b_i(\eta_1) = \alpha_{i+1}^L$, and $[g(a_i(\varepsilon)), g(b_i(\varepsilon))]$ is a chord of $\Omega_{\text{ch}}([\alpha_i^r, \alpha_{i+1}^L], *)$. Furthermore, for every $\varepsilon \in (\eta_1, \eta_2]$, there exist an integer $j = j(\varepsilon), 1 \leq j \leq k$, and $u(\varepsilon) \in (b_{j-1}(\varepsilon), a_j(\varepsilon))$ (here we put $b_0 \stackrel{\text{def}}{=} u_-^R(\varepsilon)$ and $a_k \stackrel{\text{def}}{=} u_+^L(\varepsilon)$) such that there exists a continuous function B_ε on the domain*

$$\begin{aligned} & \left(\bigcup_{i=0}^{j-2} \Omega_R(b_i(\varepsilon), a_{i+1}(\varepsilon); \varepsilon) \right) \cup \left(\bigcup_{i=1}^{j-1} \Omega_{\text{tr},R}(a_i(\varepsilon), b_i(\varepsilon); \varepsilon) \right) \cup \Omega_R(b_{j-1}(\varepsilon), u(\varepsilon); \varepsilon) \\ & \cup \Omega_{\text{ang}}(u(\varepsilon); \varepsilon) \cup \Omega_L(u(\varepsilon), a_j(\varepsilon); \varepsilon) \cup \left(\bigcup_{i=j}^{k-1} \Omega_{\text{tr},L}(a_i(\varepsilon), b_i(\varepsilon); \varepsilon) \right) \\ & \cup \left(\bigcup_{i=j}^{k-1} \Omega_L(b_i(\varepsilon), a_{i+1}(\varepsilon); \varepsilon) \right) \cup \left(\bigcup_{i=1}^{k-1} \Omega_{\text{ch}}([a_i(\varepsilon), b_i(\varepsilon)], *) \right) \end{aligned}$$

that coincides with the standard candidate inside each subdomain of the partition. Moreover, the force of B_ε coincides with $\mathfrak{F}_R(\cdot; \varepsilon)$ in the right neighborhood of $u_-^R(\varepsilon)$ and with $\mathfrak{F}_L(\cdot; \varepsilon)$ in the left neighborhood of $u_+^L(\varepsilon)$. The functions

$$\begin{aligned} \mathfrak{F}_{R,i}(\cdot; \varepsilon) &= F_R(\cdot; a_i(\varepsilon), b_i(\varepsilon); \varepsilon), \quad 1 \leq i \leq k-1, \\ \mathfrak{F}_{L,i}(\cdot; \varepsilon) &= F_L(\cdot; a_i(\varepsilon), b_i(\varepsilon); \varepsilon), \quad 1 \leq i \leq k-1, \end{aligned} \tag{4.3.6}$$

are the right and the left monotone force flows on the corresponding domains.

Proof. Consider two multitrolleybuses: $\Omega_{\text{Mtr,R}}(\{\alpha_i\}_{i=1}^k)$ and $\Omega_{\text{Mtr,L}}(\{\alpha_i\}_{i=1}^k)$. Application of Propositions 4.3.10 and 4.3.11 to these foliations gives us numbers η_2^R, η_2^L and collections of functions $\{a_j, b_j\}_{j=1}^{k-1}$ (all of them are defined on an interval $[\eta_1, \eta_2]$, where $\eta_2 = \min(\eta_2^R, \eta_2^L)$), which we call $\{a_j^R, b_j^R\}$ and $\{a_j^L, b_j^L\}$. Let $\varepsilon \in (\eta_1, \eta_2)$ be fixed.

We claim that if $b_j^R \leq b_j^L$ for some j , $1 < j < k$, then $b_{j-1}^R < b_{j-1}^L$. Indeed, we have the following chain of inequalities:

$$\begin{aligned} F_R(b_{j-1}^L; a_{j-1}^L, b_{j-1}^L; \varepsilon) &= F_L(b_{j-1}^L; a_j^L, b_j^L; \varepsilon) \\ &\geq F_L(b_{j-1}^L; a_j^R, b_j^R; \varepsilon) > F_R(b_{j-1}^L; a_{j-1}^R, b_{j-1}^R; \varepsilon). \end{aligned} \quad (4.3.7)$$

The first equality in (4.3.7) is simply the balance equation. The second inequality follows from Lemma 4.2.4 and our assumption that $b_j^R \leq b_j^L$. We note that the point b_{j-1}^L lies in the left tail of the chordal domain $\Omega_{\text{ch}}([a_j^R, b_j^R], *)$, because it lies in the left tail of the larger chordal domain $\Omega_{\text{ch}}([a_j^L, b_j^L], *)$ (see Corollary 4.2.5). Moreover, b_{j-1}^L lies on the left of the point a_j^R , which is the root of the balance equation of the forces $F_R(\cdot; a_{j-1}^R, b_{j-1}^R; \varepsilon)$ and $F_L(\cdot; a_j^R, b_j^R; \varepsilon)$, therefore the last inequality in (4.3.7) follows from Lemma 4.2.18. Inequality (4.3.7) and Lemma 4.2.4 imply that $b_{j-1}^R < b_{j-1}^L$. The claim is proved.

It follows that there exists $j = j(\varepsilon) \in \{1, 2, \dots, k\}$ such that for any i , with $0 < i < j$, the inequality $b_i^R < b_i^L$ holds true, and for any i , with $j \leq i < k$, one has $b_i^R \geq b_i^L$. We define $b_i(\varepsilon) = b_i^R(\varepsilon)$ for $0 < i < j$ and $b_i(\varepsilon) = b_i^L(\varepsilon)$ for $j \leq i < k$. We also put $a_i(\varepsilon)$, $1 \leq i \leq k-1$, in such a way that $[g(a_i(\varepsilon)), g(b_i(\varepsilon))]$ is a chord of $\Omega_{\text{ch}}([\alpha_i^r, \alpha_{i+1}^l], *)$. In what follows we use the notation (4.3.6) and also put $\mathfrak{F}_{R,0} = \mathfrak{F}_R$ and $\mathfrak{F}_{L,k} = \mathfrak{F}_L$. We only have to prove that for ε sufficiently close to η_1 , there is $u(\varepsilon) \in [b_{j(\varepsilon)-1}(\varepsilon), a_{j(\varepsilon)}(\varepsilon)]$ solving the balance equation for $\mathfrak{F}_{R,j(\varepsilon)-1}(\cdot; \varepsilon)$ and $\mathfrak{F}_{L,j(\varepsilon)}(\cdot; \varepsilon)$.

First, consider the case $1 < j(\varepsilon) < k$. Let $u_- = \max(b_{j(\varepsilon)-1}(\varepsilon), t^L)$, where t^L is the left end of the tail of $\mathfrak{F}_{L,j(\varepsilon)}(\cdot; \varepsilon)$. We claim that $\mathfrak{F}_{R,j(\varepsilon)-1}(u_-; \varepsilon) \leq \mathfrak{F}_{L,j(\varepsilon)}(u_-; \varepsilon)$. Indeed, if $u_- = t^L$, then the claim is obvious:

$$\mathfrak{F}_{L,j(\varepsilon)}(t^L; \varepsilon) = 0 \geq \mathfrak{F}_{R,j(\varepsilon)-1}(t^L; \varepsilon),$$

because t^L lies in the tail of $\mathfrak{F}_{R,j(\varepsilon)-1}(\cdot; \varepsilon)$.

If $u_- = b_{j(\varepsilon)-1}(\varepsilon)$, then $u_- < b_{j(\varepsilon)-1}^L(\varepsilon)$, therefore, due to Lemma 4.2.18, we have

$$\begin{aligned} &\mathfrak{F}_{R,j(\varepsilon)-1}(u_-; \varepsilon) - \mathfrak{F}_{L,j(\varepsilon)}(u_-; \varepsilon) \\ &\leq \mathfrak{F}_{R,j(\varepsilon)-1}(b_{j(\varepsilon)-1}^L(\varepsilon); \varepsilon) - \mathfrak{F}_{L,j(\varepsilon)}(b_{j(\varepsilon)-1}^L(\varepsilon); \varepsilon) \\ &< F_R(b_{j(\varepsilon)-1}^L; a_{j(\varepsilon)-1}^L, b_{j(\varepsilon)-1}^L; \varepsilon) - \mathfrak{F}_{L,j(\varepsilon)}(b_{j(\varepsilon)-1}^L(\varepsilon); \varepsilon) = 0, \end{aligned} \quad (4.3.8)$$

where the second inequality follows from Lemma 4.2.4 and the fact that $b_{j(\varepsilon)-1}(\varepsilon) < b_{j(\varepsilon)-1}^L(\varepsilon)$. Similarly, for $u_+ = \min(a_{j(\varepsilon)}(\varepsilon), t^R)$, where t^R is the right end of the

tail of $\mathfrak{F}_{R,j(\varepsilon)-1}(\cdot; \varepsilon)$, we have $\mathfrak{F}_{R,j(\varepsilon)-1}(u_+; \varepsilon) \geq \mathfrak{F}_{L,j(\varepsilon)}(u_+; \varepsilon)$. Therefore, there exists a root $u(\varepsilon) \in [u_-, u_+]$ of the balance equation for the forces $\mathfrak{F}_{R,j(\varepsilon)-1}(\cdot; \varepsilon)$ and $\mathfrak{F}_{L,j(\varepsilon)}(\cdot; \varepsilon)$, which is unique due to Lemma 4.2.18.

Inequality (4.3.8) implies that $u(\varepsilon) > b_{j(\varepsilon)-1}(\varepsilon)$, therefore we have that $u(\varepsilon)$ lies in $(b_{j(\varepsilon)-1}(\varepsilon), a_{j(\varepsilon)}(\varepsilon))$.

Now we consider the case $j(\varepsilon) = 1$ (the case $j(\varepsilon) = k$ is symmetric). Let $u_+ = \min(a_1(\varepsilon), t^R(\varepsilon))$, where t^R is the right end of the tail of $\mathfrak{F}_{R,0}(\cdot; \varepsilon)$. We claim that $\mathfrak{F}_{R,0}(u_+; \varepsilon) \geq \mathfrak{F}_{L,1}(u_+; \varepsilon)$. Indeed, if $u_+ = t^R$, then the argument is the same as before: $\mathfrak{F}_{R,0}(t^R; \varepsilon) = 0 \geq \mathfrak{F}_{L,1}(t^R; \varepsilon)$. If $u_+ = a_1(\varepsilon)$, then $u_+ \geq a_1^R(\varepsilon)$, and from Lemma 4.2.18 and Lemma 4.2.4, we obtain

$$\begin{aligned} \mathfrak{F}_{R,0}(u_+; \varepsilon) - \mathfrak{F}_{L,1}(u_+; \varepsilon) &\geq \mathfrak{F}_{R,0}(a_1^R; \varepsilon) - \mathfrak{F}_{L,1}(a_1^R; \varepsilon) \\ &= \mathfrak{F}_{R,0}(a_1^R; \varepsilon) - F_L(a_1^R; a_1^L, b_1^L; \varepsilon) \\ &\geq \mathfrak{F}_{R,0}(a_1^R; \varepsilon) - F_L(a_1^R; a_1^R, b_1^R; \varepsilon) = 0. \end{aligned}$$

The claim is proved.

Recall that we want to prove that, for ε sufficiently close to η_1 , there exists $u(\varepsilon)$ in $[b_0(\varepsilon), a_1(\varepsilon)]$ solving the balance equation for $\mathfrak{F}_{R,0}(\cdot; \varepsilon)$ and $\mathfrak{F}_{L,1}(\cdot; \varepsilon)$. Assume the contrary: for some sequence $\varepsilon_n \rightarrow \eta_1+$, the function $\Phi_n(\cdot) = \mathfrak{F}_{R,0}(\cdot; \varepsilon_n) - \mathfrak{F}_{L,1}(\cdot; \varepsilon_n)$ has no balance points on $(b_0(\varepsilon_n), a_1(\varepsilon_n))$. Then $\Phi_n(u_+(\varepsilon_n)) > 0$ and therefore Φ_n is positive on the intersection of the tails, hence t^L , the left end of the tail of $\mathfrak{F}_{L,1}(\cdot; \varepsilon_n)$, is not greater than $b_0(\varepsilon_n)$. The function Φ_n is strictly increasing, by Lemma 4.2.18, and $\Phi_n(a_1(\eta_1))$ tends to 0 when $n \rightarrow +\infty$. Therefore, Φ_n tends to zero uniformly on $(b_0(\eta_1), a_1(\eta_1))$ (we recall that $b_0(\eta_1) \geq b_0(\varepsilon)$ for any $\varepsilon > \eta_1$, by the definition of a monotone force flow). Thus, for $t \in (b_0(\eta_1), a_1(\eta_1))$, we have

$$\lim_{n \rightarrow +\infty} \mathfrak{F}_{L,1}(t; \varepsilon_n) - \mathfrak{R}(t) = \lim_{n \rightarrow +\infty} \mathfrak{F}_{R,0}(t; \varepsilon_n) - \mathfrak{R}(t) = \mathfrak{F}_R(t; \eta_1) - \mathfrak{R}(t).$$

The function on the right-hand side is strictly decreasing on $(b_0(\eta_1), a_1(\eta_1))$ and the functions on the left-hand side are strictly increasing on $(b_0(\eta_1), a_1(\eta_1))$ according to Remark 3.6.15. This leads to contradiction and proves the statement. ■

Remark 4.3.14. We note that $j(\varepsilon)$ in Proposition 4.3.13, indeed, could depend on ε , i.e., during the evolution, the angle could change its place between the trolleybuses. Moreover, the function $j(\cdot)$ could have an infinite number of jumps even on a bounded interval, see the example ‘‘Oscillating birdie’’ in [17, p. 123].

Remark 4.3.15. We have seen in the proof that the root $u(\varepsilon)$ of the balance equation is in the semiclosed interval $(b_{j(\varepsilon)-1}(\varepsilon), a_{j(\varepsilon)}(\varepsilon))$. It may occur that $u(\varepsilon) = a_{j(\varepsilon)}(\varepsilon)$, and in this case, the angle $\Omega_{\text{ang}}(u(\varepsilon); \varepsilon)$, the tangent domain $\Omega_L(u(\varepsilon), u(\varepsilon); \varepsilon)$, and the trolleybus $\Omega_{\text{tr},L}(u(\varepsilon), b_{j(\varepsilon)}(\varepsilon); \varepsilon)$ glue together forming a birdie, see formula (3.8.2). Moreover, this equation could be valid for ε in some interval; the birdie can shrink without disintegrating.

4.4 Global evolution

Before passing to formal statements, we describe the rules of the evolution.

Recall that in Section 3.8.2 we constructed the graph Γ corresponding to the foliation of a Bellman candidate, and its subgraph Γ^{free} corresponding to subdomains of the foliation that are not separated from the free boundary $\partial_{\text{free}}\Omega$. The vertices of the graph Γ are of two types: the vertices corresponding to linearity domains and fictitious vertices. The edges of Γ always correspond to fences: either chordal or tangent domains. The vertices of Γ^{free} correspond to linearity domains: multicups, angles, trolleybuses, multitrolleybuses, birdies, multibirdies, fictitious vertices of the first, third (corresponding to long chords), fourth, and fifth type.

To each edge \mathfrak{E} of Γ^{free} we assign a force by the formal rule described in Table 4.1. In the first column we have the type of the vertex from where the edge starts. The numerical parameters of this vertex are placed in the second column. The force that is assigned to the tangent domain lying on the left of the figure is in the third column, and the force that is assigned to the tangent domain lying on the right of the figure is in the last one.

vertex type	parameters	left force	right force
right trolleybus	$\{a, b\}$		$F_R(\cdot; a, b; \varepsilon)$
left trolleybus	$\{a, b\}$	$F_L(\cdot; a, b; \varepsilon)$	
multicup	$\{\alpha_i\}_{i=1}^k$	$F_L(\cdot; \alpha_1^l, \alpha_k^r; \varepsilon)$	$F_R(\cdot; \alpha_1^l, \alpha_k^r; \varepsilon)$
right multitrolleybus	$\{\alpha_i\}_{i=1}^k$		$F_R(\cdot; \alpha_1^l, \alpha_k^r; \varepsilon)$
left multitrolleybus	$\{\alpha_i\}_{i=1}^k$	$F_L(\cdot; \alpha_1^l, \alpha_k^r; \varepsilon)$	
fict. vert. type 1	$\{a, b\}$	$F_L(\cdot; a, b; \varepsilon)$	$F_R(\cdot; a, b; \varepsilon)$
fict. vert. type 3	$\{a, b\}$	$F_L(\cdot; a, b; \varepsilon)$	$F_R(\cdot; a, b; \varepsilon)$
fict. vert. type 4	$-\infty$		$F_R(\cdot; -\infty; \varepsilon)$
fict. vert. type 4	$+\infty$	$F_L(\cdot; +\infty; \varepsilon)$	
right fict. vert. type 5	c_i		$F_R(\cdot; c_i, c_i; \varepsilon)$
left fict. vert. type 5	c_i	$F_L(\cdot; c_i, c_i; \varepsilon)$	

Table 4.1. The vertices of Γ^{free} with the corresponding parameters and forces.

All the foliations generated during the evolution process satisfy the following rule: if $\Omega_R(u_1, u_2)$ or $\Omega_L(u_1, u_2)$ is represented by the edge \mathfrak{E} in Γ^{free} , then (u_1, u_2) belongs to the tail of the force corresponding to \mathfrak{E} . This requirement for the foliation will be called the *non-degeneracy force condition*.

Condition 4.4.1. *For each edge \mathfrak{E} in Γ^{free} that corresponds to a tangent domain $\Omega_R(u_1, u_2)$ or $\Omega_L(u_1, u_2)$, the interval (u_1, u_2) belongs to the tail of the force assigned to \mathfrak{E} .*

A short inspection of definitions shows that Condition 4.4.1 holds true for all the graphs corresponding to the standard candidates constructed. In other words, all the forces in tangent domains are strictly negative. In particular, the following remark is important.

Remark 4.4.2. The Bellman candidate constructed for a simple picture in Section 4.1 fulfills the non-degeneracy force Condition 4.4.1.

As has already been said, the main rule of the evolution is that the forces decrease (grow in absolute value), see Section 4.2.2. As a consequence, the tails strictly grow (by this we mean that the t^R increase and the t^L decrease). Thus, full chordal domains grow (Proposition 4.3.1), the multicups are stable (Proposition 4.3.2),¹ the trolley-buses shrink (Propositions 4.3.4 and 4.3.5), the angles continuously wander from side to side (Proposition 4.3.12). These figures can be described as stable. If there are multitrolleybuses or multibirdies in the foliation for a fixed ε , they immediately disintegrate (Propositions 4.3.10, 4.3.11, and 4.3.13). These figures are unstable. As for the birdie, it can shrink (see Remark 4.3.15), but in general it disintegrates. Thus, it is half-stable.

There is also one useful condition that all our graphs will satisfy. It is of structural character (and thus relies on Definition 2.1.13) and concerns mostly fictitious vertices. It is called the *leaf-root condition*.

Condition 4.4.3. *Any arc of any multigure that is not a single point coincides with one of the solid roots c_i . Numerical parameters of the fictitious vertices of the second type are some roots c_i that are single points. Each fictitious vertex of the third type corresponds to a chord $[g(a_0), g(b_0)]$ with at least one vanishing differential. If $D_R(a_0, b_0) = 0$, then $b_0 = c_i$ for some single point root c_i ; if $D_L(a_0, b_0) = 0$, then $a_0 = c_j$, where c_j must be a single point root as well. The numeric parameter of each vertex of the fifth type is also a single point root c_i .*

Remark 4.4.4. All simple graphs constructed in Section 4.1 fulfill the leaf-root Condition 4.4.3.

Definition 4.4.5. Let $\varepsilon < \varepsilon_{\max}$. We say that a graph Γ is *admissible* for f and ε if all figures corresponding to the vertices and edges of Γ satisfy their local propositions.

By “all figures corresponding to the vertices and edges of Γ satisfy their local propositions” we mean the following: for each vertex or edge in Γ , the parameters satisfy the assumptions of the proposition indicated for this vertex or edge in Table 4.2 (in the third column).

¹In a sense, they also grow: the border tangents rise; however, the numerical parameters do not change.

vertex or edge type	formulas	verification	evolution rule
right tangent domain	(3.2.11), (3.2.2)	3.3.3, 3.3.6	
left tangent domain	(3.2.11), (3.2.2)	3.3.5, 3.3.7	
chordal domain	(3.4.1)	3.4.2	4.3.1
angle	(3.6.3)	3.6.4	4.3.12
right trolleybus	(3.7.1)	3.7.2	4.3.4
left trolleybus	(3.7.1)	3.7.2	4.3.5
birdie	(3.7.1)	3.7.2	4.3.13
multicup	(3.7.6), (3.7.1)	3.7.13	4.3.2
full multicup	(3.7.6), (3.7.1)	3.7.13	4.3.3
right multitrolleybus	(3.7.6), (3.7.1)	3.7.13	4.3.10
left multitrolleybus	(3.7.6), (3.7.1)	3.7.13	4.3.11
multibirdie	(3.7.6), (3.7.1)	3.7.13	4.3.13
closed multicup	(3.7.6), (3.7.1)	3.7.13	stable
fict. vert. type 1	(3.4.1)	3.6.4	4.3.1
fict. vert. type 3, long chord	(3.4.1)	3.6.4	4.3.3
fict. vert. type 5	(3.7.4)	3.6.4	4.3.7

Table 4.2. Elementary figures, corresponding Bellman candidates, and evolution rules.

In the first column, there is the type of the vertex or edge, in the second there is a reference to formulas that are used to construct the canonical function B in the corresponding figure, and in the third column the number of the proposition that guarantees that this B is a Bellman candidate is given. Finally, the last column contains the number of the proposition that describes the local evolution of the parameters for the figure. We have omitted fictitious vertices of the second and fourth types (as well as the vertices of the third type that correspond to short chords), because the value of the function B in the domains corresponding to them is defined trivially, and these figures are stable and have no evolution scenarios.

Now we describe how to construct the function B from a graph. First, one constructs this function to be the standard candidate on all the domains corresponding to vertices and edges that participate in $\Gamma \setminus \Gamma^{\text{free}}$, because for their figures there is no additional information needed to construct B (no information from other figures). Second, we construct the function B to be a standard candidate on all the domains corresponding to vertices of Γ^{free} not being leaves (i.e., except angles). Third, we construct the standard candidates for the edges of Γ^{free} . For each such edge \mathcal{C} , the values of B in the figure corresponding to its beginning define the force function on the domain corresponding to the edge (see Table 4.1), thus one may construct B in the tangent domain corresponding to \mathcal{C} if he knows the values of B on the domain of

its source.² Finally, we construct B on the domains corresponding to leaves of Γ^{free} (i.e., on angles), because we know the values of B on the linear boundary of each such angle. Note that if Γ fulfills Condition 4.4.1, then the restriction of B to each figure is a standard Bellman candidate there. Admissibility of the graph guarantees that the force function (defined locally on each element of the foliation by the rules from Table 4.1) is a non-positive continuous function on \mathbb{R} .

Remark 4.4.6. The function B constructed from an admissible graph is a Bellman candidate.

Proof. We need to verify conditions of Definition 2.2.4 for the function B . Looking at Table 4.2, we use the corresponding verification proposition for each vertex or edge and see that the function B has the foliation on the entire Ω_ε . We also note that the function B is locally concave and C^1 -smooth not only on subdomains, but globally. ■

Since during the evolution some figures grow and angles move, several figures might collide. For example, the vertex of an angle may coincide with the right endpoint of a long chord. In such a case, we look at formula (3.8.1), and see that now they form a trolleybus. Therefore, the graph of the foliation changes at this moment ε . We call such moments *the critical points of the evolution*. The idea is that if a collision occurs, then the figures involved form a new one (according to formulas from Section 3.8.1), and we can proceed the evolution. Unfortunately, there might be infinitely many critical points (see the example in [17, p. 123], where an angle flips the direction of a trolleybus infinitely many times). However, if one focuses only on those critical points, at which the structure of the graph Γ^{fixed} changes essentially, one finds only a finite number of critical points. Such points are called *essentially critical*. The following definition is also useful.

Definition 4.4.7. We say that a graph Γ is *smooth* if there are no vertices representing full multicups, multitrolleybuses, multibirdies, fictitious vertices of the third type that represent long chords, and fictitious vertices of the fifth type in Γ .

Theorem 4.4.8. *For any $\varepsilon < \varepsilon_{\max}$, there exists a graph $\Gamma(\varepsilon)$ admissible for f and ε .*

We will not give a careful proof of the theorem because it repeats literally the proof of the same theorem for the BMO case (see [17, Theorem 4.4.15]). Here we will describe the main steps of the proof.

First, we use Theorem 4.1.2 to build a smooth admissible graph $\Gamma(\varepsilon)$ for small ε . Then we use local evolution theorems from Section 4.3 collected in Table 4.2 to show

²There is one exception: for tangent domains coming from infinity, one does not need any boundary data.

that if there exists an admissible graph $\Gamma(\eta_1)$ for some η_1 , then we can construct a smooth admissible graph $\Gamma(\varepsilon)$ for $\varepsilon, \varepsilon > \eta_1$, sufficiently close to η_1 . If for some η_2 we have smooth graphs $\Gamma(\varepsilon), \varepsilon \in (\eta_1, \eta_2)$, we can pass to the limit and construct a limit graph $\Gamma(\eta_2)$. It can happen that, in the limit graph $\Gamma(\eta_2)$, some edges “have zero length”. In this case we modify the graph using formulas from Section 3.8.1. This modified graph can be non-smooth but it is admissible and we can continue evolution starting from it. It appears that under our assumptions there is only a finite number of essentially critical points of the evolution, when the graph $\Gamma(\varepsilon)$ is not smooth. In such a way, we obtain the graph $\Gamma(\varepsilon)$ for any $\varepsilon < \varepsilon_{\max}$.

Chapter 5

Optimizers

In the previous chapter, we constructed a Bellman candidate B of a special form (Theorem 4.4.8 and Remark 4.4.6). We claim that it coincides with the Bellman function B_ε . Section 2.2.2 suggests a method to prove the claim. We have to construct an optimizer φ_x for each $x \in \Omega_\varepsilon$ (see Definition 2.2.6). Here we will follow the same strategy as when we were constructing Bellman candidates: we will first study the local behavior of the optimizers (i.e., how do optimizers vary when x runs through one figure), this is done in Section 5.2, and then “glue” these local scenarios together in Section 5.3. The optimizers for the BMO case were built in [17], and here we will follow a similar strategy for the general case. In Sections 5.1 and 5.2 we will not use evolution, so ε is fixed and we omit the subscript ε till Section 5.3. In particular, we will write Ξ instead of Ξ_ε and $\Omega = \Xi_0 \setminus \Xi$ instead of Ω_ε , see Section 2.1 for the definitions of these objects. In Section 5.4 we will consider the cases when conditions (2.1.7) and (2.1.8) are violated.

5.1 Abstract theory

We begin with an abstract description of what optimizers look like. First, as mentioned in Section 2.2.2, it is natural to construct monotone optimizers. It is not difficult to build a monotone function φ_x such that $\langle \varphi_x \rangle = x$ and $B(x) = \langle f(\varphi_x) \rangle$. The main difficulty is to verify that $\varphi_x \in \mathcal{A}_\Omega$. It was noticed in [16] that it is more natural to argue geometrically. The notion of a *delivery curve* is useful in this context.

Definition 5.1.1. Let B be a Bellman candidate on the domain Ω . Suppose $\varphi: [l, r] \rightarrow \partial_{\text{fixed}} \Omega$ is an integrable function. The curve $\gamma: (l, r] \rightarrow \Omega$, given by the formula

$$\gamma(\tau) \stackrel{\text{def}}{=} \langle \varphi \rangle_{[l, \tau]}, \quad \tau \in (l, r], \quad (5.1.1)$$

is called a *delivery curve* if $B(\gamma(\tau)) = \langle f(\varphi) \rangle_{[l, \tau]}$ for any $\tau \in (l, r]$ (in particular, $\gamma(\tau) \in \Omega$). The function φ is called the *generating function* for γ .

In other words, γ is a curve that “delivers” optimizers to the point. The word “curve” here means a parametrized curve, because the definition depends on the parametrization. The advantage of considering such a curve is that it allows to verify the condition that φ is a test function (i.e., $\varphi \in \mathcal{A}_\Omega$).

The main feature we will use is the formula

$$\gamma(\tau) + (\tau - l)\gamma'(\tau) = \varphi(\tau), \quad (5.1.2)$$

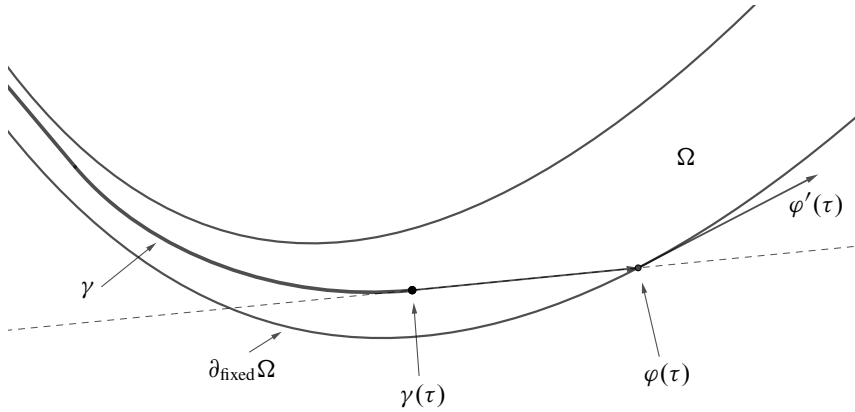


Figure 5.1. Illustration to the proof of Lemma 5.1.2.

which can be obtained by differentiation of (5.1.1). In particular, this formula shows that the tangent to γ at τ points in the direction of $\varphi(\tau)$. Thus, one can reconstruct the values of φ by looking at the points on the fixed boundary that “are indicated” by the tangents of the corresponding delivery curve. We will use this principle very often. Moreover, equation (5.1.2) allows us to reconstruct φ from γ .

Lemma 5.1.2. *A curve given by formula (5.1.1) is convex if its generating function is monotone.*

Proof. We will give a proof for the case of an increasing generating function φ . The case of decreasing function is symmetric.

Let us assume for a while that the function φ is C^2 -smooth. In such a case, we may differentiate (5.1.2) and get

$$(\tau - l)\gamma''(\tau) = -2\gamma'(\tau) + \varphi'(\tau).$$

Thus, the curvature of γ , which is $(\gamma'_1)^{-3} \det\begin{pmatrix} \gamma' \\ \gamma'' \end{pmatrix}$, has the same sign as $\det\begin{pmatrix} \gamma' \\ \varphi' \end{pmatrix}$, because $\gamma'_1 > 0$. We use (5.1.2) once again to express γ' and rewrite the determinant in the following form:

$$\det\begin{pmatrix} \gamma' \\ \varphi' \end{pmatrix} = \frac{1}{\tau - l} \det\begin{pmatrix} \varphi - \gamma \\ \varphi' \end{pmatrix}.$$

This expression is positive, because φ' is a tangent vector to $\partial_{\text{fixed}} \Omega$ at $\varphi(\tau)$, with $\varphi'_1 > 0$, and $\gamma(\tau)$ belongs to Ω , see Figure 5.1.

We only have to get rid of the smoothness assumption. One can approximate φ by smooth increasing functions $\varphi_n: [l, r] \rightarrow \partial_{\text{fixed}} \Omega$ in such a way that the curves γ_n

generated by φ_n converge to γ pointwise. Each γ_n is a convex curve (in the sense that these curves are the graphs of convex functions in the standard coordinates) and $\gamma_n \rightarrow \gamma$ pointwise. Therefore, γ is a convex curve itself. ■

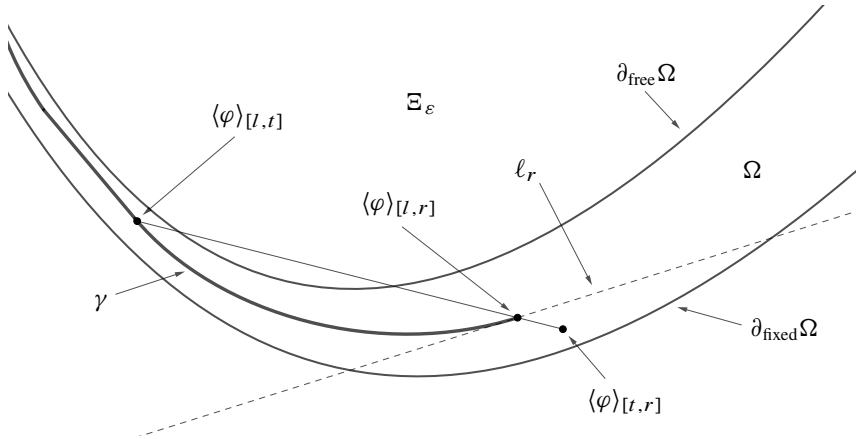


Figure 5.2. Illustration to the proof of Lemma 5.1.3.

The following lemma links the condition that γ is convex with the condition that $\varphi \in \mathcal{A}_\Omega$ (see Figure 5.2 for a visualization of the proof). The symbol $\gamma'(r)$ in the lemma below means the left derivative, which always exists due to convexity.

Lemma 5.1.3. *Suppose that γ is a curve parametrized by the interval $(l, r]$ and given by (5.1.1). Let it be convex in the sense that it is the graph of a convex function in the standard coordinates. Suppose also that the tangent line*

$$l_r = \{\gamma(r) + s\gamma'(r) \mid s \in \mathbb{R}\}$$

does not cross the domain Ξ (see Definition 2.1.1). Then, for any $t \in [l, r)$, we have $\langle \varphi \rangle_{[t,r]} \in \Omega$.

Proof. Since the curve γ is convex, it lies above the line l_r . The domain Ξ also lies above the line l_r . We note that $\langle \varphi \rangle_{[l,r]}$ is a convex combination of $\langle \varphi \rangle_{[l,t]}$ and $\langle \varphi \rangle_{[t,r]}$. Thus, $\langle \varphi \rangle_{[t,r]}$ is separated from Ξ by l_r . On the other hand, $\langle \varphi \rangle_{[t,r]}$ surely belongs to Ξ_0 , so it lies inside Ω . ■

In the following corollary, we write $\gamma'(\tau)$, $\tau \in (l, r]$, meaning any of the one-sided derivatives.

Corollary 5.1.4. *Suppose that γ is a delivery curve on $(l, r]$. Let it be convex in the sense that it is the graph of a convex function in the standard coordinates. Suppose*

also that the tangent line

$$\ell_\tau = \{\gamma(\tau) + s\gamma'(\tau) \mid s \in \mathbb{R}\}$$

does not cross the domain Ξ for any $\tau \in (l, r]$. Then the function φ that generates γ belongs to A_Ω .

Before we pass to constructing specific delivery curves, we should postulate a heuristic principle that will help us to guess them. Since a delivery curve “consists of optimizers”, it has to avoid the directions in which the Bellman candidate is non-linear. Thus, we guess that delivery curves should go either along the extremal segments or along the free boundary.

In Section 5.2 we will construct delivery curves for each elementary figure of the foliation. In the general case, a figure will have special points on the free boundary; we call them *incoming* and *outgoing nodes*. The idea is as follows: if we have special delivery curves for the incoming nodes, then we can construct a delivery curve for any point in the domain. Moreover, if the domain has outgoing nodes, then we construct special delivery curves for them. Every outgoing node for some domain is at the same time an incoming node for its neighbor domain. Continuing delivery curves along these special nodes allows us to construct optimizers for all the points in Ω .

5.2 Local behavior of optimizers

5.2.1 Optimizers for tangent domains

Consider a tangent domain $\Omega_R(u_1, u_2)$ foliated by the segments $S(u) = [g(u), w(u)]$, $u \in [u_1, u_2]$, that are tangent to the free boundary of Ω . Let B be a standard candidate on $\Omega_R(u_1, u_2)$. This domain has two linear parts of the boundary, namely, $S(u_1)$ and $S(u_2)$. The tangency points $w(u_1)$ and $w(u_2)$ on the free boundary are the *incoming* and *outgoing nodes* of $\Omega_R(u_1, u_2)$, respectively. Recall the positive-valued function λ ,

$$\lambda(u)(g(u) - w(u)) = w'(u), \quad u \in [u_1, u_2], \quad (5.2.1)$$

introduced in (2.1.6).

Suppose ψ is an optimizer for the incoming node $w(u_1)$ (see Figure 5.3 below) defined on the interval $(l, l_1]$. Our aim is to build the optimizers for all the points inside $\Omega_R(u_1, u_2)$. We start with the points $w(u)$, $u \in [u_1, u_2]$, lying on the free boundary. We look for a function φ on $(l, r]$ for some $r > l_1$ such that $\varphi = \psi$ on $(l, l_1]$ and its delivery curve γ goes along the free boundary from $w(u_1)$ to $w(u_2)$ on $[l_1, r]$. We will find a monotone function $u: [l_1, r] \rightarrow [u_1, u_2]$ such that the function

$$\varphi(t) = g(u(t)) \quad (5.2.2)$$

generates the required delivery curve $\gamma(t) = \langle \varphi \rangle_{[l,t]}$ that will coincide with $w(u(t))$, $t \in [l_1, r]$. This is equivalent to the equation

$$(t - l)w(u(t)) = \int_l^t \varphi(\tau) d\tau.$$

We differentiate this identity with respect to t and obtain

$$w(u(t)) + (t - l)w'(u(t))u'(t) = \varphi(t) = g(u(t)).$$

Using (5.2.1), we obtain $(t - l)\lambda(u(t))u'(t) = 1$. We solve this differential equation with the boundary condition $u(l_1) = u_1$ and get

$$\log \frac{t - l}{l_1 - l} = \int_{u_1}^{u(t)} \lambda(v) dv. \tag{5.2.3}$$

Since the function λ is positive and C^1 -smooth (see Section 2.1.4), equality (5.2.3) defines the required C^2 -smooth increasing function u on the interval $[l_1, r]$ (and the function φ defined by (5.2.2)), where

$$r = l + (l_1 - l) \exp\left(\int_{u_1}^{u_2} \lambda(v) dv\right).$$

We want to use Lemma 5.1.3 to verify that the function φ belongs to A_Ω . By construction, the curve $\gamma(t)$, $t \in (l_1, r]$, coincides with a part of the free boundary $\partial_{\text{free}} \Omega$. Therefore, this part of the curve is convex and its tangents do not cross the domain Ξ . In order to use Lemma 5.1.3, we need the convexity of the curve γ on the whole interval (l, r) . This consideration leads to the proposition below.

Proposition 5.2.1. *Let B be a candidate on $\Omega_R(u_1, u_2)$. Suppose that there exists a non-decreasing optimizer ψ for B at the point $w(u_1)$. Let also $\psi \leq g(u_1)$. Then there exists a non-decreasing optimizer φ_x for B at every point $x \in \Omega_R(u_1, u_2)$; moreover, $\varphi_x \leq g(u_2)$.*

Recall that the ordering \leq on $\partial_{\text{fixed}} \Omega$ was introduced in Remark 2.1.2.

Proof. We have constructed the desired function φ_x for the points $x \in \Omega_R(u_1, u_2) \cap \partial_{\text{free}} \Omega$. For any point $x \in \Omega_R(u_1, u_2)$, there exists a unique $u \in [u_1, u_2]$ such that $x \in S(u)$. We already know the optimizers at the endpoints $w(u)$ and $g(u)$ of this extremal segment. Namely, they are $\varphi_{w(u)}$ on an interval $(l, r]$ and the constant function $\varphi_{g(u)} \equiv g(u)$. We will obtain the desired optimizer φ_x extending the function $\varphi_{w(u)}$ by the constant $g(u)$ to some interval $(r, r_1]$ of appropriate length.

To prove the proposition, we need to show two things. First, we need to verify that the function φ_x lies in A_Ω . Second, we need to prove the equality $B(x) = \langle f(\varphi_x) \rangle$ for the candidate B .

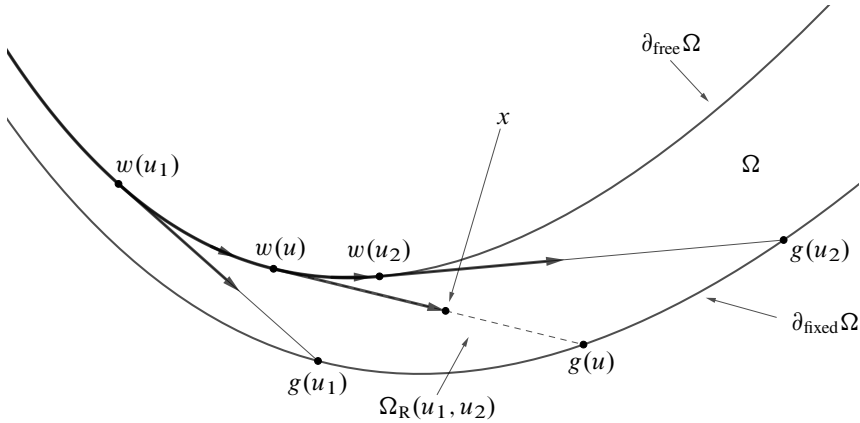


Figure 5.3. Delivery curves inside a tangent domain.

We start with the verification of convexity of the curve $\gamma = \gamma_x$ generated by $\varphi = \varphi_x$. By Lemma 5.1.2, the curve generated by ψ , i.e., $\gamma|_{(l, l_1]}$, is convex and its tangent line at the point $w(u_1)$ coincides with the tangent line to the free boundary (this is a consequence of the condition $\psi \leq g(u_1)$ and (5.1.2)). Then the curve γ goes along the free boundary, which is convex, and then it goes along the tangent line $S(u)$. Therefore, this curve γ is convex. Let $l < t < \tau \leq r_1$. If $\tau \leq l_1$, then the point $\langle \varphi \rangle_{[t, \tau]}$ lies on the curve generated by ψ , and thus it lies in Ω by the assumption. If $\tau > l_1$, then the tangent line to γ at $\gamma(\tau)$ is the tangent line to the free boundary by construction. Therefore, this tangent line do not cross the domain Ξ , and we use Lemma 5.1.3 to conclude that $\langle \varphi \rangle_{[t, \tau]}$ lies in Ω . This proves that $\varphi \in A_\Omega$.

It suffices to show that $B(\gamma(s)) = \langle f(\varphi) \rangle_{[l, s]}$ for $s \in [l_1, r_1]$, because for $s \in (l, l_1]$, this follows from the fact that ψ is an optimizer. Thus, we need to check the following identity:

$$(s - l)B(\gamma(s)) - (l_1 - l)B(\gamma(l_1)) = \int_{l_1}^s f(\varphi(\tau)) d\tau. \tag{5.2.4}$$

After differentiating with respect to s , we obtain the following equivalent identity (because (5.2.4) holds for $s = l_1$):

$$B(\gamma(s)) + (s - l)\langle \nabla B(\gamma(s)), \gamma'(s) \rangle = f(\varphi(s)) = B(\varphi(s)).$$

Since B is a Bellman candidate (see Definition 2.2.4), the gradient ∇B is constant on the extremal segment containing $\gamma(s)$ and $\varphi(s)$, therefore

$$B(\varphi(s)) - B(\gamma(s)) = \langle \nabla B, (\varphi(s) - \gamma(s)) \rangle \stackrel{(5.1.2)}{=} (s - l)\langle \nabla B, \gamma'(s) \rangle. \quad \blacksquare$$

We briefly state a symmetric proposition.

Proposition 5.2.2. *Let B be a candidate on $\Omega_L(u_1, u_2)$. Suppose that there exists a non-increasing optimizer ψ for B at the point $w(u_2)$. Let also $g(u_2) \preceq \psi$. Then there exists a non-increasing optimizer φ_x for B at every point $x \in \Omega_L(u_1, u_2)$; moreover, $g(u_1) \preceq \varphi_x$.*

The proof of similar propositions for infinite domains is slightly more complicated and requires the additional assumptions (2.1.7) and (2.1.8). We will use these assumptions together with the following technical lemma.

Lemma 5.2.3. *Let $\sigma \in \mathbb{R}$. Let ϑ and Y be two piecewise monotone continuous functions on $[\sigma, +\infty)$ with a finite number of intervals of monotonicity. Suppose that $\lim_{+\infty} \vartheta = 0$. Then the integration by parts formula is valid:*

$$\begin{aligned} \int_{\sigma}^{+\infty} \vartheta dY &= \lim_{\nu \rightarrow +\infty} \int_{\sigma}^{\nu} \vartheta dY = -Y(\sigma)\vartheta(\sigma) - \lim_{\nu \rightarrow +\infty} \int_{\sigma}^{\nu} Y d\vartheta \\ &= -Y(\sigma)\vartheta(\sigma) - \int_{\sigma}^{+\infty} Y d\vartheta, \end{aligned} \quad (5.2.5)$$

where both limits exist (finite or infinite).

Symmetrically, for piecewise monotone continuous functions ϑ and Y on $(-\infty, \sigma]$ with a finite number of intervals of monotonicity, if $\lim_{-\infty} \vartheta = 0$, then

$$\begin{aligned} \int_{-\infty}^{\sigma} \vartheta dY &= \lim_{\nu \rightarrow -\infty} \int_{\nu}^{\sigma} \vartheta dY = Y(\sigma)\vartheta(\sigma) - \lim_{\nu \rightarrow -\infty} \int_{\nu}^{\sigma} Y d\vartheta \\ &= Y(\sigma)\vartheta(\sigma) - \int_{-\infty}^{\sigma} Y d\vartheta, \end{aligned} \quad (5.2.6)$$

where both limits exist (finite or infinite).

Proof. We give the proof of (5.2.5). The proof of (5.2.6) is similar.

Since the integration by parts formula is valid on any finite interval, without loss of generality, we may assume that both Y and ϑ are monotone and do not change the sign on $[\sigma, +\infty)$. Then both integrals in (5.2.5) are monotone with respect to ν , therefore both limits exist, finite or infinite.

First, assume that $\int_{\sigma}^{+\infty} \vartheta dY$ converges. Take any $\nu > \sigma$ and write the integration by parts formula on the interval $[\sigma, \nu]$:

$$\int_{\sigma}^{\nu} \vartheta dY + Y(\sigma)\vartheta(\sigma) = Y(\nu)\vartheta(\nu) - \int_{\sigma}^{\nu} Y d\vartheta. \quad (5.2.7)$$

The left-hand side of (5.2.7) has a finite limit when $\nu \rightarrow +\infty$. The first summand on the right-hand side is of the same sign as the second one (because $\text{sign}(\vartheta') = -\text{sign}(\vartheta)$), therefore both summands have finite limits. If the limit of the first summand was nonzero, the limit of the second one would be infinite. Indeed,

$$|Y(\tau)| > \frac{\text{Const}}{|\vartheta(\tau)|}, \quad \tau \rightarrow +\infty, \quad (5.2.8)$$

therefore

$$\left| \int^{\sigma+\infty} Y(\tau) d\vartheta(\tau) \right| > \text{Const} \cdot \left| \int^{\sigma+\infty} \frac{d\vartheta(\tau)}{\vartheta(\tau)} \right| = +\infty, \quad (5.2.9)$$

because $\vartheta(+\infty) = 0$. Thus, the first summand on the right-hand side of (5.2.7) vanishes at $+\infty$. Formula (5.2.5) is proved.

Now, let us assume that $\int_{\sigma}^{+\infty} Y d\vartheta$ converges. The integral on the left-hand side of (5.2.7) is monotone with respect to ν , thus it has a finite or infinite limit when ν tends to $+\infty$. Therefore, the first term on the right-hand side of (5.2.7) also has a finite or infinite limit. If this limit is nonzero, then we have (5.2.8) and (5.2.9), which contradicts our assumption. Thus, the first summand on the right-hand side of (5.2.7) vanishes, the left-hand side of (5.2.7) has a finite limit when ν tends to $+\infty$, and (5.2.5) is proved.

Finally, if both limits in (5.2.5) are infinite, we need to show that the infinities on both sides have the same sign. We may assume that ϑ is positive and decreasing. We may also assume that Y is monotone and non-negative at infinity. If Y were bounded at infinity, then the limit on the left-hand side of (5.2.5) would be finite. Therefore, Y increases at infinity, and both sides of (5.2.5) are equal $+\infty$. ■

Lemma 5.2.4. *Suppose that condition (2.1.7) is satisfied. Then condition (2.1.13) is equivalent to the fact that β_2 given by (3.3.3), i.e.,*

$$\beta_2(v) = \mathfrak{R}(v) - \int_{-\infty}^v \exp\left(-\int_{\tau}^v \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \mathfrak{R}'(\tau) d\tau, \quad (5.2.10)$$

satisfies the inequality $\beta_2(v) < +\infty$ for any $v \in \mathbb{R}$. Moreover,

$$\begin{aligned} & \int_{-\infty}^v f'(\tau) \exp\left(-\int_{\tau}^v \lambda\right) d\tau \\ &= -(w_1(v) - g_1(v))[\kappa_3(v) + (\kappa(v) - \kappa_2(v))\beta_2(v)]. \end{aligned} \quad (5.2.11)$$

Here $\lambda = \lambda_R$, $\kappa = \kappa_R$, and $w = w_R$.

Proof. Let us modify the expression in the exponent on the right-hand side of (5.2.10). Using (3.3.2), (2.1.6), and (2.1.11), we obtain

$$\int \frac{\kappa'_2}{\kappa_2 - \kappa} = \log(\kappa_2 - \kappa) + \log(g_1 - w_1) + \int \lambda. \quad (5.2.12)$$

First, we claim that

$$\int_{-\infty}^v \frac{\kappa'_2}{\kappa_2 - \kappa} = +\infty, \quad (5.2.13)$$

which is equivalent to

$$\lim_{\tau \rightarrow -\infty} (\kappa_2(\tau) - \kappa(\tau))(g_1(\tau) - w_1(\tau)) \exp\left(-\int_{\tau}^v \lambda\right) = 0,$$

by (5.2.12). Recall that $(g_1 - w_1)\kappa = (g_2 - w_2)$, and therefore

$$\begin{aligned} & \lim_{\tau \rightarrow -\infty} \kappa(\tau)(g_1(\tau) - w_1(\tau)) \exp\left(-\int_{\tau}^v \lambda\right) \\ &= \lim_{\tau \rightarrow -\infty} (g_2(\tau) - w_2(\tau)) \exp\left(-\int_{\tau}^v \lambda\right) = 0, \end{aligned} \quad (5.2.14)$$

due to condition (2.1.7). The function $\kappa_2 = g'_2/g'_1$ is increasing by the convexity of the curve g . There are two cases. If κ_2 is bounded on $-\infty$, then

$$\lim_{\tau \rightarrow -\infty} \kappa_2(\tau)(g_1(\tau) - w_1(\tau)) \exp\left(-\int_{\tau}^v \lambda\right) = 0, \quad (5.2.15)$$

due to the same condition (2.1.7). If κ_2 is not bounded on $-\infty$, then $\kappa_2(\tau) \rightarrow -\infty$ as $\tau \rightarrow -\infty$, and then we use the fact that $0 > \kappa_2 > \kappa$, provided τ is sufficiently close to $-\infty$. Therefore, $|\kappa_2| < |\kappa|$ and (5.2.15) follows from (5.2.14). Thus, the claimed divergence in (5.2.13) is proved.

We wish to use Lemma 5.2.3 with

$$\vartheta(\tau) = \exp\left(-\int_{\tau}^v \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \quad \text{and} \quad Y(\tau) = \mathfrak{R}(\tau)$$

to integrate by parts and rewrite formula (5.2.10) for β_2 . We verify the hypotheses of Lemma 5.2.3. It is clear that $\kappa'_2 > 0$, $\kappa_2 > \kappa$, therefore the function ϑ is monotone and tends to 0 at $-\infty$ due to (5.2.13). The function Y is piecewise monotone according to Condition 2.1.11. Thus, by Lemma 5.2.3, we get

$$\begin{aligned} \beta_2(v) &= \int_{-\infty}^v \frac{\kappa'_2(\tau)}{\kappa_2(\tau) - \kappa(\tau)} \exp\left(-\int_{\tau}^v \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \mathfrak{R}(\tau) d\tau \\ &= \int_{-\infty}^v \frac{\kappa'_3(\tau)}{\kappa_2(\tau) - \kappa(\tau)} \exp\left(-\int_{\tau}^v \frac{\kappa'_2}{\kappa_2 - \kappa}\right) d\tau. \end{aligned} \quad (5.2.16)$$

Now we plug (5.2.12) into (5.2.16) and obtain

$$\begin{aligned} \beta_2(v) &= \frac{1}{\kappa_2(v) - \kappa(v)} \cdot \frac{1}{g_1(v) - w_1(v)} \\ &\quad \times \int_{-\infty}^v \kappa'_3(\tau)(g_1(\tau) - w_1(\tau)) \exp\left(-\int_{\tau}^v \lambda\right) d\tau. \end{aligned} \quad (5.2.17)$$

The next step is to integrate by parts again, using Lemma 5.2.3 once more, this time with

$$\vartheta(\tau) = (g_1(\tau) - w_1(\tau)) \exp\left(-\int_{\tau}^v \lambda\right) \quad \text{and} \quad Y(\tau) = \kappa_3(\tau).$$

We verify the hypotheses of Lemma 5.2.3. First, we have that $\vartheta \rightarrow 0$ when $\tau \rightarrow -\infty$, due to (2.1.7). Second, we calculate ϑ' using the relation $(g_1 - w_1)\lambda = w'_1$. This

yields

$$\begin{aligned}\vartheta'(\tau) &= (\lambda(\tau)(g_1(\tau) - w_1(\tau)) + g_1'(\tau) - w_1'(\tau)) \exp\left(-\int_{\tau}^v \lambda\right) \\ &\stackrel{(2.1.6)}{=} g_1'(\tau) \exp\left(-\int_{\tau}^v \lambda\right) > 0.\end{aligned}$$

Thus, ϑ is monotone. Finally, Y is piecewise monotone by Remark 2.1.12. Thus, we may apply Lemma 5.2.3 and integrate by parts in (5.2.17):

$$\begin{aligned}\beta_2(v) &= \frac{\kappa_3(v)}{\kappa_2(v) - \kappa(v)} - \frac{1}{\kappa_2(v) - \kappa(v)} \cdot \frac{1}{g_1(v) - w_1(v)} \\ &\quad \times \int_{-\infty}^v \kappa_3(\tau) g_1'(\tau) \exp\left(-\int_{\tau}^v \lambda\right) d\tau.\end{aligned}$$

Since $\kappa_3 g_1' = f'$, we get

$$\beta_2(v) = \frac{\kappa_3(v)}{\kappa_2(v) - \kappa(v)} - \frac{1}{\kappa_2(v) - \kappa(v)} \cdot \frac{1}{g_1(v) - w_1(v)} \cdot \int_{-\infty}^v f'(\tau) \exp\left(-\int_{\tau}^v \lambda\right) d\tau.$$

This completes the proof of (5.2.11).

Condition (2.1.13) states that the left-hand side of (5.2.11) is greater than $-\infty$. According to (5.2.11), this is equivalent to $\beta_2 < +\infty$. ■

Let us turn back to consideration of optimizers.

Proposition 5.2.5. *Let B be the standard candidate on $\Omega_{\mathbb{R}}(-\infty, u_2)$. There exists a non-decreasing optimizer φ_x for B at every point $x \in \Omega_{\mathbb{R}}(-\infty, u_2)$. Moreover, we have $\varphi_x \leq g(u_2)$.*

Proof. By Definition 3.3.8 of a standard candidate in the domain $\Omega_{\mathbb{R}}(-\infty, u_2)$, the function β_2 given by (3.3.3) is finite. Thus, the integral in (5.2.10) converges.

We begin with the points on the free boundary. When this is done, we will automatically get the desired optimizer φ_x for any $x \in \Omega_{\mathbb{R}}(-\infty, u_2)$. To this end, we pick any finite u_1 such that $x \in \Omega_{\mathbb{R}}(u_1, u_2)$ and use Proposition 5.2.1. Moreover, it suffices to construct an optimizer for $x = w(u_2)$ only.

Similar to the case of a bounded domain $\Omega_{\mathbb{R}}$, we would like to construct a function $\varphi: (l, r] \rightarrow \partial_{\text{fixed}} \Omega$ on some interval $(l, r]$ such that the curve $\gamma(t) = \langle \varphi \rangle_{(l, t)}$ generated by φ goes along the free boundary from the infinity to $w(u_2)$. As before, we will find a function in the form $\varphi(t) = g(u(t))$, where $u: (l, r] \rightarrow (-\infty, u_2]$ is a monotone function. The previous reasoning (see (5.2.3)) leads us to the following relation:

$$\int_{u(t_1)}^{u(t_2)} \lambda(v) dv = \log \frac{t_2 - l}{t_1 - l}, \quad l < t_1 < t_2 \leq r. \quad (5.2.18)$$

We want to have $u(r) = u_2$. We substitute $t_2 = r$ into (5.2.18) and obtain

$$\int_{u(t_1)}^{u_2} \lambda(v) dv = \log \frac{r-l}{t_1-l}, \quad l < t_1 < r. \quad (5.2.19)$$

According to condition (2.1.7), we have

$$\int_{-\infty}^{u_2} \lambda(v) dv = +\infty, \quad (5.2.20)$$

therefore (5.2.19) defines the function u on (l, r) .

We may choose the length of the segment (l, r) to be equal 1, namely, we may take $l = 0, r = 1$. We get an explicit formula for the inverse function $t = t(u)$:

$$t = \exp\left(-\int_u^{u_2} \lambda(v) dv\right). \quad (5.2.21)$$

Corollary 5.1.4 guarantees $\varphi \in A_\Omega$, provided we know that

$$\frac{1}{t} \int_0^t \varphi = w(u(t)), \quad t \in (0, 1].$$

Therefore, for any $t \in (0, 1]$, we need to check that

$$\int_0^t \varphi = t w(u(t)). \quad (5.2.22)$$

Moreover, in order to prove that φ is an optimizer for B , we also need to verify that

$$B(w(u(t))) = \langle f(\varphi) \rangle_{[0,t]} = \frac{1}{t} \int_0^t f(u(s)) ds. \quad (5.2.23)$$

We start with (5.2.22). First, we prove that the limit of the right-hand side is zero as $t \rightarrow 0$. It is equivalent to

$$\lim_{u \rightarrow -\infty} w_i(u) \exp\left(-\int_u^{u_2} \lambda(v) dv\right) = 0, \quad i = 1, 2. \quad (5.2.24)$$

Fix $i = 1, 2$. If the function w_i is bounded on $-\infty$, then (5.2.24) follows from (5.2.20).

If w_i is not bounded on $-\infty$, then we apply L'Hôpital's rule:

$$\begin{aligned} \lim_{u \rightarrow -\infty} \frac{w_i(u)}{\exp(\int_u^{u_2} \lambda)} &= - \lim_{u \rightarrow -\infty} \frac{w_i'(u)}{\lambda(u) \exp(\int_u^{u_2} \lambda)} \\ &\stackrel{(5.2.1)}{=} \lim_{u \rightarrow -\infty} (w_i(u) - g_i(u)) \exp\left(-\int_u^{u_2} \lambda\right) = 0, \end{aligned}$$

due to condition (2.1.7).

We calculate the integral on the left-hand side of (5.2.22):

$$\int_0^t \varphi(\tau) d\tau = \int_0^t g(u(\tau)) d\tau \stackrel{(5.2.21)}{=} \int_{-\infty}^{u(t)} \lambda(v) g(v) \exp\left(-\int_v^{u_2} \lambda\right) dv. \quad (5.2.25)$$

Note that

$$\begin{aligned} \int \lambda(v)g(v) \exp\left(-\int_v^{u_2} \lambda\right) dv &\stackrel{(5.2.1)}{=} \int (w'(v) + \lambda(v)w(v)) \exp\left(-\int_v^{u_2} \lambda\right) dv \\ &= w(v) \exp\left(-\int_v^{u_2} \lambda\right), \end{aligned}$$

which converges to zero as $v \rightarrow -\infty$ because of (5.2.24). This proves that the integral in (5.2.25) converges and is equal to

$$w(u(t)) \exp\left(-\int_{u(t)}^{u_2} \lambda\right) = tw(u(t)).$$

Relation (5.2.22) is now proved.

We turn to the proof of (5.2.23). The right-hand side of (5.2.23) can be rewritten using (5.2.21) in the following way:

$$\begin{aligned} \frac{1}{t} \int_0^t f(u(s)) ds &= \exp\left(\int_{u(t)}^{u_2} \lambda\right) \cdot \int_0^t f(u(s)) d\left[\exp\left(-\int_{u(s)}^{u_2} \lambda\right)\right] \\ &\stackrel{\tau=u(s)}{=} \int_{-\infty}^{u(t)} f(\tau) \lambda(\tau) \exp\left(-\int_{\tau}^{u(t)} \lambda\right) d\tau \quad (5.2.26) \\ &= f(u(t)) - \int_{-\infty}^{u(t)} f'(\tau) \exp\left(-\int_{\tau}^{u(t)} \lambda\right) d\tau. \end{aligned}$$

The integration by parts is guaranteed by Lemma 5.2.3, with

$$\vartheta(\tau) = \exp\left(-\int_{\tau}^{u(t)} \lambda\right) \quad \text{and} \quad Y = f.$$

Finally, using (5.2.11), we obtain (5.2.23), see representation (3.2.2) for the relation between B and β_2 . ■

Proposition 5.2.6. *If (2.1.7) holds but (2.1.13) fails, then $B(x; f) = +\infty$ for any $x \in \Omega \setminus \partial_{\text{fixed}} \Omega$.*

Proof. It suffices to prove $B(w(v); f) = +\infty$ for any $v \in \mathbb{R}$. Condition (2.1.7) guarantees that the function φ constructed in the proof of Proposition 5.2.5 is a test function for the point $w(v)$. The failure of (2.1.13) means

$$\int_{-\infty}^v f'(\tau) \exp\left(-\int_{\tau}^v \lambda_R\right) d\tau = -\infty,$$

whence, by Lemma 5.2.3,

$$(f(\varphi))_{[0,1]} \stackrel{(5.2.26)}{=} \int_{-\infty}^v f(\tau) \lambda_R(\tau) \exp\left(-\int_{\tau}^v \lambda_R\right) d\tau = +\infty. \quad \blacksquare$$

We state analogous propositions for the left tangent domain at $+\infty$.

Proposition 5.2.7. *Let B be the standard candidate on $\Omega_L(u_1, +\infty)$. There exists a non-increasing optimizer φ_x for B at every point $x \in \Omega_L(u_1, +\infty)$. Moreover, we have $g(u_1) \preceq \varphi_x$.*

Proposition 5.2.8. *If (2.1.8) holds but (2.1.14) fails, then $B(x; f) = +\infty$ for any $x \in \Omega \setminus \partial_{\text{fixed}} \Omega$.*

5.2.2 Optimizers for all other figures

The optimizers for chordal domains are easy to construct. Indeed, the proposition below is a straightforward consequence of formula (3.4.1).

Proposition 5.2.9. *Suppose that B is the standard candidate on a chordal domain $\Omega_{\text{ch}}([a_0, b_0], [a_1, b_1])$. Let $g(a)$ and $g(b)$ be the endpoints of a chord from this chordal domain, $a \in [a_0, a_1]$, $b \in [b_1, b_0]$. Let*

$$x = \alpha g(a) + (1 - \alpha)g(b),$$

where $\alpha \in [0, 1]$. Then the optimizer $\varphi_x: [0, 1] \rightarrow \partial_{\text{fixed}} \Omega$ is given by the formula

$$\varphi_x = g(a)\chi_{[0, \alpha]} + g(b)\chi_{(\alpha, 1]}.$$

Proof. It is obvious that $\langle \varphi_x \rangle = x$, and the linearity of B on the chord implies that $B(x) = \langle f(\varphi_x) \rangle_{[0, 1]}$. It is clear that all the averages $\langle \varphi_x \rangle_J$ lie on $[g(a), g(b)] \subset \Omega$ for $J \subset [0, 1]$, therefore, $\varphi_x \in \mathbf{A}_\Omega$. ■

Remark 5.2.10. We suggested a non-decreasing optimizer φ_x for a chordal domain. One can also construct a non-increasing optimizer $\varphi_x = g(b)\chi_{[0, 1-\alpha]} + g(a)\chi_{(1-\alpha, 1]}$.

Definition 5.2.11. The construction above works also for the special case of a long chord. In this situation the tangency point with the free boundary is called the *outgoing node* of this chord.

Let us now pass to the case of multicups.

Proposition 5.2.12. *Let B be the standard candidate on $\Omega_{\text{ClMcup}}(\{\alpha_i\}_{i=1}^k)$. Then there exists a monotone optimizer φ_x for B at any point $x \in \Omega_{\text{ClMcup}}(\{\alpha_i\}_{i=1}^k)$ and, moreover, $g(\alpha_1^l) \preceq \varphi_x \preceq g(\alpha_k^r)$ pointwise.*

Proof. Fix $x \in \Omega_{\text{ClMcup}}(\{\alpha_i\}_{i=1}^k)$ and represent it as a convex combination of three points $g(a_1), g(a_2), g(a_3)$ for some $a_1, a_2, a_3 \in \bigcup_{i=1}^k \alpha_i$:

$$x = \alpha_1 g(a_1) + \alpha_2 g(a_2) + \alpha_3 g(a_3); \quad \alpha_1 + \alpha_2 + \alpha_3 = 1, \quad \alpha_j \geq 0.$$

Without loss of generality, we may assume that $a_1 \leq a_2 \leq a_3$. We put

$$\varphi_x(\tau) = g(a_1)\chi_{[0, \alpha_1]} + g(a_2)\chi_{(\alpha_1, \alpha_1 + \alpha_2]} + g(a_3)\chi_{(\alpha_1 + \alpha_2, 1]}.$$

The equality $\langle \varphi_x \rangle = x$ is evident, and the relation $B(x) = \langle f(\varphi_x) \rangle_{[0,1]}$ follows from the linearity of B on the closed multicup. The averages $\langle \varphi_x \rangle_J$ lie in the multicup for all $J \subset [0, 1]$, thus, $\varphi_x \in A_\Omega$. ■

Proposition 5.2.13. *Let B be the standard candidate on $\Omega_{\text{Mcup}}(\{\alpha_i\}_{i=1}^k)$. Then there exists a monotone optimizer φ_x for B at any point $x \in \Omega_{\text{Mcup}}(\{\alpha_i\}_{i=1}^k)$ and, moreover, $g(\alpha_1^1) \preceq \varphi_x \preceq g(\alpha_k^r)$ pointwise.*

Proof. If $x \in \partial_{\text{fixed}}\Omega$, then $\varphi_x = x\chi_{[0,1]}$ is the optimizer. So, in what follows we assume $x \notin \partial_{\text{fixed}}\Omega$. Consider the open convex set Ω' that is the interior of the convex hull of Ξ and the points $g(\alpha_1^1)$ and $g(\alpha_k^r)$. Since $x \notin \Omega'$, by the separation theorem, there exists a line $\kappa = \kappa(x)$ that passes through x and does not intersect Ω' . Let y and z be the leftmost and the rightmost points of the intersection of the line $\kappa(x)$ with the boundary of $\Omega_{\text{Mcup}}(\{\alpha_i\}_{i=1}^k)$. For each crossing point (y and z) there are two possibilities: either it lies on an arc $g(\alpha_i)$, $i \in \{1, 2, \dots, k\}$, or on a segment $[g(\alpha_i^r), g(\alpha_{i+1}^1)]$, $i \in \{1, 2, \dots, k - 1\}$.

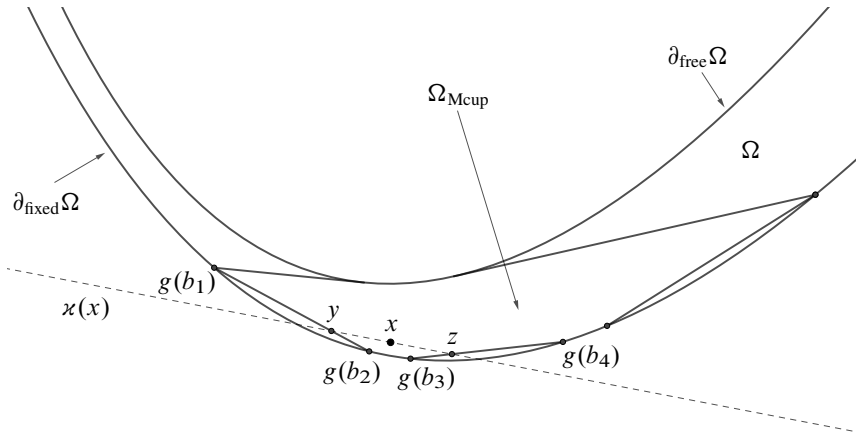


Figure 5.4. Construction of an optimizer in a multicup.

If y lies on the chord $[g(\alpha_i^r), g(\alpha_{i+1}^1)]$, then we can write

$$y = \alpha_y g(\alpha_i^r) + (1 - \alpha_y)g(\alpha_{i+1}^1) \quad \text{for some } \alpha_y \in [0, 1].$$

Similarly, if z lies on the chord $[g(\alpha_j^r), g(\alpha_{j+1}^1)]$, then

$$z = (1 - \alpha_z)g(\alpha_j^r) + \alpha_z g(\alpha_{j+1}^1) \quad \text{for some } \alpha_z \in [0, 1].$$

So, in any case, $y = \alpha_y g(b_1) + (1 - \alpha_y)g(b_2)$, $z = (1 - \alpha_z)g(b_3) + \alpha_z g(b_4)$ (see Figure 5.4), where $b_j \in \bigcup_{i=1}^k \alpha_i$ (if y is an intersection of κ with an arc, then we may

take $b_1 = b_2$; similarly with z) such that

$$g_1(b_1) \leq y_1 \leq g_1(b_2) \leq g_1(b_3) \leq z_1 \leq g_1(b_4).$$

It is convenient to define the optimizer φ_x on the segment $[y_1, z_1]$:

$$\varphi_x(t) = \begin{cases} g(b_1), & t \in J_1 = [y_1, y_1 + \alpha_y(z_1 - x_1)), \\ g(b_2), & t \in J_2 = [y_1 + \alpha_y(z_1 - x_1), z_1 - x_1 + y_1), \\ g(b_3), & t \in J_3 = [z_1 - x_1 + y_1, z_1 - \alpha_z(x_1 - y_1)), \\ g(b_4), & t \in J_4 = [z_1 - \alpha_z(x_1 - y_1), z_1]. \end{cases} \quad (5.2.27)$$

As usual, the equalities $\langle \varphi_x \rangle = x$ and $\langle f(\varphi_x) \rangle_{[y_1, z_1]} = B(x)$ are evident. However, if we draw the delivery curve for φ_x , we can see that in some cases it does not fall under the scope of Lemma 5.1.3 (the tangent may cross the free boundary), so we prove directly that φ_x lies in A_Ω .

We claim that a point $\langle \varphi_x \rangle_J$, where $J \subset [y_1, z_1]$, either belongs to one of the segments $[g(b_1), g(b_2)]$ and $[g(b_3), g(b_4)]$, or it is separated from the free boundary by κ . Once the claim is proved, we obtain $\varphi_x \in A_\Omega$.

We will consider different cases of the position of J inside $[y_1, z_1]$. If J intersects not more than two of the intervals J_1, J_2, J_3, J_4 in (5.2.27), then the claim is obvious. If J intersects all four intervals, then we may represent x as a linear combination of $\langle \varphi_x \rangle_J$, $g(b_1)$, and $g(b_4)$. Since $g(b_1)$ and $g(b_4)$ lie above κ (i.e., in the same half-plane with the free boundary), the point $\langle \varphi_x \rangle_J$ lies below κ .

So, we may suppose that J intersects three intervals from (5.2.27). Without loss of generality, we may assume that $J \cap J_4 = \emptyset$. Then $\langle \varphi_x \rangle_J$ is a convex combination of $g(b_3)$ and a point from $[y, g(b_2)]$ (since $J_2 \subset J$). Both these points lie below κ . Therefore, $\langle \varphi_x \rangle_J$ is separated from the free boundary by κ . ■

Remark 5.2.14. The optimizer φ_x we constructed for a multicup domain is non-decreasing. One can construct a non-increasing optimizer in a similar way.

Definition 5.2.15. The endpoints of the arc of the free boundary that is the part of the boundary of a multicup are called the *outgoing nodes* of the multicup.

Proposition 5.2.16. Let $\Omega_{\text{ang}}(u)$ be an angle with the vertex $g(u)$ and the boundary tangent lines S_L and S_R with the tangency points w_L and w_R , respectively. Let B be the standard candidate on $\Omega_{\text{ang}}(u)$. Let ψ_R be a non-decreasing optimizer for the point w_R , and let ψ_L be a non-increasing optimizer for the point w_L such that $\psi_R \leq g(u)$ and $g(u) \leq \psi_L$ pointwise. Then there exists an optimizer for every point $x \in \Omega_{\text{ang}}(u)$.

Proof. The proof of this proposition is very similar to the proof of Proposition 5.2.13. First, there exist numbers $\alpha_1, \alpha_2, \alpha_3$ such that

$$x = \alpha_1 w_R + \alpha_2 g(u) + \alpha_3 w_L; \quad \alpha_1 + \alpha_2 + \alpha_3 = 1, \quad \alpha_j \geq 0.$$

Definition 5.2.17. The points w_R and w_L introduced in Proposition 5.2.16 are called the *incoming nodes* of the angle.

Proposition 5.2.18. Suppose that B is the standard candidate in a right trolleybus $\Omega_{tr,R}(u_1, u_2)$. Suppose ψ is a non-decreasing optimizer for the point $w_R(u_1)$ such that $\psi \leq g(u_1)$ pointwise. Then, for any $x \in \Omega_{tr,R}(u_1, u_2)$, there exists a non-decreasing optimizer φ_x such that $\varphi_x \leq g(u_2)$.

Proof. Choose any point $x \in \Omega_{tr,R}(u_1, u_2)$. The trolleybus $\Omega_{tr,R}(u_1, u_2)$ lies inside the triangle with the vertices $g(u_1)$, $g(u_2)$, and $w_R(u_1)$. Thus, there exist $\alpha_0, \alpha_1, \alpha_2$ such that

$$x = \alpha_0 w_R(u_1) + \alpha_1 g(u_1) + \alpha_2 g(u_2); \quad \alpha_0 + \alpha_1 + \alpha_2 = 1, \quad \alpha_j \geq 0.$$

By Remark 2.1.5, we may assume that ψ is defined on $[0, \alpha_0]$. Define the optimizer φ_x on $[0, 1]$ by the formula

$$\varphi_x(\tau) = \begin{cases} \psi(\tau), & \tau \in [0, \alpha_0], \\ g(u_1), & \tau \in [\alpha_0, \alpha_0 + \alpha_1], \\ g(u_2), & \tau \in [\alpha_0 + \alpha_1, 1]. \end{cases}$$

It is clear that $\langle \varphi_x \rangle_{[0,1]} = x$ and $\langle f(\varphi_x) \rangle_{[0,1]} = B(x)$. We only have to verify $\varphi_x \in A_\Omega$, i.e., $\langle \varphi_x \rangle_J \in \Omega$ for any interval $J \subset [0, 1]$. Take a line $\kappa = \kappa(x)$ that passes through x and such that the point $g(u_2)$ together with the domain Ξ lie above this line, see Figure 5.6. As in the proof of Propositions 5.2.13 and 5.2.16, we will have to consider different cases of location of J inside $[0, 1]$.

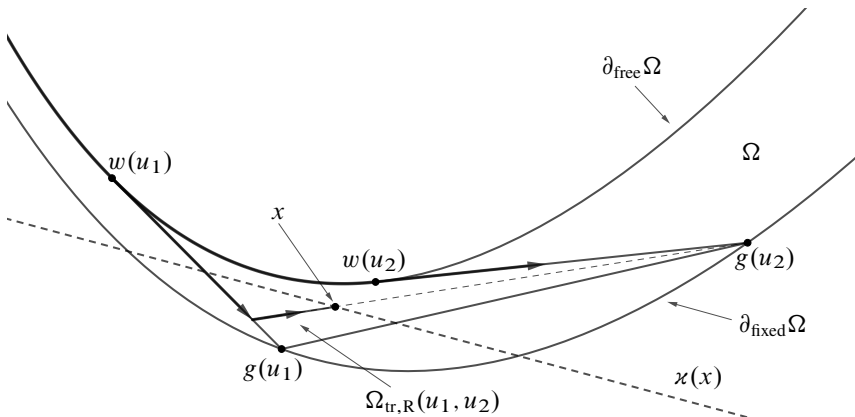


Figure 5.6. Delivery curves inside a trolleybus.

If $[\alpha_0, \alpha_0 + \alpha_1] \subset J$, then $\langle \varphi_x \rangle_J$ is separated from the free boundary by κ_x (this reasoning is completely analogous to that used in Propositions 5.2.13 and 5.2.16).

If $J \cap [\alpha_0 + \alpha_1, 1] = \emptyset$, then the situation falls under the scope of Lemma 5.1.3.

If $J \cap [0, \alpha_0] = \emptyset$, then $\langle \varphi_x \rangle_J \subset [g(u_1), g(u_2)]$.

We have considered all the cases and verified that $\varphi_x \in \mathcal{A}_\Omega$. Moreover, it follows from the construction that φ_x is non-decreasing and $\varphi_x \leq g(u_2)$. ■

As usual, we have a symmetric proposition.

Proposition 5.2.19. *Let B be the standard candidate in a left trolleybus $\Omega_{\text{tr,L}}(u_1, u_2)$. Suppose ψ is a non-increasing optimizer for the point $w_L(u_2)$ such that $\psi \geq g(u_2)$. Then, for any $x \in \Omega_{\text{tr,L}}(u_1, u_2)$, there exists a non-increasing optimizer φ_x such that $\varphi_x \geq g(u_1)$.*

It remains to construct the optimizers for multitrolleybuses, birdies, and multi-birdies. Formulas from Section 3.8.1 will help us in this task.

Proposition 5.2.20. *Suppose that B is the standard candidate in a right multitrolleybus $\Omega_{\text{Mtr,R}}(\{\alpha_i\}_{i=1}^k)$. Suppose ψ is a non-decreasing optimizer for the point $w_R(\alpha_1^1)$ such that $\psi \leq g(\alpha_1^1)$. Then, for any $x \in \Omega_{\text{Mtr,R}}(\{\alpha_i\}_{i=1}^k)$, there exists a non-decreasing optimizer that does not exceed $g(\alpha_k^r)$.*

Proof. We apply (3.8.11) and decompose a multitrolleybus in an alternating sequence of right trolleybuses and multitrolleybuses $\Omega_{\text{Mtr,R}}(\alpha_i)$ on solid roots. We consider each such multitrolleybus as a tangent domain with the candidate B on it (note that this is not a standard candidate for the tangent domain). Applying Proposition 5.2.18 to the trolleybuses and Proposition 5.2.1 to the tangent domains consecutively, we build optimizers for all the points inside $\Omega_{\text{Mtr,R}}(\{\alpha_i\}_{i=1}^k)$. ■

Proposition 5.2.21. *Suppose that B is the standard candidate in a left multitrolleybus $\Omega_{\text{Mtr,L}}(\{\alpha_i\}_{i=1}^k)$. Suppose ψ is a non-increasing optimizer for the point $w_L(\alpha_k^r)$ such that $\psi \geq g(\alpha_k^r)$. Then, for any $x \in \Omega_{\text{Mtr,L}}(\{\alpha_i\}_{i=1}^k)$, there exists a non-increasing optimizer that is not less than $g(\alpha_1^l)$.*

Definition 5.2.22. Let w_- and w_+ be the left and the right endpoints of the arc of the free boundary that is the part of the boundary of a trolleybus (or a multitrolleybus). These points are called the *incoming* and *outgoing nodes* of the trolleybus (or the multitrolleybus): for the case of $\Omega_{\text{tr,R}}$ (and $\Omega_{\text{Mtr,R}}$), the incoming node is w_- and outgoing is w_+ , and vice versa for the case of $\Omega_{\text{tr,L}}$ (and $\Omega_{\text{Mtr,L}}$).

Proposition 5.2.23. *Let B be the standard candidate in a multibirdie $\Omega_{\text{Mbird}}(\{\alpha_i\}_{i=1}^k)$. Let ψ_R be a non-decreasing optimizer for the point $w_R(\alpha_1^1)$ such that $\psi_R \leq g(\alpha_1^1)$, let ψ_L be a non-increasing optimizer for the point $w_L(\alpha_k^r)$ such that $\psi_L \geq g(\alpha_k^r)$. Then there exists an optimizer for every point $x \in \Omega_{\text{Mbird}}(\{\alpha_i\}_{i=1}^k)$.*

Proof. We apply (3.8.13) (with any choice of j) and decompose the multibirdie into two multitrolleybuses and an angle. Applying corresponding previous propositions

	outgoing	incoming
long chord	$w_- = w_+$	
$\Omega_{\text{Mcup}}(\{\alpha_i\}_{i=1}^k)$	w_-, w_+	
$\Omega_{\text{Mtr,R}}(\{\alpha_i\}_{i=1}^k)$		
$\Omega_{\text{tr,R}}(u_1, u_2)$	w_+	w_-
$\Omega_{\text{R}}(u_1, u_2)$		
$\Omega_{\text{Mtr,L}}(\{\alpha_i\}_{i=1}^k)$		
$\Omega_{\text{tr,L}}(u_1, u_2)$	w_-	w_+
$\Omega_{\text{L}}(u_1, u_2)$		
$\Omega_{\text{ang}}(w)$		
$\Omega_{\text{Mbird}}(\{\alpha_i\}_{i=1}^k)$		w_-, w_+

Table 5.1. Outgoing and incoming nodes of the domains of linearity.

for each of the domains, we build optimizers for all the points inside $\Omega_{\text{Mbird}}(\{\alpha_i\}_{i=1}^k)$. The proof is complete. ■

Definition 5.2.24. The points $w_{\text{R}}(\alpha_1^l)$ and $w_{\text{L}}(\alpha_k^r)$ in Proposition 5.2.23 are called the *incoming nodes* of the multibirdie $\Omega_{\text{Mbird}}(\{\alpha_i\}_{i=1}^k)$.

5.3 Global optimizers

Before passing to global optimizers, we collect the information about incoming and outgoing nodes for a vertex or an edge in the foliation graph. These nodes are points on the free boundary that are used to transfer the delivery curve from one domain to another. For some figures, the optimizers inside them are constructed by using the optimizer coming from the right or from the left (for example, to build optimizers in a trolleybus, we need an optimizer for a special point on the free boundary, see Propositions 5.2.18 and 5.2.19). The incoming node is the point we start the delivery curve from, whereas the outgoing node is the point where it leaves the figure. In Table 5.1, we give a precise description. We extend our notation from Definition 5.2.22 for other domains \mathcal{L} : let w_- and w_+ be the left and the right endpoints of the arc of the free boundary $\partial_{\text{free}} \Omega$ which is the part of the boundary of \mathcal{L} .

Other figures (separated from the upper boundary) have no incoming or outgoing nodes at all. Now we see that the propositions of Sections 5.2.1 and 5.2.2 are of the following form: if there is a monotone optimizer for the incoming node satisfying a certain restriction (if there are incoming nodes for the figure in question), then we can build monotone optimizers satisfying a similar restriction for all the points in the

domain, in particular, for the outgoing nodes (if there are any). These propositions are well suited for induction. To state the general theorem about optimizers, we need Definition 4.4.5 (see Remark 4.4.6 and the paragraph before it, where it is explained that an admissible graph generates the Bellman candidate).

Theorem 5.3.1. *Let $\varepsilon < \varepsilon_{\max}$, and let $\Gamma(\varepsilon)$ be an admissible graph for ε and f . The Bellman candidate B_ε generated by $\Gamma(\varepsilon)$ admits an optimizer at every point of Ω_ε .*

Proof. The proof of the theorem consists in inductive application of the propositions from Section 5.2.1 and 5.2.2. The details are the same as in the special case of the parabolic strip, see Theorem 5.3.1 in [17]. ■

Theorem 5.3.1 that we have just proved justifies the arguments given in Section 2.2.3. Also, the candidate B_ε constructed by the admissible graph $\Gamma(\varepsilon)$ coincides with the Bellman function B_ε and with the minimal locally concave function $\mathfrak{B}_{\Omega_\varepsilon, f}$.

Theorem 5.3.2. *For any $\varepsilon < \varepsilon_{\max}$, the constructed Bellman candidate B_ε coincides with B_ε and $\mathfrak{B}_{\Omega_\varepsilon, f}$.*

5.4 Remarks concerning global conditions

In this section we discuss conditions (2.1.7) and (2.1.8).

As the reader has seen, these technical conditions were used only during different steps of constructing optimizers. Here we discuss what happens when they fail. We note that construction of optimizers does not use any evolution arguments and deals with the fixed ε .

5.4.1 Behavior at $-\infty$

Since we have the right-left symmetry, let us consider the behavior at $-\infty$, namely, condition (2.1.7). In what follows, we omit the index R , i.e., we write λ, w, κ instead of λ_R, w_R, κ_R .

If (2.1.7) is not fulfilled, and the foliation for the function B contains a domain $\Omega_R(-\infty, v)$ of the right tangents coming from $-\infty$, the statement of Proposition 5.2.5 is not valid in the sense that there are no optimizers φ_x for $x \in \Omega_R(-\infty, v) \setminus \partial_{\text{fixed}} \Omega$. However, for any such x , there is a sequence of test functions $\varphi_n \in \mathcal{A}_\Omega$ such that $\langle \varphi_n \rangle = x$ and $B(x) = \lim_{n \rightarrow +\infty} \langle f(\varphi_n) \rangle$. We call such $\{\varphi_n\}_n$ an *optimizing sequence* for x . All the propositions from Section 5.2 remain true if we replace the word optimizer by optimizing sequence.

We now give some details on how to construct these functions φ_n for $x = w(v)$. Consider first sequences $v_n^-, v_n^+ \in \mathbb{R}$ such that $v_n^- < v_n^+ < v$, $v_n^\pm \rightarrow -\infty$, and the chord $[g(v_n^-), g(v_n^+)]$ is tangent to $\partial_{\text{free}} \Omega$. The tangent point is $w(v_n^+)$. Consider the

domain $\Omega_{\mathbb{R}}(v_n^+, v)$ and let γ_n be the delivery curve that starts at the point $g(v_n^-)$ and goes to $w(v_n^+)$ along the chord, and then proceeds along $\partial_{\text{free}} \Omega$ until the point $w(v)$. Build a test function φ_n on $[0, 1]$ that generates this curve γ_n :

$$\varphi_n(t) = \begin{cases} g(v_n^-), & t \in [0, \alpha_1), \\ g(v_n^+), & t \in [\alpha_1, \alpha_2), \\ g(u(t)), & t \in [\alpha_2, 1]. \end{cases} \quad (5.4.1)$$

The parameters $u = u(t)$ and $t = t(u)$ are related as in (5.2.21):

$$\log t = - \int_{u(t)}^v \lambda, \quad t \in [\alpha_2, 1], \quad (5.4.2)$$

and $u(\alpha_2) = v_n^+$, i.e.,

$$\alpha_2 = \exp\left(- \int_{v_n^+}^v \lambda\right). \quad (5.4.3)$$

The value of α_1 is determined by the relation that represents the point $w(v_n^+)$ as a convex combination of $g(v_n^-)$ and $g(v_n^+)$:

$$\frac{\alpha_1}{\alpha_2} g(v_n^-) + \frac{\alpha_2 - \alpha_1}{\alpha_2} g(v_n^+) = w(v_n^+),$$

namely,

$$\frac{\alpha_1}{\alpha_2} = \frac{g_1(v_n^+) - w_1(v_n^+)}{g_1(v_n^+) - g_1(v_n^-)}. \quad (5.4.4)$$

Remark 5.4.1. Since φ_n defined by (5.4.1) generates the delivery curve γ_n , we have $\langle g(\varphi_n) \rangle_{[0,1]} = w(v)$. Moreover, Corollary 5.1.4 guarantees φ_n lies in \mathcal{A}_{Ω} , therefore it is a test function for $w(v)$.

The following theorem generalizes Proposition 5.2.5 to the case where condition (2.1.7) fails.

Theorem 5.4.2. *Let B be the standard candidate on $\Omega_{\mathbb{R}}(-\infty, v)$. Then, for every point $x \in \Omega_{\mathbb{R}}(-\infty, v)$, there exists a sequence of non-decreasing test functions $\varphi_{n,x}$ being an optimizing sequence for B at x ; moreover, $\varphi_{n,x} \preceq g(v)$.*

The proof of Theorem 5.4.2 will be given for different cases of condition (2.1.7) failure separately, namely, we consider three cases:

- (A) $\exp(- \int_{-\infty}^v \lambda) \neq 0$, i.e., the integral $\int_{-\infty}^v \lambda$ converges,
- (B) $\int_{-\infty}^v \lambda = +\infty$, and the function

$$\vartheta_1(\tau) \stackrel{\text{def}}{=} (g_1(\tau) - w_1(\tau)) \exp\left(- \int_{\tau}^v \lambda\right) \quad (5.4.5)$$

does not tend to zero as $\tau \rightarrow -\infty$,

(C) $\int_{-\infty}^v \lambda = +\infty$, $\vartheta_1(-\infty) = 0$, and the function

$$\vartheta_2(\tau) \stackrel{\text{def}}{=} (g_2(\tau) - w_2(\tau)) \exp\left(-\int_{\tau}^v \lambda\right) \quad (5.4.6)$$

does not tend to zero as $\tau \rightarrow -\infty$.

Remark 5.4.3. For any function $\psi: \mathbb{R} \rightarrow \mathbb{R}$, we use the notation $\psi(-\infty)$ for either a finite or an infinite limit $\lim_{\tau \rightarrow -\infty} \psi(\tau)$. We will often use this notation for the functions that are monotone in a neighborhood of $-\infty$, e.g., f , κ_3 , \mathfrak{K} that are monotone by Condition 2.1.11 and Remark 2.1.12, as well as for ϑ_1 and ϑ_2 , whose monotonicity will be proved in Lemma 5.4.4.

Lemma 5.4.4. *The functions ϑ_1 and ϑ_2 , defined by (5.4.5) and (5.4.6), are monotone in a neighborhood of $-\infty$ and have finite limits at $-\infty$. Moreover, ϑ_1 and ϑ'_1 are positive, and*

$$\text{sign } \vartheta'_2 = \text{sign } \vartheta_2 = \text{sign } g'_2 = \text{sign } \kappa_2 = \text{sign } \kappa \quad (5.4.7)$$

in a neighborhood of $-\infty$.

Proof. The function ϑ_1 is positive and increasing:

$$\vartheta'_1(\tau) \stackrel{(5.2.1)}{=} g'_1(\tau) \exp\left(-\int_{\tau}^v \lambda\right) > 0. \quad (5.4.8)$$

Therefore, $\vartheta_1(-\infty)$ is finite.

The similar formula for the derivative of ϑ_2 ,

$$\vartheta'_2(\tau) = g'_2(\tau) \exp\left(-\int_{\tau}^v \lambda\right), \quad (5.4.9)$$

implies that $\text{sign } \vartheta'_2(\tau) = \text{sign } \kappa_2(\tau)$, whereas $\text{sign } \vartheta_2(\tau) = \text{sign } \kappa(\tau)$. From the geometric assumption on the domain Ω , the point $w(\tau)$ lies on a segment connecting $g(\tau)$ and some $g(\tau_-)$, $\tau_- < \tau$. It follows from the convexity of $\partial_{\text{fixed}} \Omega$ that

$$\kappa_2(\tau_-) < \kappa(\tau) < \kappa_2(\tau). \quad (5.4.10)$$

Whence, $\text{sign } \kappa_2 = \text{sign } \kappa$ in a neighborhood of $-\infty$, and $\text{sign } \vartheta'_2 = \text{sign } \vartheta_2$ as well. Therefore, the limit $\vartheta_2(-\infty)$ is finite. \blacksquare

Preparing for the proof of Theorem 5.4.2, we calculate $\langle f(\varphi_n) \rangle_{[0,1]}$:

$$\begin{aligned} \langle f(\varphi_n) \rangle_{[0,1]} &= \alpha_1 f(v_n^-) + (\alpha_2 - \alpha_1) f(v_n^+) + \int_{\alpha_2}^1 f(u(t)) dt \\ &\stackrel{(5.4.2)}{=} \alpha_1 f(v_n^-) + (\alpha_2 - \alpha_1) f(v_n^+) + \int_{v_n^+}^v f(u) \lambda(u) \exp\left(-\int_u^v \lambda\right) du \end{aligned} \quad (5.4.11)$$

$$\begin{aligned}
 &= \alpha_1 f(v_n^-) + (\alpha_2 - \alpha_1) f(v_n^+) + f(u) \exp\left(-\int_u^v \lambda\right) \Big|_{u=v_n^+}^{u=v} \\
 &\quad - \int_{v_n^+}^v f'(u) \exp\left(-\int_u^v \lambda\right) du \\
 &= \alpha_1 (f(v_n^-) - f(v_n^+)) + f(v) - \int_{v_n^+}^v f'(u) \exp\left(-\int_u^v \lambda\right) du \quad (5.4.12)
 \end{aligned}$$

$$= -\frac{f(v_n^+) - f(v_n^-)}{g_1(v_n^+) - g_1(v_n^-)} \vartheta_1(v_n^+) + f(v) - \int_{v_n^+}^v f'(u) \exp\left(-\int_u^v \lambda\right) du. \quad (5.4.13)$$

In the latter equality we use (5.4.4), (5.4.3), and (5.4.5). We apply the Cauchy mean value theorem to find $v_n \in [v_n^-, v_n^+]$ such that

$$\frac{f(v_n^+) - f(v_n^-)}{g_1(v_n^+) - g_1(v_n^-)} = \frac{f'(v_n)}{g_1'(v_n)} = \kappa_3(v_n).$$

We use this to rewrite (5.4.13):

$$\langle f(\varphi_n) \rangle_{[0,1]} = -\kappa_3(v_n) \vartheta_1(v_n^+) + f(v) - \int_{v_n^+}^v f'(u) \exp\left(-\int_u^v \lambda\right) du. \quad (5.4.14)$$

5.4.2 Case (A) : $\int_{-\infty}^v \lambda$ converges

Lemma 5.4.5. *If $\int_{-\infty}^v \lambda < +\infty$, then the limits $g(-\infty)$ and $w(-\infty)$ are finite and coincide. Thus, $\vartheta_1(-\infty) = 0$ and $\vartheta_2(-\infty) = 0$.*

Proof. We note that $\lambda > 0$, therefore the hypothesis of the lemma is equivalent to the convergence of the integral. From (5.4.8), we have

$$\vartheta_1'(\tau) \leq g_1'(\tau) \leq \vartheta_1'(\tau) \exp\left(\int_{-\infty}^v \lambda\right).$$

By Lemma 5.4.4, $\vartheta_1(-\infty)$ is finite, and therefore $\int_{-\infty}^v \vartheta_1' < +\infty$. So, $\int_{-\infty}^v g_1' < +\infty$, which means $g_1(-\infty)$ is finite.

Since the limit $g_1(-\infty)$ is finite and the curve (g_1, g_2) is convex, we see that the limit of g_2 at $-\infty$ is not $-\infty$. Therefore, $g_2(-\infty)$ is either finite or $+\infty$. If $g_2(-\infty) = +\infty$, then $g_2' < 0$ in a neighborhood of $-\infty$. From (5.4.9), we have

$$\vartheta_2'(\tau) \geq g_2'(\tau) \geq \vartheta_2'(\tau) \exp\left(\int_{-\infty}^v \lambda\right).$$

By Lemma 5.4.4, $\vartheta_2(-\infty)$ is finite, therefore $\int_{-\infty}^v \vartheta_2' > -\infty$. Thus, $\int_{-\infty}^v g_2' > -\infty$, which means g_2 has a finite limit at $-\infty$. This is a contradiction.

We have proved that $g = (g_1, g_2)$ has a finite limit at $-\infty$. From the geometric assumption on the domain Ω , any point $w(\tau)$ lies on a segment connecting $g(\tau)$ and

some $g(\tau_-)$, $\tau_- < \tau$. Since both points $g(\tau)$ and $g(\tau_-)$ tend to the same limit, the limit of $w(\tau)$ as $\tau \rightarrow -\infty$ exists and $w(-\infty) = g(-\infty)$. ■

Corollary 5.4.6. *If $\int_{-\infty}^v \lambda < +\infty$, then $(\kappa_2 - \kappa)(g_1 - w_1)$ tends to 0 at $-\infty$.*

Proof. From Lemma 5.4.5 we know that $g_1 - w_1$ tends to 0 at $-\infty$.

If $\kappa_2 > 0$ in a neighborhood of $-\infty$, then $\kappa_2(-\infty)$ is finite because $\kappa_2' > 0$. Since any point $w(\tau)$ lies on a segment connecting $g(\tau)$ and some $g(\tau_-)$, $\tau_- < \tau$, we have an inequality $\kappa_2(\tau_-) < \kappa(\tau) < \kappa_2(\tau)$. Therefore, $\kappa(-\infty) = \kappa_2(-\infty)$, and the claim is proved.

If $\kappa_2 < 0$ in a neighborhood of $-\infty$, then $0 < \kappa_2 - \kappa < -\kappa$. Therefore, by Lemma 5.4.5, we have

$$|(\kappa_2 - \kappa)(g_1 - w_1)| \leq |\kappa(g_1 - w_1)| = |g_2 - w_2| \rightarrow 0,$$

as the argument tends to $-\infty$. ■

Corollary 5.4.7. *If $\int_{-\infty}^v \lambda < +\infty$, then $\int_{-\infty}^v \frac{\kappa_2'}{\kappa_2 - \kappa} = +\infty$.*

Proof. The statement follows from (5.2.12) and Corollary 5.4.6. ■

Lemma 5.4.8. *If $\int_{-\infty}^v \lambda < +\infty$ and $f(-\infty) = -\infty$, then $\mathfrak{R}(-\infty) = -\infty$.*

Proof. Assume on the contrary that $\mathfrak{R} > C_1$ in a neighborhood of $-\infty$ for some $C_1 \in \mathbb{R}$. Then we have $\kappa_3' > C_1 \kappa_2'$ because $\kappa_2' > 0$. Integration of this inequality yields $\kappa_3 < C_1 \kappa_2 + C_2$ for some $C_2 \in \mathbb{R}$. Multiply this by positive g_1' to obtain $f' < C_1 g_2' + C_2 g_1'$. Integrating this inequality, we get $f > C_1 g_2 + C_2 g_1 + C_3$ for some $C_3 \in \mathbb{R}$. This inequality contradicts the hypothesis that $f(-\infty) = -\infty$, since the functions g_1 and g_2 have finite limits at $-\infty$ by Lemma 5.4.5. ■

Proposition 5.4.9. *If $\int_{-\infty}^v \lambda < +\infty$ and there exists a standard candidate B on $\Omega_{\mathbb{R}}(-\infty, v)$, then $f(-\infty) > -\infty$.*

Proof. Let us assume the contrary: $f(-\infty) = -\infty$. Then, by Lemma 5.4.8, we have $\mathfrak{R}(-\infty) = -\infty$, therefore $\mathfrak{R}' > 0$ in a neighborhood of $-\infty$. By (3.3.4), $\beta_2' > 0$ in the same neighborhood of $-\infty$. This contradicts the existence of the standard candidate B on $\Omega_{\mathbb{R}}(-\infty, v)$, see Definition 3.3.8. ■

Proposition 5.4.10. *If $\int_{-\infty}^v \lambda < +\infty$ and $f(-\infty) = +\infty$, then $\mathbf{B}(x; f) = +\infty$ for any $x \in \Omega \setminus \partial_{\text{fixed}} \Omega$.*

Proof. It suffices to prove $\mathbf{B}(w(v); f) = +\infty$. We construct a sequence of test functions φ_n for the point $w(v)$ by (5.4.1). The condition $\int_{-\infty}^v \lambda < +\infty$, together with (5.4.3), guarantees that α_2 is separated from 0. Therefore, from the condition $f(-\infty) = +\infty$ and (5.4.11), we obtain that $\langle f(\varphi_n) \rangle_{[0,1]}$ tends to $+\infty$, and thus $\mathbf{B}(w(v); f) = +\infty$. ■

Proposition 5.4.11. *Suppose that $\int_{-\infty}^v \lambda < +\infty$ and the limit $f(-\infty)$ is finite. Let B be the standard candidate on $\Omega_{\mathbb{R}}(-\infty, v)$. Then, for every point $x \in \Omega_{\mathbb{R}}(-\infty, v)$, there exists a sequence of non-decreasing test functions $\varphi_{n,x}$ being an optimizing sequence for B at x ; moreover, $\varphi_{n,x} \preceq g(v)$.*

Proof. As before, it suffices to prove the claim for the case $x = w(v) = w_{\mathbb{R}}(v)$. In what follows, we omit the subscript x in our notation. We use the sequence of functions φ_n defined by (5.4.1). All these functions are non-decreasing and satisfy the inequality $\varphi_n \preceq g(v)$. Therefore, we only need to verify that $\langle f(\varphi_n) \rangle_{[0,1]} \rightarrow B(w(v))$.

Recall (3.2.2):

$$B(w(v)) = f(v) + (w_1(v) - g_1(v))[\kappa_3(v) + (\kappa(v) - \kappa_2(v))\beta_2(v)], \quad (5.4.15)$$

where β_2 is given by (3.3.3), i.e.,

$$\begin{aligned} \beta_2(v) &= \mathfrak{R}(v) - \int_{-\infty}^v \exp\left(-\int_{\tau}^v \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \mathfrak{R}'(\tau) d\tau \\ &= \int_{-\infty}^v \exp\left(-\int_{\tau}^v \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \frac{\kappa'_3(\tau)}{\kappa_2(\tau) - \kappa(\tau)} d\tau. \end{aligned} \quad (5.4.16)$$

Here we have used Lemma 5.2.3 with

$$\vartheta = \exp\left(-\int_{\tau}^v \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \quad \text{and} \quad Y = \mathfrak{R}.$$

Note that $\vartheta(-\infty) = 0$ by Corollary 5.4.7.

We use (5.4.12) to evaluate the limit of $\langle f(\varphi_n) \rangle_{[0,1]}$:

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle f(\varphi_n) \rangle_{[0,1]} &= f(v) - \int_{-\infty}^v f'(u) \exp\left(-\int_u^v \lambda\right) du \\ &= f(v) - \int_{-\infty}^v \kappa_3(u) g'_1(u) \exp\left(-\int_u^v \lambda\right) du \\ &\stackrel{(5.4.8)}{=} f(v) - \int_{-\infty}^v \kappa_3(u) \vartheta'_1(u) du. \end{aligned}$$

Integrate by parts using Lemma 5.2.3 with $Y = \kappa_3$ and $\vartheta = \vartheta_1$ (by Lemma 5.4.5, we have $\vartheta_1(-\infty) = 0$) to obtain

$$\lim_{n \rightarrow \infty} \langle f(\varphi_n) \rangle_{[0,1]} = f(v) - \kappa_3(v)(g_1(v) - w_1(v)) + \int_{-\infty}^v \kappa'_3(u) \vartheta_1(u) du.$$

Thus, to check that the limit coincides with (5.4.15), it suffices to verify the identity

$$\begin{aligned} \int_{-\infty}^v \kappa'_3(u) \vartheta_1(u) du &= (\kappa(v) - \kappa_2(v))(w_1(v) - g_1(v)) \\ &\quad \times \int_{-\infty}^v \exp\left(-\int_u^v \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \frac{\kappa'_3(u)}{\kappa_2(u) - \kappa(u)} du, \end{aligned} \quad (5.4.17)$$

which follows from (5.2.12):

$$\begin{aligned} \vartheta_1(u) &= (g_1(u) - w_1(u)) \exp\left(-\int_u^v \lambda\right) \\ &= \frac{(\kappa_2(v) - \kappa(v))(g_1(v) - w_1(v))}{\kappa_2(u) - \kappa(u)} \exp\left(-\int_u^v \frac{\kappa'_2}{\kappa_2 - \kappa}\right). \end{aligned} \quad (5.4.18)$$

The proof is complete. \blacksquare

Propositions 5.4.9, 5.4.10, and 5.4.11 prove Theorem 5.4.2 in case (A).

5.4.3 Case (B) : $\int_{-\infty}^v \lambda = +\infty$, $\vartheta_1(-\infty) > 0$

Proposition 5.4.12. *If $\int_{-\infty}^v \lambda = +\infty$, $\vartheta_1(-\infty) > 0$, and $\kappa_3(-\infty) = -\infty$, then $B(x; f) = +\infty$ for any $x \in \Omega \setminus \partial_{\text{fixed}} \Omega$.*

Proof. It suffices to prove $B(w(v)) = +\infty$. We consider the sequence φ_n defined in (5.4.1); they are test functions for $w(v)$, see Remark 5.4.1. Since the sign of κ_3 coincides with the sign of f' , from (5.4.14), we see that $\langle f(\varphi_n) \rangle_{[0,1]} \rightarrow +\infty$. \blacksquare

Lemma 5.4.13. *If $\int_{-\infty}^v \lambda = +\infty$ and $\vartheta_1(-\infty) > 0$, then the limit $\kappa_2(-\infty)$ is finite and*

$$\int_{-\infty}^v \frac{\kappa'_2}{\kappa_2 - \kappa} = +\infty.$$

Proof. By Lemma 5.4.4, both $\vartheta_1(-\infty)$ and $\vartheta_2(-\infty)$ are finite. Since $\vartheta_1(-\infty) > 0$, $\kappa(-\infty) = \frac{\vartheta_2(-\infty)}{\vartheta_1(-\infty)}$ is finite as well. From (5.4.10) and the monotonicity of κ_2 , we conclude that $\kappa_2(-\infty) = \kappa(-\infty)$.

From (5.2.12), we have

$$\int_{\tau}^v \frac{\kappa'_2}{\kappa_2 - \kappa} = \log \frac{\kappa_2(v) - \kappa(v)}{\kappa_2(\tau) - \kappa(\tau)} + \log \frac{\vartheta_1(v)}{\vartheta_1(\tau)},$$

and the right-hand side tends to $+\infty$ as τ goes to $-\infty$. \blacksquare

Proposition 5.4.14. *If $\int_{-\infty}^v \lambda = +\infty$, $\vartheta_1(-\infty) > 0$, and there is a standard candidate B on $\Omega_{\mathbb{R}}(-\infty, v)$, then $\kappa_3(-\infty) < +\infty$.*

Proof. Assume on the contrary that $\kappa_3(-\infty) = +\infty$. First, we claim that $\mathfrak{R}(-\infty) = -\infty$. If not, then $\mathfrak{R} > C_1$ for some $C_1 \in \mathbb{R}$ in a neighborhood of $-\infty$, i.e., $\kappa'_3 > C_1 \kappa'_2$. Integration of this inequality yields $\kappa_3 < C_1 \kappa_2 + C_2$ for some $C_2 \in \mathbb{R}$. This contradicts to the assumption $\kappa_3(-\infty) = +\infty$ by Lemma 5.4.13. Hence, $\mathfrak{R}(-\infty) = -\infty$ and $\mathfrak{R}' > 0$ in a neighborhood of $-\infty$. By (3.3.4), $\beta'_2 > 0$ in the same neighborhood of $-\infty$, which contradicts the existence of the standard candidate B on $\Omega_{\mathbb{R}}(-\infty, v)$. We conclude that $\kappa_3(-\infty) < +\infty$. \blacksquare

Proposition 5.4.15. *Suppose that $\int_{-\infty}^v \lambda = +\infty$, $\vartheta_1(-\infty) > 0$, and $\kappa_3(-\infty)$ is finite. Let B be the standard candidate on $\Omega_{\mathbb{R}}(-\infty, v)$. Then for every point $x \in \Omega_{\mathbb{R}}(-\infty, v)$, there exists a sequence of non-decreasing test functions $\varphi_{n,x}$ being an optimizing sequence for B at x ; moreover, $\varphi_{n,x} \preceq g(v)$.*

Proof. It suffices to prove the claim for the case $x = w(v) = w_{\mathbb{R}}(v)$. In what follows, we omit the subscript x in our notation. We use the sequence of functions φ_n defined by (5.4.1). All these functions are non-decreasing and satisfy $\varphi_n \preceq g(v)$. Therefore, we only need to verify that $\langle f(\varphi_n) \rangle_{[0,1]} \rightarrow B(w(v))$.

We calculate the limit using (5.4.14):

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle f(\varphi_n) \rangle_{[0,1]} &= -\kappa_3(-\infty)\vartheta_1(-\infty) + f(v) - \int_{-\infty}^v f'(\tau) \exp\left(-\int_{\tau}^v \lambda\right) d\tau \\ &= -\kappa_3(-\infty)\vartheta_1(-\infty) + f(v) - \int_{-\infty}^v \kappa_3(\tau)\vartheta_1'(\tau) d\tau \\ &= f(v) - \kappa_3(v)(g_1(v) - w_1(v)) + \int_{-\infty}^v \kappa_3'(\tau)\vartheta_1(\tau) d\tau. \end{aligned}$$

Then, we use (5.4.15), (5.4.16), and (5.4.17) to conclude that $\langle f(\varphi_n) \rangle_{[0,1]} \rightarrow B(w(v))$. The proof is complete. \blacksquare

Propositions 5.4.12, 5.4.14, and 5.4.15 prove Theorem 5.4.2 in case (B).

5.4.4 Case (C) : $\int_{-\infty}^v \lambda = +\infty$, $\vartheta_1(-\infty) = 0$, $\vartheta_2(-\infty) \neq 0$

Lemma 5.4.16. *If $\int_{-\infty}^v \lambda = +\infty$, $\vartheta_1(-\infty) = 0$, and $\vartheta_2(-\infty) \neq 0$, then we have $\kappa_2(-\infty) = \kappa(-\infty) = -\infty$. In particular, $g_2' < 0$ in a neighborhood of $-\infty$ and $\vartheta_2(-\infty) < 0$.*

Proof. We easily obtain that $\kappa(-\infty) = \frac{\vartheta_2(-\infty)}{\vartheta_1(-\infty)}$ is infinite. From (5.4.10) and the monotonicity of κ_2 , we conclude that $\kappa_2(-\infty) = \kappa(-\infty)$ is infinite as well. Since the function κ_2 is increasing, both these limits are $-\infty$. Then $g_2' = g_1'\kappa_2 < 0$ in a neighborhood of $-\infty$, and from (5.4.7), we obtain that $\vartheta_2(-\infty) < 0$. \blacksquare

Proposition 5.4.17. *If $\int_{-\infty}^v \lambda = +\infty$, $\vartheta_1(-\infty) = 0$, $\vartheta_2(-\infty) \neq 0$, and $\mathfrak{K}(-\infty) = +\infty$, then $B(x; f) = +\infty$ for any $x \in \Omega \setminus \partial_{\text{fixed}} \Omega$.*

Proof. It suffices to prove that $B(w(v)) = +\infty$. We consider the sequence φ_n defined in (5.4.1); they are test functions for $w(v)$, see Remark 5.4.1. Using (5.4.6), we rewrite (5.4.12) in the form

$$\begin{aligned} \langle f(\varphi_n) \rangle_{[0,1]} &= -\frac{f(v_n^+) - f(v_n^-)}{g_2(v_n^+) - g_2(v_n^-)} \vartheta_2(v_n^+) + f(v) - \int_{v_n^+}^v f'(u) \exp\left(-\int_u^v \lambda\right) du \\ &= -\frac{f'(v_n)}{g_2'(v_n)} \vartheta_2(v_n^+) + f(v) - \int_{v_n^+}^v f'(u) \exp\left(-\int_u^v \lambda\right) du, \quad (5.4.19) \end{aligned}$$

for some $v_n \in (v_n^-, v_n^+)$. Since we have $\vartheta_2(-\infty) < 0$ and $g'_2 < 0$ in a neighborhood of $-\infty$ (see Lemma 5.4.16), it suffices to prove that $\frac{f'}{g'_2}(-\infty) = +\infty$. This follows from the identity $\frac{f'}{g'_2} = \frac{\kappa_3}{\kappa_2}$ and the L'Hôpital rule: $\kappa_2(-\infty) = -\infty$, and $\frac{\kappa'_3}{\kappa'_2}(-\infty) = \mathfrak{K}(-\infty) = +\infty$. We see that $\langle f(\varphi_n) \rangle_{[0,1]} \rightarrow +\infty$. ■

Proposition 5.4.18. *If $\int_{-\infty}^v \lambda = +\infty$, $\vartheta_1(-\infty) = 0$, $\vartheta_2(-\infty) \neq 0$, and there is a standard candidate B on $\Omega_{\mathbb{R}}(-\infty, v)$, then $\mathfrak{K}(-\infty) > -\infty$.*

Proof. Assume on the contrary that $\mathfrak{K}(-\infty) = -\infty$. Then $\mathfrak{K}' > 0$ in a neighborhood of $-\infty$. By (3.3.4), we see that $\beta'_2 > 0$, which is impossible for the standard candidate B , see Definition 3.3.8. ■

Proposition 5.4.19. *Let $\int_{-\infty}^v \lambda = +\infty$, $\vartheta_1(-\infty) = 0$, and $\vartheta_2(-\infty) \neq 0$, and let $\mathfrak{K}(-\infty)$ be finite. If B is the standard candidate on $\Omega_{\mathbb{R}}(-\infty, v)$, then, for every point $x \in \Omega_{\mathbb{R}}(-\infty, v)$, there exists a sequence of non-decreasing test functions $\varphi_{n,x}$ being an optimizing sequence for B at x ; moreover, $\varphi_{n,x} \leq g(v)$.*

Proof. It suffices to prove the claim for the case $x = w(v) = w_{\mathbb{R}}(v)$. In what follows, we omit the subscript x in our notation. We use the sequence of functions φ_n defined by (5.4.1). All these functions are non-decreasing and satisfy $\varphi_n \leq g(v)$. Therefore, it suffices to verify that $\langle f(\varphi_n) \rangle_{[0,1]} \rightarrow B(w(v))$.

First, we note that $\frac{f'}{g'_2} = \frac{\kappa_3}{\kappa_2}$ and $\kappa_2(-\infty) = -\infty$, by Lemma 5.4.16, and therefore, by the L'Hôpital rule,

$$\frac{f'}{g'_2}(-\infty) = \frac{\kappa'_3}{\kappa'_2}(-\infty) = \mathfrak{K}(-\infty).$$

We calculate the limit using (5.4.19):

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle f(\varphi_n) \rangle_{[0,1]} &= -\mathfrak{K}(-\infty)\vartheta_2(-\infty) + f(v) - \int_{-\infty}^v f'(\tau) \exp\left(-\int_{\tau}^v \lambda\right) d\tau \\ &= -\mathfrak{K}(-\infty)\vartheta_2(-\infty) + f(v) - \int_{-\infty}^v \kappa_3(\tau)\vartheta'_1(\tau) d\tau \\ &= -\mathfrak{K}(-\infty)\vartheta_2(-\infty) + f(v) - \kappa_3(v)(g_1(v) - w_1(v)) \\ &\quad + \int_{-\infty}^v \kappa'_3(\tau)\vartheta_1(\tau) d\tau, \end{aligned}$$

where in the latter equality, we integrate by parts using Lemma 5.2.3 with $Y = \kappa_3$ and $\vartheta = \vartheta_1$. Then, we use (5.4.17) to rewrite this formula in the form

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle f(\varphi_n) \rangle_{[0,1]} &= f(v) + (w_1(v) - g_1(v))\kappa_3(v) + (w_1(v) - g_1(v))(\kappa(v) - \kappa_2(v)) \\ &\quad \times \left(\frac{-\mathfrak{K}(-\infty)\vartheta_2(-\infty)}{(w_1(v) - g_1(v))(\kappa(v) - \kappa_2(v))} + \int_{-\infty}^v \exp\left(-\int_{\tau}^v \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \frac{\kappa'_3(\tau)}{\kappa_2(\tau) - \kappa(\tau)} d\tau \right). \end{aligned}$$

We need to verify that the expression in the large parentheses coincides with β_2 given by (3.3.3). We have the following formula for β_2 :

$$\begin{aligned} \mathfrak{K}(v) &= \int_{-\infty}^v \exp\left(-\int_{\tau}^v \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \mathfrak{K}'(\tau) d\tau \\ &= \mathfrak{K}(-\infty) \exp\left(-\int_{-\infty}^v \frac{\kappa'_2}{\kappa_2 - \kappa}\right) + \int_{-\infty}^v \exp\left(-\int_{\tau}^v \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \frac{\kappa'_3(\tau)}{\kappa_2(\tau) - \kappa(\tau)} d\tau. \end{aligned}$$

Therefore, it suffices to verify that

$$\begin{aligned} -\vartheta_2(-\infty) &\stackrel{?}{=} (g_1(v) - w_1(v))(\kappa_2(v) - \kappa(v)) \exp\left(-\int_{-\infty}^v \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \\ &= (g_1(v) - w_1(v))(\kappa_2(v) - \kappa(v)) \lim_{u \rightarrow -\infty} \exp\left(-\int_u^v \frac{\kappa'_2}{\kappa_2 - \kappa}\right) \\ &\stackrel{(5.4.18)}{=} \lim_{u \rightarrow -\infty} (\kappa_2(u) - \kappa(u)) \vartheta_1(u) = \lim_{u \rightarrow -\infty} \left(\frac{\kappa_2(u)}{\kappa(u)} - 1\right) \vartheta_2(u), \quad (5.4.20) \end{aligned}$$

where all the limits exist. Since the limit $\vartheta_2(-\infty)$ is finite and nonzero, it suffices to prove that $\frac{\kappa_2}{\kappa}(-\infty) = 0$; note that this limit exists. We rewrite κ_2 and κ in terms of ϑ_1 and ϑ_2 :

$$\kappa = \frac{\vartheta_2}{\vartheta_1}, \quad \kappa_2 = \frac{g'_2}{g'_1} = \frac{\vartheta'_2}{\vartheta'_1},$$

therefore

$$\frac{\kappa_2}{\kappa}(-\infty) = \frac{(\log |\vartheta_2|)'}{(\log \vartheta_1)' }(-\infty). \quad (5.4.21)$$

Since $\log \vartheta_1(-\infty) = -\infty$, and the limit (5.4.21) exists, we finish the proof by the L'Hôpital rule:

$$0 = \frac{\log |\vartheta_2|}{\log \vartheta_1}(-\infty) = \frac{(\log |\vartheta_2|)'}{(\log \vartheta_1)' }(-\infty) = \frac{\kappa_2}{\kappa}(-\infty). \quad \blacksquare$$

Remark 5.4.20. As opposed to cases (A) and (B), in case (C), we have

$$\int_{-\infty}^v \frac{\kappa'_2}{\kappa_2 - \kappa} < \infty,$$

which follows from (5.4.20).

Propositions 5.4.17, 5.4.18, and 5.4.19 prove Theorem 5.4.2 in case (C). Thus, we have completed the proof of the theorem.

5.4.5 Behavior at $+\infty$

We formulate the analog of Theorem 5.4.2 for the behavior at $+\infty$ that generalizes Proposition 5.2.7 for the case when condition (2.1.8) fails.

Theorem 5.4.21. *Let B be the standard candidate on $\Omega_L(v, +\infty)$. Then, for every point $x \in \Omega_L(v, +\infty)$, there exists a sequence of non-increasing test functions $\varphi_{n,x}$ being an optimizing sequence for B at x ; moreover, $g(v) \preceq \varphi_{n,x}$.*

Since the situation is absolutely symmetric, we omit the proof of this theorem.

5.4.6 When $B = +\infty$?

We collect the conditions on the behavior of the function f at both infinities that guarantee $B(x; f) = +\infty$ for any $x \in \Omega \setminus \partial_{\text{fixed}} \Omega$.

Propositions 5.4.10, 5.4.12, 5.4.17, and 5.2.6 describe the conditions at $-\infty$, see Table 5.2.

condition on Ω	condition on f
(A) $\exp(-\int_{-\infty}^v \lambda_R) \neq 0$	$f(-\infty) = +\infty$
(B) $\exp(-\int_{-\infty}^v \lambda_R) = 0,$ $\vartheta_1(-\infty) \neq 0$	$\kappa_3(-\infty) = -\infty$
(C) $\exp(-\int_{-\infty}^v \lambda_R) = 0,$ $\vartheta_1(-\infty) = 0, \vartheta_2(-\infty) \neq 0$	$\mathfrak{R}(-\infty) = +\infty$
(2.1.7) $\exp(-\int_{-\infty}^v \lambda_R) = 0,$ $\vartheta_1(-\infty) = 0, \vartheta_2(-\infty) = 0$	$\int_{-\infty}^v f'(\tau) \exp(-\int_{\tau}^v \lambda_R) d\tau = -\infty$

Table 5.2. Conditions at $-\infty$ for $B = +\infty$.

Here $v \in \mathbb{R}$ is arbitrary, the conditions do not depend on v . We underline that Condition 2.1.11 is of crucial importance for these results. If Condition 2.1.11 is violated, the problem of characterizing the functions f for which B is infinite, is more involved, see [17, Lemma 6.1.4].

Remark 5.4.22. In case (A), the conditions $\int_{-\infty}^v f'(\tau) \exp(-\int_{\tau}^v \lambda_R) d\tau = -\infty$ and $f(-\infty) = +\infty$ are equivalent. In cases (B) and (C), the conditions on f look different, but in case (C), the condition $\mathfrak{R}(-\infty) = +\infty$ is equivalent to $\frac{f'}{g_2}(-\infty) = +\infty$, which is symmetric to the condition $\frac{f'}{g_1}(-\infty) = \kappa_3(-\infty) = -\infty$ that appears in case (B).

As usual, we omit the symmetrical statements for the case of $+\infty$ but collect the corresponding results in Table 5.3. Therein,

$$\vartheta^L(\tau) = (g(\tau) - w_L(\tau)) \exp\left(\int_v^{\tau} \lambda_L\right), \quad \tau \in \mathbb{R},$$

where v is a fixed parameter, $v \in \mathbb{R}$.

condition on Ω	condition on f
(A) $\exp(\int_v^{+\infty} \lambda_L) \neq 0$	$f(+\infty) = +\infty$
(B) $\exp(\int_v^{+\infty} \lambda_L) = 0,$ $\vartheta_1^L(+\infty) \neq 0$	$\kappa_3(+\infty) = +\infty$
(C) $\exp(\int_v^{+\infty} \lambda_L) = 0,$ $\vartheta_1^L(+\infty) = 0, \vartheta_2^L(+\infty) \neq 0$	$\mathfrak{K}(+\infty) = +\infty$
(2.1.8) $\exp(\int_v^{+\infty} \lambda_L) = 0,$ $\vartheta_1^L(+\infty) = 0, \vartheta_2^L(+\infty) = 0$	$\int_v^{+\infty} f'(\tau) \exp(\int_v^\tau \lambda_L) d\tau = +\infty$

Table 5.3. Conditions at $+\infty$ for $B = +\infty$.

Remark 5.4.23. The function $B(\cdot; f)$ is infinite on $\Omega \setminus \partial_{\text{fixed}} \Omega$ if the pair (Ω, f) satisfies one of the conditions listed in the tables presented in this subsection. If (Ω, f) satisfies none of the conditions, then the function $B(\cdot; f)$ is finite on Ω and can be constructed using the procedure described in this memoir.

5.4.7 Examples of domains

In Section 2.1.1 we have seen several examples of the domains that satisfy conditions (2.1.7) and (2.1.8). Now we present the domains that satisfy conditions (A), (B), or (C). We provide examples of the required behavior when t tends to $-\infty$. The case of $+\infty$ is obviously symmetric.

$$\begin{aligned}
 \text{(A)} \quad & g(t) = (e^t + e^{t/2}, e^{2t} + 2e^{3t/2}), \quad w(t) = (e^t, e^{2t}), \quad \lambda(t) = e^{t/2}, \\
 \text{(B)} \quad & g(t) = (t, (1+t+t^2)e^{-t^2}), \quad w(t) = (-t^2, e^{-t^2}), \quad \lambda(t) = -\frac{2}{t+1}, \\
 \text{(C)} \quad & g(t) = \left(t + \frac{t^2}{1+t^2}, \frac{e^{-t}}{1+t^2}\right), \quad w(t) = (t, e^{-t}), \quad \lambda(t) = 1 + \frac{1}{t^2}.
 \end{aligned}$$

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Bellman Functions on Simple Non-Convex Domains in the Plane

The present memoir provides a generalization of the authors' previous work on Bellman functions for integral functionals on BMO. Those Bellman functions are the minimal locally concave functions on parabolic strips in the plane. We describe an algorithm for constructing minimal locally concave functions on a planar domain that is a difference of two unbounded convex domains. This leads to many sharp estimates for functions in classes such as BMO, A_p , or the Gehring classes.

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