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Compression Molding II: Existence of the Solution for a Hele-Shaw Type Model

Ming Fang and R. P. Gilbert

Abstract. We discuss an idealized model for compression molding, assuming a compressible flow. Existence theorems are established for this system.

Keywords: *Hele-Shaw flows, compressible flows, nonlinear partial differential equations, weak solutions*

MSC 2000: Primary 35M10, 35J70, secondary 35K55, 35Q35, 76D27

1. Introduction

Compression molding is a manufacturing process where a material is squeezed into a desired shape by the application of heat and pressure to the material. Ideally this is done by placing the object between two parallel plates. The pressures generated during a squeezing flow are often large [24, p. 504] and give rise to the possible necessity of taking compressibility into consideration as Cole, Batchelor, and many other scholars have suggested in the deep oceans and other circumstances [8] [6, p. 56]. In this paper we study a model where the flow is compressible. The resulting equations, only caricatures of the true physics, nevertheless they allow a rigorous and detailed mathematical analysis, which gives the essential properties of the flow.

Section 2 recounts the derivation of our model from [11], and in particular clarifies the correction terms which rely on the equation of state and explains some simplifying physical hypotheses. In sections 3 and 4, we prove the existence of weak solutions to the resulting problems 1 and 2:

Ming Fang: Department of Mathematics, Norfolk State University, Norfolk, VA 23504 USA; mfang@nsu.edu
R. P. Gilbert: Department of Mathematical Science, University of Delaware, Newark, DE 19716 U.S.A.; gilbert@math.udel.edu

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Problem 1. Find functions θ and p defined in Ω such that

$$-\Delta\theta = k(\theta,\lambda)|\nabla p|^r + k(\theta,\lambda)|p|^r + f \quad in \quad \Omega \tag{1.1}$$

$$-div\{k(\theta,\lambda)|\nabla p|^{r-2}\nabla p\} + k(\theta,\lambda)|p|^{r-2}p = g \quad in \quad \Omega \tag{1.2}$$

$$\theta = \theta_0 \quad on \quad \partial\Omega \tag{1.3}$$

$$p = p_0 \quad on \quad \Gamma_0 \tag{1.4}$$

$$-k(\theta,\lambda)|\nabla p|^{r-2}\frac{\partial p}{\partial \nu} = l \quad on \quad \Gamma_1. \tag{1.5}$$

Problem 2. Find functions θ and p defined in Ω_T such that

$$\theta_t - \Delta\theta = k(\theta,\lambda,t)|\nabla p|^r + k(\theta,\lambda,t)|p|^r + f \quad in \quad \Omega_T \tag{1.6}$$

$$-div\{k(\theta,\lambda,t)|\nabla p|^{r-2}\nabla p\} + k(\theta,\lambda,t)|p|^{r-2}p = g \quad in \quad \Omega_T \tag{1.7}$$

$$\theta = \theta_0 \quad on \quad \partial\Omega \times (0,T) \tag{1.8}$$

$$\theta = \varphi \quad on \quad \Omega \times \{0\} \tag{1.9}$$

$$p = p_0 \quad on \quad \Gamma_0 \times (0,T) \tag{1.10}$$

$$-k(\theta,\lambda,t)|\nabla p|^{r-2}\frac{\partial p}{\partial \nu} = l \quad on \quad \Gamma_1 \times (0,T). \tag{1.11}$$

The given functions g results from the forced deformation in the vertical direction. A derivation is given in Section 2 (we leave aside the problem of the free contact surface). Here we assume that Ω is a bounded domain in R^n with C^1 boundary and $\partial\Omega$ is decomposed as $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, where n is a natural number that is greater or equal to 2, Γ_0 and Γ_1 are C^1 manifolds with $\Gamma_0 \cap \Gamma_1 = \emptyset$. The outward unit normal of $\partial\Omega$ is denoted by ν. For a given time interval (0,T), let $\Omega_T = \Omega \times (0,T)$. We assume also that f, θ_0, p_0, l, φ, and k are given functions and $k(\theta,\lambda,t)$ is continuous in time, while r is a given positive constant related to the power law index n; p is the pressure of the flow and θ is the temperature. The value of λ at which the shear stresses $s^h_{31} = s^h_{32} = 0$ vanish is not known a priori. One can find λ by satisfying the no-slip upper boundary condition as suggested at the end of section 2.

Problem 1 is a model for a stationary flow and Problem 2 is a model for the time-dependent flow. Although the physical models are two dimensional, we generalize our proofs in the case of N dimension.
In this paper, for $s > 1$, let

$$H^{1,s}_{\Gamma_0}(\Omega) = \{v; v \in H^{1,s}(\Omega), v = 0 \quad \text{on } \Gamma_0\}$$

denote the usual Sobolev space equipped with the standard norm. Let

$$\sigma = \begin{cases} \frac{n}{n-1} & \text{if } 1 < r < n \\ \frac{n+1}{n} & \text{if } r = n \\ r^* & \text{if } r > n. \end{cases} \tag{1.12}$$

where $r^* = \frac{r}{r-1}$. We assume that the boundary values θ_0 and p_0 for Problem 1 and 2 can be extended to functions defined on Ω such that

$$\theta_0 \in H^{1,\sigma}(\Omega) \quad \text{and} \quad p_0 \in H^{1,\tau}(\Omega), \tag{1.13}$$

where τ is a fixed number that is greater than r. We further assume that

$$f \in L^{\sigma_1}(\Omega), \quad g \in L^{\sigma_2}(\Omega), \quad \text{and} \quad l \in L^{\sigma_3}(\Gamma_1), \tag{1.14}$$

where $\sigma_i,\ i = 1, 2, 3$ satisfy

$$\sigma_1 > \frac{n}{2}, \quad \sigma_2 > \left(\frac{nr}{n-r}\right)^*, \quad \text{and} \quad \sigma_3 > \left(\frac{(n-1)r}{n-r}\right)^* \text{ if } 1 < r < n. \tag{1.15}$$

Otherwise, we assume that

$$f, g \in L^{\sigma_4}(\Omega), l \in L^{\sigma_4}(\Gamma_1),\ (\sigma_4 > 1) \qquad \text{if } r = n \tag{1.16}$$

$$f, g \in L^1(\Omega),\ \ l \in L^1(\Gamma_1), \qquad \text{if } r > n. \tag{1.17}$$

Finally, we assume that there exist positive constants $k_2 > k_1 > 0$ such that

$$k_1 < k(\theta, \lambda) < k_2\ , \quad k_1 < k(\theta, \lambda, t) < k_2 \qquad \forall \theta \in R^1,\ t \geq 0 \tag{1.18}$$

The principle diffculty of the proof lies in overcoming the critical growth $|\nabla p|^r$ and nonlinear correction terms in both systems.

Remarks. Many of the subsequent calculations, both rigorous and formal, are inspired by ideas originating with the injection molding problem, as studied in [13, 10]. It is worthwhile to mention that compressibility has already been considered in injection-molding (e.g. [7, 15]).

Some other related papers are Aronsson-Evans [3], Advani-Sozer [4] and Jackson-Advani-Tucker [16].

2. Formulation of the problem

This section provides a reconstruction of the derivation in [11]. Instead of asymptotic analysis and general form of state equations, as done in [11], we derive the systems from several simplifying asumptions and some restrictions on state equations.

a. Notations. Indices with Greek letters range from 1 to 2 while indicies with Roman letters range from 1 to 3. For example, we use $(x_\alpha) := (x_1, x_2)$ to designate two coordinates and $(x_i) := (x_1, x_2, x_3)$ to designate three coordinates. In addition, the summation convention will be in effect.

We suppose that at time t, the compressed plastic lies between two infinite horizontal plates, the lower at height zero and the upper at height $h(t) > 0$. We assume

$$\dot{h}(t) < 0 \quad \text{for} \quad 0 \le t < T, \tag{2.1}$$

$T \le \infty$ being the time when the two plates meet. Ω denotes the open subregion in R^2 with a C^1 boundary above which the polymer lies.

b. Velocity, pressure, temperature, strain and stress. The full flow equations read

$$\rho^h \frac{Dv^h}{Dt} = \text{div}\ \sigma^h + \rho^h \vec{f^h} \tag{2.2}$$

$$\rho^h c^h \frac{DT^h}{Dt} = \frac{\partial}{\partial x_i}\left(K^h \frac{\partial T^h}{\partial x_i}\right) + \sigma^h_{ij} d^h_{ij} \tag{2.3}$$

$$\frac{D\rho^h}{Dt} + \rho^h \text{div}\ v^h = 0, \tag{2.4}$$

where $\frac{D}{Dt}$ denote the *material derivative*, $v^h = (v^h_1, v^h_2, v^h_3)$ is the velocity field, $\sigma^h = (\sigma^h_{ij})$ is the *Cauchy stress tensor*, ρ^h is the *density* of the fluid, $\vec{f^h}$ is the *volume force density*, c^h is the *specific heat*, K^h is the *thermal conductivity*, and

$$d^h = (d^h_{ij}) \qquad \text{with} \quad d^h_{ij} = \frac{1}{2}\left(\frac{\partial v^h_i}{\partial x_j} + \frac{\partial v^h_j}{\partial x_i}\right) \tag{2.5}$$

denotes the *strain rate tensor*. Repeated indices are used for the summation convention.

The stress tensor is gorvened by the *power-law* model

$$\sigma^h_{ij} = -p^h \delta_{ij} + s^h_{ij} \qquad \text{with} \quad s^h_{ij} = k^h(T^h)\dot{\gamma}_h^{n-1} d^h_{ij}, \tag{2.6}$$

where $s^h = (s^h_{ij})$ is the viscous part of stress tensor σ^h, p^h is the pressure, $\dot{\gamma}_h$ is the *strain rate* given by $\dot{\gamma}_h = 2\sqrt{(d^h_{ij} d^h_{ij})}$, and n is the *power-law* index, and k^h is a given positive function. The compressible *power-law* structure (2.6) has been studied in both engineering and mathematics literatures (e.g. [19, 21, 20]).

c. Continuity. We simplify the continuity equation (2.4) by additionally hypothesizing that the fluid's density changes are very small, in accordance with most of compressible fluids. In particular, convective term, $v^h \cdot \nabla \rho^h$, in (2.4) can be neglected, (2.4) then reduces to

$$\text{div}\, v^h = -\frac{1}{\rho^h}\frac{\partial \rho^h}{\partial t}. \tag{2.7}$$

This simplification is consistent with equations of injection-molding corresponding to Chung and Hieber [7, 15] (see also [2]).

We choose ρ^h as a function of p^h and T^h, $\rho^h = f(p^h, T^h)$, as in the state equation postulate [2] (see also [8, 6, 22, 18]). As an illustrative example, we set

$$f(p^h, T^h) = \rho_0 e^{\int_0^t k(T^h)^{-\frac{1}{n}} |p^h|^{\frac{1}{n}-1} p^h ds},$$

where ρ_0 is the initial density. That is $\frac{1}{\rho^h}\frac{\partial \rho^h}{\partial t} = k(T^h)^{-\frac{1}{n}}|p^h|^{\frac{1}{n}-1}p^h$. Therefore the continuity equation (2.4) becomes

$$\operatorname{div} v^h = -k(T^h)^{-\frac{1}{n}}|p^h|^{\frac{1}{n}-1}p^h. \tag{2.8}$$

Since we will assume $0 < h \ll 1$, we expect the first two components of the velocity vector v^h to be physically most important. To eliminate the x_3 direction dependence, we integrate the continuity equation (2.8) in the x_3 direction

$$\frac{\partial \overline{v}^h_\alpha}{\partial x_\alpha} = -\frac{\dot{h}}{h} - k(\overline{T}^h)^{-\frac{1}{n}}|p^h|^{\frac{1}{n}-1}p^h, \tag{2.9}$$

where we have replaced T^h by its average $\overline{T}^h$ over the interval $(0, h)$ and

$$\overline{v}^h_\alpha = \frac{1}{h(x_1, x_2)} \int_0^{h(x_1,x_2)} v^h_\alpha(x_1, x_2, x_3)\, dx_3.$$

d. Hele-Shaw approximations. We next assume that viscosity effects and pressure gradient effects predominate. In particular, we drop the inertial term $\frac{Dv^h}{Dt}$, the body force $\vec{f}^h$ in (2.2). Then (2.2) and (2.6) imply

$$\nabla p = \operatorname{div}(k^h(T^h)\dot{\gamma}_h^{n-1} d^h_{ij}). \tag{2.10}$$

We further simplify by assuming the pressure p^h does not depend on x_3 and that v^h_3 may henceforth be taken to be zero in computing d^h_{ij}. Additionally, the velocity components vary much more rapidly in the x_3-direction than the lateral directions and consequently $\frac{\partial v^h_\alpha}{\partial x_\beta}$ may be ignored within the stretching tensor $\{d^h_{ij}\}$. Incorporating all these simplifying hypotheses into (2.10) yields the identities

$$\frac{\partial p^h}{\partial x_\alpha} = \frac{1}{2}\frac{\partial}{\partial x_3}\left\{ k(T^h) \left(\frac{\partial v^h_\alpha}{\partial x_3}\frac{\partial v^h_\alpha}{\partial x_3} \right)^{\frac{n-1}{2}} \frac{\partial v^h_\alpha}{\partial x_3} \right\}. \tag{2.11}$$

As p^h, and so $\frac{\partial p^h}{\partial x_\alpha}$, don't depend on x_3, we conclude

$$2(x_3 - \lambda)\frac{\partial p^h}{\partial x_\alpha} = k(T^h) \left(\frac{\partial v^h_\alpha}{\partial x_3}\frac{\partial v^h_\alpha}{\partial x_3} \right)^{\frac{n-1}{2}} \frac{\partial v^h_\alpha}{\partial x_3}, \tag{2.12}$$

where λ is the value of x_3 at which the shear stresses $s^h_{3\alpha} = 0$, that is, $\frac{\partial v^0_\alpha}{\partial y_3} = 0$. We shall find λ by satisfying the no-slip upper boundary condition $v^h_\alpha = 0$ on $x_3 = h$ given at the end of this section. Summations with repeated indices are used here. (2.12) implies

$$\left(\frac{\partial v^h_\alpha}{\partial x_3}\frac{\partial v^h_\alpha}{\partial x_3}\right)^{\frac{n-1}{2}} = \frac{|2x_3 - 2\lambda|^{1-\frac{1}{n}}}{k(T^h)^{1-\frac{1}{n}}}|\nabla p^h|^{1-\frac{1}{n}}. \tag{2.13}$$

We insert this equality into (2.12) and integrate, to deduce

$$v^h_\alpha = -\frac{\partial p^h}{\partial x_\alpha}|\nabla p^h|^{\frac{1}{n}-1}\frac{(2\lambda)^{\frac{1}{n}+1} - |2x_3 - 2\lambda|^{\frac{1}{n}+1}}{2\left(\frac{1}{n}+1\right)k(T^h)^{\frac{1}{n}}} \ . \tag{2.14}$$

Hence

$$\overline{v}^h_\alpha = -m(\overline{T}^h, \lambda, t)|\nabla p^h|^{\frac{1}{n}-1}\frac{\partial p^h}{\partial x_\alpha}, \tag{2.15}$$

where

$$m(\overline{T}^h, \lambda, t) = [k(\overline{T}^h)]^{-\frac{1}{n}}\frac{1}{2\left(\frac{1}{n}+1\right)h(t)}\int_0^{h(t)}\left(|2\lambda|^{\frac{1}{n}+1} - |2x_3 - 2\lambda|^{\frac{1}{n}+1}\right)dx_3 \ .$$

Recalling then the continuity condition (2.9) we conclude

$$\frac{\partial}{\partial x_\alpha}\left(m(\overline{T}^h, \lambda, t)|\nabla p^h|^{\frac{1}{n}-1}\frac{\partial p^h}{\partial x_\alpha}\right) = \frac{\dot{h}}{h} + k(\overline{T}^h)^{-\frac{1}{n}}|p^h|^{\frac{1}{n}-1}p^h. \tag{2.16}$$

e. Rescaling time. For simplicity, we can change variables in time by writing $t = \theta(s)$ $(0 \le s < \infty)$, θ solving the ordinary differential equation

$$\begin{cases} \theta'(s) = -\frac{h(\theta(s))}{\dot{h}(\theta(s))} & \text{if } 0 \le s < \infty \\ \theta(0) \ = \ 0 \ . \end{cases}$$

If we reinterpret the derivative $\dot{} = \frac{d}{ds}$ and $h = h(\theta(s))$, the partial differential equation (2.16) now becomes

$$-\frac{\partial}{\partial x_\alpha}\left(m(\overline{T}^h, \lambda, s)|\nabla p^h|^{\frac{1}{n}-1}\frac{\partial p^h}{\partial x_\alpha}\right) = 1 - k(\overline{T}^h)^{-\frac{1}{n}}|p^h|^{\frac{1}{n}-1}p^h \ . \tag{2.17}$$

f. Energy equation. We now switch to the energy equation (2.3) by assuming that the temperature change in the x_3-direction is insignificant compared with the lateral directions. This amounts to saying that

$$\rho^h c\frac{\partial T^h}{\partial t} = \frac{\partial}{\partial x_\alpha}\left(K\frac{\partial T^h}{\partial x_\alpha}\right) + k(T^h)\left(\frac{\partial v^h_\alpha}{\partial y_3}\frac{\partial v^h_\alpha}{\partial x_3}\right)^{\frac{n+1}{2}} - p^h \operatorname{div} v^h \ . \tag{2.18}$$

We now assume that the surface of the mold is insulated. This translates to

$$\frac{\partial T^h}{\partial x_3} = 0 \quad \text{at} \quad x_3 = 0 \quad \text{and} \quad h(t). \tag{2.19}$$

Taking the average on both sides of (2.18) and making use of (2.13) and (2.19), we obtain

$$\rho c \frac{\partial \overline{T}^h}{\partial t} = \frac{\partial}{\partial x_\alpha}\left(K \frac{\partial \overline{T}^h}{\partial x_\alpha}\right) + \kappa m(\overline{T}^h, \lambda, t)|\nabla p^h|^{\frac{n+1}{n}} + k(\overline{T}^h)^{-\frac{1}{n}}|p^h|^{\frac{1}{n}+1} \tag{2.20}$$

for a constant κ. Here we have replaced T^h by its average $\overline{T}^h$ over the interval $(0, h)$.

Next, let us transform notation, so as to be consistent with the mathematics references. We introduce the parameter r according to

$$r = \frac{n+1}{n}$$

and then write $\theta = \overline{T}^h$ to denote the average temperature. Dropping superscript "h" from all variables, it is easy to see that (1.6) and (1.7) are non-dimensional forms of (2.20) and (2.17), with the non-homogeneous extension f and 1 replacing by g.

Once the pressure and temperature distribution is known, λ may be found from (2.14) by one of no-slip boundary conditions, i.e. $v_\alpha = 0$ at $x_3 = h(t)$ can be used to find λ.

3. Problem 1

This section consists of two subsections. We will study the existence, uniqueness, stability, and continuity of solution p to the nonlinear equation (1.2) in the first subsection. The second subsection is devoted to Problem 1 based on the results of the first subsection.

3.1. A mixed boundary value problem. We study the following mixed boundary value problem:

$$-\text{div}\{k(\theta, \lambda)|\nabla p|^{r-2}\nabla p\} + k(\theta, \lambda)|p|^{r-2}p = g \quad \text{in } \Omega \tag{3.1}$$

$$p = p_0 \quad \text{on } \Gamma_0 \tag{3.2}$$

$$-k(\theta, \lambda)|\nabla p|^{r-2}\frac{\partial p}{\partial \nu} = l \quad \text{on } \Gamma_1\ . \tag{3.3}$$

Definition 3.1. We say that $p_\theta - p_0 \in H^{1,r}_{\Gamma_0}(\Omega)$ is a *weak solution to* (3.1) - (3.3) if

$$\theta \in H^{1,\sigma}_0(\Omega) + \theta_0, \tag{3.4}$$

and for all $\xi \in H^{1,r}_{\Gamma_0}(\Omega)$

$$\int_\Omega k(\theta,\lambda)(|\nabla p_\theta|^{r-2}\nabla p_\theta \nabla\xi + |p_\theta|^{r-2}p_\theta\xi)dx + \int_{\Gamma_1} l\xi ds = \int_\Omega g\xi dx. \tag{3.5}$$

Remark. We define $|\nabla p|^{r-2}\nabla p = 0$ on the set where $\nabla p = 0$ and $|p|^{r-2}p = 0$ on the set where $p = 0$.

Theorem 3.2. *Assume that the given g, σ, θ_0, l, and $k(\theta,\lambda)$ satisfy* (1.12) - (1.18). *Then there exists a unique weak solution p_θ to the mixed boundary value problem* (3.1) - (3.3) *in the sense of Definition* 3.1. *In addition, the solution p_θ satisfies the following properties:*

1) *It holds*

$$\|p_\theta\|_{H^{1,r}(\Omega)} \le C, \tag{3.6}$$

where C is a constant independent of θ and p_θ.

2) *Suppose that $k(\theta_m,\lambda) \to k(\theta,\lambda)$ a.e. in Ω if $\theta_m \to \theta$ a.e. in Ω where $\theta_m, \theta \in H^{1,\sigma}_0(\Omega) + \theta_0$. Then when $\theta_m \to \theta$ a.e. in Ω,*

$$p_{\theta_m} \to p_\theta \quad \text{strongly} \quad \text{in} \quad H^{1,r}(\Omega). \tag{3.7}$$

Proof. First we prove the existence and uniqueness of the solution. It is easy to see that the weak solutions of the mixed boundary value problem (3.1) - (3.3) correspond to critical points of the functional

$$I(p) = \int_\Omega \left[k(\theta,\lambda)\left(\frac{|\nabla p|^r}{r} + \frac{|p|^r}{r}\right) - gp\right]dx + \int_{\Gamma_1} lp\, ds. \tag{3.8}$$

According to the remark before Theorem 3.2, the functional belongs to C^1. Gâteaux derivative exists for all $\xi \in H^{1,r}_0(\Omega)$. From (1.14) - (1.18), the Sobolev imbedding theorem and Young's inequality with ϵ, we have

$$\begin{aligned} I(p) &\ge \frac{k_1}{r}\|p\|^r_{H^{1,r}(\Omega)} - \|g\|_{L^{\sigma_2}(\Omega)}\|p\|_{L^{\sigma_2^*}(\Omega)} - \|l\|_{L^{\sigma_3}(\Gamma_1)}\|p\|_{L^{\sigma_3^*}(\Gamma_1)} \\ &\ge \frac{k_1}{r}\|p\|^r_{H^{1,r}(\Omega)} - C_1\|g\|_{L^{\sigma_2}(\Omega)}\|p\|_{H^{1,r}(\Omega)} - C_2\|l\|_{L^{\sigma_3}(\Gamma_1)}\|p\|_{H^{1,r}(\Omega)} \\ &\ge \left(\frac{k_1}{r} - 2\epsilon\right)\|p\|^r_{H^{1,r}(\Omega)} - B(\epsilon) \end{aligned}$$

when $r < n$. We leave the estimate of the other cases, namely, $r \ge n$, to interested readers. Therefore $I(p)$ is coercive. Thus there exists at least one critical point p_θ of $I(p)$ which satisfies (3.5).

For a given θ, assume that there exists another solution p_θ^1. Then we have that

$$\int_\Omega k(\theta,\lambda)\Big[\big(|\nabla p_\theta|^{r-2}\nabla p_\theta - |\nabla p_\theta^1|^{r-2}\nabla p_\theta^1\big)\nabla\xi + \big(|p_\theta|^{r-2}p_\theta - |p_\theta^1|^{r-2}p_\theta^1\big)\xi\Big]\,dx = 0.$$

If we take $\xi = p_\theta - p_\theta^1$ in above equation, then we obtain $p_\theta = p_\theta^1$ from the well-known inequality (see, for example, p. 550 in [13])

$$(|x|^{r-2}x - |y|^{r-2}y)(x-y) \geq \begin{cases} a|x-y|^r & \text{if } r \geq 2 \\ \dfrac{a|x-y|^2}{(b+|x|+|y|)^{2-r}} & \text{if } 1 < r < 2, \end{cases} \tag{3.9}$$

where $a > 0$ and $b > 0$ are certain constants.

Next we prove 1). Taking $\xi = p_\theta - p_0$, we can rewrite (3.5) as

$$\begin{aligned}\int_\Omega k(\theta,\lambda)(|\nabla p_\theta|^r + |p_\theta|^r)\,dx \\ = \int_\Omega k(\theta,\lambda)(|\nabla p_\theta|^{r-2}\nabla p_\theta\cdot\nabla p_0 + |p_\theta|^{r-2}p_\theta p_0)\,dx \\ - \int_{\Gamma_1} l(p_\theta - p_0)\,ds + \int_\Omega g(p_\theta - p_0)\,dx\end{aligned}$$

Rel. (3.6) follows from (1.12) - (1.18), the Hölder inequality, the Sobolev imbedding theorem and Young's inequality with ϵ.

Finally, we prove 2). From (3.5), we know that

$$\begin{aligned}\int_\Omega k(\theta_m,\lambda)\left(|\nabla p_{\theta_m}|^{r-2}\nabla p_{\theta_m}\nabla\xi + |p_{\theta_m}|^{r-2}p_{\theta_m}\xi\right)\,dx \\ = \int_\Omega k(\theta,\lambda)\left(|\nabla p_\theta|^{r-2}\nabla p_\theta\nabla\xi + |p_\theta|^{r-2}p_\theta\xi\right)\,dx\end{aligned} \tag{3.10}$$

Recall that $\theta_m \to \theta$ a.e. in Ω. From (3.6), there exist $p \in H^{1,r}(\Omega)$ and a subsequence in the sequence $\{p_{\theta_{m_j}}\}$ such that

$$p_{\theta_{m_j}} \rightharpoonup p \quad \text{weakly in } H^{1,r}_{\Gamma_0}(\Omega)$$

as $j \to \infty$. We choose in (3.10) the test function $\xi = p_{\theta_{m_j}} - p$ to obtain

$$\begin{aligned}
&\int_\Omega k(\theta_{m_j},\lambda)\Big[\big(|\nabla p_{\theta_{m_j}}|^{r-2}\nabla p_{\theta_{m_j}} - |\nabla p|^{r-2}\nabla p\big)\nabla(p_{\theta_{m_j}} - p)\\
&\qquad + \big(|p_{\theta_{m_j}}|^{r-2}p_{\theta_{m_j}} - |p|^{r-2}p\big)(p_{\theta_{m_j}} - p)\Big]dx\\
&= \int_\Omega \big(k(\theta,\lambda) - k(\theta_{m_j},\lambda)\big)\Big[|\nabla p|^{r-2}\nabla p\cdot\nabla(p_{\theta_{m_j}} - p)\\
&\qquad + |p|^{r-2}p(p_{\theta_{m_j}} - p)\Big]dx\\
&\qquad + \int_\Omega k(\theta,\lambda)\Big[\big(|\nabla p_\theta|^{r-2}\nabla p_\theta - |\nabla p|^{r-2}\nabla p\big)\cdot\nabla(p_{\theta_{m_j}} - p)\\
&\qquad + \big(|p_\theta|^{r-2}p_\theta - |p|^{r-2}p\big)(p_{\theta_{m_j}} - p)\Big]dx
\end{aligned} \tag{3.11}$$

Since $k(\theta_{m_j},\lambda) \to k(\theta,\lambda)$ a.e. in Ω as $\theta_{m_j} \to \theta$ a.e. in Ω and $\{p_{\theta_{m_j}}\}$ are bounded in $H^{1,r}(\Omega)$, the right-hand side approaches zero as $j \to \infty$ due to Egoroff's theorem and the fact that $p_{\theta_{m_j}} \rightharpoonup p$ and $\nabla p_{\theta_{m_j}} \rightharpoonup \nabla p$ weakly in $L^r(\Omega)$ as $j \to \infty$. Hence

$$\lim_{j\to\infty}\int_\Omega e_{m_j}dx = 0 \tag{3.12}$$

where

$$\begin{aligned}
e_{m_j} \;&=\; k(\theta_{m_j},\lambda)\left[(|\nabla p_{\theta_{m_j}}|^{r-2}\nabla p_{\theta_{m_j}} - |\nabla p|^{r-2}\nabla p)\nabla(p_{\theta_{m_j}} - p)\right.\\
&\qquad \left.+(|p_{\theta_{m_j}}|^{r-2}p_{\theta_{m_j}} - |p|^{r-2}p)(p_{\theta_{m_j}} - p)\right].
\end{aligned}$$

Using (3.9) and the Hölder inequality we obtain

$$p_{\theta_{m_j}} \to p \quad \text{strongly in } H^{1,r}(\Omega).$$

This allows us to pass to the limit in the equation

$$\int_\Omega k(\theta_{m_j},\lambda)(|\nabla p_{\theta_{m_j}}|^{r-2}\nabla p_{\theta_{m_j}}\nabla\xi + |p_{\theta_{m_j}}|^{r-2}p_{\theta_{m_j}}\xi)dx + \int_{\Gamma_1} l\xi ds = \int_\Omega g\xi dx$$

to obtain that $p = p_\theta$, where $\xi \in H^{1,r}_{\Gamma_0}(\Omega)$. Since p is independent of the choice of subsequence, (3.7) is proved. Theorem 3.2 is thereby proved. ∎

3.2. Problem 1. In this subsection, we study Problem 1.

Definition 3.3. We say that $\{\theta, p\}$ is a *weak solution to* Problem 1 if

$$\theta - \theta_0 \in H^{1,\sigma}_0(\Omega), \quad p - p_0 \in H^{1,r}_{\Gamma_0}(\Omega)$$

and for all $v \in C_0^\infty(\Omega)$

$$\int_\Omega \nabla\theta\nabla v dx = \int_\Omega (k(\theta,\lambda)|\nabla p|^r + k(\theta,\lambda)|p|^r + f)v\, dx\ , \tag{3.13}$$

and for all $\xi \in H^{1,r}_{\Gamma_0}(\Omega)$

$$\int_\Omega k(\theta,\lambda)(|\nabla p|^{r-2}\nabla p\cdot\nabla\xi + |p|^{r-2}p\xi)\, dx + \int_{\Gamma_1} l\xi\, ds = \int_\Omega g\xi\, dx. \tag{3.14}$$

Next we shall bound the critical growth of $|\nabla p|^r$ and the non-linear correction term $k(\theta,\lambda)|p|^r$ on the right-hand side of (3.13).

Lemma 3.4. *Suppose that* (1.12) - (1.18) *hold. Suppose that* θ *and* p *satisfy*

$$\theta - \theta_0 \in H_0^{1,\sigma}(\Omega),\ p - p_0 \in H^{1,r}_{\Gamma_0}(\Omega)$$

and (3.14). *Then for all* $v \in C^1(\overline{\Omega})$

$$\begin{aligned}\int_\Omega & k(\theta,\lambda)(|\nabla p|^r + |p|^r)v\, dx \\ &= \int_\Omega k(\theta,\lambda)|\nabla p|^{r-2}\nabla p\cdot\nabla p_0 v\, dx \\ &\quad - \int_\Omega k(\theta,\lambda)(p-p_0)|\nabla p|^{r-2}\nabla p\cdot\nabla v\, dx \\ &\quad + \int_\Omega k(\theta,\lambda)|p|^{r-2}pp_0 v\, dx \\ &\quad - \int_{\Gamma_1} l(p-p_0)v\, dx + \int_\Omega g(p-p_0)v\, dx.\end{aligned} \tag{3.15}$$

Moreover, there exists a polynomial F *that is independent of* θ *and* p *such that*

$$\int_\Omega k(\theta,\lambda)(|\nabla p|^r + |p|^r)v dx \le F(\|p\|_{H^{1,r}(\Omega)})\|v\|_{H^{1,\sigma^*}(\Omega)}. \tag{3.16}$$

Proof. We first show (3.15). Letting $\xi = v(p-p_0)$ in (3.14), we obtain

$$\begin{aligned}&\int_\Omega k(\theta,\lambda)|\nabla p|^{r-2}\nabla p\cdot[v\,\nabla(p-p_0) + (p-p_0)\nabla v]dx \\ &\quad + \int_\Omega k(\theta,\lambda)|p|^{r-2}pv(p-p_0)\, dx + \int_{\Gamma_1} lv(p-p_0)ds \ = \ \int_\Omega gv(p-p_0)\, dx\ .\end{aligned}$$

This yields exactly (3.15) after straightforward computation.

We now show (3.16). We denote the five terms on the right-hand side of equation (3.15) by I, II, III, IV, and V, respectively. We shall use a general

Hölder inequality [14, p. 146] and Sobolev inequalities to estimate I, II, III, IV, and V.

For I we get

$$|I| \leq k_2 \|\nabla p\|_{L^r(\Omega)}^{r-1} \|\nabla p_0\|_{L^\tau(\Omega)} \|v\|_{L^{\zeta_1}(\Omega)}$$

where $\zeta_1 = \frac{\tau r}{\tau - r}$ satisfies $\frac{r-1}{r} + \frac{1}{\tau} + \frac{1}{\zeta_1} = 1$.

We estimte II in three different cases:

`Case 1:` $1 < r < n$.

$$|II| \leq k_2 \|p - p_0\|_{L^{\frac{nr}{n-r}}(\Omega)} \|\nabla p\|_{L^r(\Omega)}^{r-1} \|\nabla v\|_{L^n(\Omega)}.$$

`Case 2:` $r = n$.

$$|II| \leq k_2 \|p - p_0\|_{L^{n(n+1)}(\Omega)} \|\nabla p\|_{L^r(\Omega)}^{r-1} \|\nabla v\|_{L^{n+1}(\Omega)}.$$

`Case 3:` $r > n$.

$$|II| \leq k_2 C \|\nabla p\|_{L^r(\Omega)}^{r-1} \|\nabla v\|_{L^r(\Omega)}.$$

We estimte III in two different cases:

`Case 1:` $1 < r < n$.

$$|III| \leq k_2 \|p_0\|_{L^{\frac{nr}{n-r}}(\Omega)} \|p\|_{L^r(\Omega)}^{r-1} \|v\|_{L^n(\Omega)}.$$

`Case 2:` $r \geq n$.

$$|III| \leq C \|p\|_{L^r(\Omega)}^{r-1} \|v\|_{L^r(\Omega)}.$$

We estimte IV in three different cases:

`Case 1:` $1 < r < n$.

$$|IV| \leq \|l\|_{L^{\sigma_3}(\Gamma_1)} \|p - p_0\|_{L^{\frac{(n-1)r}{n-r}}(\Gamma_1)} \|v\|_{L^{\zeta_3}(\Gamma_1)}.$$

`Case 2:` $r = n$.

$$|IV| \leq C \|l\|_{L^{\sigma_4}(\Gamma_1)} \|p - p_0\|_{L^{\sigma_4^*}(\Gamma_1)}.$$

`Case 3:` $r > n$.

$$|IV| \leq C \|l\|_{L^1(\Gamma_1)}$$

where $\frac{1}{\zeta_3} + \frac{1}{\sigma_3} + \frac{n-r}{(n-1)r} = 1$.

We estimte V in three different cases:

`Case 1:` $1 < r < n$.

$$|V| \leq \|g\|_{L^{\sigma_2}(\Omega)} \|p - p_0\|_{L^{\frac{nr}{n-r}}(\Omega)} \|v\|_{L^{\zeta_4}(\Omega)};$$

where $\frac{1}{\sigma_2} + \frac{n-r}{nr} + \frac{1}{\zeta_4} = 1$.

`Case 2:` $r = n$.

$$|V| \leq C\|g\|_{L^{\sigma_2}(\Omega)}\|p - p_0\|_{L^{\sigma_2^*}(\Omega)}$$

`Case 3:` $r > n$.

$$|V| \leq C\|g\|_{L^1(\Omega)}.$$

These estimates together with Sobolev imbedding theorems lead to

$$|I| + |II| + |III| + |IV| + |V| \leq F(\|p\|_{H^{1,r}(\Omega)})\|v\|_{H^{1,\sigma^*}(\Omega)}$$

for some polynomial F. ∎

Theorem 3.5. *Assume that* (1.12) - (1.18) *hold and* $k(\theta_m, \lambda) \to k(\theta, \lambda)$ *a.e. if* $\theta_m \to \theta$ *a.e. in* Ω. *Then there exists a weak solution to Problem* 1 *in the sense of Definition* 3.3.

Proof. We will construct a mapping Λ whose fixed points will be solutions to the problem. Here we only present the proof for the case where $1 < r < n$. For $r \geq n$, the same proof goes through with slight modification. Recall that $\sigma = \frac{n}{n-1}$ in this case.

Let $z \in H^{1,\sigma}(\Omega) + \theta_0$, and let $p_z \in H^{1,r}_{\Gamma_0}(\Omega) + p_0$ be the unique solution of the problem

$$\int_\Omega k(z,\lambda)(|\nabla p_z|^{r-2}\nabla p_z \nabla \xi + |p_z|^{r-2}p_z\xi)dx + \int_{\Gamma_1} l\xi ds = \int_\Omega g\xi dx,$$

for all $\xi \in H^{1,r}_{\Gamma_0}(\Omega)$. Theorem 3.2 implies that

$$\|p_z\|_{H^{1,r}(\Omega)} \leq C. \tag{3.17}$$

Next, using Lemma 3.4, we can define a linear functional $F_z \in (H^{1,\sigma^*}(\Omega))^*$ determined by

$$\begin{aligned} \langle F_z, v\rangle = &-\int_\Omega k(\theta,\lambda)((p_z - p_0)|\nabla p_z|^{r-2}\nabla p_z \cdot \nabla v + |\nabla p|^{r-2}\nabla p \cdot \nabla p_0 v)\, dx \\ &+ \int_\Omega k(\theta,\lambda)|p|^{r-2}pp_0 v\, dx - \int_{\Gamma_1} l(p - p_0)v\, dx \\ &+ \int_\Omega g(p - p_0)v\, dx + \int_\Omega fv\, dx \end{aligned} \tag{3.18}$$

for all $v \in H^{1,\sigma^*}(\Omega)$. By virtue of (3.16), F_z is well defined, and there exists a constant $C > 0$ independent of z such that

$$|\langle F_z, v\rangle| \leq C\|v\|_{H^{1,\sigma^*}(\Omega)}. \tag{3.19}$$

Thus, we defined a mapping

$$F_z = \Lambda_1 z : z \in \theta_0 + H^{1,\sigma}_0(\Omega) \to F_z \in (H^{1,\sigma^*}(\Omega))^*. \tag{3.20}$$

Let $w_z - \theta_0 \in H_0^{1,\sigma}(\Omega)$, the Poisson equation

$$\int_\Omega \nabla w_z \nabla v dx = \langle F_z, v\rangle \qquad \forall v \in H^{1,\sigma^*}(\Omega) \tag{3.21}$$

exists a unique solution w_z for any given $F_z \in (H^{1,\sigma^*}(\Omega))^*$. So, we can define an isomorphism between $H^{1,\sigma}(\Omega)$ and $(H^{1,\sigma^*}(\Omega))^*$:

$$w_z = \Lambda_2 F_z : \ (H^{1,\sigma^*}(\Omega))^* \to H^{1,\sigma}(\Omega). \tag{3.22}$$

Next we show that the composition

$$\Lambda := \Lambda_2\Lambda_1 z : H^{1,\sigma}(\Omega) + \theta_0 \to H^{1,\sigma}(\Omega) + \theta_0 \tag{3.23}$$

is continuous under the weak topology of $H^{1,\sigma}(\Omega)$.

Our investigation is achieved in two steps.

`Step 1:` Weak continuity of Λ_1. Let $z_m \in \theta_0 + H_0^{1,\sigma}(\Omega)$ with

$$z_m \rightharpoonup z \qquad \text{weakly in } H^{1,\sigma}(\Omega) \tag{3.24}$$

$$z_m \to z \qquad \text{a.e. in } \Omega. \tag{3.25}$$

From the proof of part 2) in Theorem 3.2, we see that

$$p_m \to p_z \qquad \text{strongly in } H^{1,r}(\Omega), \quad \text{as } m \to \infty\ . \tag{3.26}$$

The standard argument, after passing to the limit, obtains

$$\lim_{j\to\infty} \left\langle F_{z_{m_j}}, v\right\rangle = \langle F_z, v\rangle \quad \forall v \in H^{1,\sigma^*}(\Omega). \tag{3.27}$$

`Step 2:` Weak continuity of Λ_2. Suppose $F_{z_m} \rightharpoonup F_z$ weakly in $(H^{1,\sigma^*}(\Omega))^*$. Suppose w_{z_m} and w_z are the unique solutions of the equations

$$\int_\Omega \nabla w_{z_m} \nabla v dx = \langle F_{z_m}, v\rangle \qquad \forall v \in H_0^{1,\sigma^*}(\Omega)$$

and

$$\int_\Omega \nabla w_z \nabla v dx = \langle F_z, v\rangle \qquad \forall v \in H_0^{1,\sigma^*}(\Omega)\ ,$$

respectively. It is easy to see that

$$\int_\Omega \nabla(w_{z_m} - w_z)\nabla v dx = \langle F_{z_m} - F_z, v\rangle \to 0$$

since (3.21) is linear. Thus the weak continuity of Λ is proved.

Invoking (3.19) and (3.21), it follows that

$$\|\Lambda z\|_{H^{1,\sigma}(\Omega)} \leq c$$

for some constant c independent of z. This proves that Λ maps the ball

$$B \equiv \{z : z \in H_0^{1,\sigma}(\Omega) + \theta_0, \|z\|_{H^{1,\sigma}(\Omega)} \leq c\}$$

into itself. By Tychonoff's Fixed Point theorem, there exists a z such that

$$z = \Lambda z,$$

that is,

$$\int_\Omega \nabla\theta\nabla v dx = \int_\Omega k(\theta,\lambda)(|\nabla p|^r + |p|^r)v\,dx + \int_\Omega fv\,dx.$$

Theorem 3.5 is completed. ∎

In the next section we shall use the extension of Theorem 3.5 which we state below.

Problem 3. Find functions θ and p defined in Ω such that

$$a\theta - \Delta\theta = k(\theta,\lambda)|\nabla p|^r + k(\theta,\lambda)|p|^r 6 + f \quad \text{in } \Omega, \tag{3.28}$$

$$-div\{k(\theta,\lambda)|\nabla p|^{r-2}\nabla p\} + k(\theta,\lambda)|p|^{r-2}p = g \quad \text{in } \Omega, \tag{3.29}$$

$$\theta = \theta_0 \quad \text{on } \partial\Omega, \tag{3.30}$$

$$p = p_0 \quad \text{on } \Gamma_0, \tag{3.31}$$

$$-k(\theta,\lambda)|\nabla p|^{r-2}\frac{\partial p}{\partial\nu} = l \quad \text{on } \Gamma_1. \tag{3.32}$$

where $a > 0$ is a constant.

Definition 3.6. We say that $\{\theta, p\}$ is a *weak solution of* Problem 3.2 if

$$\theta - \theta_0 \in H_0^{1,\sigma}(\Omega), \quad p - p_0 \in H_{\Gamma_0}^{1,r}(\Omega)$$

and for all $v \in C_0^\infty(\Omega)$

$$\begin{aligned}\int_\Omega \nabla\theta\nabla v\,dx + a\int_\Omega \theta v\,dx \\ = \int_\Omega k(\theta,\lambda)|\nabla p|^r v\,dx + \int_\Omega k(\theta,\lambda)|p|^r v\,dx + \int_\Omega fv\,dx\end{aligned} \tag{3.33}$$

and for all $\xi \in H_{\Gamma_0}^{1,r}(\Omega)$

$$\int_\Omega k(\theta,\lambda)(|\nabla p|^{r-2}\nabla p\cdot\nabla\xi + |p|^{r-2}p\xi)\,dx + \int_{\Gamma_1} l\xi\,ds = \int_\Omega g\xi\,dx. \tag{3.34}$$

Theorem 3.7. *Assume the conditions of Theorem* 3.5 *hold. Then there exists a weak solution to Problem* 3.2 *in the sense of Definition* 3.6.

The proof is only a slight modification of the proof of Theorem 3.5. We leave the details for the interested readers.

4. Problem 2

Here we study initial-boundary problems of type 2. We shall show that Problem 2 has a weak solution for $1 < r < n$ and $n = 2$. For purposes of exposition, we simplify the assumption about the data as specified in (1.12) - (1.17), namely we make it time independent, although it is only a technical argument to extend our methodology to the case where it is time dependent. As a further assumption, the initial temperature φ is to satisfy

$$\varphi \in H^{1,2}(\Omega) \ . \tag{4.1}$$

Definition 4.1. For $1 < r < n$ and $n = 2$, we say that $\{\theta, p\}$ is a *weak solution of* Problem 2 if

$$\theta - \theta_0 \in L^2(0,T; H_0^{1,2}(\Omega)), \quad p - p_0 \in L^r(0,T; H_{\Gamma_0}^{1,r}(\Omega)) \tag{4.2}$$

and for all $v \in C_0^\infty(\overline{\Omega_T})$ with $v = 0$ on $\partial\Omega \times (0,T) \cup \Omega \times \{T\}$,

$$\begin{aligned} -\int_{\Omega_T} (\theta v_t - \nabla\theta \cdot \nabla v)\, dx\, dt \\ = \int_{\Omega_T} k(\theta, \lambda, t)|\nabla p|^r v\, dx\, dt + \int_{\Omega_T} k(\theta, \lambda, t)|p|^r v\, dx\, dt \\ + \int_{\Omega_T} f v\, dx\, dt + \int_\Omega \varphi v(x,0)\, dx \ , \end{aligned} \tag{4.3}$$

and for all $\xi \in L^r(0,T; H_{\Gamma_0}^{1,r}(\Omega))$ and for almost all $t \in (0,T)$,

$$\begin{aligned} \int_\Omega k(\theta, \lambda, t)|\nabla p|^{r-2}\nabla p \cdot \nabla\xi\, dx \\ + \int_\Omega k(\theta, \lambda, t)|p|^{r-2} p\xi\, dx + \int_{\Gamma_1} l\xi\, ds = \int_\Omega g\xi\, dx \ . \end{aligned} \tag{4.4}$$

We use Rothe's method of time discretization to prove the following main result of this section.

Theorem 4.2. *Assume that* (1.12) - (1.18), (4.1) *hold and* $k(\theta_m) \to k(\theta)$ *a.e. if* $\theta_m \to \theta$ *a.e. in* Ω. *Then there exists a weak solution to Problem* 2 *in the sense of Definition* 4.1.

4.1. Notations and Preliminary. The time step is $\delta = T/N$ where N is some suitably large integer. For each fixed $m = 1, 2, \ldots, N$, $\{\theta_m^N, p_m^N\}$ are weak solutions to the stationary problem

$$\frac{\theta_m^N - \theta_{m-1}^N}{\delta} - \Delta\theta_m^N - k(\theta_m^N, \lambda, (m - \frac{1}{2})\delta)|\nabla p_m^N|^r - k(\theta_m^N, \lambda, (m - \frac{1}{2})\delta)|p_m^N|^r = f \tag{4.5}$$

$$-div\{k(\theta_m^N, \lambda, (m - \frac{1}{2})\delta)|\nabla p_m^N|^{r-2}\nabla p_m^N\} + k(\theta_m^N, \lambda, (m - \frac{1}{2})\delta)|p_m^N|^{r-2}p_m^N = g \tag{4.6}$$

$$\theta_m^N = \theta_0 \quad \text{on } \partial\Omega \tag{4.7}$$

$$p_m^N = p_0 \quad \text{on } \Gamma_0 \tag{4.8}$$

$$-k(\theta_m^N, \lambda, (m - \frac{1}{2})\delta)|\nabla p_m^N|^{r-2}\frac{\partial p_m^N}{\partial \nu} = l \qquad \text{on } \Gamma_1. \tag{4.9}$$

To start the time marching procedure, set

$$\theta_0^N = \varphi\ . \tag{4.10}$$

It follows from Theorem 3.7, for each m the problem (4.5) - (4.9) has a solution in the distributional sense of Definition 3.6. Note that for $1 < r < n$ and n=2, $\theta_m^N \in H^{1,2}(\Omega)$ since $\sigma = 2$ from equation (1.12). Let $\overline{\theta}_N$ and $\overline{k}_N(\theta_N, \lambda, t)$ be the function defined by

$$\overline{\theta}_N(x,t) = \theta_m^N(x) \qquad \text{if } (m-1)\delta \le t < m\delta, \quad (x,t) \in \Omega_T$$

$$k_N(\overline{\theta}_N, \lambda, t) = k(\theta_m^N, \lambda, (m - \frac{1}{2})\delta) \quad \text{if } (m-1)\delta \le t < m\delta.$$

To recover a solution to the time dependent problem, we define $\{\theta_N, p_N\}$ on Ω_T via

$$\theta_N(x,t) = \frac{\theta_m^N - \theta_{m-1}^N}{\delta}[t - (m-1)\delta] + \theta_{m-1}^N \quad \text{if } (m-1)\delta \le t < m\delta \tag{4.11}$$

$$p_N(x,t) = p_m^N \qquad \text{if } (m-1)\delta \le t < m\delta. \tag{4.12}$$

Next we state a compactness lemma which we shall use.

Lemma 4.3 (Aubin-Lions-Simon). *Let X, B, and Y be Banach spaces with $X \subset B \subset Y$. X is compactly imbedded in B. Let $1 \le q < \infty$ and F be a bounded subset of $L^q(0,T;X)$. Moreover, the set*

$$\frac{\partial F}{\partial t} = \left\{\frac{\partial f}{\partial t}; f \in F\right\}$$

is bounded in $L^1(0,T;Y)$, where the partial derivative is a distributional derivative for vector valued functions. Then F is compact in $L^q(0,T;B)$.

For the proof see [5, 17] and [23, Corollary 4, p. 85].

4.2. Proof of Theorem 4.1. First, we extend $\overline{\theta}_N$ to $\Omega\times[-\delta,T]$ by setting $\overline{\theta}_N=\varphi$ if $-\delta\le t\le 0$. Using Theorem 3.7, $\{\overline{\theta}_N,\,p_N\}$ satisfies the weak form

$$\overline{\theta}_N-\theta_0\in L^2(0,T;H_0^{1,2}(\Omega)),\qquad p_N-p_0\in L^2(0,T;H_{\Gamma_0}^{1,r}(\Omega)),\tag{4.13}$$

and for all $v\in C_0^\infty(\overline{\Omega}_T)$ with $v=0$ on $\partial\Omega\times(0,T)\cup\Omega\times T$

$$\begin{aligned}&\frac{1}{\delta}\int_{\Omega_T}(\overline{\theta}_N(x,t)-\overline{\theta}_N(x,t-\delta))v\,dx\,dt+\int_{\Omega_T}\nabla\overline{\theta}_N(x,t)\nabla v\,dx\,dt\\&=\int_{\Omega_T}k_N(\overline{\theta}_N,\lambda,t)(|\nabla p_N(x,t)|^r+|p_N(x,t)|^r)v\,dx\,dt+\int_{\Omega_T}fv\,dx\,dt,\end{aligned}\tag{4.14}$$

and for all $\xi\in H_{\Gamma_0}^{1,r}(\Omega_T)$ and $0<\tau\le T$

$$\begin{aligned}&\int_{\Omega_\tau}k_N(\overline{\theta}_N,\lambda,t)(|\nabla p_N(x,t)|^{r-2}\nabla p_N(x,t)\cdot\nabla\xi\\&\quad+|p_N(x,t)|^{r-2}p_N(x,t)\xi)\,dx\,dt+\int_{\Gamma_1\times(0,\tau)}l\xi\,ds\,dt=\int_{\Omega_\tau}g\xi\,dx\,dt.\end{aligned}\tag{4.15}$$

where $\Omega_\tau=\Omega\times(0,\tau)$. Let $\overline{t}=t-\delta$, we have

$$\begin{aligned}\int_{\Omega_T}\overline{\theta}_N(x,t-\delta)v\,dx\,dt&=\int_{-\delta}^{T-\delta}\int_\Omega\overline{\theta}_N(x,\overline{t})v(x,\overline{t}+\delta)\,dx\,d\overline{t}\\&=\int_{-\delta}^{T-\delta}\int_\Omega\overline{\theta}_N(x,t)v(x,t+\delta)\,dx\,dt\end{aligned}$$

which implies

$$\begin{aligned}&\frac{1}{\delta}\int_{\Omega_T}[\overline{\theta}_N(x,t)-\overline{\theta}_N(x,t-\delta)]v\,dx\,dt\\&\qquad=\frac{1}{\delta}\int_{T-\delta}^{T}\int_\Omega\overline{\theta}_Nv\,dx\,dt-\frac{1}{\delta}\int_{-\delta}^{0}\int_\Omega\varphi v(x,t+\delta)\,dx\,dt\\&\qquad\quad+\frac{1}{\delta}\int_0^{T-\delta}\int_\Omega\overline{\theta}_N(v(x,t)-v(x,t+\delta))\,dx\,dt\ .\end{aligned}\tag{4.16}$$

Considering integrations in Ω and Γ_1 instead of Ω_τ and $\Gamma_1\times(0,\tau)$, respectively, and setting $\xi=p_N-p_0$ in equation (4.15), similar to part 1) of Theorem 3.2, we obtain

$$\|p_N\|_{L^\infty(0,T;H^{1,r}(\Omega))}\le c\ .\tag{4.17}$$

We combine inequality (4.17) and Lemma 3.4 to obtain

$$\begin{aligned} \left| \int_\Omega k(\overline{\theta}_N, \lambda, t)(|\nabla p_N|^r + |p_N|^r) v \ \mathrm{dx} \right| &\leq F(\|p_N\|_{H^{1,r}(\Omega)}) \|v\|_{H^{1,2}(\Omega)} \\ &\leq F(c)\|v\|_{H^{1,2}(\Omega)}, \end{aligned} \tag{4.18}$$

where F is a polynomial that is independent of $\{\overline{\theta}_N, p_N\}$.

Next we show an analog of Lemma 3.4. Setting $\xi = (p_N - p_0)v$ in (4.15), we obtain

$$\begin{aligned} \int_{\Omega_T} & k_N(\overline{\theta}_N,\lambda, t)|\nabla p_N|^r v \, dx \, dt \\ &= \int_{\Omega_T} k_N(\overline{\theta}_N, \lambda, t)|\nabla p_N|^{r-2} \nabla p_N \cdot \nabla p_0 v \, dx \, dt \\ &\quad - \int_{\Omega_T} k_N(\overline{\theta}_N, \lambda, t)(p_N - p_0)|\nabla p_N|^{r-2} \nabla p_N \cdot \nabla v \, dx \, dt \\ &\quad - \int_{\Omega_T} k_N(\overline{\theta}_N, \lambda, t)|p_N|^{r-2} p_N (p_N - p_0) v \, dx \, dt \\ &\quad - \int_{\Gamma_1 \times (0,T)} l(p_N - p_0) v \, ds dt + \int_{\Omega_T} g(p_N - p_0) v \, dx \, dt. \end{aligned} \tag{4.19}$$

Using the Cauchy's inequality and (4.18) we obtain

$$\begin{aligned} \left| \int_{\Omega_T} k_N(\overline{\theta}_N, \lambda, t)|\nabla p_N|^r v \, dx \, dt \right| &\leq \int_0^T F(c)\|v\|_{H_0^{1,2}(\Omega)} \ \mathrm{dt} \\ &\leq F(c)\sqrt{T}\left(\int_0^T \|v\|^2_{H^{1,2}(\Omega)} \ \mathrm{dt} \right)^{\frac{1}{2}} \\ &\leq c\|v\|_{L^2(0,T;H^{1,2}(\Omega))}. \end{aligned} \tag{4.20}$$

This allows that the test functions in (4.14) can be taken from the space $L^2(0, T; H^{1,2}(\Omega))$. In particular, set $v = \overline{\theta}_N(x, t) - \theta_0$. This leads to

$$\begin{aligned} \frac{1}{\delta} \int_{\Omega_T} (\overline{\theta}_N(x, t) - \overline{\theta}_N(x, t - \delta))(\overline{\theta}_N - \theta_0) dx dt + \int_{\Omega_T} \nabla \overline{\theta}_N \nabla (\overline{\theta}_N - \theta_0) dx dt \\ = \int_{\Omega_T} k_N(\overline{\theta}_N, \lambda, t) \left(|\nabla p_N|^r + |p_N|^r + f\right) \left(\overline{\theta}_N - \theta_0\right) dx \, dt. \end{aligned} \tag{4.21}$$

Using the inequality

$$2\theta_m^N(\theta_m^N - \theta_{m-1}^N) \geq (\theta_m^N)^2 - (\theta_{m-1}^N)^2$$

we have

$$\begin{aligned}\int_{\Omega_T} |\nabla\overline{\theta}_N|^2\,dx\,dt \le & \int_{\Omega_T} k_N(\overline{\theta}_N,\lambda,t)\,(|\nabla p_N|^r + |p_N|^r + f)\,(\overline{\theta}_N - \theta_0)\,dx\,dt \\ & + \int_{\Omega_T} \nabla\overline{\theta}_N\cdot\nabla\theta_0\,dx\,dt + \int_\Omega \overline{\theta}_N(x,T-\frac{1}{2}\delta)\theta_0\,dx \\ & - \int_\Omega \varphi\theta_0\,dx + \frac{1}{2}\int_\Omega [(\varphi)^2 - (\theta_N^N)^2]\,dx\ .\end{aligned}$$

The estimate (4.18) with $v = \overline{\theta}_N - \theta_0$ and Young's inequality with ϵ yield

$$\int_{\Omega_T} |\nabla\overline{\theta}_N|^2\,dx\,dt \le c, \tag{4.22}$$

where c is independent of N. Therefore there exists a subsequence $\{\overline{\theta}_{N_j}\}$ such that

$$\overline{\theta}_{N_j} \rightharpoonup \theta \quad \text{weakly in } L^2(0,T;H^{1,2}(\Omega)) \text{ as } j\to\infty\ . \tag{4.23}$$

In order to pass to the limit in (4.14) - (4.16), we need to show that (use the same subsequence notation again)

$$p_{N_j} \;\to\; p \quad \text{strongly in } L^r(0,T;[H^{1,r}(\Omega)]^n) \tag{4.24}$$

$$\overline{\theta}_{N_j} \;\to\; \theta \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \tag{4.25}$$

as $j\to\infty$. In fact, the proof of (4.24) is only a slight variation of the proof of part 2) of Theorem 3.2 in that

(i) instead of (3.5), we begin with (4.15) with $\tau = T$

(ii) integrations and inequalities are considered in Ω_T instead of Ω.

Next we show the compactness result (4.25) via θ_N. Squaring both sides of (4.11) and integrating the results over $(0,T)$ we obtain

$$\begin{aligned}&\int_{\Omega_T} |\nabla\theta_N|^2\,dx\,dt \\ &\quad = \delta\sum_{m=1}^N \int_\Omega \Big[(\nabla\theta_m^N - \nabla\theta_{m-1}^N)\nabla\theta_{m-1}^N + |\nabla\theta_{m-1}^N|^2 \\ &\qquad + \frac{1}{3}|\nabla\theta_m^N - \nabla\theta_{m-1}^N|^2\Big]\,dx \\ &\quad = \int_{\Omega_T} \Big[(\nabla\overline{\theta}_N(x,t) - \nabla\overline{\theta}_N(x,t-\delta))\nabla\overline{\theta}_N(x,t-\delta) \\ &\qquad + |\nabla\overline{\theta}_N(x,t-\delta)|^2 + \frac{1}{3}|\nabla\overline{\theta}_N(x,t) - \nabla\overline{\theta}_N(x,t-\delta)|^2\Big]\,dx\,dt\ .\end{aligned} \tag{4.26}$$

Hence

$$\begin{aligned}\int_{\Omega_T} |\nabla\theta_N|^2\,dx\,dt &= \frac{1}{3}\int_{\Omega_T}\Big[|\nabla\overline{\theta}_N(x,t)|^2 + |\nabla\overline{\theta}_N(x,t-\delta)|^2 \\ &\quad + \nabla\overline{\theta}_N(x,t-\delta)\cdot\nabla\overline{\theta}_N(x,t)\Big]\,dx\,dt \\ &\le \frac{1}{2}\int_{\Omega_T}(|\nabla\overline{\theta}_N(x,t)|^2 + |\nabla\overline{\theta}_N(x,t-\delta)|^2)\,dx\,dt \\ &\le c + \|\nabla\varphi\|_{L^2(\Omega)} \le c\,.\end{aligned} \tag{4.27}$$

By the lemmas 4.4 and 4.5 below, we can achieve strong convergence of $\{\overline{\theta}_{N_j}\}$ in $L^2(0,T;L^2(\Omega))$.

By virtue of (4.23), (4.25), and (4.24), we can now pass to the limit as $j\to\infty$ in (4.14) - (4.16) and conclude that the limit functions $\{\theta,p\}$ satisfy Definition 4.1. Thus, Theorem 4.2 is proved. ∎

Let us now prove the following lemmas.

Lemma 4.4. *A subsequence of* $\{\theta_N\}$ *converges in the norm of* $L^2(0,T;L^2(\Omega))$.

Proof. Equation (4.14) can be rewritten in the form

$$\begin{aligned}\int_{\Omega_T}\frac{\partial\theta_N}{\partial t}v\,dx\,dt + \int_{\Omega_T}\nabla\overline{\theta}_N(x,t)\nabla v\,dx\,dt \\ = \int_{\Omega_T} k_N(\overline{\theta}_N,\lambda,t)(|\nabla p_N(x,t)|^r + |p_N(x,t)|^r)v\,dx\,dt \\ + \int_{\Omega_T} fv\,dx\,dt.\end{aligned} \tag{4.28}$$

Next we show that there exists a constant $c>0$, independent of N, such that

$$\left|\int_{\Omega_T}\frac{\partial\theta_N}{\partial t}v\ \mathrm{dx\,dt}\right| \le c\|v\|_{L^2(0,T;H_0^{1,2}(\Omega))}. \tag{4.29}$$

Using the Cauchy's inequalities and inequality (4.17), we obtain

$$\begin{aligned}\int_{\Omega_T}\nabla\overline{\theta}_N(x,t)\nabla v\,dx\,dt &\le \int_0^T\left(\int_\Omega|\nabla\overline{\theta}_N|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla v|^2\,dx\right)^{\frac{1}{2}}dt \\ &\le \left(\int_{\Omega_T}|\nabla\overline{\theta}_N|^2\,dx\,dt\right)^{\frac{1}{2}}\left(\int_{\Omega_T}|\nabla v|^2\,dx\,dt\right)^{\frac{1}{2}} \\ &\le c\|v\|_{L^2(0,T;H_0^{1,2}(\Omega))}.\end{aligned} \tag{4.30}$$

The estimate (4.29) then follows from (4.28), (4.30) and (4.18). Note that $C_0^\infty([0,T];H_0^{1,2}(\Omega))$ is dense in $L^2(0,T;H_0^{1,2}(\Omega))$. Hence, $\frac{\partial\theta_N}{\partial t}$ can be extended uniquely as a bounded linear functional on $L^2(0,T;H_0^{1,2}(\Omega))$. Using duality pairing between $L^2(0,T;H_0^{1,2}(\Omega))^*$ and $L^2(0,T;H_0^{1,2}(\Omega))$, (4.29) then implies

$$\left|\left\langle \frac{\partial \theta_N}{\partial t}, v\right\rangle\right| \le c\|v\|_{L^2(0,T;H_0^{1,2}(\Omega))} \quad \forall v \in L^2(0,T;H_0^{1,2}(\Omega)). \tag{4.31}$$

Introduce the notations

$$X = H_0^{1,2}(\Omega), \qquad B = L^2(\Omega), \qquad Y = H^{-1,2}(\Omega)$$

and let

$$F = \{\theta_N - \theta_0; N = 1, 2, 3 \ldots\}.$$

Clearly $X \subset B \subset Y$, and X is compactly imbedded into B. Inequality (4.27) states that F is a bounded subset of $L^2(0,T;X)$. Moreover, from (4.31) it follows that $\partial F/\partial t$ is a bounded subset of $L^2(0,T;Y)$. By applying Aubin-Lions-Simon's lemma (see Lemma 4.3), we know that F is compact in $L^2(0,T;X)$. Consequently, Lemma 4.4 is proved. ∎

Lemma 4.5. *$\{\overline{\theta}_{N_j}\}$ converges to θ strongly in $L^2(0,T;L^2(\Omega))$ if and only if $\{\theta_{N_j}\}$ converges to θ strongly in $L^2(0,T;L^2(\Omega))$.*

Proof. Setting $v = \overline{\theta}_N(x,t) - \overline{\theta}_N(x,t-\delta)$ in (4.14) and multiplying both sides by δ, we obtain

$$\begin{aligned}
&\int_{\Omega_T} \big(\overline{\theta}_N(x,t) - \overline{\theta}_N(x,t-\delta)\big)^2\,dx\,dt \\
&\qquad + \delta \int_{\Omega_T} \nabla\overline{\theta}_N(x,t)\nabla\big(\overline{\theta}_N(x,t) - \overline{\theta}_N(x,t-\delta)\big)\,dx\,dt \\
&\quad = \delta \int_{\Omega_T} \Big[k_N\big(\overline{\theta}_N,\lambda,t\big)(|\nabla p_N(x,t)|^r \\
&\qquad + |p_N(x,t)|^r)(\overline{\theta}_N(x,t) - \overline{\theta}_N(x,t-\delta))\Big]\,dx\,dt \\
&\qquad + \delta \int_{\Omega_T} f(x,t)\big(\overline{\theta}_N(x,t) - \overline{\theta}_N(x,t-\delta)\big)\,dx\,dt.
\end{aligned} \tag{4.32}$$

Using the Cauchy's inequality and the estimates (4.18) and (4.22), one can show that

$$\begin{aligned}
&\left|\delta \int_{\Omega_T} \nabla\overline{\theta}_N(x,t)\nabla(\overline{\theta}_N(x,t) - \overline{\theta}_N(x,t-\delta))\,dx\,dt\right| \\
&\qquad \le \delta\Big[2\int_{\Omega_T} (\nabla\overline{\theta}_N(x,t))^2\,dx\,dt + \int_{\Omega} (\nabla\overline{\theta}_N(x,t-\delta))^2\,dx\,dt\Big] \\
&\qquad \le 3\delta \int_{\Omega_T} (\nabla\overline{\theta}_N)^2\,dx\,dt + \delta\|\varphi\|^2_{H^{1,2}(\Omega)} \le c\delta
\end{aligned}$$

and

$$\left|\delta \int_{\Omega_T} k_N(\overline{\theta}_N, \lambda, t)\big(|\nabla p_N(x,t)|^r + |p_N(x,t)|^r\big)\big(\overline{\theta}_N(x,t) - \overline{\theta}_N(x,t-\delta)\big)\, dx\, dt\right|$$
$$\leq 2\delta F(c)\|\overline{\theta}_N\|_{L^2(0,T;H^{1,2}(\Omega))} + \delta\|\varphi\|^2_{H^{1,2}(\Omega)} \ .$$

Hence

$$\int_{\Omega_T} (\overline{\theta}_N(x,t) - \overline{\theta}_N(x,t-\delta))^2\, dx\, dt = O(\sqrt{\delta}). \tag{4.33}$$

Using Cauchy's inequality and the definitions of θ_N and $\overline{\theta}_N$, it is easy to establish the following relations:

$$\begin{aligned}
&\int_{\Omega_T} (\theta_N - \theta)^2\, dx\, dt \\
&\qquad = \frac{1}{3}\int_{\Omega_T} \big(\overline{\theta}_N(x,t) - \overline{\theta}_N(x,t-\delta)\big)^2\, dx\, dt \\
&\qquad\quad + \int_0^{T-\delta}\int_\Omega (\overline{\theta}_N - \theta)^2\, dx\, dt \qquad\qquad (4.34)\\
&\qquad\quad + 2\sum_{m=1} N \int_\Omega \left(\frac{\theta_m^N - \theta_{m-1}^N}{\delta}\right)\big(t - (m-1)\delta\big)(\theta_{m-1}^N - \theta)\, dx\, dt \\
&\qquad \leq \frac{2}{3}\int_{\Omega_T} \big(\overline{\theta}_N(x,t) - \overline{\theta}_N(x,t-\delta)\big)^2\, dx\, dt \\
&\qquad\quad + 2\int_0^{T-\delta}\int_\Omega (\overline{\theta}_N - \theta)^2\, dx\, dt\ , \qquad\qquad (4.35)
\end{aligned}$$

where we have used

$$\int_{(m-1)\delta}^{m\delta} (t - (m-1)\delta)^2\, dt = \frac{\delta^3}{3}$$

in the last term on the right-hand side of (4.34). Thus, (4.33) implies that $\{\theta_{N_j}\}$ converges to θ in $L^2(0,T;L^2(\Omega))$ provided $\{\overline{\theta}_{N_j}\}$ converges to θ strongly in $L^2(0,T;L^2(\Omega))$.

From Cauchy's inequality, Young's inequality with ϵ, and relation (4.34), we have

$$\begin{aligned}
&\int_0^{T-\delta}\int_\Omega (\overline{\theta}_N - \theta)^2\ \mathrm{dx}\,\mathrm{dt} \\
&\qquad \leq C\int_{\Omega_T} (\theta_N - \theta)^2\ \mathrm{dx}\,\mathrm{dt} + C(\epsilon)\int_{\Omega_T} (\overline{\theta}_N(x,t) - \overline{\theta}_N(x,t-\delta))^2\, dx\, dt.
\end{aligned}$$

Therefore, Lemma 4.5 follows from (4.33). ∎

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