

Solved and Unsolved Problems

Michael Th. Rassias (University of Zürich, Switzerland)

The calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more unequivocally than anything else the inception of modern mathematics; and the system of mathematical analysis, which is its logical development, still constitutes the greatest technical advance in exact thinking.

John von Neumann (1903–1957)

The column in this issue is devoted to fundamentals of mathematical analysis.

Mathematical analysis (or simply analysis) is an enormous field and arguably one of the most central in all of mathematics, appearing in the most abstract of research as well as an extremely wide range of applicable areas like physics, engineering, finance, sociology and biology, to name just a few.

In mathematics, in principle, one can study two categories of structures and phenomena: *discrete*¹ and *continuous*. Generally speaking, the study of the continuous lies at the heart of analysis. The origin of analysis as an independent field of mathematics traces back to the 17th century, with the discovery of the differential by Isaac Newton, and Gottfried Wilhelm Leibniz playing a central role in its genesis. We must note, though, that several important mathematical concepts of analysis were introduced even earlier. For example, the concept of an *integral* traces back to Eudoxus (ca. 390–337 BC) and Archimedes (ca. 287–212 BC). Some of the central generative discoveries of analysis arose from the effort to answer fundamental questions in disciplines such as astronomy, optics and engineering, as well as from the effort to determine mathematical methods for the calculation of areas, volumes, centres of gravity, etc., for both theoretical and practical applications.

Since analysis, as mentioned above, is a vast field of mathematics with several subfields, we shall devote future individual columns to subfields like real analysis, complex analysis, harmonic analysis, etc.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

187. Let $(a_n)_{n \geq 1}$, $(b_n)_{n \geq 1}$ and $(c_n)_{n \geq 0}$ be sequences such that $a_n > 0$, $b_n > 0$ and $c_n > 0$ for $n \geq 1$ and:

(G1) $c_0 = 0$ and c_n is increasing,

(G2) $c_{n+1} - c_n$ is decreasing for $n \geq 0$,

(G3) $c_k \left(\frac{a_{k+1}}{a_k} - 1 \right) \geq c_n \left(\frac{b_{n+1}}{b_n} - 1 \right)$ for $1 \leq k < n$.

Given a function f , let

$$A_n = \frac{1}{c_{n-1}} \sum_{k=1}^{n-1} f\left(\frac{a_k}{b_n}\right), \quad n \geq 2.$$

Then, if f is real, convex increasing and non-negative on an interval $[D, E]$ that includes all the points $\frac{a_k}{b_n}$ for $k < n$, prove that A_n increases with n .

(Shoshana Abramovich, University of Haifa, Israel)

188. For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a positive integer n , we denote by f^n the function defined by $f^n(x) = (f(x))^n$.

(a) Show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function that has an antiderivative then $f^n : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the intermediate value property for any $n \geq 1$.

(b) Give an example of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that has an antiderivative and for which $f^n : \mathbb{R} \rightarrow \mathbb{R}$ has no antiderivatives for any $n \geq 2$.

(Dorin Andrica, Babeş Bolyai University, Cluj-Napoca, Romania)

189.

(a) Let $\{f_n\}_{n=1}^\infty$ be an increasing sequence of continuous real-valued functions on a compact metric space X that converges pointwisely to a continuous function f . Show that the convergence must be uniform.

(b) Show by a counterexample that the compactness of X in (a) is necessary.

(c) Determine whether (a) remains valid if the sequence $\{f_n\}_{n=1}^\infty$ is not monotone.

(W. S. Cheung, University of Hong Kong, Pokfulam, Hong Kong)

190. Let $\{a_n\}$ be a sequence of positive numbers. In the ratio test, we know that the condition

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$$

is not sufficient to determine whether the series $\sum_{n=1}^\infty a_n$ is convergent or divergent. For example, if $a_n = 1/n$ then

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} = 1 - \frac{1}{n+1} = 1 - \frac{n+1}{(n+1)^2}$$

and if $a_n = 1/n^2$ then

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} = 1 - \frac{2n+1}{(n+1)^2}.$$

Hence, the coefficient a in the expression $1 - \frac{an+1}{(n+1)^2}$ plays an important role in the convergence of $\sum a_n$. In this question, we would like to study it more closely.

Let a be a non-negative real number and let $\{a_n\}$ be a sequence with $a_n > 0$, satisfying

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{an+1}{(n+1)^2} \tag{1}$$

for all $n \geq n_0 := \lceil [2-a] + 1 \rceil$, where $[x]$ is the integral part of x .

(i) Show that if $a > 0$ then

$$\lim_{n \rightarrow \infty} a_n = 0.$$

If $a = 0$, for any $\lambda > 0$, find an example such that

$$\lim_{n \rightarrow \infty} a_n = \lambda.$$

(ii) Show that if $a > 1$ then

$$\sum_{n=1}^{\infty} a_n$$

is convergent. Is this still true when $a = 1$?

(Stephen Choi and Peter Lam, Simon Fraser University, Burnaby B.C., Canada)

191. Show that for any $a, b > 0$, we have

$$\frac{1}{2} \left(1 - \frac{\min\{a, b\}}{\max\{a, b\}} \right)^2 \leq \frac{b-a}{a} - \ln b + \ln a \leq \frac{1}{2} \left(\frac{\max\{a, b\}}{\min\{a, b\}} - 1 \right)^2.$$

(Silvestru Sever Dragomir, Victoria University, Melbourne City, Australia)

192. Let $a, b, c, d \in \mathbb{R}$ with $bc > 0$. Calculate

$$\lim_{n \rightarrow \infty} \begin{pmatrix} \cos \frac{a}{n} & \sin \frac{b}{n} \\ \frac{c}{n} & \cos \frac{d}{n} \end{pmatrix}^n.$$

(Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania)

II Open Problems: Some questions related to the Heisenberg uncertainty principle, by Larry Guth
(Massachusetts Institute of Technology, Department of Mathematics, Cambridge, Massachusetts, USA)

The Heisenberg uncertainty principle is a fundamental idea in Fourier analysis. It loosely says that f and \hat{f} cannot both be concentrated into small regions. When I was a student, I thought of the Heisenberg uncertainty principle as a single inequality, which was proven almost a hundred years ago. But this idea that f and \hat{f} cannot both be concentrated into small regions can be made precise in many ways. So there are many cousins of the Heisenberg uncertainty principle. In a recent paper, Jean Bourgain and Semyon Dyatlov proved a striking new variant of the Heisenberg uncertainty principle called the fractal uncertainty principle – see [BD] for the original paper and [D] for an expository survey paper. After looking at that paper and talking with Semyon, I have been wondering about different variations of the Heisenberg uncertainty principle. I think there is probably a great deal that we don't know yet about the Heisenberg uncertainty principle and here are some questions in that spirit.

Some of the questions seem cleanest in the setting of functions on \mathbb{Z}_N – the integers modulo N . Suppose that

$$f : \mathbb{Z}_N \rightarrow \mathbb{C}.$$

Recall that $\hat{f} : \mathbb{Z}_N \rightarrow \mathbb{C}$ is defined by

$$\hat{f}(m) = \frac{1}{N} \sum_{n \in \mathbb{Z}_N} f(n) e^{-2\pi i \frac{nm}{N}}.$$

Then, we have the Fourier inversion theorem

$$f(n) = \sum_{m \in \mathbb{Z}_N} \hat{f}(m) e^{2\pi i \frac{nm}{N}}.$$

Suppose that f is supported in a set $X \subset \mathbb{Z}_N$ and \hat{f} is supported in $Y \subset \mathbb{Z}_N$. What can we say about X and Y ? One classical version of the Heisenberg uncertainty principle says that $|X||Y| \geq N$. On the other hand, if $|X| + |Y| > N$ then there is always a non-zero function f so that f is supported in X and \hat{f} is supported in Y . The set of all such functions is a linear subspace of $L^2(\mathbb{Z}_N)$ defined by $2N - |X| - |Y|$ equations, and if $|X| + |Y| > N$ then the dimension of this subspace is at least 1. In the case when N is prime, we have a complete characterisation. The result was proven independently by Tao [T], by Biro and by Meshulam (see [T] for more references).

Theorem 1 Suppose that N is prime and $X, Y \subset \mathbb{Z}_N$. Then, there is a non-zero function f with the support of f in X and the support of \hat{f} in Y if and only if $|X| + |Y| > N$.

On the other hand, if N is composite then the situation is quite different because of the subgroups of \mathbb{Z}_N . For instance, if $N = M^2$ is a square and if f is the characteristic function of the multiples of M then the support of f has cardinality M and the support of \hat{f} is also a subgroup of cardinality M . This saturates the bound $|X||Y| \geq N$.

193*. For each N , give a complete characterisation of possible pairs $X, Y \subset \mathbb{Z}_N$ admitting a non-zero function f with the support of f in X and the support of \hat{f} in Y .

Theorem 1 gives the result when N is prime. The square of a prime could be a good next case. I believe the state of the art is in a paper of Meshulam [M].

It would also be interesting to prove more quantitative versions of the Heisenberg uncertainty principle. Instead of asserting that there is no function f so that f is supported in X and \hat{f} is supported in Y , it would be nice to say that there is no function f so that f is concentrated in X and \hat{f} is concentrated in Y . For any sets $X, Y \subset \mathbb{Z}_N$, define

$$H(X, Y) := \max_{f: \|f\|_{L^2(\mathbb{Z}_N)} = 1} \|f\|_{L^2(X)} \|\hat{f}\|_{L^2(Y)}.$$

Based on the examples above, we expect that $H(X, Y)$ could be big in cases related to subgroups of \mathbb{Z}_N and probably also in cases related to approximate subgroups of \mathbb{Z}_N such as arithmetic progressions. It would be interesting to better understand what happens in other cases that are far from these. One class of examples is random examples.

194*. Suppose $0 < \alpha < 1$ and suppose that $X, Y \subset \mathbb{Z}_N$ are independent random subsets chosen uniformly among all subsets of cardinality N^α . Estimate the expected value of $H(X, Y)$.

For this question, I think it would even be interesting to find a conjecture.

In additive combinatorics, there are several ways of saying that a set X is far from being an approximate subgroup. One such way uses the idea of additive energy. Recall that the energy of X is defined by

$$E(X) := |\{(x_1, x_2, x_3, x_4) \in X^4 : x_1 + x_2 = x_3 + x_4\}|.$$

If X is a subgroup of \mathbb{Z}_N then there is a unique choice of x_4 for each x_1, x_2, x_3 and so $E(X) = |X|^3$, which is the maximum possible value of $E(X)$. On the other hand, if X is a random subset of \mathbb{Z}_N of cardinality N^α then $E(X) \sim |X|^3 N^{\alpha-1} + |X|^2$.

195*. Given $|X|, |Y|, E(X), E(Y)$, what is the maximum possible size of $H(X, Y)$?

There are a lot of parameters in this question, so let me highlight one particular case that seems interesting to me.

196*. Suppose that $|X| = |Y| \leq 2N^{1/2}$ but $E(X), E(Y) \leq |X|^{2.1}$. Estimate the maximum possible value of $H(X, Y)$.

References

- [BD] J. Bourgain and S. Dyatlov, Spectral gaps without the pressure condition, arXiv:1612.09040.
- [D] S. Dyatlov, Control of eigenfunctions on hyperbolic surfaces: An application of fractal uncertainty principle, arXiv:1710.08762.
- [M] R. Meshulam, An uncertainty inequality for finite abelian groups. *European J. Combin.* **27** (2006), 63–67.
- [T] T. Tao, An uncertainty principle for cyclic groups of prime order. *Math. Res. Lett.* **12** (2005), 121–127.

III Solutions

179. Let $p = p_1 p_2 \dots p_n$ and $q = q_1 q_2 \dots q_n$ be two permutations. We say that they are *colliding* if there exists at least one index i so that $|p_i - q_i| = 1$. For instance, 3241 and 1432 are colliding (choose $i = 3$ or $i = 4$), while 3421 and 1423 are not colliding. Let S be a set of *pairwise colliding* permutations of length n . Is it true that $|S| \leq \binom{n}{\lfloor n/2 \rfloor}$?

(Miklós Bóna, Department of Mathematics, University of Florida, Gainesville, FL 32608, USA)

Solution by the proposer. Yes. Let $p \in S$, let $q \in S$ and let p' (resp. q') denote p (resp. q) modulo 2. As p and q are colliding, there is no index i so that $p'_i = q'_i$. Therefore, if we consider all elements of S modulo 2, we get a set of $|S|$ *different* vectors of length n that have only zeros and ones as coordinates, and in which the number of zeros is $\lfloor n/2 \rfloor$. The number of such vectors is $\binom{n}{\lfloor n/2 \rfloor}$, hence that number is an upper bound for $|S|$. This proof is due to János Körner and Claudia Malvenuto.

Remark: It is not known if the presented upper bound is optimal, though empirically it is for $n \leq 7$. □

Also solved by Mihaly Bencze (Brasov, Romania), Souvik Dey (Kolkata, India), Jim K. Kelesis (Athens, Greece), Panagiotis T. Krasopoulos (Athens, Greece), Alexander Vauth² (Lübbbecke, Germany).

180. Let us say that a word w over the alphabet $\{1, 2, \dots, n\}$ is *n-universal* if w contains all $n!$ permutations of the symbols $1, 2, \dots, n$ as a subword, not necessarily in consecutive positions. For instance, the word 121 is 2-universal as it contains both 12 and 21, while the word 1232123 is 3-universal. Let $n \geq 3$. Does an *n-universal* word of length $n^2 - 2n + 4$ exist?

(Miklós Bóna, Department of Mathematics, University of Florida, Gainesville, FL 32608, USA)

Solution by the proposer. Yes. Write down n copies of 1 and, between two consecutive copies of 1, insert any permutation of the set $\{2, 3, \dots, n\}$. This is a string of $n^2 - n + 1$ entries. Call the strings between two consecutive copies of 1 *segments*. Now remove $n - 3$ entries as follows. Let $1 < i < n - 1$. Moving left to right, remove an entry j from the i th segment. Then, in segment $i - 1$, move the entry j of that segment into the last position. In segment $i + 1$, move the entry j of that segment into the first position. The result is a string with the desired length that contains all permutations of length n . This construction is due to S. P. Mohanty.

Remark: While this construction is optimal for $n \leq 7$, it is not optimal in general. For $n \geq 13$, one can construct an *n-universal* word of length $\lceil n^2 - \frac{7n}{3} + \frac{19}{3} \rceil$. □

Also solved by Mihaly Bencze (Brasov, Romania), Jim K. Kelesis (Athens, Greece), Sotirios E. Louridas (Athens, Greece), Socratis Varelogiannis (Paris, France).

181. Given natural numbers m and n , let $[m]^n$ be the collection of all n -letter words, where each letter is taken from the alphabet $[m] = \{1, 2, \dots, m\}$. Given a word $w \in [m]^n$, a set $S \subseteq [n]$ and $i \in [m]$, let $w(S, i)$ be the word obtained from w by replacing the j th letter with i for all $j \in S$. The Hales–Jewett theorem then says that for any natural numbers m and r , there exists a natural number n such that every r -colouring of $[m]^n$ contains a monochromatic combinatorial line, that is, a monochromatic set of the form $\{w(S, 1), w(S, 2), \dots, w(S, m)\}$ for some $S \subseteq [n]$. Show that for $m = 2$, it is always possible to take S to be an interval in this theorem, while for $m = 3$, this is not the case.

(David Conlon, Mathematical Institute, University of Oxford, Oxford, UK)

Solution by the proposer. For the $m = 2$ case, consider the following $r + 1$ words of length r :

$$\begin{aligned} w_0 &= 111 \dots 11, \\ w_1 &= 111 \dots 12, \\ w_2 &= 111 \dots 22, \\ &\vdots \\ w_{r-1} &= 122 \dots 22, \\ w_r &= 222 \dots 22. \end{aligned}$$

That is, w_i is 1 for the first $r - i$ letters and 2 from then on. By the pigeonhole principle, since there are $r + 1$ words but only r colours, two of these words, say w_i and w_j with $i < j$, receive the same colour. But then, taking $S = [r - j + 1, r - i]$, we see that $w_i = w_i(S, 1)$ and $w_j = w_i(S, 2)$, as required.

For $m = 3$, given a word w , let $n(w)$ be the number of consecutive pairs of letters in w that differ from one another and let χ be the 3-colouring of the words in $[3]^n$ given by $\chi(w) = n(1w1) \pmod{3}$, where $1w1$ is the $(n + 2)$ -letter word formed by adding a single 1 before and after w . Suppose now that the words in $[3]^n$ have been coloured with χ and there is a monochromatic combinatorial line defined by a word $w \in [3]^n$ and an interval $S \subseteq [n]$. Suppose also that the letter in w that immediately precedes S is a , while the letter that immediately follows S is b (note that we added the dummy 1's above so that these are always defined). If now, for example, $a = 1$ and $b = 2$, it is easy to check that $\chi(w(S, 1)) \neq \chi(w(S, 3))$, since changing $\dots 1|11 \dots 1|2 \dots$ to $\dots 1|33 \dots 3|2 \dots$ adds one to the number of consecutive pairs of letters that differ from one another. Therefore, this case cannot occur. Similarly, one can easily verify that none of the other possible choices of a and b can occur. Therefore, S cannot have been an interval. □

Also solved by Mihaly Bencze (Brasov, Romania), Socratis Varelogiannis (Paris, France).

182. (A) Let A_1, A_2, \dots be finite sets, no two of which are disjoint. Must there exist a finite set F such that no two of $A_1 \cap F, A_2 \cap F, \dots$ are disjoint?
 (B) What happens if all of the A_i are the same size?

(Imre Leader, Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge, UK)

Solution by the proposer. (A) No. Just make the A_i meet “further and further to the right”. For example, take the sets

$$\{2, 4, 5\}, \{1, 3, 5, 6\}, \{2, 4, 6, 7\}, \{1, 3, 5, 7, 8\}, \{2, 4, 6, 8, 9\}, \dots$$

(B) There does have to be such a set. We fix one set A in our family and group the other sets according to how they intersect A – so we write $F(I)$ for the sets in our family that intersect A equal I (for each non-empty subset I of A). More precisely, let us write $G(I)$ for the family formed by each set in $F(I)$ but with I removed. So, to be done, we would like that for each I and J (disjoint subsets of A), there exists a finite set on which all of $G(I)$ meet all of $G(J)$.

This looks a lot like the original statement. So, instead, we prove the stronger statement “for any r and s , if we have some r -sets and some s -sets and each of the r -sets meets each of the s -sets then there is a finite set on which each r -set meets each s -set”. And the above argument does prove this, by induction on, say, $r + s$. \square

Also solved by John N. Daras (Athens, Greece), Souvik Dey (Kolkata, India), Jean Moulin-Ollagnier (Palaiseau, France), Alexander Vauth (Lübbecke, Germany).

183. The following is from the 2012 Green Chicken maths contest between Middlebury and Williams Colleges. A graph G is a collection of vertices V and edges E connecting pairs of vertices. Consider the following graph. The vertices are the integers $\{2, 3, 4, \dots, 2012\}$. Two vertices are connected by an edge if they share a divisor greater than 1; thus, 30 and 1593 are connected by an edge as 3 divides each but 30 and 49 are not. The colouring number of a graph is the smallest number of colours needed so that each vertex is coloured *and* if two vertices are connected by an edge then those two vertices are not coloured the same. The Green Chicken says the colouring number of this graph is at most 9. Prove he is wrong and find the correct colouring number.

(Steven J. Miller, Department of Mathematics and Statistics, Williams College, Williamstown, MA, USA)

Solution by the proposer. The colouring number is at least 10, as the vertices $2, 4, 8, 16, 32, \dots, 1024 = 2^{10}$ are all connected to each other, and thus we need at least 10 colours. Why? This is a complete graph with 10 vertices, and its colouring number is 10. As this subgraph of our graph has colouring number 10, the entire graph has colouring number at least 10.

We can get a very good lower bound easily. Instead of looking at powers of 2, we can look at the even numbers. There are 1006 even numbers and each even number is connected to every other. Thus, we have a complete graph with 1006 vertices, implying the colouring number is at least 1006.

It’s easy to see the colouring number is at most $2012 - \pi(2012) + 1$, where $\pi(2012)$ is the number of primes at most 2012. Why? We can colour all the primes the same colour, as none are connected to any other. That’s our plus 1; the $2012 - \pi(2012)$ comes from a trivial bounding, using a different colour for each remaining vertex.

Interestingly, our lower bound is the answer: the colouring number is 1006. To see this, choose 1006 colours and colour each even number with one of these colours, never using the same colour twice. Note we *have* to do this, as no two even numbers can share a colour. We are left with colouring the odd numbers $3, 5, 7, 9, \dots, 2011$. We colour the vertex $2k + 1$ with the colour of vertex $2k$. Note $2k + 1$ and $2k$ can’t share a factor d greater than 1 and are thus not connected. (If they shared a factor, it would have to divide their difference, which is 1.) Since vertex $2k$ is the only vertex that has the colour we want

to use for vertex $2k + 1$, we see that we have a valid colouring. We showed the colouring number must be at least 1006; since we’ve found a colouring that works with 1006 colours, we know this must be the answer. \square

Also solved by Mihaly Bencze (Brasov, Romania), Jim K. Kelesis (Athens, Greece), Panagiotis T. Krasopoulos (Athens, Greece), Alexander Vauth (Lübbecke, Germany), Socratis Varelogiannis (Paris, France).

184. There are n people at a party. They notice that for every two of them, the number of people at the party that they both know is *odd*. Prove that n is an odd number.³

(Benny Sudakov, Department of Mathematics, ETH Zürich, Zürich, Switzerland)

Solution by the proposer. Let G be a graph whose vertices are the people at the party and two are connected if they know each other. Then this graph has the property that every pair of vertices have an odd number of common neighbours. Let N_v be the set of neighbours of some vertex v and let $G[N_v]$ be the subgraph of G induced by this set. Then, all degrees of $G[N_v]$ are odd, since these are exactly the number of common neighbours that v has with its neighbours. Since the sum of the degrees of the vertices in $G[N_v]$ is twice its number of edges (an even number) we have that $|N_v|$ is even. Therefore, all vertices in G have even degree.

Consider now the adjacency matrix A of G over the field with two elements (addition and multiplications are modulo 2). This is an n by n symmetric matrix whose rows and columns are indexed by the vertices of G and $a_{uv} = 1$ if u, v are adjacent and 0 otherwise. Note that the sum of the columns of A is 0 (modulo 2), since every row has an even number of 1’s. Therefore, A does not have full rank. Consider $B = A^2$. It is easy to check that the diagonal of B consists of the degrees of the vertices (modulo 2) of G and b_{uv} is the number of common neighbours of u and v (modulo 2). Therefore, A^2 has 0’s on the diagonal and 1’s everywhere else. When n is even, such a matrix has full rank, since it has a non-zero determinant (which is easy to compute). Since A does not have full rank, neither does A^2 . This implies that n is odd. \square

Also solved by Mihaly Bencze (Brasov, Romania), Jim K. Kelesis (Athens, Greece), Jean Moulin-Ollagnier (Palaiseau, France).

Notes

1. For the “Solved and Unsolved Problems” column devoted to discrete mathematics, the reader is referred to Issue 105, September 2017, EMS Newsletter, p. 55.
2. Alexander Vauth made the interesting remark that this problem has also been affirmatively proven in the paper: Körner, J., and Malvenuto, C., *Pairwise colliding permutations and the capacity of infinite graphs*, SIAM Journal on Discrete Mathematics, 20(1) (2006), 203-212.
3. This problem appeared a long time ago in the Tournament of the Towns.

We would like you to submit solutions to the proposed problems and ideas on the open problems. Send your solutions by email to Michael Th. Rassias, Institute of Mathematics, University of Zürich, Switzerland, michail.rassias@math.uzh.ch.

We also solicit your new problems with their solutions for the next “Solved and Unsolved Problems” column, which will be devoted to probability theory.