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On the Cauchy Problem for Systems Containing Locally Explicit Equations

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Abstract. In this paper we consider so-called locally explicit equations involving nonlinear differentials. Such equations are characterized by certain continuity and semigroup properties of the corresponding quasiflow and arise typically in the mathematical modelling of non-smooth mechanical and physical systems. Under some natural hypotheses, we prove the local solvability of the corresponding Cauchy problem by applying Schauder's fixed point principle to a suitable equivalent integral equation. Afterwards, we illustrate the abstract existence result by means of an application to an automatic regulation system involving a hysteresis element of stop type.

Keywords: *Locally explicit equation, Cauchy problem, quasiflow, semigroup property, local solvability, hysteresis nonlinearity, non-smooth mechanical system*

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1. Locally explicit equations

The purpose of this note is to prove a local existence theorem for solutions of the Cauchy problem for systems which contain so-called *locally explicit equations* with nonlinear differentials. An equation involving *nonlinear differentials* (or *quasi-differential equation*, see e.g. [1,2]) has the form

$$u(t+dt) - u(t) = D(t, u(t), dt) + o(dt). \tag{1}$$

In what follows, we suppose throughout that the function $(t, u) \mapsto D(t, u, dt)$ is defined on some set $U \subseteq \mathbb{R} \times \mathbb{R}^m$, and the function $dt \mapsto D(t, u, dt)$ on some interval $[0, \alpha(t, u)]$. The range of the function D in (1) lies in $\mathbb{R}^m$, and we assume in addition that $D(t, u, 0) = 0$.

A *solution* of equation (1) is, by definition, a function $u = \varphi(t)$ which is left-continuous on some interval I and satisfies for all $t \in \tilde{I} = I \setminus \{\sup I\}$ the

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relation

$$\lim_{dt\to+0}\frac{1}{dt}\left[\varphi(t+dt)-\varphi(t)-D(t,\varphi(t),dt)\right]=0.$$

A solution φ is called a *strong solution* if for any $t\in\tilde{I}$ one can find a $\delta>0$ such that

$$\varphi(t+dt)-\varphi(t)-D(t,\varphi(t),dt)=0 \qquad (0\le dt<\delta).$$

Consider the *quasiflow* [3] generated by equation (1), i.e.,

$$\gamma_t^{t+dt}u=u+D(t,u,dt).$$

The equation (1) is called *locally explicit* [4] if its quasiflow is left-continuous w.r.t. dt and has the following *semigroup property*: for all $(t,u)\in U$ there exists $\delta>0$ such that for all $t_1\in[t,t+\delta)$ one can find $\delta_1>0$ such that

$$\gamma_{t_1}^{t_2}\gamma_t^{t_1}u=\gamma_t^{t_2}u \qquad (t_1\le t_2<t_1+\delta_1).$$

Given $(t,u)\in U$, we fix a corresponding δ and denote it by $\Delta=\Delta(t,u)$.

The following assertion on the existence of a strong solution to the locally explicit equation (1), subject to the initial condition

$$u(t_0)=u_0, \tag{2}$$

was announced in [5]; for the reader's ease we give a short proof.

Proposition 1. *For* $(t_0,u_0)\in U$*, the function* $\varphi(t)=\gamma_{t_0}^t u_0$ *is a strong solution of the problem* (1)/(2) *on the interval* $\big[t_0,t_0+\Delta(t_0,u_0)\big)$.

Proof. The initial condition (2) is an obvious consequence of the equality $D(t_0,u_0,0)=0$. Moreover, for $t\in[t_0,t_0+\Delta(t_0,u_0))$ we have

$$\begin{aligned}\varphi(t+dt)-\varphi(t)-D(t,\varphi(t),dt)&=\gamma_{t_0}^{t+dt}u_0-\gamma_{t_0}^{t}u_0-D(t,\varphi(t),dt)\\&=\gamma_{t_0}^{t+dt}u_0-\gamma_{t}^{t+dt}\gamma_{t_0}^{t}u_0.\end{aligned}$$

By the definition of a locally explicit equation, the last term is zero for sufficiently small $dt>0$, and so φ is a strong solution of equation (1). ∎

2. Closed systems with locally explicit equations

Denoting $\Delta u=u(t+dt)-u(t)$, consider the system

$$\dot{x}=f(t,u,x) \tag{3}$$

$$\sigma=p(x) \tag{4}$$

$$\Delta u=E(t,u,\sigma_t^{t+dt},dt)+o(dt) \tag{5}$$

subject to the initial conditions (2) for u, and the additional initial condition

$$x(t_0) = x_0 \tag{6}$$

for x. Given any function σ whose domain of definition contains the interval $[t, t+dt]$, we denote by σ_t^{t+dt} its restriction to this interval. Throughout the following, we make the following assumptions:

(H1) The function $f : \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3 \to \mathbb{R}^n$ is continuous, where $\mathcal{D}_1 \subseteq \mathbb{R}$ is some neighborhood of t_0, $\mathcal{D}_3 \subseteq \mathbb{R}^n$ is some neighborhood of x_0, and $\mathcal{D}_2 \subseteq \mathbb{R}^m$.

(H2) The function $p : \mathcal{D}_3 \to \mathbb{R}$ is continuous.

(H3) The $\mathbb{R}^m$-valued function $(t, u, \sigma, dt) \mapsto E(t, u, \sigma_t^{t+dt}, dt)$ is defined for $t \in \mathcal{D}_1, u \in \mathcal{D}_2, \sigma \in C[t_0, T]$ (for some $T > t_0$), and $dt \in [0, +\infty)$, and the function $(\sigma, dt) \mapsto E(t, u, \sigma_t^{t+dt}, dt)$ is continuous.

(H4) For any continuous function $\sigma : [t_0, T] \to \mathbb{R}$, the formula (5) defines a locally explicit equation with the function $D(t, u, dt) = E(t, u, \sigma_t^{t+dt}, dt)$.

Under these assumptions, we may now prove our main local existence result.

Theorem 1. *If the hypostheses* (H1) – (H4) *are satisfied, the problem* (3) – (6) *has a solution on some interval* $[t_0, t_0 + h]$ $(h > 0)$.

Proof. For any continuous function $x : [t_0, T] \to \mathbb{R}^n$ we consider the function u given by $u(t) = u_0 + E(t_0, u_0, \sigma_{t_0}^t, t - t_0)$, where σ is defined through x as in (4). Then the function u is a solution of equation (5) on some interval $[t_0, t_0 + \Delta)$, where Δ depends on (t_0, u_0) and on the choice of x, and so utakes its values in $\mathcal{D}_2$. Consequently, one may rewrite equation (3) in the form

$$\dot{x} = \widetilde{f}(t, x_{t_0}^t), \tag{7}$$

where

$$\widetilde{f}\big(t, x_{t_0}^t\big) = f(t, u_0 + E[t_0, u_0, (p(x(\tau)) : t_0 \le \tau \le t), t - t_0], x(t)\big).$$

From our hypotheses (H1) – (H4) we conclude that $\widetilde{f}$ is continuous w.r.t. both t and x.

It is not hard to see that the initial problem (7)/(6) is equivalent, as usual, to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} \widetilde{f}(s, x_{t_0}^s)\, ds. \tag{8}$$

The right-hand side of (8) defines an integral operator

$$(Jx)(t) = x_0 + \int_{t_0}^{t} \widetilde{f}(s, x_{t_0}^s)\, ds$$

which obviously maps $C[t_0, T]$ into itself. We claim that this operator is continuous. In fact, let $x_n \in C[t_0, T]$ be a sequence with $x_n \to \overline{x}$. The continuous map $(t, x) \mapsto \widetilde{f}(t, x_{t_0}^t)$ is uniformly continuous on the compact set $[t_0, T] \times (\{x_1, x_2, x_3, \ldots\} \cup \{\overline{x}\})$. Consequently, the functions $\widetilde{f}(t, (x_n)_{t_0}^t)$ converge uniformly on $[t_0, T]$ to the function $\widetilde{f}(t, \overline{x}_{t_0}^t)$. This shows that $Jx_n \to J\overline{x}$, as $n \to \infty$, and so J is continuous as claimed.

By $\overline{x}_0$ we denote a function which coincides with x_0 on $[t_0, T]$. Again by the continuity of the map $(t, x) \mapsto \widetilde{f}(t, x_{t_0}^t)$ we can find a $\delta > 0$ such that

$$\|\widetilde{f}(t, x_{t_0}^t) - \widetilde{f}(t_0, (\overline{x}_0)_{t_0}^t)\| < 1 \qquad (|t - t_0| < \delta,\ \|x - x_0\| < \delta).$$

Consider the closed ball $B(\overline{x}_0, \delta) = \{x \in C[t_0, T] : \|x - x_0\| \leq \delta\}$, where $T - t_0 < \delta$. For $x \in B(\overline{x}_0)$ we get then

$$\|Jx - \overline{x}_0\| \leq \int_{t_0}^{T} \|\widetilde{f}(s, (\overline{x}_0)_{t_0}^s)\|\, ds \leq \big(1 + \|\widetilde{f}(t_0, (\overline{x}_0)_{t_0}^{t_0})\|\big)(T - t_0).$$

So if we choose $T \leq \delta(1 + \|\widetilde{f}(t_0, (\overline{x}_0)_{t_0}^{t_0})\|)^{-1} + t_0$, then certainly $Jx \in B(\overline{x}_0, \delta)$, and so the ball $B(\overline{x}_0, \delta)$ is invariant under J.

We show that the family $\{Jx : x \in B(\overline{x}_0, \delta)\}$ is equicontinuous on $[t_0, T]$. In fact, for $x \in B(\overline{x}_0, \delta)$ and $t_0 \leq t_1 \leq t_2 \leq T$ we have

$$\|(Jx)(t_1) - (Jx)(t_2)\| \leq \int_{t_0}^{T} \|\widetilde{f}(s, (\overline{x}_0)_{t_0}^s)\|\, ds \leq (1 + \|\widetilde{f}(t_0, (\overline{x}_0)_{t_0}^{t_0})\|)(t_2 - t_1),$$

and so this family is equicontinuous. The classical Arzelà-Ascoli theorem implies that the set $J(B(\overline{x}_0, \delta))$ is relatively compact. Consequently, from Schauder's fixed point theorem we may conclude that the completely continuous operator J has a fixed point x, which is then a solution of equation (7) and satisfies (6). Choosing Δ by means of this solution x, and putting $h > \min\{\Delta, T - t_0\}$, we arrive at a pair of functions $(u(t), x(t))$ which solves the problem (2) – (6) on the interval $[t_0, t_0 + h]$. ∎

3. Example: Hysteresis-type systems with stop

The mathematical modelling of some automatic regulation systems containing *hysteresis elements of stop type* lead to the problem (3) – (5). More precisely, the so-called *stop-converter* (see [6]) associated to an arbitrary continuous function $\sigma(t)$ may be described by equation (1) with function $D(t, u, dt) =$

$E(t, u, \sigma_t^{t+dt}, dt)$, where (see [4])

$$E(t, u, \sigma_t^{t+dt}, dt)) = \begin{cases} \sigma(t + dt) - \sigma(t) & \text{if } u \in (0, 1) \\ \sigma(t + dt) - \max\limits_{t \le s \le t+dt} \sigma(s) & \text{if } u = 1 \\ \sigma(t + dt) - \min\limits_{t \le s \le t+dt} \sigma(s) & \text{if } u = 0. \end{cases} \tag{9}$$

Clearly, the function $(t, u, \sigma, dt) \mapsto E(t, u, \sigma_t^{t+dt}, dt)$ is continuous w.r.t. (σ, dt).

Proposition 2. *For any continuous input function σ, the stop equation is locally explicit.*

Proof. Being a composition of continuous functions, the map $dt \mapsto D(t, u, dt)$ is left-continuous on $(0, T - t)$. We distinguish the three cases for u occurring in (9). First, for $u \in (0, 1)$ we have $\gamma_t^{t+dt} u = u + \sigma(t + dt) - \sigma(t)$. In this case we put

$$\Delta = \begin{cases} T - t & \text{if} \quad u + \sigma(\tau) - \sigma(t) \in (0, 1) \qquad \text{for } \tau \in [t, T), \\ \min\{\tau \in [t, T) : u + \sigma(\tau) - \sigma(t) \in \{0, 1\}\} - t & \text{otherwise.} \end{cases}$$

Clearly, $\Delta > 0$. Now, for $t \le t_1 \le t_2 < t + \Delta$ we have $u + \sigma(t_1) - \sigma(t) \in (0, 1)$, hence

$$\gamma_{t_1}^{t_2} \gamma_t^{t_1} u = \gamma_{t_1}^{t_2} [u + \sigma(t_1) - \sigma(t)] \tag{10}$$

$$= u + \sigma(t_1) - \sigma(t) + \sigma(t_2) - \sigma(t_1) \tag{11}$$

$$= \gamma_t^{t_2} u. \tag{12}$$

This is the desired semigroup property, where the term $t + \Delta - t_1$ plays here the role of the δ_1 occurring in the definition of a locally explicit equation.

In case $u = 1$ we get $\gamma_t^{t+dt} u = 1 + \sigma(t + dt) - \max\limits_{t \le s \le t+dt} \sigma(s)$. In this case we put

$$\Delta = \begin{cases} T - t & \text{if} \quad \sigma(\tau) - \max\limits_{t \le s \le \tau} \sigma(s) > -1 \qquad \text{for } \tau \in [t, T) \\ \min\{\tau \in [t, T) : \sigma(\tau) - \max\limits_{t \le s \le \tau} \sigma(s) = -1\} - t & \text{otherwise.} \end{cases}$$

Suppose that $t \le t_1 < t + \Delta$. Then either

$$\begin{array}{ll} \text{either} & 1 + \sigma(t_1) - \max\limits_{t \le s \le t_1} \sigma(s) = 1 \\ \text{or} & 1 + \sigma(t_1) - \max\limits_{t \le s \le t_1} \sigma(s) \in (0, 1). \end{array}$$

In the first case we obtain $\sigma(t_1) = \max_{t\leq s\leq t_1} \sigma(s)$, hence $\max_{t_1\leq s\leq t_2} \sigma(s) = \max_{t\leq s\leq t_2} \sigma(s)$. This shows that, for $t_2 \in [t_1, t_1 + \delta_1)$ (with $\delta_1 = t + \Delta - t_1$), we get

$$\gamma_{t_1}^{t_2}\gamma_t^{t_1}u = \gamma_{t_1}^{t_2}1 = 1 + \sigma(t_2) - \max_{t_1\leq s\leq t_2} \sigma(s) = \gamma_t^{t_2}u.$$

On the other hand, in the second case we obtain $\sigma(t_1) < \max_{t\leq s\leq t_1} \sigma(s)$. By continuity we find $\delta_1 > 0$ such that $\sigma(\tau) < \max_{t\leq s\leq t_1} \sigma(s)$ for $\tau \in [t_1, t_1 + \delta_1)$, and thus

$$\begin{aligned}\gamma_{t_1}^{t_2}\gamma_t^{t_1}u &= \gamma_{t_1}^{t_2}[1 + \sigma(t_1) - \max_{t\leq s\leq t_1} \sigma(s)]\\ &= 1 + \sigma(t_1) - \max_{t\leq s\leq t_1} \sigma(s) + \sigma(t_2) - \sigma(t_1)\\ &= \gamma_t^{t_2}u.\end{aligned}$$

In both cases the desired semigroup property for the quasiflow follows again. Finally, the remaining case $u = 0$ is proved similarly. Summarizing, we have shown that the local solvability result for the Cauchy problem proved in the preceding section applies to the system (3) – (5), where the function E is given by (9), and so we are done. ∎

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