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Boundary Data Smoothness for Solutions of Three Point Boundary Value Problems for Second Order Ordinary Differential Equations

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Abstract. Under certain conditions, solutions of the boundary value problem, $y'' = f(x, y, y')$, $a < x < b$, $y(x_1) = y_1, y(x_3) - y(x_2) = y_2$, $a < x_1 < x_2 < x_3 < b$, are differentiated with respect to the boundary conditions.

Keywords: *Boundary value problem, ordinary differential equation*

AMS subject classification: Primary 34B10, secondary 34B15

1. Introduction

In this paper, we will be concerned with differentiating solutions of certain three point boundary value problems with respect to boundary data for the second order ordinary differential equation,

$$y'' = f(x, y, y'), \quad a < x < b, \tag{1.1}$$

satisfying

$$y(x_1) = y_1, \qquad y(x_3) - y(x_2) = y_2, \tag{1.2}$$

where $a < x_1 < x_2 < x_3 < b$, and $y_1, y_2 \in \mathbb{R}$, and where we assume:

(i) $f(x, u_1, u_2) : (a, b) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous

(ii) $\frac{\partial f}{\partial u_i}(x, u_1, u_2) : (a, b) \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, $i = 1, 2$,

(iii) solutions of initial value problems for (1.1) extend to (a, b).

We remark that condition (iii) is not necessary for the spirit of this work's results, however, by assuming (iii), we avoid continually making statements in terms of solutions' maximal intervals of existence.

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Under uniqueness assumptions on solutions of (1.1) and (1.2), we will establish analogues of a result that Hartman [8] attributes to Peano concerning differentiation of solutions of (1.1) with respect to initial conditions. For our differentiation with respect to boundary conditions results, given a solution $y(x)$ of (1.1), we will give much attention to the *variational equation for* (1.1) *along* $y(x)$, which is defined by

$$z'' = \frac{\partial f}{\partial u_1}\Big(x, y(x),\ y'(x)\Big)z + \frac{\partial f}{\partial u_2}\Big(x, y(x), y'(x)\Big)z'. \tag{1.3}$$

Interest in multipoint boundary value problems for second order ordinary differential equations has been ongoing for several years, with much attention given to positive solutions. To see only few of these papers, we refer the reader to papers by Bai and Fang [1], Gupta and Trofimchuk [7], Ma [14, 15] and Yang [22].

Likewise, many papers have been devoted to smoothness of solutions of boundary value problems in regard to smoothness of the differential equation's nonlinearity, as well as the smoothness of the boundary conditions. For a view of how this work has evolved, involving not only boundary value problems for ordinary differential equations, but also discrete versions, functional differential equations versions and smoothness versions concerning solutions of dynamic equations on time scales, we suggest the manifold results in the papers [2] - [6], [8] - [11], [13], [16] - [20].

One instance in which the three point boundary value problem (1.1), (1.2) arises would involve the case when $f < 0$ and $y_1 = y_2 = 0$. Such a situation could describe the path of a projectile fired from ground level at time x_1, then later exiting the atmosphere followed by re-entry of the atmosphere at the same level at the respective times x_2 and x_3. The projectile's path smoothness with respect to boundary data would be the same smoothness as that of f.

The theorem for which we seek an analogue and attributed to Peano by Hartman can be stated in the context of (1.1) as follows:

Theorem 1.1 (Peano). *Assume that with respect of* (1.1)*, conditions* (i)-(iii) *are satisfied. Let* $x_0 \in (a, b)$ *and* $y(x) \equiv y(x, x_0, c_1, c_2)$ *denote the solution of* (1.1) *satisfying the initial conditions* $y(x_0) = c_1, y'(x_0) = c_2$. *Then,*

(a) $\frac{\partial y}{\partial c_1}$ *and* $\frac{\partial y}{\partial c_2}$ *exist on* (a, b)*, and* $\alpha_i \equiv \frac{\partial y}{\partial c_i}$*,* $i = 1, 2$*, are solutions of the variational equation* (1.3) *along* $y(x)$ *satisfying the respective initial conditions,*

$$\begin{aligned} \alpha_1(x_0) &= 1, \quad \alpha_1'(x_0) = 0, \\ \alpha_2(x_0) &= 0, \quad \alpha_2'(x_0) = 1. \end{aligned}$$

(b) $\frac{\partial y}{\partial x_0}$ *exists on* (a, b), *and* $\beta \equiv \frac{\partial y}{\partial x_0}$ *is the solution of the variational equation* (1.3) *along* $y(x)$ *satisfying the initial conditions,*

$$\begin{aligned}\beta(x_0) &= -y'(x_0),\\ \beta'(x_0) &= -y''(x_0).\end{aligned}$$

(c) $\frac{\partial y}{\partial x_0}(x) = -y'(x_0)\frac{\partial y}{\partial c_1}(x) - y''(x_0)\frac{\partial y}{\partial c_2}(x)$.

In addition, our analogue of Theorem 1.1 depends on uniqueness of solutions of (1.1), (1.2), a condition we list as an assumption:

(iv) Given $a < x_1 < x_2 < x_3 < b$, if

$$y(x_1) = z(x_1), \qquad y(x_3) - y(x_2) = z(x_3) - z(x_2),$$

where $y(x)$ and $z(x)$ are solutions of (1.1), then $y(x) \equiv z(x)$.

We will also make extensive use of a similar uniqueness condition on (1.3) along solutions $y(x)$ of (1.1).

(v) Given $a < x_1 < x_2 < x_3 < b$ and a solution $y(x)$ of (1.1), if

$$u(x_1) = 0, \qquad u(x_3) - u(x_2) = 0,$$

where $u(x)$ is a solution of (1.3) along $y(x)$, then $u(x) \equiv 0$.

2. An analogue of Peano's theorem for (1.1), (1.2)

In this section, we derive our analogue of Theorem 1.1 for the boundary value problem (1.1), (1.2). For such a differentiation result, we need continuous dependence of solutions on boundary conditions. Such continuity was established recently in [12], which we state here.

Theorem 2.1. *Assume conditions* (i)-(iv) *are satisfied with respect to* (1.1). *Let* $u(x)$ *be a solution of* (1.1) *on* (a, b), *and let* $a < c < x_1 < x_2 < x_3 < d < b$ *be given. Then, there exists a* $\delta > 0$ *such that, for* $|x_i - t_i| < \delta$, $i = 1, 2, 3$, *and* $|u(x_1) - y_1| < \delta$, $|u(x_3) - u(x_2) - y_2| < \delta$, *there exists a unique solution* $u_\delta(x)$ *of* (1.1) *such that* $u_\delta(t_1) = y_1$, $u_\delta(t_3) - u_\delta(t_2) = y_2$, *and* $\{u_\delta^{(j)}(x)\}$ *converges uniformly to* $u^{(j)}(x)$, *as* $\delta \to 0$, *on* $[c, d]$, *for* $i = 0, 1$.

We now present the result of the paper.

Theorem 2.2. *Assume conditions* (i)-(v) *are satisfied. Let* $u(x)$ *be a solution of* (1.1) *on* (a, b). *Let* $a < x_1 < x_2 < x_3 < b$ *be given, so that* $u(x) = u(x, x_1, x_2, x_3, u_1, u_2)$, *where* $u(x_1) = u_1$ *and* $u(x_3) - u(x_2) = u_2$. *Then,*

(a) $\frac{\partial u}{\partial u_1}$ *and* $\frac{\partial u}{\partial u_2}$ *exist on* (a,b), *and* $y_i \equiv \frac{\partial u}{\partial u_i}$, $i = 1, 2$, *are solutions of* (1.3) *along* $u(x)$ *and satisfy the respective boundary conditions,*

$$y_1(x_1) = 1, \qquad y_1(x_3) - y_1(x_2) = 0$$
$$y_2(x_1) = 0, \qquad y_2(x_3) - y_2(x_2) = 1.$$

(b) $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}$ *exist on* (a,b), *and* $z_i \equiv \frac{\partial u}{\partial x_i}$, $i = 1, 2, 3$, *are solutions of* (1.3) *along* $u(x)$ *and satisfy the respective boundary conditions,*

$$\begin{aligned} z_1(x_1) &= -u'(x_1), & z_1(x_3) - z_1(x_2) &= 0 \\ z_2(x_1) &= 0, & z_2(x_3) - z_2(x_2) &= u'(x_2) \\ z_3(x_1) &= 0, & z_3(x_3) - z_3(x_2) &= -u'(x_3). \end{aligned}$$

(c) *The partial derivatives satisfy,*

$$\begin{aligned} \frac{\partial u}{\partial x_1}(x) &= -u'(x_1)\frac{\partial u}{\partial u_1}(x) \\ \frac{\partial u}{\partial x_2}(x) + \frac{\partial u}{\partial x_3}(x) &= (u'(x_2) - u'(x_3))\frac{\partial u}{\partial u_2}(x). \end{aligned}$$

Proof. For part (a) we will give the argument for $\frac{\partial u}{\partial u_1}$, since the argument for $\frac{\partial u}{\partial u_2}$ is somewhat similar. Let $\delta > 0$ be as in Theorem 2.1. Let $0 < |h| < \delta$ be given and define

$$y_{1h}(x) = \frac{1}{h}[u(x, x_1, x_2, x_3, u_1 + h, u_2) - u(x, x_1, x_2, x_3, u_1, u_2)].$$

Note that $u(x_1, x_1, x_2, x_3, u_1 + h, u_2) = u_1 + h$, and $u(x_1, x_1, x_2, x_3, u_1, u_2) = u_1$, so that for every $h \neq 0$,

$$y_{1h}(x_1) = \frac{1}{h}[u_1 + h - u_1] = 1,$$

and

$$y_{1h}(x_3) - y_{1h}(x_2) = u_2 - u_2 = 0.$$

Let

$$\beta_2 = u'(x_1, x_1, x_2, x_3, u_1, u_2)$$
$$\epsilon_2 = \epsilon_2(h) = u'(x_1, x_1, x_2, x_3, u_1 + h, u_2) - \beta_2.$$

By Theorem 2.1, $\epsilon_2 = \epsilon_2(h) \to 0$, as $h \to 0$. Using the notation of Theorem 1.1 for solutions of initial value problems for (1.1) and viewing the solutions u as solutions of initial value problems, we have

$$y_{1h}(x) = \frac{1}{h}[y(x, x_1, u_1 + h,\ \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2)].$$

Then, by utilizing a telescoping sum, we have

$$\begin{aligned} y_{1h}(x) &= \frac{1}{h}\Big[\{y(x, x_1, u_1 + h,\ \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2 + \epsilon_2)\} \\ &\qquad + \{y(x, x_1, u_1, \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2)\}\Big]. \end{aligned}$$

By Theorem 1.1 and the Mean Value Theorem, we obtain

$$\begin{aligned} y_{1h}(x) &= \frac{1}{h}\alpha_1\big(x, y(x, x_1, u_1 + \bar{h},\ \beta_2 + \epsilon_2)\big)(u_1 + h - u_1) \\ &\quad + \frac{1}{h}\alpha_2\big(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2)\big)(\beta_2 + \epsilon_2 - \beta_2), \end{aligned}$$

where $\alpha_i(x, y(\cdot)),\ i = 1, 2$, is the solution of the variational equation (1.3) along $y(\cdot)$ and satisfies in each case,

$$\begin{aligned} \alpha_1(x_1) &= 1, \qquad \alpha_1'(x_1) = 0, \\ \alpha_2(x_1) &= 0, \qquad \alpha_2'(x_1) = 1. \end{aligned}$$

Furthermore, $u_1 + \bar{h}$ is between u_1 and $u_1 + h$, and $\beta_2 + \bar{\epsilon}_2$ is between β_2 and $\beta_2 + \epsilon_2$. Now simplifying,

$$\begin{aligned} y_{1h}(x) = \alpha_1\big(&x, y(x, x_1, u_1 + \bar{h},\ \beta_2 + \epsilon_2)\big) \\ &+ \frac{\epsilon_2}{h}\alpha_2\big(x, y(x, x_1, u_1,\ \beta_2 + \bar{\epsilon}_2)\big). \end{aligned}$$

Thus, to show $\lim_{h\to 0} y_{1h}(x)$, exists, it suffices to show $\lim_{h\to 0}\frac{\epsilon_2}{h}$ exists.

Now $\alpha_2(x, y(\cdot))$ is a nontrivial solution of (1.3) along $y(\cdot)$, and $\alpha_2(x_1, y(\cdot)) = 0$. So, by assumption (v),

$$\alpha_2\big(x_3, y(\cdot)\big) - \alpha_2\big(x_2, y(\cdot)\big) \neq 0.$$

However, we observed that $y_{1h}(x_3) - y_{1h}(x_2) = 0$, from which we obtain

$$\frac{\epsilon_2}{h} = \frac{-[\alpha_1\big(x_3, y(x, x_1, u_1 + \bar{h},\ \beta_2 + \epsilon_2)\big) - \alpha_1\big(x_2, y(x, x_1, u_1 + \bar{h},\ \beta_2 + \epsilon_2)\big)]}{[\alpha_2\big(x_3, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2)\big) - \alpha_2\big(x_2, y(x, x_1, u_1,\ \beta_2 + \bar{\epsilon}_2)\big)]}.$$

As a consequence of continuous dependence, we can let $h \to 0$, so that

$$\begin{aligned} \lim_{h\to 0}\frac{\epsilon_2}{h} &= \frac{-[\alpha_1\big(x_3, y(x, x_1, u_1, \beta_2)\big) - \alpha_1\big(x_2, y(x, x_1, u_1, \beta_2)\big)]}{[\alpha_2\big(x_3, y(x, x_1, u_1, \beta_2)\big) - \alpha_2\big(x_2, y(x, x_1, u_1\beta_2)\big)]} \\ &= \frac{-[\alpha_1\big(x_3, u(\cdot)\big) - \alpha_1\big(x_2, u(\cdot)\big)]}{[\alpha_2\big(x_3, u(\cdot)\big) - \alpha_2\big(x_2, u(\cdot)\big)]} \\ &\equiv D. \end{aligned}$$

Let $y_1(x) = \lim_{h \to 0} y_{1h}(x)$, and note by construction of $y_{1h}(x)$,

$$y_1(x) = \frac{\partial u}{\partial u_1}(x, x_1, x_2, x_3, u_1, u_2).$$

Furthermore,

$$\begin{aligned} y_1(x) &= \lim_{h \to 0} y_{1h}(x) \\ &= \alpha_1(x, y(x, x_1, u_1, \beta_2)) + D\alpha_2(x, y(x, x_1, u_1, \beta_2)) \\ &= \alpha_1(x, u(x, x_2, x_2, x_3, u_1, u_2)) + D\alpha_2(x, u(x, x_1, x_2, x_3, u_1, u_2)), \end{aligned}$$

which is a solution of the variational equation (1.3) along $u(x)$. In addition because of the boundary conditions satisfied by $y_{1h}(x)$, we also have,

$$y_1(x_1) = 1, \qquad y_1(x_3) - y_1(x_2) = 0.$$

This completes the argument for $\frac{\partial u}{\partial u_1}$.

In part (b) of the theorem, we will produce the details for $\frac{\partial u}{\partial x_2}$, with the arguments for $\frac{\partial u}{\partial x_1}$ and $\frac{\partial u}{\partial x_3}$ being somewhat along the same lines. So, let $\delta > 0$ be as in Theorem 2.1, let $0 < |h| < \delta$ be given, and define

$$z_{2h}(x) = \frac{1}{h}[u(x, x_1, x_2 + h, x_3, u_1, u_2) - u(x, x_1, x_2, x_3, u_1, u_2)].$$

Note that $z_{2h}(x_1) = \frac{1}{h}[u_1 - u_1] = 0$, for every $h \neq 0$. Next, let

$$\begin{aligned} \beta_2 &= u'(x_1, x_1, x_2, x_3, u_1, u_2), \\ \epsilon_2 = \epsilon_2(h) &= u'(x_1, x_1, x_2 + h, x_3, u_1, u_2). \end{aligned}$$

By Theorem 2.1, $\epsilon_2 = \epsilon_2(h) \to 0$, as $h \to 0$. As in part (a), we use the notation of Theorem 1.1 for solutions of initial value problems for (1.1) and viewing the solutions u as solutions of initial value problems, we have

$$z_{2h}(x) = \frac{1}{h}[y(x, x_1, u_1, \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2))].$$

By the Mean Value Theorem,

$$z_{2h}(x) = \frac{1}{h}\alpha_2(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2))(\beta_2 + \epsilon_2 - \beta_2),$$

where $\alpha_2(x, y(\cdot))$ is the solution of (1.3) along $y(\cdot)$ and satisfies

$$\alpha_2(x_1) = 0, \quad \alpha_2'(x_1) = 1,$$

and moreover, $\beta_2+\bar{\epsilon}_2$ lies between β_2 and $\beta_2+\epsilon_2$. As before, to show $\lim_{h\to 0} z_{2h}(x)$ exists, it suffices to show $\lim_{h\to 0} \frac{\epsilon_2}{h}$ exists.

Since $\alpha_2(x, y(\cdot))$ is a nontrivial solution of (1.3) along $y(\cdot)$ and $\alpha_2(x_1, y(\cdot)) = 0$, it follows from assumption (v) that

$$\alpha_2(x_3, y(\cdot)) - \alpha_2(x_2, y(\cdot)) \neq 0.$$

Hence,

$$\frac{\epsilon_2}{h} = \frac{z_{2h}(x_3) - z_{2h}(x_2)}{\alpha_2(x_3, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2)) - \alpha_2(x_2, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2))}.$$

We look in more detail at the numerator of this quotient. In particular, by the Mean Value Theorem for integrals,

$$\begin{aligned}
z_{2h}(x_3) - z_{2h}(x_2) &= \frac{1}{h}\Big[\{u(x_3, x_1, x_2 + h, x_3, u_1, u_2) \\
&\qquad\qquad -u(x_2, x_1, x_2 + h, x_3, u_1, u_2)\} \\
&\quad -\{u(x_3, x_1, x_2, x_3, u_1, u_2) - u(x_2, x_1, x_2, x_3, u_1, u_2)\}\Big] \\
&= \frac{1}{h}\Big[\{u(x_3, x_1, x_2 + h, x_3, u_1, u_2) \\
&\qquad\qquad -u(x_2, x_1, x_2 + h, x_3, u_1, u_2)\} \\
&\quad -\{u(x_3, x_1, x_2 + h, x_3, u_1, u_2) \\
&\qquad\qquad -u(x_2 + h, x_1, x_2 + h, x_3, u_1, u_2)\}\Big] \\
&= \frac{1}{h}\Big[u(x_2 + h, x_1, x_2 + h, x_3, u_1, u_2) \\
&\quad -u(x_2, x_1, x_2 + h, x_3, u_1, u_2)\Big] \\
&= \frac{1}{h}\int_{x_2}^{x_2+h} u'(s, x_1, x_2 + h, x_3, u_1, u_2)ds \\
&= \frac{1}{h}u'(c_h, x_1, x_2 + h, x_3, u_1, u_2)(x_2 + h - x_2) \\
&= u'(c_h, x_1, x_2 + h, x_3, u_1, u_2),
\end{aligned}$$

for some c_h inclusively between x_2 and x_2+h. By Theorem 2.1, we can compute the limit,

$$\begin{aligned}
\lim_{h\to 0}(z_{2h}(x_3) - z_{2h}(x_2)) &= \lim_{h\to 0} u'(c_h, x_1, x_2 + h, u_1, u_2) \\
&= u'(x_2).
\end{aligned}$$

As a consequence, when we return to the quotient defining $\frac{\epsilon_2}{h}$, we can now compute the limit,

$$\begin{aligned}\lim_{h\to 0}\frac{\epsilon_2}{h} &= \frac{u'(x_2)}{\alpha_2\big(x_3, y(x,x_1,u_1,\beta_2)\big) - \alpha_2\big(x_2, y(x,x_1,u_1,\beta_2)\big)} \\ &= \frac{u'(x_2)}{\alpha_2\big(x_3, u(\cdot)\big) - \alpha_2\big(x_2, u(\cdot)\big)} \\ &\equiv E.\end{aligned}$$

From the above expression,

$$z_{2h}(x) = \frac{\epsilon_2}{h}\alpha_2\big(x, y(x,x_1,u_1,\beta_2+\bar{\epsilon}_2)\big),$$

we can evaluate the limit as $h \to 0$, and if we let $z_2(x) = \lim_{h\to 0} z_{2h}(x)$, we have $z_2(x) = \frac{\partial u}{\partial x_2}$, and we obtain

$$\begin{aligned}z_2(x) &= \lim_{h\to 0} z_{2h}(x) \\ &= E\alpha_2\big(x, y(x,x_1,u_1,\beta_2)\big) \\ &= E\alpha_2\big(x, u(x,x_1,x_2,x_3,u_1,u_2)\big),\end{aligned}$$

which is a solution of (1.3) along $u(x)$. In addition, from above observations, $z_2(x)$ satisfies the boundary conditions,

$$\begin{aligned}z_2(x_1) &= \lim_{h\to 0} z_{2h}(x_1) = 0 \\ z_2(x_3) - z_2(x_2) &= \lim_{h\to 0}\big(z_{2h}(x_3) - z_{2h}(x_2)\big) = u'(x_2).\end{aligned}$$

This completes the proof of part (b).

Part (c) of the theorem is immediate given by verifying that each side of the respective equations are solutions of (1.3) along $u(x)$ and satisfy the same three point boundary conditions, and then assumption (v) establishes the equalities. ∎

Remark. A result broadly extending Theorems 1.1 and 2.1 can be proved within the context of linear nonresonant multipoint boundary conditions for n-th order differential equations with smooth right hand sides. It suffices to start with assumptions on the equation $y^{(n)} = 0$ along with the multipoint boundary conditions, and then transform the problem into an integral equation of the form

$$u = N(u, \alpha), \tag{2.1}$$

where α is a vector of parameters involving the boundary conditions. We suppose:

(A) Equation (2.1) has a unique solution for some fixed parameter α_0.

(B) The linear operator $E - N_u(u(\alpha_0), \alpha_0)$ is invertible (in other words, the variational equation has a unique solution along $u(\alpha_0)$).

Then by the implicit function theorem, the equation (2.1) can be solved with respect to u. Thus we will obtain a smooth function $u = u(\alpha)$ defined in some neighborhood of α_0. Finally, additional properties of $\partial u(\alpha)/\partial\alpha$ can be deduced from the given boundary conditions. In fact, Ehme [3] has obtained some very nice results for n-th order boundary value problems by utilizing assumptions like (A) and (B) in conjunction with the implicit function theorem.

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References

[1] Bai, C. and J. Fang: *Existence of multiple positive solutions for nonlinear m-point boundary value problems.* J. Math. Anal. Appl. 281 (2003), 76 – 85.

[2] Datta, A.: *Differences with respect to boundary points for right focal boundary conditions.* J. Diff. Equations Appl. 4 (1998), 571 – 578.

[3] Ehme, J.: *Differentiation of solutions of boundary value problems with respect to nonlinear boundary conditions.* J. Diff. Equations 101 (1993), 139 – 147.

[4] Ehme, J., Eloe, P. W. and J. Henderson: *Differentiability with respect to boundary conditions and deviating argument for functional-differential systems.* Diff. Equations Dynam. Systems. 1 (1993), 59 – 71.

[5] Ehme, J. and J. Henderson: *Functional boundary value problems and smoothness of solutions.* Nonlinear Anal. 26 (1996), 139 – 148.

[6] Ehme, J. and B. Lawrence: *Linearized problems and continuous dependence for finite difference equations.* Panamer. Math. J. 10 (2000), 13 – 24.

[7] Gupta, C. P. and S.I. Trofimchuk: *Solvability of a multi-point boundary value problem and related a priori estimates.* Canad. Appl. Math. Quart. 6 (1998), 45 – 60.

[8] Hartman, P.: *Ordinary Differential Equations.* New York, Wiley, 1964.

[9] Henderson, J.: *Right focal point boundary value problems for ordinary differential equation and variational equations.* J. Math. Anal. Appl. 98 (1984), 363 – 377.

[10] Henderson, J.: *Disconjugacy, disfocality and differentiation with respect to boundary conditions.* J. Math. Anal. Appl. 121 (1987), 1 – 9.

[11] Henderson, J. and B. Lawrence: *Smooth dependence on boundary matrices.* J. Diff. Equations Appl. 2 (1996), 161 – 166.

[12] Henderson, J., Karna, B. and C.C. Tisdell: *Uniqueness implies existence for multipoint boundary value problems second order equations.* Proc. Amer. Math. Soc., in press.

[13] Lawrence, B.: *A variety of differentiability results for a multi-point boundary value problem.* J. Comput. Appl. Math. 141 (2002), 237 – 248.

[14] Ma, R.: *Existence theorems for a second-order three-point boundary value problems.* J. Math. Anal. Appl. 212 (1997), 430 – 442.

[15] Ma, R.: *Existence and uniqueness of solutions to first-order three-point boundary value problems.* Appl. Math. Lett. 15 (2002), 211 – 216.

[16] Peterson, A. C.: *Comparison theorems and existence theorems for ordinary differential equations.* J. Math. Anal. Appl. 55 (1976), 773-784.

[17] Peterson, A. C.: *Existence-uniqueness for ordinary differential equations.* J. Math. Anal. Appl. 64 (1978), 166 – 172.

[18] Peterson, A. C.: *Existence-uniqueness for focal point boundary value problems.* SIAM J. Math. Anal. 12 (1981), 173 – 185.

[19] Peterson, A. C.: *Existence and uniqueness theorems for nonlinear difference equations.* J. Math. Anal. Appl. 125 (1987), 185 – 191.

[20] Spencer, J.: *Relations between boundary value functions for a nonlinear differential equation and its variational equation.* Canad. Math. Bull. 18 (1975), 269 – 276.

[21] Sukup, D.: *On the existence of solutions to multipoint boundary value problems.* Rocky Mtn. J. Math. 6 (1976), 357 – 375.

[22] Yang, B.: *Boundary Value Problems for Ordinary Differential Equations.* (Ph.D. dissertation, Mississippi State University.) Mississippi State, MS, 2002.

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