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On the Regularity of Weak Solutions of Quasi-Linear Elliptic Transmission Problems on Polyhedral Domains

Dorothee Knees

Abstract. The regularity of weak solutions of quasi-linear elliptic boundary transmission problems of p-structure on polyhedral domains Ω is considered. Ω is divided into polyhedral subdomains Ω_i and it is assumed that the growth properties of the differential operator vary from subdomain to subdomain. We prove higher regularity of weak solutions up to the transmission surfaces, provided that the differential operators are distributed quasi-monotonely with respect to the subdomains Ω_i. The proof relies on a difference quotient technique which is based on the ideas of C. Ebmeyer and J. Frehse.

Keywords: *Regularity, quasi-linear elliptic transmission problem, nonsmooth domains, difference quotient technique*

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1. Introduction

This paper is concerned with the study of the global regularity of weak solutions of boundary transmission problems for nonlinear elliptic systems with p–structure, $1 < p < \infty$. The systems are defined in polygonal or polyhedral domains $\Omega = \cup_{i=1}^{M} \Omega_i \subset \mathbb{R}^d$, $d \geq 2$, and have the following form for $u : \Omega \to \mathbb{R}^m$, $u_i = u\big|_{\Omega_i}$:

$$\operatorname{div}_x (D_A W_i(\nabla u_i)) + f_i = 0 \quad \text{in } \Omega_i,\ 1 \leq i \leq M, \tag{1}$$

$$u_i - u_j = 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega_j, \tag{2}$$

$$D_A W_i(\nabla u_i)\vec{n}_{ij} + D_A W_j(\nabla u_j)\vec{n}_{ji} = 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega_j, \tag{3}$$

$$u = g \quad \text{on } \Gamma_D, \tag{4}$$

$$D_A W_i(\nabla u_i)\vec{n}_i = h \quad \text{on } \Gamma_N. \tag{5}$$

Dorothee Knees, Institut für Angewandte Analysis und Numerische Simulation, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany;
knees@ians.uni-stuttgart.de

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The functions $W_i : \mathbb{R}^{m\times d} \to \mathbb{R}$ can be interpreted as energy densities and satisfy growth conditions which will be specified in Section 3. $D_A W_i(A)$ denotes the gradient of $W_i(A)$ for $A \in \mathbb{R}^{m\times d}$. It is admitted that the energy densities W_i have different growth properties on each subdomain. We assume mixed boundary conditions on $\partial\Omega$ which are given by the equations (4)–(5). Equations (2)–(3) describe the transmission or interface conditions which connect $u\big|_{\Omega_i}$ and $u\big|_{\Omega_j}$ on the interface $\partial\Omega_i \cap \partial\Omega_j$. In particular, the differential equations (1) can be

$$\operatorname{div}\left(\mu(x)\left|\nabla u\right|^{p(x)-2}\nabla u\right) + f = 0,$$

where μ, $p : \Omega \to \mathbb{R}$ are piecewise constant with respect to the partition of Ω. The main result, Theorem 4.1, states the following: If the energy densities W_i are distributed quasi-monotonely (see Section 4.1), then the weak solution $u\big|_{\Omega_i}$ is in $W^{\frac{3}{2}-\epsilon,r_i}(\Omega_i)$ for a suitable $r_i \in [p_i, 2]$ if $p_i \in (1, 2]$, and $u\big|_{\Omega_i}$ is in $W^{1+\frac{1}{p_i}-\epsilon,p_i}(\Omega_i)$ if $p_i > 2$. This result generalizes known results for linear and quasi-linear elliptic transmission problems.

In the case of linear elliptic systems on polyhedral domains as well as for linear elliptic transmission problems it is well known that the behavior of weak solutions in the neighborhood of corners, edges or cross points can be completely described by an asymptotic expansion. Linear elliptic systems on a single domain are treated in [3, 12, 15, 16, 19, 22], whereas transmission problems are investigated in [24, 25, 26]. The singular exponents in the asymptotic expansions characterize the regularity of weak solutions and depend on the structure of the differential operators, the geometry and the type of the boundary conditions. In the papers [2, 14, 20, 26, 28], the authors derive estimates for the singular exponents of solutions of Poisson's equation and for the equations of linear, isotropic elasticity with constant or piecewise constant coefficients. It turned out that a quasi-monotone distribution of these coefficients in combination with some constraints on the geometry of Ω leads to piecewise $H^{\frac{3}{2}}-$regularity of weak solutions. On the other hand there are various examples which show that the regularity can get very low (i.e. $H^{1+\epsilon}$, $\epsilon > 0$ small) if these conditions are violated, see e.g. [13].

Let us note that our main result, Theorem 4.1, can be applied to transmission problems for Poisson's equation and also to coupled linear, isotropic elasticity and is in accordance with the results cited above. Moreover, our results cover more general linear elliptic transmission problems such as coupled anisotropic heat equations or composites of different anisotropic linear elastic materials and provide new estimates for the singular exponents.

There are only few results concerning the regularity of weak solutions of quasi-linear elliptic transmission problems: In [18], the author describes the regularity of weak solutions of a scalar quasi-linear elliptic transmission problem with two subdomains. Here, it is assumed that the interface is smooth and does not intersect with the exterior boundary $\partial\Omega$. Due to the smoothness of the interface, the author derives a higher regularity result compared to our main theorem. In [10], a transmission problem on a polyhedral domain which is divided into two polyhedral subdomains with a plane interface is investigated. It is essential for the proof that the energy densities W_i, $i = 1, 2$, behave "nearly quadratic", i. e. the energy densities satisfy certain growth properties with $p_1 = p_2 = 2$. It shall be emphasized that in our main theorem there is no restriction on the number of subdomains which come together at a crossing point, and we deal also with the case when the growth properties of the energy densities W_i vary from subdomain to subdomain.

Our main result is proved with the help of a difference quotient technique. This technique is widely used in order to derive interior regularity results, see for example [4, 21, 23, 29, 32], and was improved by C.Ebmeyer and J.Frehse in order to obtain global regularity results on polyhedral domains, [6, 8, 9]. In the proof of the main theorem, test functions of the form $\xi_l(x) = \varphi^2(x)(u(x+he_l) - u(x))$, $1 \le l \le d$, are inserted into the weak formulation. Here, u denotes a weak solution, φ is a cut-off function, $h > 0$ and $\{e_1, \dots, e_d\}$ is a basis of $\mathbb{R}^d$. The difficulty is that the differences are taken across the transmission boundaries and due to the different growth properties of the differential operators on the subdomains, the functions ξ_l are not admissible test functions in general. From the assumption that the energy densities W_i are distributed quasi-monotonely one can deduce the existence of a basis $\{e_1, \dots, e_d\}$ such that the functions $\xi_l, 1 \le l \le d$, are admissible test functions. This is the key for the proof of our main theorem.

The quasi-monotonicity condition, which will be introduced in this paper, is a considerable modification and generalization of the original definition by M. Dryja, M. V. Sarkis and O. B. Widlund. In [5] they defined quasi-monotonicity for the distribution of the parameters in Poisson's equation with piecewise constant coefficients. In the present paper, we change the point of view and define quasi-monotonicity for the distribution of the energy densities W_i which correspond to the transmission problem. The relation between the definition in [5] and our definition is discussed in Chapter 4.

The paper is organized as follows: In Section 2, the domains and function spaces are defined following the approach in [18]. The weak formulation of the transmission problem and existence results are presented briefly in Section 3. Here, the main theorem of monotone operators plays a crucial role. In Section 4, the quasi-monotonicity is introduced and illustrated by various examples for two and three dimensional domains. The main theorem is stated and proved in Section 4 using the difference-quotient technique. The paper closes with an appendix, where some essential inequalities are given, which follow from the growth properties and convexity of the energy densities W_i.

2. Domains and function spaces

Throughout the whole article it is assumed that $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded polygonal or polyhedral domain with Lipschitz-boundary. It is further assumed that there exists a finite number of pairwise disjoint polyhedral domains $\Omega_i \subset \Omega$, $1 \leq i \leq M$, with Lipschitz-boundaries such that

$$\overline{\Omega} = \bigcup_{i=1}^{M} \overline{\Omega_i}, \quad \Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j.$$

On each of these subdomains a differential operator will be given and the growth properties of these operators may vary from subdomain to subdomain. Therefore, the following function spaces are introduced, which take into account the splitting of Ω (analogously to [18]):

For $1 \leq i \leq M$ let $p_i \in (1, \infty)$, $\vec{p} = (p_1, \ldots, p_M)$, $p_{\min} = \min\{p_i, 1 \leq i \leq M\}$. Then

$$L^{\vec{p}}(\Omega) := \left\{ u \in L^{p_{\min}}(\Omega) : \ u\big|_{\Omega_i} \in L^{p_i}(\Omega_i) \right\}$$
$$W^{1,\vec{p}}(\Omega) := \left\{ u \in W^{1,p_{\min}}(\Omega) : \ u\big|_{\Omega_i} \in W^{1,p_i}(\Omega_i) \right\},$$

where $u\big|_{\Omega_i}$ is the restriction of u to the subdomain Ω_i. These spaces are endowed with the following norms:

$$\|u\|_{L^{\vec{p}}(\Omega)} := \sum_{i=1}^{M} \left\| u\big|_{\Omega_i} \right\|_{L^{p_i}(\Omega_i)}$$
$$\|u\|_{W^{1,\vec{p}}(\Omega)} := \sum_{i=1}^{M} \left\| u\big|_{\Omega_i} \right\|_{W^{1,p_i}(\Omega_i)}.$$

Note that we do not distinguish in the notation between scalar and vector valued functions or spaces. The next lemma states some essential properties of these spaces.

Lemma 2.1. [18] *Let $p_i \in (1, \infty)$ for $1 \leq i \leq M$. Then:*

1. *$L^{\vec{p}}(\Omega)$ is a reflexive Banach space and the dual space is given by $\left(L^{\vec{p}}(\Omega)\right)' = L^{\vec{q}}(\Omega)$, where $\vec{q} = (q_1, \ldots, q_M)$ and $q_i = p_i'$, i.e. $\frac{1}{p_i} + \frac{1}{q_i} = 1$.*
2. *$W^{1,\vec{p}}(\Omega)$ is a reflexive Banach space.*
3. *$\mathcal{C}^\infty(\overline{\Omega})$ is dense in $L^{\vec{p}}(\Omega)$ and also in $W^{1,\vec{p}}(\Omega)$.*

Since $W^{1,\vec{p}}(\Omega)$ is contained in $W^{1,p_{\min}}(\Omega)$, the trace operator

$$W^{1,\vec{p}}(\Omega) \to W^{1-\frac{1}{p_{\min}},p_{\min}}(\partial\Omega) : \ u \to u\big|_{\partial\Omega}$$

is well defined, linear and continuous [12]. Analogously to [18], the space of traces of functions from $W^{1,\vec{p}}(\Omega)$ is defined as

$$W^{\frac{\vec{p}-1}{\vec{p}},\vec{p}}(\partial\Omega) := \left\{ u\big|_{\partial\Omega} :\ u \in W^{1,\vec{p}}(\Omega) \right\},$$

where $\frac{\vec{p}-1}{\vec{p}} := (1-\frac{1}{p_1},\ldots,1-\frac{1}{p_M})$. The trace theorem [12] also shows that the latter space is a subspace of

$$\{u \in L^1(\partial\Omega) :\ u\big|_{(\partial\Omega\cap\partial\Omega_i)} \in W^{1-\frac{1}{p_i},p_i}(\partial\Omega\cap\partial\Omega_i)\}.$$

For the description of mixed boundary value problems, the following spaces are useful: Let $\partial\Omega = \overline{\Gamma_D}\cup\overline{\Gamma_N}$, where Γ_D and Γ_N are open and disjoint. Then

$$\begin{aligned}
V^{\vec{p}}(\Omega) &= \left\{ u \in W^{1,\vec{p}}(\Omega) :\ u\big|_{\Gamma_D} = 0 \right\} \\
W^{(\vec{p}-1)/\vec{p}}(\Gamma_D) &= \left\{ u\big|_{\Gamma_D} :\ u \in W^{\frac{\vec{p}-1}{\vec{p}},\vec{p}}(\partial\Omega) \right\} \\
\tilde{W}^{(\vec{p}-1)/\vec{p}}(\Gamma_N) &= \left\{ u\big|_{\Gamma_N} :\ u \in V^{\vec{p}}(\Omega) \right\} \\
&= \left\{ u\big|_{\Gamma_N} :\ u \in W^{\frac{\vec{p}-1}{\vec{p}},\vec{p}}(\partial\Omega) \text{ and } u\big|_{\Gamma_D} = 0 \right\}.
\end{aligned}$$

Finally, there is an equivalent characterization of the space $W^{1,\vec{p}}(\Omega)$.

Lemma 2.2. *Let $p_i \in (1,\infty)$ for $1 \le i \le M$. Then*

$$W^{1,\vec{p}}(\Omega) = \left\{ u \in L^{\vec{p}}(\Omega) :\ u\big|_{\Omega_i} \in W^{1,p_i}(\Omega_i),\ \left(u\big|_{\Omega_i}\right)\big|_{\Gamma_{ij}} = \left(u\big|_{\Omega_j}\right)\big|_{\Gamma_{ij}} \right\}. \quad (6)$$

Moreover, $W^{1,\vec{p}}(\Omega)$ is a closed subspace of $\left\{ u \in L^{\vec{p}}(\Omega) :\ u\big|_{\Omega_i} \in W^{1,p_i}(\Omega_i) \right\}$.

In other words, the space $W^{1,\vec{p}}(\Omega)$ consists of all functions which are piecewise in $W^{1,p_i}(\Omega_i)$ and which do not jump at the interfaces Γ_{ij}.

Proof. Let $u \in W^{1,\vec{p}}(\Omega)$ be a scalar-valued function and $\varphi \in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^d) = \{v : \Omega \to \mathbb{R}^d :\ v \in \mathcal{C}^\infty(\Omega), \operatorname{supp} v \subset \Omega\}$. Since $u \in W^{1,p_{\min}}(\Omega)$, there holds

$$\begin{aligned}
0 &= \langle \nabla u, \varphi\rangle - \int_\Omega \nabla u\cdot\varphi \,\mathrm{d}x \\
&= -\int_\Omega u \operatorname{div}\varphi \,\mathrm{d}x - \int_\Omega \nabla u\cdot\varphi\,\mathrm{d}x \\
&= -\sum_{i=1}^M \int_{\Omega_i} \operatorname{div}(u_i\varphi)\,\mathrm{d}x \\
&\overset{\text{Gauss}}{=} -\sum_{i=1}^M \int_{\partial\Omega_i} u\,(\varphi\cdot\vec{n}_i)\,\mathrm{d}s \\
&= -\sum_{i=1}^M\sum_{j=1}^{i-1} \int_{\Gamma_{ij}} \left((u\big|_{\Omega_i})\big|_{\Gamma_{ij}} - (u\big|_{\Omega_j})\big|_{\Gamma_{ij}} \right)(\varphi\cdot\vec{n}_{ij})\,\mathrm{d}s
\end{aligned}$$

for the distributional derivative of u. Since $\varphi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^d)$ is arbitrary, it follows that $\left(u\big|_{\Omega_i}\right)\big|_{\Gamma_{ij}} - \left(u\big|_{\Omega_j}\right)\big|_{\Gamma_{ij}} = 0$ on Γ_{ij}, and "$\subset$ " is proved in (6). In order to prove the inverse relation one has to show that functions from the space on the right hand side in (6) are elements of $W^{1,p_{\min}}(\Omega)$. To prove this, one has to calculate the distributional derivative of these functions. With the help of Gauss' Theorem the assertion follows. ∎

The Sobolev embedding theorems can be carried over directly to the $W^{1,\vec{p}}(\Omega)$ spaces, see [18], and consequently there is also an inequality of Poincaré-Friedrichs' type.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain with Lipschitz boundary which is decomposed into M pairwise disjoint polyhedral subdomains with Lipschitz boundaries; $1 < p_i < \infty$ for $1 \le i \le M$. If $V \subset W^{1,\vec{p}}(\Omega)$ is a closed subspace with the property*

$$u \in V,\ \nabla u = 0 \text{ in } \Omega \implies u = 0 \text{ in } \Omega,$$

then there exists a constant $c > 0$ such that for every $u \in V$: $\|u\|_{L^{\vec{p}}(\Omega)} \le c\,\|\nabla u\|_{L^{\vec{p}}(\Omega)}$.

Proof. This lemma can be proved (as in the case $M = 1, p = 2$, [35]) by contradiction using that the embedding $W^{1,\vec{p}}(\Omega) \to L^{\vec{p}}(\Omega)$ is compact. ∎

Difference quotients of weak solutions will be estimated in the proof of the regularity results. Therefore we introduce the Nikolskii space, which takes difference quotients into account explicitly.

Definition 2.1 (Nikolskii space). [1, 27] Let $\Omega \subset \mathbb{R}^d$ be an open domain, $s = m + \sigma$, where $m \ge 0$ is an integer and $0 < \sigma < 1$. For $1 < p < \infty$

$$\mathcal{N}^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \ \|u\|_{\mathcal{N}^{s,p}(\Omega)} < \infty \right\} \tag{7}$$

is a *Nikolskii space*, where

$$\|u\|^p_{\mathcal{N}^{s,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + \sum_{|\alpha|=m} \sup_{\substack{\eta>0 \\ h\in\mathbb{R}^d \\ 0<|h|<\eta}} \int_{\Omega_\eta} \frac{|D^\alpha u(x+h) - D^\alpha u(x)|^p}{|h|^{\sigma p}}\,\mathrm{d}x \tag{8}$$

and $\Omega_\eta = \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \eta\}$.

The relation between Nikolskii spaces and Sobolev-Slobodeckij spaces is described in the next lemma.

Lemma 2.4. [1, 27, 33, 34] *Let s, p be as in Definition 2.1. If $\Omega = \mathbb{R}^d$ or if $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary, then the following embeddings are continuous for every $\varepsilon > 0$:*

$$\mathcal{N}^{s+\varepsilon,p}(\Omega) \subset W^{s,p}(\Omega) \subset \mathcal{N}^{s,p}(\Omega).$$

Proof. If $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary, then there exist linear and continuous extension operators $E_1 : W^{s,p}(\Omega) \to W^{s,p}(\mathbb{R}^d)$ and $E_2 : \mathcal{N}^{s,p}(\Omega) \to \mathcal{N}^{s,p}(\mathbb{R}^d)$ for $s > 0$ and $1 < p < \infty$ (see [12, Theorem 1.4.1.3] for $W^{s,p}$ and [27, p. 381] for $\mathcal{N}^{s,p}$). Furthermore, the restriction operators from $\mathbb{R}^d$ to Ω are continuous as well. Therefore it suffices to prove Lemma 2.4 for the case $\Omega = \mathbb{R}^d$.

For s, p as in Definition 2.1 and $1 \leq r \leq \infty$ we denote by $B^s_{p,r}(\mathbb{R}^d)$ the Besov spaces on $\mathbb{R}^d$. For the definition see, e.g., [31, 33]. There holds $B^s_{p,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d)$ and $B^s_{p,\infty}(\mathbb{R}^d) = \mathcal{N}^{s,p}(\mathbb{R}^d)$, see [33, Sections 1.3 and 2.2.9]. The following embeddings are continuous for $\epsilon > 0$, [34, Section 2.3.2, Proposion 2] and [33, Section 2.1.1]:

$$\mathcal{N}^{s+\epsilon,p}(\mathbb{R}^d) = B^{s+\epsilon}_{p,\infty}(\mathbb{R}^d) \subset B^s_{p,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d) \subset B^s_{p,\infty}(\mathbb{R}^d) = \mathcal{N}^{s,p}(\mathbb{R}^d).$$

This completes the proof. Note that in Lemma 2.4 the assumptions on Ω can be weakened: Lemma 2.4 is valid for domains for which continuous extension operators E_1 and E_2 exist. ∎

For inner products and norms of matrices $A, B \in \mathbb{R}^{m\times d}$, $m \geq 1, d \geq 2$, the following abbreviations are used:

$$A : B = \operatorname{tr}(B^T A) = \operatorname{tr}(AB^T) = \sum_{i=1}^{m}\sum_{j=1}^{d} A_{ij}B_{ij},$$

$$|A| = \sqrt{A : A} = \left(\sum_{i=1}^{m}\sum_{j=1}^{d} A_{ij}^2\right)^{\frac{1}{2}}.$$

For $R > 0$ and $x \in \mathbb{R}^d$, $B_R(x)$ denotes the open ball with center x and radius R: $B_R(x) = \{y \in \mathbb{R}^d : |x - y| < R\}$ and $\partial B_R(x) = \{y \in \mathbb{R}^d : |x - y| = R\}$.

3. Weak formulation of the transmission problem and existence of solutions

In this section we describe the assumptions on the structure of the boundary transmission problem (1)–(5) and give some short comments on the existence of weak solutions.

Let $\Omega \subset \mathbb{R}^d$ be a polygonal or polyhedral domain with Lipschitz boundary which is decomposed into M pairwise disjoint Lipschitz-polyhedra Ω_i (compare Section 2). $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, Γ_D and Γ_N open and disjoint; by $\vec{n}_{ij}$ we denote the exterior normal vector of Ω_i with respect to Γ_{ij}, $\vec{n}_{ij} = -\vec{n}_{ji}$ and $\vec{n}_i$ is the exterior normal vector of Ω_i with respect to $\partial\Omega_i \cap \partial\Omega$. Let $m \geq 1$ and assume that there are given M functions $W_i : \mathbb{R}^{m\times d} \to \mathbb{R}$. The boundary transmission problem reads as follows:

$$\left\{\begin{array}{l} \text{Find } u : \Omega \to \mathbb{R}^m,\ u\big|_{\Omega_i} = u_i \text{ such that:} \\[1em] \begin{aligned} \operatorname{div}_x\left(D_A W_i(\nabla u_i)\right) + f_i &= 0 \quad \text{in } \Omega_i,\ 1 \leq i \leq M && (9)\\ u_i - u_j &= 0 \quad \text{on } \Gamma_{ij} && (10)\\ D_A W_i(\nabla u_i)\vec{n}_{ij} + D_A W_j(\nabla u_j)\vec{n}_{ji} &= 0 \quad \text{on } \Gamma_{ij} && (11)\\ u &= g \quad \text{on } \Gamma_D && (12)\\ D_A W_i(\nabla u_i)\vec{n}_i &= h \quad \text{on } \Gamma_N. && (13) \end{aligned} \end{array}\right.$$

Here and in the sequel the following notations are used. Let $A, B, C \in \mathbb{R}^{m\times d}$, then

$$\begin{aligned} \left(D_A W_i(A)\right)_{k,l} &= \frac{\partial W_i(A)}{\partial A_{kl}} \\ D_A W_i(A) : B &= \sum_{k=1}^{m}\sum_{l=1}^{d} \frac{\partial W_i(A)}{\partial A_{kl}} B_{kl} \\ D_A^2 W_i(A)[B,C] &= \sum_{k,j=1}^{m}\sum_{s,t=1}^{d} \frac{\partial^2 W_i(A)}{\partial A_{ks}\partial A_{jr}} B_{ks} C_{jr} \\ \left|D_A^2 W_i(A)\right| &= \left(\sum_{k,j=1}^{m}\sum_{s,t=1}^{d} \left(\frac{\partial^2 W_i(A)}{\partial A_{ks}\partial A_{jr}}\right)^2\right)^{\frac{1}{2}} \\ \left(\operatorname{div}_x\left(D_A W_i(\nabla u(x))\right)\right)_j &= \sum_{l=1}^{d} \frac{\partial}{\partial x_l}\Big(\left(D_A W_i(\nabla u(x))\right)_{jl}\Big) \end{aligned}$$

with $D_A W_i(A) \in \mathbb{R}^{m\times d}$ and $\operatorname{div}_x(D_A W_i(\nabla u(x)) \in \mathbb{R}^m$. In this paper, it is assumed that the functions W_i are of p–structure which means that the functions W_i and their derivatives satisfy the following growth properties (compare also [7, 8]). Let $p_i \in (1, \infty)$:

(H0): $W_i \in \mathcal{C}^1(\mathbb{R}^{m\times d}) \cap \mathcal{C}^2(\mathbb{R}^{m\times d}\backslash\{0\})$.

(H1): There exist $c_0^i \in \mathbb{R}, c_1^i, c_2^i > 0$ such that for every $A \in \mathbb{R}^{m\times d}$:

$$c_0^i + c_1^i \left|A\right|^{p_i} \leq W_i(A) \leq c_2^i \left(1 + \left|A\right|^{p_i}\right).$$

(H2): There exists $c^i > 0$ such that for every $A \in \mathbb{R}^{m\times d}$:

$$|D_A W_i(A)| \le c^i \left(1 + |A|^{p_i - 1}\right).$$

(H3): There exists $c^i > 0$ such that for every $A \in \mathbb{R}^{m\times d}\backslash\{0\}$:

$$\left|D_A^2 W_i(A)\right| \le c^i \left(1 + |A|^{p_i - 2}\right).$$

(H4): (Ellipticity condition, convexity of W_i.) There exist $c_i > 0$ and $\kappa_i \in \{0,1\}$ such that for every $A, B \in \mathbb{R}^{m\times d}, A \neq 0$:

$$D_A^2 W_i(A)[B,B] \ge c_i \left(\kappa_i + |A|\right)^{p_i - 2} |B|^2 .$$

We are now able to describe in which sense equations (9)–(13) shall be solved.

Definition 3.1. Let $\Omega \subset \mathbb{R}^d, d \ge 2$, with $\overline{\Omega} = \bigcup_{i=1}^M \overline{\Omega_i}$ be a polygonal or polyhedral domain as introduced above, $m \in \mathbb{N}$. Assume that the functions $W_i : \mathbb{R}^{m\times d} \to \mathbb{R}$ satisfy (H0)–(H4) with $p_i \in (1,\infty)$. Let $\vec{p} = (p_1, \ldots, p_M)$, $\vec{q} = (q_1, \ldots, q_M)$ with $q_i = p_i' = \frac{p_i}{p_i - 1}$ and $f \in L^{\vec{q}}(\Omega, \mathbb{R}^m)$, $g \in W^{(\vec{p}-1)/\vec{p}}(\Gamma_D, \mathbb{R}^m)$ and $h \in \left(\tilde{W}^{(\vec{p}-1)/\vec{p}}(\Gamma_N, \mathbb{R}^m)\right)'$. A function $u : \Omega \to \mathbb{R}^m$, $u \in W^{1,\vec{p}}(\Omega)$, is a *weak solution* of the boundary transmission problem (9)–(13) if $u\big|_{\Gamma_D} = g$ and if for every $v \in V^{\vec{p}}(\Omega, \mathbb{R}^m)$

$$\sum_{i=1}^M \int_{\Omega_i} D_A W_i(\nabla u_i(x)) : \nabla v_i(x)\,\mathrm{d}x = \sum_{i=1}^M \int_{\Omega_i} f_i(x) v_i(x)\,\mathrm{d}x + \langle h, v\rangle. \qquad (14)$$

Thereby, $\langle \cdot, \cdot \rangle$ denotes the *dual pairing* between elements of $\left(\tilde{W}^{(\vec{p}-1)/\vec{p}}(\Gamma_N)\right)'$ and $\tilde{W}^{(\vec{p}-1)/\vec{p}}(\Gamma_N)$.

If a weak solution u and the right hand sides f, g, h in equation (14) are smooth enough, then u satisfies equations (9)–(13).

Remark. The functions W_i can be interpreted as energy density functions. Furthermore, equation (14) is the weak Euler-Lagrange equation which is associated with the following minimizing problem:

$$\left\{ \begin{array}{l} \text{Find } u \in W^{1,\vec{p}}(\Omega) \text{ with } u\big|_{\Gamma_D} = g \text{ such that for every} \\ v \in W^{1,\vec{p}}(\Omega) \text{ with } v\big|_{\Gamma_D} = g\text{:} \\ \qquad\qquad J(u) \le J(v), \end{array} \right.$$

where $J(v) = \sum_{i=1}^M \int_{\Omega_i} W_i(\nabla v)\,\mathrm{d}x - \int_\Omega f v\,\mathrm{d}x - \langle h, v\rangle$.

Remark. Note that the coupling of linear homogeneously elliptic systems of second order with constant coefficients, where in addition the principal parts of the differential operators coincide with the differential operators themselves and which are Euler-Lagrange equations for minimizing problems, is also included here as a special case.

It shall be emphasized that different exponents p_i for the functions W_i on each subdomain Ω_i are possible. The following existence result is a direct consequence of the theorem on monotone operators, see e.g. [36].

Theorem 3.1 (Existence). *Let $\Omega \subset \mathbb{R}^d$ be a polyhedral domain with Lipschitz boundary $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ and assume that it is decomposed into M polyhedral subdomains Ω_i as introduced in section 2. For $1 \le i \le M$ let $p_i \in (1, \infty)$ and assume that $W_i : \mathbb{R}^{m\times d} \to \mathbb{R}$ satisfies (H0)–(H4). Furthermore, let $f \in L^{\vec{q}}(\Omega)$, where $q_i = p_i'$, $g \in W^{(\vec{p}-1)/\vec{p}}(\Gamma_D)$ and $h \in \big(\tilde{W}^{(\vec{p}-1)/\vec{p}}(\Gamma_N)\big)'$. For $\Gamma_D = \emptyset$ the following solvability condition shall be satisfied for every constant function v:*

$$\int_\Omega f v \,\mathrm{d}x + \langle h, v \rangle = 0. \tag{15}$$

Then there exists a weak solution $u \in W^{1,\vec{p}}(\Omega)$ of problem (14) with $u\big|_{\Gamma_D} = g$. If $\Gamma_D = \emptyset$, then u is unique, else u is unique up to constants.

Proof. The theorem can be proved with the main theorem of monotone operators, see for example [36]. Hypotheses (H0)–(H4), inequality (49) in the Appendix and Poincaré-Friedrichs' inequality guarantee that the nonlinear operator, which is related to the weak formulation, satisfies the assumptions of the main theorem of monotone operators. In particular, the operator $W^{1,\vec{p}}(\Omega) \to \big(W^{1,\vec{p}}(\Omega)\big)' : u \to \sum_{i=1}^M \int_{\Omega_i} D_A W_i(\nabla u_i(x)) : \nabla(\,\cdot\,) \,\mathrm{d}x$ is continuous and monotone on $W^{1,\vec{p}}(\Omega)$ and coercive on $V^{\vec{p}}(\Omega)$ if $\Gamma_D \neq \emptyset$. ∎

Remark (Physically nonlinear elasticity). Let $m = d \in \{2, 3\}$ and assume that $D_A W_i(B)$ is symmetric if $B \in \mathbb{R}^{d\times d}$ is symmetric. It is reasonable to consider the following equation instead of equation (14):

$$\sum_{i=1}^M \int_{\Omega_i} D_A W_i(\varepsilon(u_i(x))) : \varepsilon(v_i(x)) \,\mathrm{d}x = \sum_{i=1}^M \int_{\Omega_i} f_i(x) v_i(x) \,\mathrm{d}x + \langle h, v \rangle, \tag{16}$$

where $\varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$ is the linearized strain tensor corresponding to the displacement field u. For this equation, the statements of Theorem 3.1 hold without any changes when $\Gamma_D \neq \emptyset$. In the case of $\Gamma_D = \emptyset$, one has to require that the solvability condition (15) is satisfied for every $v \in \ker\varepsilon$, which is the set of rigid body motions.

4. Regularity results for polyhedral domains

In this section, the main result on regularity of weak solutions of transmission problems on polyhedral domains is formulated. The main result in Theorem 4.1 states: If the energy densities W_i satisfy a quasi-monotonicity condition, then $u_i \in W^{\frac{3}{2}-\varepsilon,r_i}(\Omega_i)$ for a suitable $r_i \in [p_i,2]$ for $p_i \in (1,2]$ and $u\big|_{\Omega_i} \in W^{1+\frac{1}{p}-\epsilon,p}(\Omega_i)$ if $p_i > 2$. As a special case, the theorem includes the earlier derived results for Poisson's equation and Lamé's equation with piecewise constant coefficients, see [14]. The quasi-monotone distribution of the energy densities W_i is the essential assumption for our main theorem. The definition will be given in Section 4.1 and is inspired by the definition of M. Dryja, M. V. Sarkis and O. B. Widlund in [5] for the distribution of the coefficients in Poisson's equation with piecewise constant coefficients. Let us remark that our definition of quasi-monotonicity is a generalization of the definition in [5] and can be applied to a large class of linear and nonlinear boundary transmission problems.

The proof of the main result uses a difference quotient technique for polyhedra, which was developed by C. Ebmeyer and J. Frehse in [7, 9], where they investigated the global regularity of weak solutions of nonlinear elliptic systems of $p-$structure on polyhedral domains.

Throughout the whole section various examples illustrate the condition of quasi-monotonicity. Furthermore, the obtained regularity results will be compared with known results for linear elliptic transmission problems.

4.1. Quasi-monotone distribution of energy densities. In the proof of the main theorem, $\overline{\Omega} = \bigcup_{i=1}^{M} \overline{\Omega_i}$ will be divided into a finite number of model domains, where it is assumed that each of these model domains coincides with the intersection of a ball with a collection of N suitable polyhedral cones (N depends on the model domain). This motivates the next definition.

Definition 4.1 (Polyhedral cone). A set $\mathcal{K} \subset \mathbb{R}^d$, is a *polyhedral cone with tip in* $S \in \mathbb{R}^d$ if

1. there exists $\mathcal{C} \subset \partial B_1(0)$, $\mathcal{C}$ open and not empty such that

$$\mathcal{K} = \left\{ x \in \mathbb{R}^d : \ \frac{x-S}{|x-S|} \in \mathcal{C} \right\}$$

2. there is a finite number of hyperplanes E_i, $1 \le i \le n$ such that

$$\partial\mathcal{K} = \bigcup_{i=1}^{n} \overline{E_i \cap \partial\mathcal{K}}.$$

Note that $\mathcal{K}$ is open and $S \notin \mathcal{K}$.

Definition 4.2 (Quasi-monotonicity with respect to interior cross points). Let $\mathcal{K}_1, \ldots, \mathcal{K}_N \subset \mathbb{R}^d$ be pairwise disjoint polyhedral cones with tip in 0 such that $\mathbb{R}^d = \cup_{i=1}^N \overline{\mathcal{K}_i}$. For $s \in \mathbb{N}$ we consider N functions $W_i : \mathbb{R}^s \to \mathbb{R} \cup \{\pm\infty\}$, $1 \le i \le N$. The functions W_i are *distributed quasi-monotonely with respect to the cones* $\mathcal{K}_i$ if there exist numbers $k_1, \cdots, k_N \in \mathbb{R}$ and a basis $\{e_1, \ldots e_d\} \subset \mathbb{R}^d$ with $|e_l| = 1$ such that for every $h > 0$, $1 \le l \le d$ and $1 \le i, j \le N$ there holds the implication

$$(\mathcal{K}_i + he_l) \cap \mathcal{K}_j \neq \emptyset \implies W_j(A) + k_j \ge W_i(A) + k_i \quad \text{for every } A \in \mathbb{R}^s. \tag{17}$$

Here, $\mathcal{K}_i + he_l = \left\{x \in \mathbb{R}^d : \; x = y + he_l, \, y \in \mathcal{K}_i\right\}$.

In the two dimensional case, this definition can be reformulated in a more illustrative way. Let $d = 2$ and assume that the polygonal cones $\mathcal{K}_i$ in Definition 4.2 are given as follows: There are angles $\Phi_0 < \Phi_1 < \ldots < \Phi_N = \Phi_0 + 2\pi$ such that $\mathcal{K}_i = \{x \in \mathbb{R}^2 : \; 0 < r, \, \Phi_{i-1} < \varphi < \Phi_i\}$. Here, polar coordinates are used.

Lemma 4.1. *Let $d = 2$. The functions $W_i : \mathbb{R}^s \to \mathbb{R}$ are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ if and only if the following two conditions are satisfied:*

1. *There exist numbers $k_i \in \mathbb{R}$ and indices $i_{min}, i_{max} \in \{1, \ldots, N\}$ such that for every $A \in \mathbb{R}^s$ (the indices are numbered modulo N):*

$$\begin{aligned} W_{i_{max}}(A) + k_{i_{max}} &\ge W_{i_{max}+1}(A) + k_{i_{max}+1} \ge \ldots \\ &\ge W_{i_{min}-1}(A) + k_{i_{min}-1} \ge W_{i_{min}}(A) + k_{i_{min}} \\ W_{i_{min}}(A) + k_{i_{min}} &\le W_{i_{min}+1}(A) + k_{i_{min}+1} \le \ldots \\ &\le W_{i_{max}-1}(A) + k_{i_{max}-1} \le W_{i_{max}}(A) + k_{i_{max}}. \end{aligned}$$

2. *There exists a vector $\vec{t} \in \mathbb{R}^2$, $|\vec{t}| = 1$, such that $\vec{t} \in \mathcal{K}_{i_{max}}$ and $-\vec{t} \in \mathcal{K}_{i_{min}}$.*

The second condition in the previous lemma states that $\mathcal{K}_{i_{\min}}$ and $\mathcal{K}_{i_{\max}}$ are lying opposite, see also Figure 1, where $i_{\max} = 1$.

Proof. If $\mathcal{K}_i$ and W_i satisfy conditions *1.* and *2.* in lemma 4.1, then it is easy to see that the functions W_i are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ in the sense of definition 4.2: Choose $e_1 = \vec{t}$. From 2. in Lemma 4.1 and from the assumption that the cones $\mathcal{K}_i$ are open, it follows that there exists a vector $\vec{t}_* \neq \vec{t}$ with $\vec{t}_* \in \mathcal{K}_{i_{\max}}$ and $-\vec{t}_* \in \mathcal{K}_{i_{\min}}$. Choose $e_2 = \vec{t}_*$. With this choice, relation (17) is satisfied.

It remains to prove that conditions 1. and 2. of Lemma 4.1 can be deduced from Definition 4.2. Assume that $e_1 = \binom{1}{0}$ and that the cones $\mathcal{K}_i$, $1 \le i \le N$, are numbered counterclockwise in such a way that the intersection of $\mathcal{K}_1$ with

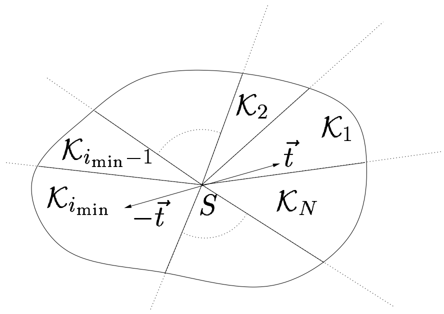

Figure 1: Example for the geometric condition at an interior cross point S

the upper half plane is not empty and that $e_1 \in \overline{\mathcal{K}_1}$. It follows from (17) that there holds

$$\mathcal{K}_i + e_1 \cap \mathcal{K}_j \neq \emptyset \implies j \leq i \text{ and } W_j(A) + k_j \geq W_i(A) + k_i \ \forall A \in \mathbb{R}^s$$

for every $\mathcal{K}_i$ which has a nonempty intersection with the the upper half plane. On the other hand, there holds

$$\mathcal{K}_i + e_1 \cap \mathcal{K}_j \neq \emptyset \implies j \geq i \text{ and } W_j(A) + k_j \geq W_i(A) + k_i \ \forall A \in \mathbb{R}^s$$

for every $\mathcal{K}_j$ which has a nonempty intersection with the lower half plane. It follows that there exist $n \in \{1, \ldots, N\}$ and $\tilde{n} \in \{n, n+1\}$ such that for every $A \in \mathbb{R}^s$

$$W_1(A) + k_1 \geq W_2(A) + k_2 \geq \ldots \geq W_n(A) + k_n \tag{18}$$

$$W_{\tilde{n}}(A) + k_{\tilde{n}} \leq W_{\tilde{n}+1}(A) + k_{\tilde{n}+1} \leq \ldots \leq W_N(A) + k_N \tag{19}$$

holds. In order to find $i_{\min}$, $i_{\max}$ and $\vec{t}$, several cases have to be distinguished.

`Case 1.` $n = \tilde{n}$ and $e_1 \in \mathcal{K}_1$, i.e. the positive x_1-axis is contained in $\mathcal{K}_1$ and the negative x_1-axis is contained in $\mathcal{K}_n$. Then $i_{\min} = n$, $i_{\max} = 1$ and $\vec{t} = e_1$.

`Case 2.` $\tilde{n} = n + 1$ and $e_1 \in \mathcal{K}_1$, i.e. the negative x_1-axis is the interface between $\mathcal{K}_n$ and K_{n+1}. It follows from the assumptions (see Definition 4.2) that $W_N(A)+k_N \leq W_1(A)+k_1$ and therefore $i_{\max} = 1$. To find $i_{\min}$, assume without loss of generality that $e_2 \cdot \binom{0}{1} > 0$. Then it follows that $\mathcal{K}_{n+1} + e_2 \cap \mathcal{K}_n \neq \emptyset$ and therefore, by the assumptions of Definition 4.2, $i_{\min} = n + 1$. Furthermore there exists $\theta \in (0, 1)$ such that $\vec{t} := \theta e_1 + (1 - \theta)e_2$ satisfies condition 2. of Lemma 4.1.

The remaining two cases, where either only the positive x_1-axis or the whole x_1-axis is part of the boundaries of $\mathcal{K}_1$ or $\mathcal{K}_n$, can be treated similarly. ∎

The following corollary is essential in the proof of the regularity results.

Corollary. *Let $\mathcal{K}_1, \ldots, \mathcal{K}_N \subset \mathbb{R}^d$ be polyhedral cones as in definition 4.2. Assume that the functions $W_i : \mathbb{R}^{m\times d} \to \mathbb{R}$ are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ and that they satisfy (H0)-(H1) for some $p_i \in (1,\infty)$. Let $\{e_1, \ldots, e_d\} \subset \mathbb{R}^d$ be the basis in Definition 4.2. Then there holds for every $h > 0$, $1 \le l \le d$, $1 \le i, j \le N$ the implication*

$$(\mathcal{K}_i + he_l) \cap \mathcal{K}_j \neq \emptyset \implies p_j \ge p_i \ .$$

Furthermore, if $u \in W^{1,\vec{p}}(\mathbb{R}^d)$ and has compact support, then also $u(\cdot + he_l) \in W^{1,\vec{p}}(\mathbb{R}^d)$.

Proof. From $\mathcal{K}_i + he_l \cap \mathcal{K}_j \neq \emptyset$ it follows that $W_j(A) + k_j \ge W_i(A) + k_i$ for every $A \in \mathbb{R}^{m\times d}$ and therefore, by (H1) it holds

$$c_2^j(1 + |A|^{p_j}) + k_j \ge c_0^i + c_1^i\,|A|^{p_i} + k_i \qquad \forall\, A \in \mathbb{R}^{m\times d}.$$

This is only possible if $p_j \ge p_i$.

We prove the second assertion: Let $u \in W^{1,\vec{p}}(\mathbb{R}^d)$ with compact support. Then, by the definition of the space $W^{1,\vec{p}}(\mathbb{R}^d)$ we have $u \in W^{1,p_{\min}}(\mathbb{R}^d)$ and $u\big|_{\mathcal{K}_i} \in W^{1,p_i}(\mathcal{K}_i)$. Obviously, $u(\cdot + he_l) \in W^{1,p_{\min}}(\mathbb{R}^d)$ for $h > 0$. It remains to show that $u(\cdot + he_l)\big|_{\mathcal{K}_i} \in W^{1,p_i}(\mathcal{K}_i)$. Note that $u(x + he_l)\big|_{\mathcal{K}_i} = u(y)\big|_{\mathcal{K}_i + he_l}$ with $y = x + he_l$. Furthermore, $\mathcal{K}_i + he_l = \bigcup_{j=1}^N \overline{\mathcal{K}_i + he_l \cap \mathcal{K}_j}$. Assume that $\mathcal{K}_i + he_l \cap \mathcal{K}_j \neq \emptyset$. By the definition of $W^{1,\vec{p}}(\mathbb{R}^d)$, there holds $u\big|_{\mathcal{K}_i + he_l \cap \mathcal{K}_j} \in W^{1,p_j}(\mathcal{K}_i + he_l \cap \mathcal{K}_j)$ and, due to the first assertion of Corollary 4.1, $p_j \ge p_i$. Since u has compact support, Hölder's inequality yields $u\big|_{\mathcal{K}_i + he_l \cap \mathcal{K}_j} \in W^{1,p_i}(\mathcal{K}_i + he_l \cap \mathcal{K}_j)$ for every j with $\mathcal{K}_i + he_l \cap \mathcal{K}_j \neq \emptyset$. Since $u \in W^{1,p_{\min}}(\mathbb{R}^d)$, the assertion follows by arguments which are similar to those in the proof of Lemma 2.2. ∎

The next examples describe some possible choices for the functions W_i and cones $\mathcal{K}_i$ for $d = 2, 3$.

Example 4.1. For $\Phi_0 < \Phi_1 < \ldots < \Phi_N = \Phi_0 + 2\pi$ let $\mathcal{K}_i = \{x \in \mathbb{R}^2 : 0 < r,\ \Phi_{i-1} < \varphi < \Phi_i\}$. Consider the functions $W_i : \mathbb{R}^2 \to \mathbb{R} : A \to \frac{\mu_i}{2}\,|A|^2$ with $\mu_i > 0$. The functions W_i are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ if there exists $i_{\min} \in \{2, \ldots, N\}$ such that

$$\mu_1 \ge \mu_2 \ge \ldots \ge \mu_{i_{\min}} \le \mu_{i_{\min}+1} \le \cdots \le \mu_N \le \mu_1 \tag{20}$$

and $-\mathcal{K}_1 \cap \mathcal{K}_{i_{\min}} \neq \emptyset$, see Figure 1. The constants k_i in Definition 4.2 can be chosen as 0.

The transmission problem, which corresponds to the functions W_i, is Poisson's equation with piecewise constant coefficients μ_i on $\mathcal{K}_i$. Historically, quasi-monotonicity was first defined by Dryja, Sarkis and Widlund in [5] for the distribution of these coefficients. In contrast to our definition they did not require the geometric assumption $-\mathcal{K}_1 \cap \mathcal{K}_{i_{\min}} \neq \emptyset$, which is hidden in Definition 4.2.

Example 4.2. Let $\mathcal{K}_i \subset \mathbb{R}^2$, $1 \leq i \leq N$ be as in Example 4.1 and assume that the functions $W_i : \mathbb{R}^{m\times d} \to \mathbb{R}$ satisfy (H0) and (H1) for some $p_i \in (1, \infty)$ with $p_i \neq p_j$ for $i \neq j$ and $p_1 = \max\{p_i,\, 1 \leq i \leq N\}$. The functions W_i are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ if and only if there exists $i_{\min} \in \{2, \dots, N\}$ such that $-\mathcal{K}_1 \cap \mathcal{K}_{i_{\min}} \neq \emptyset$ and

$$p_1 > p_2 > \ldots > p_{i_{\min}-1} > p_{i_{\min}} < p_{i_{\min}+1} < \ldots < p_N < p_1.$$

Example 4.3. Let $\mathcal{K}_i \subset \mathbb{R}^2$, $1 \leq i \leq N$ be as in Example 4.1 and consider the functions $W_i : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$, $W_i(A) = \frac{1}{2}(\lambda_i + \mu_i)\,|\mathrm{tr}\,A|^2 + \mu_i\,|A^D|^2$, where $\mu_i > 0$, $\lambda_i + \mu_i > 0$ and $A^D = A - \frac{1}{2}(\mathrm{tr}\,A)I$. The functions W_i describe the elastic energy density for homogeneous, isotropic, linear elastic materials with Lamé constants λ_i, μ_i if A is replaced by $\varepsilon(u)$. If there exists an index $i_{\min} \in \{2, \dots, N\}$ such that

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{i_{\min}} \leq \mu_{i_{\min}+1} \leq \cdots \leq \mu_N \leq \mu_1,$$
$$\lambda_1 + \mu_1 \geq \ldots \geq \lambda_{i_{\min}} + \mu_{i_{\min}} \leq \ldots \leq \lambda_N + \mu_N \leq \lambda_1 + \mu_1$$

and $-\mathcal{K}_1 \cap \mathcal{K}_{i_{\min}} \neq \emptyset$, then the functions W_i are distributed quasi-monotonely. This generalizes the definition of quasi-monotonicity for the coefficients of the Lamé equation in [14, Definition 5.1].

Example 4.4. Let $\mathcal{K}_i \subset \mathbb{R}^2$, $1 \leq i \leq N$ be as in Example 4.1. Consider the functions $W_i : \mathbb{R}^s \to \mathbb{R}$ with $W_i(A) = C_i A \cdot A$, where $C_i \in \mathbb{R}^{s\times s}$ is symmetric and positive definite. Let λ_i be the smallest and Λ_i the largest eigenvalue of C_i. If there exists $i_{\min} \in \{2, \dots, N\}$ such that

$$\begin{aligned}\lambda_1 \geq \Lambda_2 \geq \lambda_2 \geq \Lambda_3 \geq \lambda_3 \geq \cdots \geq \lambda_{i_{\min}-1} \geq \Lambda_{i_{\min}}&\\ \Lambda_{i_{\min}} \leq \lambda_{i_{\min}+1} \leq \Lambda_{i_{\min}+1} \leq \ldots \lambda_N \leq \Lambda_N \leq \lambda_1&\end{aligned} \quad (21)$$

and $-\mathcal{K}_1 \cap \mathcal{K}_{i_{\min}} \neq \emptyset$, then the functions W_i are distributed quasi-monotonely. Condition (21) can be weakened if more details are known on the eigenvectors of the matrices C_i. Note that Example 4.3 is a special case of this example.

If $s = 2$, then the corresponding boundary transmission problem reads as follows for $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$: $\operatorname{div}(C_i \nabla u) + f = 0$ in Ω_i together with boundary and transmission conditions. These equations describe transmission problems for anisotropic Laplace operators.

Example 4.5. Consider a cube which is decomposed into two subdomains as in Figure 2 (left). Any two functions $W_i : \mathbb{R}^s \to \mathbb{R}$ which satisfy either a) or b) here below are quasi-monotonely distributed:

a) $\exists k_1, k_2 \in \mathbb{R}$: $\forall\, A \in \mathbb{R}^s$: $W_1(A) + k_1 \geq W_2(A) + k_2$
b) $\exists k_1, k_2 \in \mathbb{R}$: $\forall\, A \in \mathbb{R}^s$: $W_1(A) + k_1 \leq W_2(A) + k_2$

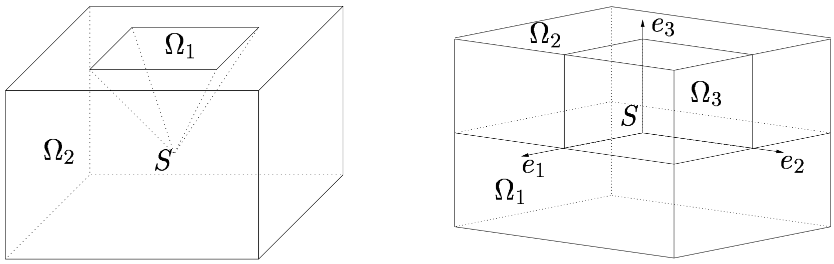

Figure 2: Examples for interior cross points

In the filled Fichera-corner, see Figure 2 (right), the quasi-monotonicity condition is satisfied, if e.g. $W_1(A) + k_1 \leq W_2(A) + k_2 \leq W_3(A) + k_3$ for every $A \in \mathbb{R}^s$. For this case, a possible choice of the vectors e_i is indicated in Figure 2.

The next definition describes quasi-monotonicity for the case, when the cones $\mathcal{K}_i$ do not fill $\mathbb{R}^d$ completely. The definition depends on the kind of the prescribed boundary conditions. For this, let $\mathcal{K}_i \subset \mathbb{R}^d$, $1 \leq i \leq N$, be pairwise disjoint polyhedral cones with tip in 0, $\mathcal{C}_i = \mathcal{K}_i \cap \partial B_1(0)$. We set $\mathcal{C} := \operatorname{int}\left(\bigcup_{i=1}^N \overline{\mathcal{C}_i}\right)$ and assume that $\mathcal{C}_0 := \partial B_1(0) \backslash \overline{\mathcal{C}}$ is not the empty set. Furthermore $\mathcal{K} := \{x \in \mathbb{R}^d : \frac{x}{|x|} \in \mathcal{C}\}$ and $\mathcal{K}_0 := \{x \in \mathbb{R}^d : \frac{x}{|x|} \in \mathcal{C}_0\}$.

Definition 4.3 (Quasi-monotonicity for cross points on the boundary). It is supposed that $\mathcal{K}$ has a Lipschitz boundary and that N functions $W_i : \mathbb{R}^s \to \mathbb{R}$ for $1 \leq i \leq N$ and a fixed $s \geq 2$ are given.

(1) Dirichlet conditions on $\partial\mathcal{K}$: Choose $W_0(A) := \infty$ for $A \in \mathbb{R}^s$. The functions $W_i : \mathbb{R}^s \to \mathbb{R}$, $1 \leq i \leq N$, are *distributed quasi-monotonely with respect to the cones* $\mathcal{K}_i$, $1 \leq i \leq N$, if the functions $W_0, W_1, \ldots, W_N$ are distributed quasi-monotonely with respect to the cones $\mathcal{K}_0, \ldots, \mathcal{K}_N$ in the sense of Definition 4.2.

(2) Neumann conditions on $\partial\mathcal{K}$: Choose $W_0(A) := -\infty$ for $A \in \mathbb{R}^s$. The functions $W_i : \mathbb{R}^s \to \mathbb{R}$, $1 \leq i \leq N$, are *distributed quasi-monotonely with respect to the cones* $\mathcal{K}_i$, $1 \leq i \leq N$, if the functions $W_0, W_1, \ldots, W_N$ are distributed quasi-monotonely with respect to the cones $\mathcal{K}_0, \ldots, \mathcal{K}_N$ in the sense of Definition 4.2.

(3) Mixed conditions on $\partial\mathcal{K}$: Assume that $\partial\mathcal{C} = \overline{\gamma_D} \cup \overline{\gamma_N}$, where γ_D and γ_N are nonempty, open and disjoint sets; $\Gamma_D = \{x \in \mathbb{R}^d : \frac{x}{|x|} \in \gamma_D\}$, $\Gamma_N = \{x \in \mathbb{R}^d : \frac{x}{|x|} \in \gamma_N\}$. The functions $W_1, \ldots, W_N : \mathbb{R}^s \to \mathbb{R}$ are *distributed quasi-monotonely with respect to the cones* $\mathcal{K}_i$ and the splitting of the boundary into Γ_D and Γ_N if there exist two disjoint polyhedral cones $\mathcal{K}_{-\infty}, \mathcal{K}_\infty$ with $\overline{\mathcal{K}_0} = \overline{\mathcal{K}_{-\infty}} \cup \overline{\mathcal{K}_\infty}$ and $\Gamma_D \subset \partial\mathcal{K}_\infty, \Gamma_N \subset \partial\mathcal{K}_{-\infty}$ such that the functions $W_{-\infty}, W_\infty, W_1, \ldots, W_N$ with $W_{-\infty}(A) = -\infty$,

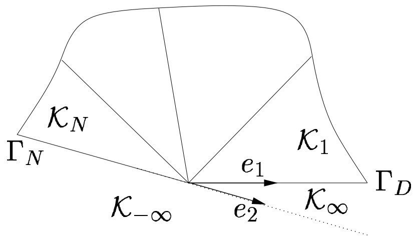

Figure 3: Two dimensional domain with mixed boundary conditions

$W_\infty(A) = \infty$, are distributed quasi-monotonely with respect to the cones $\mathcal{K}_\infty, \mathcal{K}_{-\infty}, \mathcal{K}_1, \ldots, \mathcal{K}_N$ in the sense of Definition 4.2.

Remark. It follows from Definition 4.3 that

$$\begin{aligned} x + he_l &\notin \mathcal{K} \quad \text{for every } x \in \Gamma_D, \\ x + he_l &\in \overline{\mathcal{K} \cup \mathcal{K}_\infty} \quad \text{for every } x \in \Gamma_N. \end{aligned}$$

for every $h > 0, 1 \leq l \leq d$.

The next lemma reformulates Definition 4.3 for the two dimensional case. Assume that $\mathcal{K} \subset \mathbb{R}^2$ is given in the following way (polar coordinates): There exist angles $\Phi_0 < \Phi_1 < \ldots < \Phi_N < \Phi_0 + 2\pi$ such that $\mathcal{K}_i = \{x \in \mathbb{R}^2 : r > 0, \Phi_{i-1} < \varphi < \Phi_i\}$, $\mathcal{K} = \{x \in \mathbb{R}^2 : r > 0, \Phi_0 < \varphi < \Phi_N\}$ and $\mathcal{K}_0 = \{x \in \mathbb{R}^2 : r > 0, \Phi_N < \varphi < \Phi_0 + 2\pi\}$.

Lemma 4.2. *Consider N functions $W_i : \mathbb{R}^s \to \mathbb{R}$, $1 \leq i \leq N$.*

Dirichlet conditions on $\partial\mathcal{K}$: *Let $\partial\mathcal{K} \subset \Gamma_D$. The functions W_i are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ if and only if*

1. *there exist constants $k_1, \ldots, k_N \in \mathbb{R}$ and $i_{min} \in \{1, \ldots, N\}$ such that for every $A \in \mathbb{R}^s$*

$$W_1(A) + k_1 \geq \ldots \geq W_{i_{min}}(A) + k_{i_{min}} \leq \ldots \leq W_N(A) + k_N$$

2. *there exists $\vec{t} \in \mathbb{R}^2$ such that $\vec{t} \in \mathcal{K}_{i_{min}}$ and $-\vec{t} \in \mathcal{K}_0$.*

Neumann conditions on $\partial\mathcal{K}$: *Let $\partial\mathcal{K} \subset \Gamma_N$. The functions W_i are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ if and only if*

1. *there exist constants $k_1, \ldots, k_N \in \mathbb{R}$ and $i_{max} \in \{1, \ldots, N\}$ such that for every $A \in \mathbb{R}^s$:*

$$W_1(A) + k_1 \leq \ldots \leq W_{i_{max}}(A) + k_{i_{max}} \geq \ldots \geq W_N(A) + k_N$$

2. *there exists $\vec{t} \in \mathbb{R}^2$ such that $\vec{t} \in \mathcal{K}_{i_{max}}$ and $-\vec{t} \in \mathcal{K}_0$.*

Mixed conditions on $\partial\mathcal{K}$: *Assume that $\partial\mathcal{K}\cap\partial\mathcal{K}_1\subset\Gamma_D$ and $\partial\mathcal{K}_N\cap\partial\mathcal{K}\subset\Gamma_N$. The functions W_i are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ if and only if*

1. *there exist constants $k_i\in\mathbb{R}$ such that for every $A\in\mathbb{R}^s$*

$$W_1(A)+k_1\geq W_2(A)+k_2\geq\ldots\geq W_N(A)+k_N$$

2. $\measuredangle(\Gamma_D,\Gamma_N)=\Phi_N-\Phi_0<\pi$, *$\measuredangle$ denotes the interior opening angle.*

Proof. The assertions for the case of pure Dirichlet or Neumann conditions on $\partial\mathcal{K}$ follow directly from Definition 4.3 in combination with Lemma 4.1.

In the case of mixed boundary conditions we assume that 1. and 2. in Lemma 4.2 hold. Then a possible choice for e_1,e_2 and $\mathcal{K}_\infty,\mathcal{K}_{-\infty}$ is given by $e_1=\binom{\cos\Phi_0}{\sin\Phi_0}$, $e_2=\binom{\cos(\Phi_N+\pi)}{\sin(\Phi_N+\pi)}$ and $\mathcal{K}_{-\infty}=\{x:\,r>0,\Phi_N<\varphi<\Phi_N+\pi\}$, $\mathcal{K}_\infty=\{x:\,r>0,\Phi_N+\pi<\varphi<\Phi_0+2\pi\}$ (see also Figure 3).

On the other hand, if the functions W_i satisfy Definition 4.3, part 3., for some cones $\mathcal{K}_\infty,\mathcal{K}_{-\infty}$ and a basis e_1,e_2, then $\operatorname{int}\overline{\mathcal{K}_\infty\cup\mathcal{K}_{-\infty}}=\{x:\,r>0,\Phi_N<\varphi<\Phi_0+2\pi\}$, and Lemma 4.1 states that there exists $\vec{t}\in\mathbb{R}^2$ with $\vec{t}\in\mathcal{K}_\infty$ and $-\vec{t}\in\mathcal{K}_{-\infty}$. This shows that $\Phi_0+2\pi-\Phi_N>\pi$. The remaining part of Lemma 4.2 again follows by Lemma 4.1 with $\mathcal{K}_{i_{\max}}=\mathcal{K}_\infty$, $\mathcal{K}_{i_{\min}}=\mathcal{K}_{-\infty}$. ∎

Example 4.6. Assume that $\mathcal{K},\mathcal{K}_i\subset\mathbb{R}^2$, $1\leq i\leq N$, are given as in Lemma 4.2 and that the numbering is counterclockwise. Consider the functions $W_i(A)=\frac{\mu_i}{2}|A|^2$, $\mu_i>0$, $A\in\mathbb{R}^2$. These functions are distributed quasi-monotonely if there exists $i_0\in\{1,\ldots,N\}$ such that

$$\mu_1\geq\ldots\geq\mu_{i_0}\leq\ldots\leq\mu_N \qquad \text{in the Dirichlet case}$$
$$\mu_1\leq\ldots\leq\mu_{i_0}\geq\ldots\geq\mu_N \qquad \text{in the Neumann case}$$

and $-\mathcal{K}_{i_0}\cap\mathcal{K}_0\neq\emptyset$. In the case of mixed boundary conditions with $\Gamma_D\subset\partial\mathcal{K}_1$ and $\Gamma_N\subset\partial\mathcal{K}_N$ the parameters μ_i are distributed quasi-monotonely if

$$\mu_1\geq\mu_2\geq\ldots\geq\mu_N$$

and $\measuredangle(\Gamma_D,\Gamma_N)<\pi$, where $\measuredangle$ denotes the interior opening angle.

In the same way, Examples 4.2 - 4.4 can be carried over to the case of a cross point on the boundary.

Example 4.7. (Mixed boundary conditions on one subdomain, $d=3$.) Consider the pyramid $\mathcal{K}$, given by A,B,C,D,S, in Figure 4 with $AB\parallel CD$, $BC\parallel AD$ and let $N=1$ (only one subdomain). Assume that the faces ABS and BCS are parts of the Dirichlet boundary and CDS and DAS are parts of the Neumann boundary. Let $W:\mathbb{R}^{m\times3}\to\mathbb{R}$ satisfy (H1). Then one can

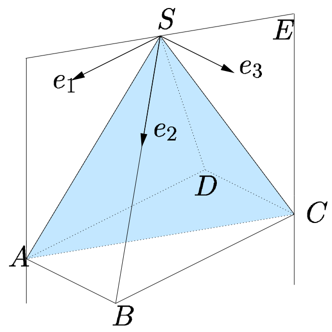

Figure 4: Example for mixed boundary conditions

find a basis $e_1, \ldots, e_3$ and cones $\mathcal{K}_{-\infty}, \mathcal{K}_\infty$ such that the assumptions in Definition 4.3, part 3. are satisfied with $N = 1$. A possible choice is plotted in Figure 4, where $e_1 \parallel BC$, $e_3 \parallel AB$ and $e_2 \parallel SB$. $\mathcal{K}_{-\infty}$ can be chosen as the complementary of $\mathcal{K}$ in the rear half space with respect to the plane E. Furthermore $\mathcal{K}_\infty = \mathbb{R}^3 \backslash \overline{\mathcal{K} \cup \mathcal{K}_{-\infty}}$. This example shows that for $N = 1$ and mixed boundary conditions the assumptions in Definition 4.3 for this case are slightly weaker than the assumptions in [7, 8]. There, for $d = 3$ at most three faces may intersect at points S with changing boundary conditions.

4.2. Regularity of weak solutions of the transmission problem. We consider the transmission problem (14). The assumptions for the main theorem are as follows:

(A1) $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a polygonal or polyhedral domain with Lipschitz boundary, $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, Γ_D and Γ_N open and disjoint. Furthermore, $\overline{\Omega} = \cup_{i=1}^M \overline{\Omega_i}$, where Ω_i is a polyhedral domain with Lipschitz boundary, $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$.

(A2) For $1 \leq i \leq M$, $W_i : \mathbb{R}^{m\times d} \to \mathbb{R}$ satisfies (H0) – (H4) for some $p_i \in (1, \infty)$ and $\kappa_i \in \{0, 1\}$.

(A3) There exists a finite number of balls $B_l(x_l)$ with center $x_l \in \overline{\Omega}$ such that $\Omega \subset \bigcup_l B_l(x_l)$ and $\Omega \cap B_l(x_l)$ coincides with an appropriate polyhedral cone $\mathcal{K}_l$ with tip in x_l, i.e. $\overline{\Omega} \cap B_l(x_l) = \overline{\mathcal{K}_l} \cap B_l(x_l)$. Let $\Omega_{l,1}, \ldots, \Omega_{l,N(l)}$ be those subdomains of Ω with $x_l \in \overline{\Omega_{l,j}}$, $1 \leq j \leq N(l)$, and $W_{l,1}, \ldots, W_{l,N(l)}$ the corresponding energy densities. We assume that there exist $N(l)$ pairwise disjoint polyhedral cones $\mathcal{K}_{l,j}$ with tip in x_l such that

$$\overline{\mathcal{K}_l} = \bigcup_{j=1}^{N(l)} \overline{\mathcal{K}_{l,j}} \text{ and } \overline{\mathcal{K}_{l,j}} \cap B_l(x_l) = \overline{\Omega_{l,j}} \cap B_l(x_l) \quad \text{for } 1 \leq j \leq N(l).$$

On each of the composed cones $\mathcal{K}_l$, the corresponding energy densities $W_{l,j}$, $1 \leq j \leq N(l)$, are distributed quasi-monotonely.

(A4) $f \in L^{\vec{q}}(\Omega)$ where $q_i = p_i' = \frac{p_i}{p_i - 1}$.

(A5) Dirichlet-datum: $u\big|_{\Gamma_D} = g\big|_{\Gamma_D}$ where g is an element of $W^{2,(\vec{p},p_{\max})}(\hat{\Omega})$ with $\nabla g \in L^\infty(\hat{\Omega})$ for some domain $\hat{\Omega} \supset\supset \Omega$. The space $W^{2,(\vec{p},p_{\max})}(\hat{\Omega})$ is defined as follows for $p_{\max} = \max_i\{p_i\}$:

$$W^{2,(\vec{p},p_{\max})}(\hat{\Omega}) = \Big\{ g \in W^{2,p_{\min}}(\hat{\Omega}) : \; g\big|_{\Omega_i} \in W^{2,p_i}(\Omega_i), \\ g\big|_{\hat{\Omega}\setminus\Omega} \in W^{2,p_{\max}}(\hat{\Omega}\setminus\Omega) \Big\}.$$

(A6) Neumann-datum: $H \in W^{1,\vec{q}}(\Omega, \mathbb{R}^{m\times d}) \cap L^\infty(\Omega, \mathbb{R}^{m\times d})$ and $D_A W_i(\nabla u)\vec{n} = H\vec{n}$ on Γ_N.

The assumption that the Dirichlet-datum g is defined on a larger region $\hat{\Omega} \supset\supset \Omega$ is for technical reasons. Note that for the Neumann-datum no extension to $\hat{\Omega}$ is needed.

Theorem 4.1 (Main Theorem). *Assume that assumptions* (A1) - (A6) *are satisfied and that* $u \in W^{1,\vec{p}}(\Omega)$ *is a weak solution of problem* (14). *Then for every* $\epsilon, \delta > 0$ *and* $1 \le i \le M$, *there holds*

$$u\big|_{\Omega_i} \in \mathcal{N}^{1+\frac{1}{p_i},p_i}(\Omega_i) \subset W^{1+\frac{1}{p_i}-\epsilon,p_i}(\Omega_i) \quad \text{if } p_i \in [2,\infty) \tag{22}$$

$$u\big|_{\Omega_i} \in \mathcal{N}^{\frac{3}{2},r_i-\epsilon}(\Omega_i) \cap W^{\frac{3}{2}-\delta,r_i}(\Omega_i) \quad \text{if } p_i \in (1,2] \tag{23}$$

with $r_i = \frac{2dp_i}{2d-2+p_i}$. *Note that* $p_i \le r_i \le 2$ *for* $p_i \in (1,2]$. *Furthermore, if* $p_i \in [2,\infty)$ *and* $\kappa_i = 1$ *in* (H4), *then*

$$u\big|_{\Omega_i} \in \mathcal{N}^{\frac{3}{2},2}(\Omega_i) \cap \mathcal{N}^{1+\frac{1}{p_i},p_i}(\Omega_i). \tag{24}$$

If $p_i \in (1,2]$ *for every* $i \in \{1,\dots,M\}$, *then*

$$u \in \mathcal{N}^{\frac{3}{2},r_{min}-\epsilon}(\Omega) \tag{25}$$

holds globally, where $r_{min} = \frac{2dp_{min}}{2d-2+p_{min}}$.

Before we prove the main theorem in Section 5, we first give some corollaries and remarks and compare the results in Theorem 4.1 with known results for linear elliptic boundary-transmission problems.

Remark. Theorem 4.1 has local character that means: If there is a subset $\tilde{\Omega} \subset \Omega$, for which the assumptions of Theorem 4.1 are satisfied, then $u\big|_{\tilde{\Omega}}$ has the regularity which is given in Theorem 4.1.

Corollary. *Let the assumptions be the same as in Theorem* 4.1 *with* $p_i \in (1,2]$ *for every* $i \in \{1,\dots,M\}$ *and assume that* $d = 2$. *Then by Lemma* 2.4 *and the standard embedding theorems for Sobolev-Slobodeckij spaces it holds*

$$u \in W^{\frac{3}{2}-\epsilon, \frac{4p_{min}}{2+p_{min}}}(\Omega) \subset \mathcal{C}(\overline{\Omega}) \qquad \forall \epsilon > 0 \ (small).$$

Remark. In the case $M = 1$, i.e. the problem reduces to a boundary value problem on a single domain, the result of Theorem 4.1 is well known for $p_i \in (1,2]$ (if $g = 0$ and $\kappa = 0$ in (H4)) and is derived by C. Ebmeyer and J. Frehse in [9, 8]. For $p > 2$, Theorem 4.1 sharpens the results in [8]. In the proof, Ebmeyer and Frehse developed and applied a difference quotient technique, which will be adapted for the proof of theorem 4.1. In the case of two coupled nonlinear elliptic systems with a plane interface, $p_1 = p_2 = 2$ and pure Dirichlet conditions, Theorem 4.1 is a special case of the results in [10]. There, the authors require a geometric condition, but they do not need a quasi-monotone distribution of the energy densities W_i.

Remark. Assume that $m = d$ and that $D_A W_i(B)$ is symmetric for symmetric $B \in \mathbb{R}^{d\times d}$. Then Theorem 4.1 also holds if in equation (14) ∇u is replaced by $\varepsilon(u)$. The necessary changes in the proof will be indicated. Therefore, transmission problems for linear and special classes of physically nonlinear elastic materials are covered as well by Theorem 4.1.

Remark. There exist higher local regularity results and results for smooth interfaces, see for example [29, 21], where for the case $\kappa_i = 0$ in assumption (H4) and $1 < p_i < 2$ the regularity $u\big|_{\tilde{\Omega}_i} \in W^{2, \frac{dp_i}{d-2+p_i}}(\tilde{\Omega}_i)$ is derived for $\tilde{\Omega}_i \subset\subset \Omega_i$. The same result is obtained at plane parts of the boundary of Ω_i, if assumption (H3) is replaced by (see [30])

(H3)' : $|D_A^2 W_i(A)| \leq c^i \, |A|^{p_i - 2} \quad \forall A \in \mathbb{R}^{m\times d} \backslash \{0\}$.

4.3. Comparison to results for linear elliptic boundary-transmission problems. For simplicity we assume that $d = 2$ and $m \in \{1,2\}$. Let $\Omega \subset \mathbb{R}^2$, $\Omega = \cup_{i=1}^M \Omega_i$, be a polygonal domain and choose $B_i \in \mathrm{Lin}(\mathbb{R}^{m\times 2}, \mathbb{R}^{m\times 2})$ symmetric and positive definite. For $u_i : \Omega_i \to \mathbb{R}^m$ set

$$W_i(u_i) := \begin{cases} \frac{1}{2} B_i(\nabla u_i)\cdot \nabla u_i & \text{if } m = 1 \\ \frac{1}{2} B_i(\varepsilon(u_i)) : \varepsilon(u_i) & \text{if } m = 2 \end{cases}, \quad F_i(Du_i) := \begin{cases} B_i \nabla u_i & \text{if } m = 1 \\ B_i(\varepsilon(u_i)) & \text{if } m = 2. \end{cases}$$

Due to the assumptions on B_i, the operator $\operatorname{div} F_i(Du_i)$ is linear and elliptic. With f, g, h as in Theorem 4.1 ($p_i = 2$) consider the following boundary transmission problem:

$$\begin{aligned} \operatorname{div} F_i(Du_i) + f &= 0 \quad \text{in } \Omega_i, \\ u_i - u_j &= 0 \quad \text{on } \Gamma_{ij}, \\ F_i(Du_i)\vec{n}_{ij} + F_j(Du_j)\vec{n}_{ji} &= 0 \quad \text{on } \Gamma_{ij}, \\ u_i &= g \quad \text{on } \partial\Omega_i \cap \Gamma_D, \\ F_i(Du_i)\vec{n}_i &= h \quad \text{on } \partial\Omega_i \cap \Gamma_N. \end{aligned}$$

For $m = 2$ these equations can be interpreted as the field equations of coupled linear elastic bodies with elasticity matrices B_i. The regularity theory for linear elliptic boundary transmission problems states that every weak solution $u \in W^{1,2}(\Omega)$ with $u_i = u\big|_{\Omega_i}$ has an asymptotic expansion of the following form in the neighborhood of interior cross points S or cross points on the boundary (polar coordinates r, φ with respect to S are used) [12, 15, 16, 19, 22]:

$$\eta^S u = \eta^S u_{\mathrm{reg}} + \eta^S \sum_{\operatorname{Re}\alpha \in (0,1)} r^\alpha v_\alpha^S(\ln r, \varphi), \tag{26}$$

where η^S is a cut-off function, $\eta^S u_{\mathrm{reg}}\big|_{\Omega_i} \in W^{2,2}(\Omega_i)$ and α is an eigenvalue of a corresponding eigenvalue problem, for details see e.g. [3, 24, 25, 26]. The functions $v_\alpha^S(\ln r, \varphi)$ contain in general powers of $\ln r$ and generalized eigenfunctions. It holds that $r^\alpha v_\alpha^S\big|_{\Omega_i} \in W^{1+\operatorname{Re}\alpha-\epsilon,2}(\Omega_i)$ for arbitrary $\epsilon > 0$, see [12, Theorem 1.4.5.3].

Assume now that the matrices B_i are distributed quasi-monotonely with respect to the cross point S. A sufficient condition for this is described in Example 4.4. Then by Theorem 4.1: $\eta^S u\big|_{\Omega_i} \in W^{\frac{3}{2}-\epsilon,2}(\Omega_i)$ and $\eta^S u \in W^{\frac{3}{2}-\epsilon,2}(\Omega)$ for every $\epsilon > 0$. It follows that Re $\alpha \geq \frac{1}{2}$ in the asymptotic expansion (26). In an earlier work [14], estimates for the eigenvalues were derived for Poisson's and Lamé's equations with piecewise constant coefficients. There, the same assumptions as in Theorem 4.1 were used and by a homotopy argument it was proved that Re $\alpha > \frac{1}{2}$. This indicates that the results in Theorem 4.1 are nearly optimal (up to ϵ).

The following linear example shows that if the assumptions of Theorem 4.1 are violated, then one cannot expect the regularity $\eta^S u_i \in W^{\frac{3}{2}-\epsilon,2}(\Omega_i)$.

Example 4.8. Consider a domain $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2} \subset \mathbb{R}^2$, where Ω_1 and Ω_2 coincide in the neighborhood of $S = (0,0)$ with the cones in polar coordinates (see Figure 5)

$$\begin{aligned} \mathcal{K}_1 &= \{x \in \mathbb{R}^2 : |x| > 0,\ 0 < \varphi < \frac{\pi}{2}\}, \\ \mathcal{K}_2 &= \{x \in \mathbb{R}^2 : |x| > 0, \frac{\pi}{2} < \varphi < \frac{\pi}{2} + \Phi\}, \quad \Phi > 0. \end{aligned}$$

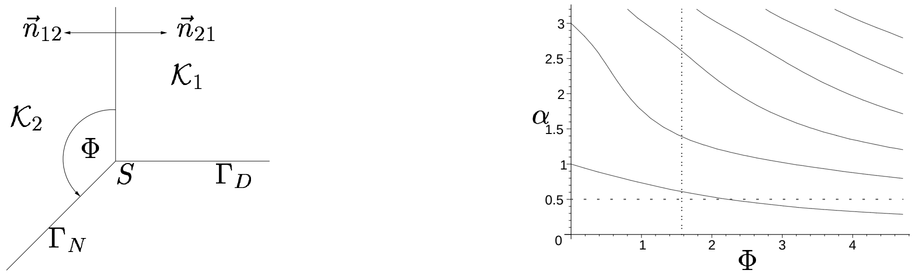

Figure 5: Domain and singular exponents for Example 4.8

Dirichlet-conditions are prescribed on $\partial\Omega \cap \partial\mathcal{K}_1$, Neumann-conditions on $\partial\Omega \cap \partial\mathcal{K}_2$. The problem under consideration is to find a solution of the following linear boundary transmission problem for the Poisson equation with piecewise constant coefficients $\mu_1, \mu_2 > 0$:

$$\begin{aligned} \mu_i \triangle u_i + f_i &= 0 \quad \text{in } \Omega_i,\ i = 1,2, \\ u &= g \quad \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \vec{n}} &= h \quad \text{on } \Gamma_N, \\ u_1 - u_2 &= 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2, \\ \mu_1 \frac{\partial u_1}{\partial \vec{n}_{12}} + \mu_2 \frac{\partial u_2}{\partial \vec{n}_{21}} &= 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2. \end{aligned}$$

Let the data f_i, g, h satisfy the assumptions of Theorem 4.1 with $p_1 = p_2 = 2$. Weak solutions of this boundary transmission problem admit an asymptotic expansion of the following type near the cross point S, [26]:

$$\eta^S(x)u(x) = u_{\text{reg}}(x) + \eta^S(x) \sum_{0<\alpha<1} c_\alpha \,|x|^\alpha\, v_\alpha(\varphi),$$

where η^S is a cut-off function with respect to S, $u_{\text{reg}}\big|_{\Omega_i} \in W^{2,2}(\Omega_i)$, c_α are constants which are determined by the data f_i, g, h; α is the singular exponent and v_α the corresponding eigenfunction. Note that the singular exponents are real numbers in our special case and that there are no logarithmic terms in the singular expansion. The singular exponents α solve the following equation, [26]:

$$-\mu_2 \sin(\alpha\Phi)\sin(\alpha\frac{\pi}{2}) + \mu_1 \cos(\alpha\Phi)\cos(\alpha\frac{\pi}{2}) = 0.$$

Choose $\mu_1 = 1, \mu_2 = \frac{1}{2}$. For $\Phi < \frac{\pi}{2}$, the quasi-monotonicity condition in Theorem 4.1 is satisfied and therefore the smallest positive singular exponent $\alpha_{\min}$ is larger than or equal to $\frac{1}{2}$. For $\Phi \geq \frac{\pi}{2}$, the quasi-monotonicity condition is violated and if Φ is large enough, one obtains $\alpha_{\min} < \frac{1}{2}$. In this case, one can guarantee $u\big|_{\Omega_i} \in W^{1+\alpha_{\min}-\epsilon,2}(\Omega_i)$, only. The behavior of the singular exponents is illustrated in Figure 5, where the exponents α are plotted versus the opening angle Φ of subdomain Ω_2.

5. Proof of Theorem 4.1

In the proof of the main theorem, a difference quotient technique is used. This technique is frequently applied to derive interior regularity results, [23, 32, 4, 29, 21], and is modified by C. Ebmeyer and J. Frehse, [9, 7], in order to prove global regularity results on polygonal or polyhedral domains.

The main idea is to insert test functions of the form $\xi_j(x) = \varphi^2(u(x + he_j) - u(x))$ into the weak formulation and to apply the convexity inequality (49) from the Appendix. This leads to estimates in Nikolskii-spaces and by the embedding-lemma 2.4 to regularity results in Sobolev-Slobodeckij spaces.

The main problem is that the differences $u(x+he_j) - u(x)$ are taken across the interfaces and one has to check whether ξ_j is an admissible test function in $V^{\vec{p}}(\Omega)$. Due to the quasi-monotonicity condition, there exists a basis $\{e_j,\ 1 \leq l \leq d\} \subset \mathbb{R}^d$, such that the functions ξ_j are indeed admissible test functions. Furthermore, in the proof occur differences of the form $W_i(\nabla u(x)) - W_j(\nabla u(x))$, which have to be estimated in an appropriate way. Here, the quasi-monotonicity condition is also very useful.

The proof is organized as follows: The case of pure Dirichlet-conditions will be proved in detail. For the remaining cases (Neumann, mixed and pure interface problems) the necessary changes in the proof will be indicated.

Cross point on the boundary of Ω with pure Dirichlet conditions. Let $S \subset \partial\Omega$ and assume that there exists $R > 0$ such that $B_R(S) \subset \hat{\Omega}$ and $\Omega \cap B_R(S) = \mathcal{K} \cap B_R(S)$, where $\mathcal{K}$ is an appropriate polyhedral cone with tip in S and $\partial\mathcal{K} \cap B_R(S) \subset \Gamma_D$. Assume further that for every $j \in \{1, \ldots, M\}$ with $\Omega_j \cap B_R(S) \neq \emptyset$ there exists a polyhedral cone $\mathcal{K}_j$ with tip in S such that $\Omega_j \cap B_R(S) = \mathcal{K}_j \cap B_R(S)$. Note that $\overline{\mathcal{K}} = \bigcup_{i=1}^{N} \overline{\mathcal{K}_i}$ after a suitable renumbering, see also Figure 6. Due to the assumptions in Theorem 4.1, the cones $\mathcal{K}_i$ and functions W_i, $1 \leq i \leq N$, satisfy the quasi-monotonicity conditions in Definition 4.3, part *1.* with $\mathcal{K}_0 := \mathbb{R}^d \backslash \overline{\mathcal{K}}$.

Let $u \in W^{1,\vec{p}}(\Omega)$ be a weak solution of problem (14) with right hand sides g, f, h as in Theorem 4.1; $R''' = \frac{R}{2}, h_0 = R'' = \frac{R}{4}, R' = \frac{R}{8}$. Choose $\varphi \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$ with $\operatorname{supp}\varphi \subset B_{R''}(S)$, $\varphi\big|_{B_{R'}(S)} = 1$ and $0 \leq \varphi \leq 1$. Let further be e_l one of the basis vectors given by Definition 4.3. For the definition of an

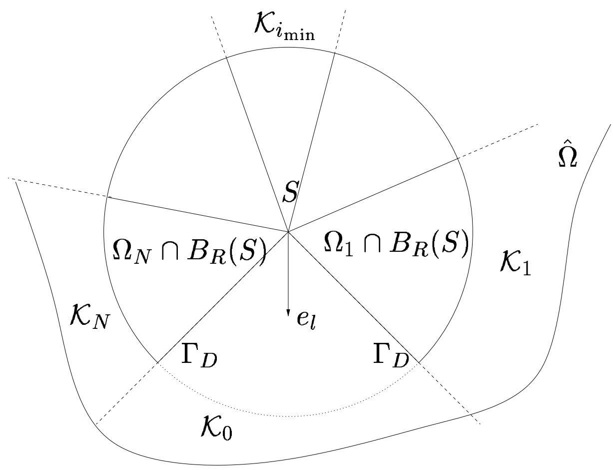

Figure 6: Example for the notation with Dirichlet conditions

appropriate test function, an extension of u across the Dirichlet boundary is needed:

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ g(x) & \text{if } x \in \hat{\Omega} \backslash \Omega. \end{cases} \tag{27}$$

For the extended function $\tilde{u}$ it holds

$$\varphi^2 \tilde{u} \in W^{1,(\vec{p},p_{\max})}(B_R(S)) = \Big\{ v \in W^{1,p_{\min}}(B_R(S)) : \\ v\big|_{\Omega_i \cap B_R(S)} \in W^{1,p_i}(\Omega_i \cap B_R(S)), v\big|_{\mathcal{K}_0 \cap B_R(S)} \in W^{1,p_{\max}}(\mathcal{K}_0 \cap B_R(S)) \Big\}.$$

This follows since $\varphi^2 \tilde{u}\big|_{\mathcal{K}_i} \in W^{1,p_i}(\mathcal{K}_i)$ for $1 \le i \le N$, $\varphi^2 \tilde{u}\big|_{\mathcal{K}_0} \in W^{1,p_{\max}}(\mathcal{K}_0)$ and since, by the definition of $\tilde{u}$, $\varphi^2 \tilde{u}$ does not jump across interfaces:

$$\left(\varphi^2 \tilde{u}\big|_{\mathcal{K}_i} \right) |_{\Gamma_{ij}} = \left(\varphi^2 \tilde{u}\big|_{\mathcal{K}_j} \right) |_{\Gamma_{ij}}$$

for $0 \le i, j, \le N$.

The regularity results (23) and (22) will be derived in two steps. In a first step we prove inequality (28) here after. This is the essential inequality from which we deduce in a second step estimates for Nikolskii-norms of $\tilde{u}$ and u.

`First step.` We prove the following inequality: There is a constant $c > 0$ such that for $1 \le l \le d$ and $0 < h < h_0$

$$\sum_{i=1}^{N} \int_{\Omega_i} \varphi^2(x) \big(\kappa_i + |\nabla \tilde{u}(x + he_l)| \\ + |\nabla \tilde{u}(x)| \big)^{p_i - 2} |\tilde{u}(x + he_l) - \tilde{u}(x)|^2 \, \mathrm{d}x \;\le\; ch \tag{28}$$

holds with κ_i from (H4).

To proof the inequality (28) we define on Ω

$$\begin{aligned}\xi(x) &= \varphi^2(x)\,(\tilde{u}(x+he_l) - g(x+he_l) - (\tilde{u}(x) - g(x)))\\ &\equiv \varphi^2(x)\triangle_h(\tilde{u}(x) - g(x))\end{aligned}$$

as test function for $0 < h < h_0$. From the quasi-monotonicity assumptions and by Corollary 4.1 it follows that $\xi \in W^{1,\vec{p}}(\Omega)$. Furthermore, $\xi\big|_{\Gamma_D} = 0$ and therefore $\xi \in V^{\vec{p}}(\Omega)$ is an admissible test function. Inserting ξ into the variational formulation (14) and rearranging the terms yields

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega_i} &\varphi^2 D_A W_i(\nabla u)) : \nabla(\triangle_h\tilde{u})\,\mathrm{d}x\\ &= \int_\Omega f\xi\,\mathrm{d}x + \sum_{i=1}^{N}\int_{\Omega_i}\varphi^2 D_A W_i(\nabla u) : \triangle_h\nabla g\,\mathrm{d}x \\ &\quad -\sum_{i=1}^{N}\int_{\Omega_i} D_A W_i(\nabla u) : (\triangle_h(\tilde{u}-g)\otimes\nabla\varphi^2)\,\mathrm{d}x.\end{aligned} \tag{29}$$

For $a \in \mathbb{R}^m, b \in \mathbb{R}^d$, $a \otimes b = (a_i b_j)_{ij} \in \mathbb{R}^{m\times d}$ denotes the tensor product. Inequality (49) with $A = \nabla\tilde{u}(x+he_l)$, $B = \nabla\tilde{u}(x) = \nabla u(x)$ for $x \in \Omega$, applied to the left hand side of equation (29) results in ($c > 0$ is independent of h)

$$\begin{aligned} c\sum_{i=1}^{N}\int_{\Omega_i} &\varphi^2(\kappa_i + |\nabla\tilde{u}(x+he_l)| + |\nabla\tilde{u}(x)|)^{p_i-2}\,|\triangle_h\nabla\tilde{u}(x)|^2\,\mathrm{d}x\\ &\overset{(49)}{\leq} \sum_{i=1}^{N}\int_{\Omega_i}\varphi^2\triangle_h W_i(\nabla\tilde{u})\,\mathrm{d}x - \sum_{i=1}^{N}\int_{\Omega_i}\varphi^2 D_A W_i(\nabla u) : \triangle_h\nabla\tilde{u}\,\mathrm{d}x\\ &\overset{(29)}{=} \sum_{i=1}^{N}\int_{\Omega_i}\varphi^2\triangle_h W_i(\nabla\tilde{u}))\,\mathrm{d}x - \int_\Omega f\xi\,\mathrm{d}x\\ &\qquad -\sum_{i=1}^{N}\int_{\Omega_i}\varphi^2 D_A W_i(\nabla u) : \triangle_h\nabla g\,\mathrm{d}x\\ &\qquad +\sum_{i=1}^{N}\int_{\Omega_i} D_A W_i(\nabla u) : (\triangle_h(\tilde{u}-g)\otimes\nabla\varphi^2)\,\mathrm{d}x\\ &= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{30}$$

In the next steps, the integrals $I_1, \ldots, I_4$ will be estimated. By Hölder's inequality one gets

$$|I_2| \leq \sum_{i=1}^{N} \|\varphi f\|_{L^{q_i}(\Omega_i)} \|\varphi \triangle_h (\tilde{u} - g)\|_{L^{p_i}(\Omega_i)} .$$

Put $\tilde{\Omega}_i := \{x \in \mathbb{R}^d : x = y + he_l, 0 \leq h < h_0, y \in \Omega_i\} \supset \Omega_i$. Due to the quasi-monotonicity and the special choice of the extension of u to $\tilde{u}$, it is $(\tilde{u} - g)\big|_{\tilde{\Omega}_i} \in W^{1,p_i}(\tilde{\Omega}_i)$. This follows by arguments which are similar to those in the proof of Lemma 2.2. By [11, Lemma 7.23] one obtains

$$\|\varphi \triangle_h(\tilde{u} - g)\|_{L^{p_i}(\Omega_i)} \leq \|\triangle_h(\tilde{u} - g)\|_{L^{p_i}(\Omega_i \cap \operatorname{supp}\varphi)} \leq ch \|\nabla(\tilde{u} - g)\|_{L^{p_i}(\tilde{\Omega}_i \cap \operatorname{supp}\varphi)} ,$$

where the constant c depends on the vector e_l but is independent of h. Therefore

$$|I_2| \leq ch \sum_{i=1}^{N} \|\varphi f\|_{L^{q_i}(\Omega_i)} \|\nabla(\tilde{u} - g)\|_{L^{p_i}(\tilde{\Omega}_i \cap \operatorname{supp}\varphi)} . \tag{31}$$

The same considerations can be made for I_3 and I_4 using assumption (H2) which yields $D_A W_i(\nabla u) \in L^{q_i}(\Omega_i)$. One finally gets

$$|I_3| \leq ch \sum_{i=1}^{N} \|\varphi D_A W_i(\nabla u)\|_{L^{q_i}(\Omega_i)} \left\|D^2 g\right\|_{L^{p_i}(\tilde{\Omega}_i \cap \operatorname{supp}\varphi)} \tag{32}$$

$$|I_4| \leq ch \sum_{i=1}^{N} \|\varphi D_A W_i(\nabla u)\|_{L^{q_i}(\Omega_i)} \|\nabla(\tilde{u} - g)\|_{L^{p_i}(\tilde{\Omega}_i \cap \operatorname{supp}\varphi)} . \tag{33}$$

Again, c is a constant which is independent of h. It remains to estimate I_1. Here, it is essential that the functions W_i are distributed quasi-monotonely. Let $k_1, \ldots, k_N$ be the numbers from Definition 4.3. (Do not confuse k_i from Definition 4.3 with κ_i from (H4).) It is

$$I_1 \equiv \sum_{i=1}^{N} \int_{\Omega_i} \varphi^2 \triangle_h (W_i(\nabla \tilde{u}) + k_i) \, \mathrm{d}x.$$

By $\triangle_h(fg)(x) = (\triangle_h f)(x)g(x) + f(x+he_l)\triangle_h g(x)$ (product rule for differences) it follows

$$\begin{aligned} I_1 = & \sum_{i=1}^{N} \int_{\Omega_i} \triangle_h \left(\varphi^2(W_i(\nabla \tilde{u}) + k_i)\right) \mathrm{d}x \\ & - \sum_{i=1}^{N} \int_{\Omega_i} (\triangle_h \varphi^2)(W_i(\nabla \tilde{u}(x + he_l)) + k_i) \, \mathrm{d}x \\ = & \; I_{11} + I_{12}. \end{aligned}$$

By assumption (H1) and the fact that $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, it holds

$$\begin{aligned} |I_{12}| &\leq ch \sum_{i=1}^{N} \big(\|\nabla \tilde{u}(\cdot + he_l)\|_{L^{p_i}(\Omega_i)}^{p_i} + k_i\, |\Omega_i| \big) \\ &\leq ch \sum_{i=1}^{N} \big(\|\nabla \tilde{u}\|_{L^{p_i}(\tilde{\Omega}_i)}^{p_i} + k_i\, |\Omega_i| \big). \end{aligned} \tag{34}$$

with a constant c which is independent of h. In the next estimates, it is used the notation $\Omega_0 = \mathcal{K}_0 \cap B_R(S)$. It holds

$$\Omega_i \cap \operatorname{supp}\varphi \cap \left(\bigcup_{j=0}^{N} \overline{\Omega_j + he_l} \right) = \Omega_i \cap \operatorname{supp}\varphi \tag{35}$$

$$(\Omega_i + he_l) \cap \operatorname{supp}\varphi \cap \left(\bigcup_{j=0}^{N} \overline{\Omega_j} \right) = \Omega_i + he_l \cap \operatorname{supp}\varphi. \tag{36}$$

Note that for $1 \leq i \leq N$, $0 < h < h_0$. It follows that

$$\begin{aligned} I_{11} &= \sum_{i=1}^{N} \int_{\Omega_i + he_l} \varphi^2 \left(W_i(\nabla \tilde{u})) + k_i \right) \mathrm{d}x - \int_{\Omega_i} \varphi^2 \left(W_i(\nabla \tilde{u})) + k_i \right) \mathrm{d}x \\ &= \sum_{i=1}^{N} \int_{\Omega_i + he_l \setminus \Omega_i} \varphi^2 \left(W_i(\nabla \tilde{u})) + k_i \right) \mathrm{d}x \\ &\quad - \int_{\Omega_i \setminus \Omega_i + he_l} \varphi^2 \left(W_i(\nabla \tilde{u})) + k_i \right) \mathrm{d}x \\ &\overset{(35)}{=} \sum_{i=1}^{N} \sum_{\substack{j=0, \\ j \neq i}}^{N} \int_{\Omega_i + he_l \cap \Omega_j} \varphi^2 \left(W_i(\nabla \tilde{u})) + k_i \right) \mathrm{d}x \\ &\quad - \int_{\Omega_i \cap \Omega_j + he_l} \varphi^2 \left(W_i(\nabla \tilde{u})) + k_i \right) \mathrm{d}x \\ &= \sum_{i=1}^{N} \int_{\Omega_i + he_l \cap \Omega_0} \varphi^2 \left(W_i(\nabla \tilde{u})) + k_i \right) \mathrm{d}x \\ &\quad - \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 \left(W_i(\nabla \tilde{u})) + k_i \right) \mathrm{d}x \\ &\quad + \sum_{\substack{i,j=1, \\ j \neq i}}^{N} \int_{\Omega_i + he_l \cap \Omega_j} \varphi^2 \left(W_i(\nabla \tilde{u}) + k_i - W_j(\nabla \tilde{u}) - k_j \right) \mathrm{d}x \\ &= I_{111} + I_{112}. \end{aligned} \tag{37}$$

Since the functions W_i are distributed quasi-monotonely it follows that $\Omega_i \cap \Omega_0 + he_l = \emptyset$ for $h > 0$ and $1 \leq i \leq N$, compare Definition 4.3. It remains

$$I_{111} = \sum_{i=1}^{N} \int_{\Omega_i + he_l \cap \Omega_0} \varphi^2 (W_i(\nabla \tilde{u}) + k_i) \, \mathrm{d}x \overset{(27)}{=} \sum_{i=1}^{N} \int_{\Omega_i + he_l \cap \Omega_0} \varphi^2 (W_i(\nabla g) + k_i) \mathrm{d}x$$

and hence

$$I_{111} \overset{(A5)}{\leq} c \sum_{i=1}^{N} |\Omega_i + he_l \cap \Omega_0 \cap \operatorname{supp} \varphi| \leq ch.$$

taking into account the definition of $\tilde{u}$ and (H1) and (A5). Again due to the quasi-monotonicity of the functions W_i it holds: If $\Omega_i + he_l \cap \Omega_j \neq \emptyset$ then $W_j(A) + k_j \geq W_i(A) + k_i$ for every $A \in \mathbb{R}^{m \times d}$. Therefore $I_{112} \leq 0$. Collecting these estimates finally yields $I_1 \leq ch$, where $c > 0$ is a constant which is independent of h. This finishes the proof of inequality (28).

`Second step.` In this step, we derive estimates for the Nikolskii-norms of u on the basis of inequality (28).

Since the addends on the left hand side of inequality (28) are nonnegative, it holds

$$\int_{\Omega_i} \varphi^2 \left(\kappa_i + |\nabla \tilde{u}(x + he_l)| + |\nabla \tilde{u}(x)| \right)^{p_i - 2} |\triangle_h \nabla \tilde{u}(x)|^2 \, \mathrm{d}x \leq ch. \tag{38}$$

for $1 \leq i \leq N$. Applying inequality (50) with $\alpha_i = \frac{p_i}{2}$ to each subdomain separately yields

$$\int_{\Omega_i} \varphi^2 \left| \triangle_h (\kappa_i + |\nabla \tilde{u}_i|)^{\frac{p_i}{2}} \right|^2 \mathrm{d}x \leq ch$$

for $1 \leq i \leq N$. Since $\varphi\big|_{B_{R'}(S)} = 1$ it follows for

$$\Omega'_{i,\eta} := \{ x \in B_{R'}(S) \cap \Omega_i : \operatorname{dist}(x, \partial(B_{R'}(S) \cap \Omega_i)) > \eta \}$$

that

$$\sup_{\substack{\eta > 0 \\ 0 < h < \eta}} \int_{\Omega'_{i,\eta}} h^{-1} \left| \triangle_h (\kappa_i + |\nabla u_i|)^{\frac{p_i}{2}} \right|^2 \mathrm{d}x \leq c$$

and therefore

$$(\kappa_i + |\nabla u_i|)^{\frac{p_i}{2}} \in \mathcal{N}^{\frac{1}{2}, 2}(\Omega_i \cap B_{R'}(S)).$$

Assume first that $p_i \in (1, 2]$. The remaining part of the proof for this case follows exactly the considerations in [8] and is given here for completeness. Lemma 2.4 and the embedding theorems for Sobolev Slobodeckij spaces state that

$$(\kappa_i + |\nabla u_i|)^{\frac{p_i}{2}} \in W^{\frac{1}{2} - \delta, 2}(\Omega'_i) \subset L^{\frac{2d}{d-1} - \epsilon}(\Omega'_i) \tag{39}$$

for every δ and $\epsilon = \epsilon(\delta) > 0$, where $\Omega_i' = \Omega_i \cap B_{R'}(S)$. Thus, $\nabla u_i \in L^{\frac{dp_i}{d-1}-\varepsilon}(\Omega_i')$. By standard embedding theorems, the space $W^{1,p_i}(\Omega_i')$ is continuously embedded in $L^{\frac{dp_i}{d-1}-\epsilon}(\Omega_i')$. This together with the previous estimate for ∇u_i shows that $u_i \in W^{1,\frac{dp_i}{d-1}-\epsilon}(\Omega_i')$ for every $\epsilon > 0$. Choose $\sigma_i = r_i - \delta$ for arbitrary $\delta > 0$, where $r_i = \frac{2dp_i}{2d-2+p_i}$ as in Theorem 4.1. For $1 < p_i \leq 2$ it is $1 < \sigma_i \leq \frac{dp_i}{d-1}$ and therefore $u_i \in W^{1,\sigma_i}(\Omega_i')$. Thus for $0 < h < \eta < h_0$, $1 \leq l \leq d$ and $M_h := \{x \in \Omega_i' : \nabla\tilde{u}(x+he_l) = \nabla\tilde{u}(x) = 0\}$ it holds (apply Hölder's inequality)

$$
\begin{aligned}
&\int_{\Omega_{i,\eta}'} |h^{-\frac{1}{2}}\triangle_h \nabla u|^{\sigma_i}\,\mathrm{d}x \\
&\quad= \int_{\Omega_{i,\eta}'\setminus M_h} |h^{-\frac{1}{2}}\triangle_h\nabla u_i|^{\sigma_i}\left(\kappa_i + |\nabla u_i(x)| + |\nabla u_i(x+he_l)|\right)^{\frac{\sigma_i(p_i-2)}{2}} \\
&\qquad\qquad \times \left(\kappa_i + |\nabla u_i(x)| + |\nabla u_i(x+he_l)|\right)^{-\frac{\sigma_i}{2}(p_i-2)}\,\mathrm{d}x \\
&\quad\leq \left(\int_{\Omega_{i,\eta}'} \left|h^{-\frac{1}{2}}\triangle_h\nabla u\right|^2 (\kappa_i + |\nabla u_i(x+he_l)| + |\nabla u(x)|)^{p_i-2}\,\mathrm{d}x\right)^{\frac{\sigma_i}{2}} \\
&\qquad\qquad \times \left(\int_{\Omega_{i,\eta}'} (\kappa_i + |\nabla u_i(x)| + |\nabla u_i(x+he_l)|)^{\frac{\sigma_i(2-p_i)}{2-\sigma_i}}\,\mathrm{d}x\right)^{\frac{2-\sigma_i}{2}} .
\end{aligned}
$$

By inequality (38) the first factor is bounded independently of h and η. Furthermore, $1 < \frac{\sigma_i(2-p_i)}{2-\sigma_i} < \frac{dp_i}{d-1}$ and thus the second term is bounded independently of h and η as well. It follows

$$
\sup_{\substack{\eta>0,\\ 0<h<\eta}} \int_{\Omega_{i,\eta}'} \left|h^{-\frac{1}{2}}\triangle_h\nabla u_i\right|^{\sigma_i}\,\mathrm{d}x \leq c\,,
$$

and relation (23) of Theorem 4.1 is proved for $p_i \in (1,2]$. For the proof of the global result (25) note that for arbitrary $A, B \in \mathbb{R}^{m\times d}$: $(|A| + |B|)^{p_i-2} \geq (1+|A|+|B|)^{p_i-2} \geq (1+|A|+|B|)^{p_{\min}-2}$ and proceed as subsequent to equation (28) with Ω_i replaced by $\operatorname{supp}\varphi \cap \Omega$.

We assume now that $p_i > 2$. The following two inequalities can be deduced from (38):

$$
\int_{\Omega_i} \varphi^2\,|\triangle_h\nabla\tilde{u}(x)|^{p_i}\,\mathrm{d}x \leq ch \tag{40}
$$

$$
\int_{\Omega_i} \varphi^2\,|\triangle_h\nabla\tilde{u}(x)|^2\,\mathrm{d}x \leq ch \qquad \text{if } \kappa_i = 1. \tag{41}
$$

This yields the assertions (22) and (24) and completes the proof of the Dirichlet case.

Cross point on the boundary of Ω with pure Neumann conditions. Note first that it follows

$$\langle H\vec{n}, v\rangle = \int_\Omega \left(H^T v\right)\vec{n}\,\mathrm{d}s = \int_\Omega H^T : \nabla v\,\mathrm{d}x + \int_\Omega (\operatorname{div} H) v\,\mathrm{d}x \quad \forall v \in V^{\vec{p}}(\Omega).$$

by the special structure of the Neumann data in (A6). Therefore, the weak formulation (14) is equivalent to: For every $v \in V^{\vec{p}}(\Omega)$ it holds

$$\begin{aligned}\sum_{i=1}^M \int_{\Omega_i} D_A W_i(\nabla u_i) : \nabla v_i\,\mathrm{d}x \\ = \sum_{i=1}^M \int_{\Omega_i} (f_i + \operatorname{div} H_i) v_i\,\mathrm{d}x + \sum_{i=1}^M \int_{\Omega_i} H_i^T : \nabla v_i\,\mathrm{d}x.\end{aligned} \tag{42}$$

Let $S \subset \partial\Omega$ and assume that there exists $R > 0$ such that $B_R(S) \subset \hat{\Omega}$ and $\Omega \cap B_R(S) = \mathcal{K} \cap B_R(S)$, where $\mathcal{K}$ is an appropriate polyhedral cone with tip in S and $\partial\mathcal{K} \cap B_R(S) \subset \Gamma_N$. Assume further that for every $j \in \{1, \dots, M\}$ with $\Omega_j \cap B_R(S) \neq \emptyset$ there exists a polyhedral cone $\mathcal{K}_j$ with tip in S such that $\Omega_j \cap B_R(S) = \mathcal{K}_j \cap B_R(S)$. Note that $\overline{K} = \bigcup_{i=1}^N \overline{\mathcal{K}_i}$. Due to the assumptions in Theorem 4.1, the cones $\mathcal{K}_i$ and functions W_i, $1 \leq i \leq N$, satisfy the conditions in Definition 4.3, part 2.; $\mathcal{K}_0 = \mathbb{R}^d \backslash \overline{\mathcal{K}}$.

Let $u \in W^{1,\vec{p}}(\Omega)$ be a weak solution of problem (14), $R''' = \frac{R}{2}, h_0 = R'' = \frac{R}{4}, R' = \frac{R}{8}$ and choose $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R})$ with $\operatorname{supp}\varphi \subset B_{R''}(S)$, $\varphi\big|_{B_{R'}(S)} = 1$ and $0 \leq \varphi \leq 1$. Let further be e_l one of the basis vectors given by Definition 4.3. For $0 < h < h_0$ the function

$$\xi(x) := \varphi^2(x)(u(x + he_l) - u(x)) = \varphi^2(x)\triangle_h u(x), \qquad x \in \Omega$$

is an admissible test function in $V^{\vec{p}}(\Omega)$. This is due to the quasi-monotonicity condition, compare also Corollary 4.1 and Remark 4.1. Note that no extension of u across the Neumann boundary is needed. The next goal is to prove that inequality (28) also holds in the case of pure Neumann conditions (with u instead of $\tilde{u}$). Inserting ξ into equation (42) and rearranging the terms yields

$$\begin{aligned}\sum_{i=1}^N \int_{\Omega_i} \varphi^2 D_A W_i(\nabla u) : \triangle_h \nabla u\,\mathrm{d}x \\ = \sum_{i=1}^N \int_{\Omega_i} (f + \operatorname{div} H)\xi\,\mathrm{d}x + \sum_{i=1}^N \int_{\Omega_i} H^T : \nabla\xi\,\mathrm{d}x \\ - \sum_{i=1}^N \int_{\Omega_i} D_A W_i(\nabla u) : (\triangle_h u \otimes \nabla\varphi^2)\,\mathrm{d}x.\end{aligned} \tag{43}$$

Applying inequality (49) (see Appendix) to (43) results in

$$\begin{aligned} &c\sum_{i=1}^{N}\int_{\Omega_i}\varphi^2(\kappa_i+|\nabla u(x+he_l)|+|\nabla u(x)|)^{p_i-2}\,|\triangle_h\nabla u|^2\;\mathrm{d}x \\ &\overset{(49)}{\leq} \sum_{i=1}^{N}\int_{\Omega_i}\varphi^2\triangle_h W_i(\nabla u)\,\mathrm{d}x-\sum_{i=1}^{N}\int_{\Omega_i}\varphi^2 D_A W_i(\nabla u):\triangle_h\nabla u\;\mathrm{d}x \\ &\overset{(43)}{=} \sum_{i=1}^{N}\int_{\Omega_i}\varphi^2\triangle_h W_i(\nabla u)\;\mathrm{d}x-\sum_{i=1}^{N}\int_{\Omega_i}(f+\operatorname{div}H)\xi\,\mathrm{d}x \\ &\quad -\sum_{i=1}^{N}\int_{\Omega_i}H^T:\nabla\xi\;\mathrm{d}x+\sum_{i=1}^{N}\int_{\Omega_i}D_A W_i(\nabla u):(\triangle_h u\otimes\nabla\varphi^2)\;\mathrm{d}x \\ &= I_1+I_2+I_3+I_4. \end{aligned} \tag{44}$$

The constant c is independent of h. The integrals I_2 and I_4 can be estimated as in the case of pure Dirichlet conditions, compare (31) - (33), and one gets $|I_2|+|I_4|\leq ch$ for some $c>0$ which is independent of h. Let $k_1,\dots k_N$ be the numbers from Definition 4.3. Then by the product rule for differences we get

$$I_1=\sum_{i=1}^{N}\int_{\Omega_i}\triangle_h\left((\varphi^2(W_i(\nabla u)+k_i)\right)\;\mathrm{d}x-\int_{\Omega_i}(\triangle_h\varphi^2)(W_i(\nabla u)(x+he_l)+k_i)\,\mathrm{d}x$$

which is $I_1=I_{11}+I_{12}$. As in (34) it follows that $|I_{12}|\leq ch$. Furthermore, with $\Omega_0=\mathcal{K}_0\cap B_R(S)$, I_{11} can be transformed analogously to (37), that is

$$\begin{aligned} I_{11} &= \sum_{i=1}^{N}\int_{\Omega_i+he_l\cap\Omega_0}\varphi^2(W_i(\nabla u)+k_i)\;\mathrm{d}x \\ &\quad -\int_{\Omega_i\cap\Omega_0+he_l}\varphi^2(W_i(\nabla u)+k_i)\;\mathrm{d}x \\ &\quad +\sum_{\substack{i,j=1\\ j\neq i}}^{N}\int_{\Omega_i+he_l\cap\Omega_j}\varphi^2(W_i(\nabla u)+k_i-(W_j(\nabla u)+k_j))\;\mathrm{d}x. \end{aligned} \tag{45}$$

Due to the quasi-monotonicity condition, it is $W_i(\nabla u)+k_i-(W_j(\nabla u)+k_j)\leq 0$ if $\Omega_i+he_l\cap\Omega_j\cap\operatorname{supp}\varphi\neq\emptyset$ and in addition $\Omega_i+he_l\cap\Omega_0=\emptyset$. Therefore it remains

$$I_{11}\leq-\sum_{i=1}^{N}\int_{\Omega_i\cap\Omega_0+he_l}\varphi^2(W_i(\nabla u)+k_i)\;\mathrm{d}x$$

and hence

$$I_{11} \overset{(H1)}{\leq} -\sum_{i=1}^{N} \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 \left(c_1^i \left| \nabla u \right|^{p_i} + c_0^i + k_i \right) \, \mathrm{d}x. \tag{46}$$

Now we consider the estimation of I_3. By the product rule for differences we get

$$\begin{aligned} I_3 = &-\sum_{i=1}^{N} \int_{\Omega_i} \triangle_h (\varphi^2 H^T : \nabla u) \, \mathrm{d}x \\ &+ \sum_{i=1}^{N} \int_{\Omega_i} (\triangle_h \varphi^2) H^T(x + he_l) : \nabla u(x + he_l) \, \mathrm{d}x \\ &+ \sum_{i=1}^{N} \int_{\Omega_i} \varphi^2 \triangle_h H^T : \nabla u(x + he_l) \, \mathrm{d}x - \sum_{i=1}^{N} \int_{\Omega_i} H^T : (\triangle_h u \otimes \nabla \varphi^2) \, \mathrm{d}x \\ = &\; I_{31} + I_{32} + I_{33} + I_{34}. \end{aligned}$$

By the usual arguments, compare (31)-(33), it follows $|I_{32}| + |I_{33}| + |I_{34}| \leq ch$, where c is independent of h. Analogously to the considerations in (37), keeping in mind that $\Omega_i + he_l \cap \Omega_0 \cap \operatorname{supp} \varphi = \emptyset$, one obtains

$$\begin{aligned} I_{31} = &-\sum_{i=1}^{N} \left(\int_{\Omega_i + he_l \cap \Omega_0} \varphi^2 H^T : \nabla u \, \mathrm{d}x - \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 H^T : \nabla u \, \mathrm{d}x \right) \\ &- \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \left(\int_{\Omega_i + he_l \cap \Omega_j} \varphi^2 H^T : \nabla u \, \mathrm{d}x - \int_{\Omega_i \cap \Omega_j + he_l} \varphi^2 H^T : \nabla u \, \mathrm{d}x \right) \\ = &\sum_{i=1}^{N} \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 H^T : \nabla u \, \mathrm{d}x - 0 \end{aligned}$$

since $\Omega_i + he_l \cap \Omega_0 = \emptyset$, see also Definition 4.3 and Remark 4.1. By Hölder's and Young's inequality and since $H \in L^\infty(\Omega)$ it follows

$$\begin{aligned} |I_{31}| &\leq \sum_{i=1}^{N} \delta_i^{-1} \left\| \varphi^{\frac{2}{q_i}} H^T \right\|_{L^{q_i}(\Omega_i \cap \Omega_0 + he_l)} \delta_i \left\| \varphi^{\frac{2}{p_i}} \left| \nabla u \right| \right\|_{L^{p_i}(\Omega_i \cap \Omega_0 + he_l)} \\ &\leq c \sum_{i=1}^{N} \left(\delta_i^{-q_i} \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 \left| H^T \right|^{q_i} \, \mathrm{d}x + \delta_i^{p_i} \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 \left| \nabla u \right|^{p_i} \, \mathrm{d}x \right) \\ &\overset{(A6)}{\leq} \tilde{c} h \sum_{i=1}^{N} \delta_i^{-q_i} + \sum_{i=1}^{N} c \delta_i^{p_i} \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 \left| \nabla u \right|^{p_i} \, \mathrm{d}x. \end{aligned} \tag{47}$$

for arbitrary $\delta_i > 0$ ($c, \tilde{c}$ independent of h). For $1 \leq i \leq N$ choose $\delta_i = \left(\frac{c_1^i}{c}\right)^{\frac{1}{p_i}}$ where c_1^i is the constant from assumption (H1). Then with (46) and (47) we get

$$\begin{aligned} I_{11} + |I_{31}| &\leq \tilde{c}h \sum_{i=1}^{N} \delta_i^{-q_i} + \sum_{i=1}^{N} \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 \left(c\delta_i^{p_i} |\nabla u|^{p_i} - c_1^i |\nabla u|^{p_i} - k_i - c_0^i\right) \mathrm{d}x \\ &\leq \tilde{c}h \sum_{i=1}^{N} \delta_i^{-q_i} + \sum_{i=1}^{N} (k_i + |c_0^i|) \, |\Omega_i \cap \Omega_0 + he_l| \\ &\leq c^* h, \end{aligned}$$

where c^* is independent of h. Collecting the estimates, one obtains for (44)

$$\sum_{i=1}^{N} \int_{\Omega_i} \varphi^2 (\kappa_i + |\nabla u(x + he_l)| + |\nabla u(x)|)^{p_i - 2} \, |\triangle_h \nabla u|^2 \, \mathrm{d}x \leq ch.$$

The remaining part of the proof is completely analogous to the considerations in the second step for the Dirichlet problem.

Cross point on the boundary with mixed boundary conditions. Let $S \in \partial\Omega$ be a cross point with mixed boundary conditions in its neighborhood and $e_1, \ldots, e_d$ be a basis as in Definition 4.3, part 3. Assume that $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ is a suitable cut-off function. For the choice of the test function ξ one has to distinguish two cases, see also Remark 4.1. If $\operatorname{supp} \varphi \cap (\Gamma_D + he_l) \subset \Omega$ for $0 < h < h_0$, then choose ξ as in the case of pure Neumann boundary conditions. Else choose ξ as in the case of pure Dirichlet conditions. Proceeding analogously to these two cases yields the assertion.

Interior cross point. Here we choose $\xi(x) = \varphi^2(x)(u(x + he_l) - u(x))$ as test function, where φ is a suitable cut-off function with $\operatorname{supp} \varphi \subset \Omega$, and proceed analogous to the case of pure Neumann conditions. This completes the proof of Theorem 4.1. ∎

Remark. If in the weak formulation (14) ∇u is replaced by $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$, then the proof of the regularity result for equation (16) is completely analogous to the one of equation (14), one has to replace ∇u by $\varepsilon(u)$, only, and (28) changes to the following inequality:

$$\sum_{i=1}^{N} \int_{\Omega_i} \varphi^2 \left(\kappa_i + |\varepsilon(\tilde{u}(x + he_l))| + |\varepsilon(\tilde{u}(x))|\right)^{p_i - 2} |\triangle_h \varepsilon(\tilde{u}(x))|^{p_i - 2} \, \mathrm{d}x \leq ch.$$

This leads to $\varepsilon(u_i) \in L^{\frac{dp_i}{d-1} - \epsilon}(\Omega_i')$. By Korn's inequality, the estimates can be carried over to ∇u and considerations analogous to those in the second step of the proof for the Dirichlet problem can be carried out in the case $p_i \in (1,2]$. In the case $p_i > 2$, the argumentation is similar to (40) – (41) and again the estimates can be carried over to ∇u by Korn's inequality.

A. Some essential inequalities

Lemma A.1.

1. *For $A, B \in \mathbb{R}^s$, $|B| \geq |A|$ and $t \in [0, \frac{1}{4}]$ it holds,* [32, *Formula* (2.20)]*:*

$$4\,|B + t(A - B)| \geq |A| + |B| \,. \tag{48}$$

2. *Assume that $W : \mathbb{R}^{m\times d} \to \mathbb{R}$, $d \geq 2$, satisfies* (H0) *and* (H4) *for some $p \in (1, \infty)$ and $\kappa \in \{0, 1\}$. Then there exists $c > 0$ such that*

$$W(A) - W(B) \geq D_A W(B) : (A - B) + c\,(\kappa + |B| + |A|)^{p-2}\,|A - B|^2 \tag{49}$$

for every $A, B \in \mathbb{R}^{m\times d}$.

3. *Let $\kappa \in \{0, 1\}$, $\alpha > 0$. There exists a constant $c > 0$ such that*

$$|(\kappa + |x|)^\alpha - (\kappa + |y|)^\alpha| \leq c\,(\kappa + |x| + |y|)^{\alpha-1}\,|x - y| \,. \tag{50}$$

for every $x, y \in \mathbb{R}^s$.

Remark. For the case $1 < p < 2$ and $W(A) = |A|^p$ inequality (49) is proved in [17, Lemma 4.2].

Proof. `Inequality (48).` For $0 \leq t \leq \frac{1}{4}$ and $A, B \in \mathbb{R}^s$ with $|B| \geq |A|$ it holds

$$|B + t(A - B)| \geq |(1 - t)\,|B| - t\,|A|| \geq \left| \frac{3}{4}\,|B| - \frac{1}{4}\,|A| \right| \geq \frac{1}{2}\,|B| \geq \frac{1}{4}\,|B| + \frac{1}{4}\,|A| \,.$$

`Inequality (49).` For $t \in [0, 1]$ set $f(t) = W(B + t(A - B))$. Assume first that $B + t(A - B) \neq 0$ for every $t \in [0, 1]$. In this case,

$$\begin{aligned} &W(A) - W(B) \\ &\quad \overset{(H4)}{\geq} D_A W(B) : (A - B) \\ &\qquad + c \int_0^1 (1 - t)\,(\kappa + |B + t(A - B)|)^{p-2}\,\mathrm{d}t\,|A - B|^2 \end{aligned} \tag{51}$$

If $1 < p \leq 2$, then

$$\begin{aligned} &W(A) - W(B) \\ &\overset{1<p\leq 2}{\geq} D_A W(B) : (A - B) + c \int_0^1 (1 - t)\,(\kappa + t\,|A| + (1 - t)\,|B|)^{p-2}\,\mathrm{d}t |A - B|^2 \\ &\geq D_A W(B) : (A - B) + c\,(\kappa + |B| + |A|)^{p-2}\,|A - B|^2 \,. \end{aligned}$$

In the case $p > 2$, it follows

$$\begin{aligned} &W(A) - W(B) \\ &\geq D_A W(B) : (A - B) + c \int_0^{\frac{1}{4}} (1-t)\,(\kappa + |B + t(A-B)|)^{p-2}\,\mathrm{d}t\,|A-B|^2 \\ &\overset{(48)}{\geq} D_A W(B) : (A - B) + \frac{c}{4^{p-2}} \int_0^{\frac{1}{4}} (1-t)\,\mathrm{d}t\,(\kappa + |B| + |A|)^{p-2}\,|A-B|^2 . \end{aligned}$$

from (51) by inequality (48) for $|B| \geq |A|$. On the other hand, if $|A| \geq |B|$, then a change of variables, $t = 1 - s$, and reasoning similarly to the case $|B| \geq |A|$ yields the assertion.

If there exists $t_0 \in (0, 1]$ with $B + t_0(A - B) = 0$, then consider $A_\delta := A + \delta C$ for $\delta > 0, C \in \mathbb{R}^{m\times d}\backslash\{0\}$. Note that $B + t(A_\delta - B) \neq 0$ for every $t \in [0, 1]$ and by the first step, inequality (49) holds for A_δ and B for every $\delta > 0$. Taking the limit $\delta \to 0$ yields the assertion.

`Inequality` (50). Assume first that $\alpha > 1$. For $|x| > |y| \geq 0$, Taylor's expansion yields

$$\begin{aligned} 0 \leq (\kappa + |x|)^\alpha - (\kappa + |y|)^\alpha &\leq \int_0^1 \alpha\,(\kappa + t\,|x| + (1-t)\,|y|)^{\alpha-1}\,|x-y|\,\mathrm{d}t \\ &\leq \alpha \int_0^1 (\kappa + |x| + |y|)^{\alpha-1}\,\mathrm{d}t\,|x-y| \end{aligned}$$

and (50) is proved for $\alpha > 1$. Assume now that $0 < \alpha < 1$ and $|x| > |y| \geq 0$. Then

$$\begin{aligned} 0 &\leq (\kappa + |x|)^{\alpha+1} - (\kappa + |y|)^{\alpha+1} + (\kappa + |x|)(\kappa + |y|) \\ &\leq c(\kappa + |x| + |y|)^\alpha\,|x-y| \leq c(2\kappa + |x| + |y|)^\alpha\,|x-y| \end{aligned}$$

from the first step and since $((\kappa + |x|)^{\alpha-1} - (\kappa + |y|)^{\alpha-1}) \leq 0$. The lemma is proved. ∎

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