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Quadratic Spline Collocation for Volterra Integral Equations

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Abstract. In the traditional step-by-step collocation method with quadratic splines for Volterra integral equations an initial condition is replaced by a not-a-knot boundary condition at the other end of the interval. Such a nonlocal method gives the uniform boundedness of collocation projections for all parameters $c \in (0,1)$ characterizing the position of collocation points between spline knots. For $c = 1$ the projection norms have linear growth and, therefore, for any choice of c some general convergence theorems may be applied to establish the convergence with two-sided error estimates. The numerical tests supporting the theoretical results are also presented.

Keywords: *quadratic spline collocation, spline projections, Volterra integral equations, stability and convergence of spline collocation method*

MSC 2000: Primary 65D07, 65R20, secondary 41A15

1. Introduction

Probably the most widely used in practice and theoretically studied class of methods for initial value problems of ordinary differential equations is that of Runge-Kutta methods. The generalization of Runge-Kutta methods for Volterra integral equations is given already by Pouzet [19] and Bel'tjukov [3]. An important development could be find in [4], see also [5]. Runge-Kutta methods are fully discretized and give a finite number of approximate values (in grid points) to the exact solution. Spline collocation for integral equations requires the evaluation of integrals but gives a function as approximate solution which is principal advantage in comparison with Runge-Kutta methods. In addition, spline method allows to speak about retainment of smoothness proper to exact solution. Let us mention that, with suitable evaluation of integrals by interpolatory quadratures, in some special cases like, e.g., the equations generated by ordinary differential equations, the spline collocation is equivalent to Runge-Kutta methods. For details, see [5].

The step-by-step collocation methods with piecewise polynomials have been

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studied in detail for different kind of equations under various assumptions and, as a rule, the convergence results are positive, see, e.g., [6]. The stability analysis in this case (see [16] and references therein) gives an additional explication why it is so. But piecewise polynomials, being only continuous in best configurations of collocation parameters, do not preserve the smoothness of the exact solution. Our research is justified by the need to have a C^1 smoothness preserving stable method for solving Volterra integral equations. Here the quadratic splines as approximate solutions are a natural choice.

The polynomial spline collocation with step-by-step implementation is one of the most practical methods for solving Volterra integral equations of the second kind. This method is known to be unstable for cubic and higher order smooth splines (see [7, 9, 16, 17]). In the case of quadratic splines of class C^1 the stability region consists only of one point [16]. In this paper we replace the first derivative condition, which is required by the standard quadratic spline collocation, by a not-a-knot boundary condition at the other end of the interval. This method cannot be any more implemented step-by-step and, in the case of linear integral equation, needs the solution of a linear system which can be successfully done by Gaussian elimination. On the other hand, we get the stability in the whole interval of collocation parameter. For comparison, the nonlocal method with cubic splines (see [18]) gives the stability in the same interval of collocation parameter as in the case of the traditional collocation with linear splines [16]. We also point out that our nonlocal method does not need the use of derivative of the exact solution (or an approximant of it) at starting point. Solving, for example, a weakly singular equation where the solution is typically only continuous and the derivative is unbounded in the neighborhood of a starting point, we cannot use the traditional step-by-step collocation.

We will see that the stability of the method implies the convergence in uniform norm. This is due to general convergence theorems for operator equations. The main assumption in the classical convergence theorem for the second kind operator equations is the compactness of the operator, and this allows to apply these results to a quite wide class of equations, including, for example, those with weakly singular kernel.

Choosing the collocation parameter $c \in (0, 1)$ we prove the uniform in the number of knots boundedness of projection operators. This allows us to apply the classical convergence theorem. In the case $c = 1$ the sequence of projection operators is unbounded. Nevertheless, we succeeded in proving the regular convergence of operators in approximate equations. This implies two-sided error estimates which guarantee the convergence for smooth solutions. To this end, we explore another technique and a different representation of quadratic splines.

An m-stage Runge-Kutta method has the rate of convergence (at grid points) $O(h^m)$, and this may be somewhat increased with special choice of parameters. Collocation with quadratic splines (step-by-step or nonlocal) corresponds to $m = 1$ but this has the rate $O(h^3)$ on the whole interval of integration, due to the approximation properties of quadratic splines. Remark that all the methods under discussion here have the same complexity $O(h^{-2})$.

2. Description of the method

Consider the Volterra integral equation

$$y(t) = \int_0^t \mathcal{K}(t, s, y(s))ds + f(t), \qquad t \in [0, T] \tag{2.1}$$

with given functions $f : [0, T] \to \mathbb{R}$, $\mathcal{K} : S \times \mathbb{R} \to \mathbb{R}$ and the set $S = \{(t, s) : 0 \le s \le t \le T\}$.

A mesh $\Delta_N : 0 = t_0 < t_1 < \ldots < t_N = T$ representing spline knots will be used and as the process $N \to \infty$ is allowed, the knots t_i depend on N. Denote $h_i = t_i - t_{i-1}$. Then, for given collocation parameter $c \in (0, 1]$, define collocation points $\tau_i = t_{i-1} + ch_i$, $i = 1, \ldots, N$. In order to determine the approximate solution u of the equation (2.1) as a quadratic spline of class C^1 (denote this space by $S_2(\Delta_N)$), we impose the following collocation conditions:

$$u(\tau_i) = \int_0^{\tau_i} \mathcal{K}(\tau_i, s, u(s))ds + f(\tau_i), \quad i = 1, \ldots, N\,. \tag{2.2}$$

Since $\dim S_2(\Delta_N) = N + 2$ it is necessary to give two additional conditions which we choose

$$\begin{aligned} u(0) &= y(0), \\ u''(t_{N-1} - 0) &= u''(t_{N-1} + 0). \end{aligned} \tag{2.3}$$

Let the operator $P_N : C[0, T] \to C[0, T]$ be such that for any $v \in C[0, T]$ we have $P_N v \in S_2(\Delta_N)$ and

$$\left.\begin{aligned} (P_N v)(0) &= v(0), \\ (P_N v)(\tau_i) &= v(\tau_i), \quad i = 1, \ldots, N, \\ (P_N v)''(t_{N-1} - 0) &= (P_N v)''(t_{N-1} + 0). \end{aligned}\right\} \tag{2.4}$$

Let us introduce, for a moment, the vector of knots

$$\begin{aligned} \sigma :\ s_1 = s_2 = s_3 = t_0 < s_4 = t_1 < \ldots < s_{N+1} = t_{N-2} \\ < s_{N+2} = t_N = s_{N+3} = s_{N+4} \end{aligned}$$

and corresponding B-splines $B_{1,2,\sigma}, \ldots, B_{N+1,2,\sigma}$, which are linearly independent functions. These B-splines form a basis in the spline space

$$S_2\ (\tilde{\Delta}_N)\ = S_2(\Delta_N) \cap \{f : f''(t_{N-1}-0) = f''(t_{N-1}+0)\},$$

where $\tilde{\Delta}_N$ means the grid $t_0 < t_1 < \ldots < t_{N-2} < t_N$. The Schoenberg-Whitney conditions [21, p.171] applied in the quadratic case ensure that the interpolation problem (2.4) to determine $P_N v = \sum_{1\le i\le N+1} c_i B_{i,2,\sigma}$ has a unique solution. Thus, the operator P_N is correctly defined. It is clear that P_N is a linear projection onto the space $S_2(\tilde{\Delta}_N)$.

We consider also the integral operator defined by

$$(Ku)(t) = \int_0^t \mathcal{K}(t, s, u(s))ds, \quad t \in [0, T]. \tag{2.5}$$

Lemma 2.1. *The spline collocation problem* (2.2), (2.3) *is equivalent to the equation*

$$u = P_N K u + P_N f, \quad u \in S_2(\tilde{\Delta}_N). \tag{2.6}$$

Proof. The proof is a standard calculation based on the property of P_N that $P_N v = 0$ if and only if $v(0) = 0$, $v(\tau_i) = 0$, $i = 1, \ldots, N$. Indeed, then (2.6) is equivalent to the equalities $(u - Ku - f)(0) = 0$ and $(u - Ku - f)(\tau_i) = 0$, $i = 1, \ldots, N$. The first one of them is equivalent to $u(0) = f(0)$ or $u(0) = y(0)$ because $y(0) = f(0)$. Using the definition of the operator K, the equalities in τ_i are just (2.2). ∎

3. Uniform boundedness of projections

We would apply general convergence theorems for operator equations. In the classical case, one of the assumptions is the convergence of the sequence of approximating operators P_N to the identity or injection operator. Thus, the uniform boundedness of the sequence P_N is the key problem in the study of the collocation method (2.2), (2.3).

Fix a number $c \in (0, 1)$. Given any function $f \in C[0, T]$, let us consider $S = P_N f \in S_2(\Delta_N)$ determined by the conditions

$$\left.\begin{aligned} S(0) &= f(0), \\ S(\tau_i) &= f(\tau_i), \quad i = 1, \ldots, N \\ S''(t_{N-1}-0) &= S''(t_{N-1}+0). \end{aligned}\right\} \tag{3.1}$$

Denote $S_{i,c} = S(\tau_i)$ and $m_i = S'(t_i)$. Using $t = t_{i-1} + \tau h_i$, we have the representation of S for $t \in [t_{i-1}, t_i]$

$$S(t) = S_{i,c} + \frac{h_i}{2}(\tau - c)\Big(\big(2 - (c + \tau)\big)m_{i-1} + (c + \tau)m_i\Big). \tag{3.2}$$

The continuity of S in the knots, i.e. $S(t_i - 0) = S(t_i + 0)$, gives

$$\begin{aligned}(1-c)^2 h_i m_{i-1} + \big((1-c^2)h_i + c(2-c)h_{i+1}\big)m_{+} c^2 h_{i+1} m_{i+1} \\ = 2\big(f(\tau_{i+1}) - f(\tau_i)\big), \quad i = 1, \ldots, N-1.\end{aligned} \tag{3.3}$$

The initial condition $S(0) = f(0)$ adds the equation

$$c(2-c)h_1 m_0 + c^2 h_1 m_1 = 2\big(f(\tau_1) - f(0)\big), \tag{3.4}$$

and the not-a-knot requirement at t_{N-1} could be written in the form

$$h_N m_{N-2} - (h_{N-1} + h_N)m_{N-1} + h_{N-1} m_N = 0. \tag{3.5}$$

The equation (3.5) yields

$$m_N = \Big(1 + \frac{h_N}{h_{N-1}}\Big) m_{N-1} - \frac{h_N}{h_{N-1}} m_{N-2}. \tag{3.6}$$

Then, eliminating m_N in (3.3) using (3.6), we write (3.4) and (3.3) as follows:

$$\left.\begin{aligned} \beta_0 m_0 + \gamma_0 m_1 = g_0 \\ \alpha_i m_{i-1} + \beta_i m_i + \gamma_i m_{i+1} = g_i, \quad i = 1, \ldots, N-2 \\ \alpha_{N-1} m_{N-2} + \beta_{N-1} m_{N-1} = g_{N-1}\ , \end{aligned}\right\} \tag{3.7}$$

where we denote

$$\beta_0 = \frac{2-c}{2(1-c)}\ ,\ \gamma_0 = \frac{c}{2(1-c)}\ ,\ g_0 = \frac{1}{c(1-c)}\ \frac{f(\tau_1) - f(0)}{h_1}$$

and

$$\left.\begin{aligned} \lambda_i &= \frac{h_i}{h_i + h_{i+1}} \\ \mu_i &= 1 - \lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}} \end{aligned}\right\} \quad i = 1, \ldots, N-1$$

$$\left.\begin{aligned} \alpha_i &= \frac{1-c}{2c}\ \lambda_i \\ \beta_i &= \frac{1+c}{2c}\ \lambda_i + \frac{2-c}{2(1-c)}\ \mu_i \\ \gamma_i &= \frac{c}{2(1-c)}\ \mu_i \end{aligned}\right\} \quad i = 1, \ldots, N-2$$

$$\alpha_{N-1} = \frac{1-c}{2c}\ \lambda_{N-1} - \frac{c}{2(1-c)}\ \frac{h_N}{h_{N-1}}\ \mu_{N-1}$$

$$\beta_{N-1} = \frac{1+c}{2c}\ \lambda_{N-1} + \frac{2-c}{2(1-c)}\ \mu_{N-1} + \frac{c}{2(1-c)}\ \frac{h_N}{h_{N-1}}\ ,$$

and finally

$$g_i = \frac{1}{c(1-c)} \frac{f(\tau_{i+1}) - f(\tau_i)}{h_i + h_{i+1}} , \quad i = 1, \ldots, N-1.$$

It is straightforward to check that, in the system (3.7), the difference of domination in rows is 1 and even greater than 1 in the last equation. Hence,

$$\max_{0 \le i \le N-1} |m_i| \le \max_{0 \le i \le N-1} |g_i| \tag{3.8}$$

and, in addition,

$$|m_N| \le \left(1 + 2\frac{h_N}{h_{N-1}}\right) \max_{0 \le i \le N-1} |g_i|. \tag{3.9}$$

Our aim now is to estimate the norms of projections P_N in the space $C[0,T]$. In this section, in the sequel, we assume that the sequence of meshes Δ_N is quasi-uniform, i.e. there is a constant r such that $h_{\max}/h_{\min} \le r$ where $h_{\max} = \max_{1 \le i \le N} h_i$ and $h_{\min} = \min_{1 \le i \le N} h_i$. Then, for any function $f \in C[0,T]$, we have

$$|g_i| \le \frac{1}{c(1-c)h_{\min}} \, \|f\|_{C[0,T]}, \quad i = 1, \ldots, N-1$$
$$|g_0| \le \frac{2}{c(1-c)h_{\min}} \, \|f\|_{C[0,T]}.$$

The representation (3.2), the quasi-uniformity of the meshes and the obtained estimates (3.8), (3.9) allow to get

$$\begin{aligned} \|P_N f\|_{C[0,T]} &= \max_{1 \le i \le N} \max_{t \in [t_{i-1}, t_i]} |S(t)| \\ &\le \|f\|_{C[0,T]} + h_{\max} \max_{0 \le i \le N} |m_i| \\ &\le \text{const } \|f\|_{C[0,T]} , \end{aligned}$$

where the constant is independent of N and h, but it depends on c and r. We have proved the following

Proposition 3.1. *For $c \in (0,1)$, in the case of quasi-uniform meshes, the projections P_N defined by (2.4) are uniformly bounded in the space $C[0,T]$.*

Note that similar quadratic spline projections are studied in [11]. Let us mention that quadratic spline projections on an arbitrary sequence of meshes could be not uniformly bounded in the space $C[0,T]$ (see [23]).

Next, we will study the behavior of $\|P_N\|$ in the space $C[0,T]$ for $c = 1$. We restrict ourselves to the case of uniform mesh, i.e. we suppose that $h_i = h =$

$T/N, \quad i = 1, \dots, N$. Assume that the mesh Δ_N is complemented with knots $t_i = ih$ for $i = -2, \ -1$ and $i = N+1, \ N+2$. We will use the B-splines

$$B_i(t) = \frac{1}{h^2} \begin{cases} (t - t_{i-1})^2, & t \in [t_{i-1}, t_i] \\ 2h^2 - (t_{i+1} - t)^2 - (t - t_i)^2, & t \in [t_i, t_{i+1}] \\ (t_{i+2} - t)^2, & t \in [t_{i+1}, t_{i+2}]. \end{cases}$$

They are normalized with the condition

$$\sum_{i=-1}^{N} B_i(t) = 2, \quad t \in [0, T].$$

Given any function $f \in C[0,T]$, let us consider $u = P_N f = \sum_{-1 \le j \le N} c_j B_j$, which is equivalent to the conditions

$$\begin{aligned} u(t_i) &= f(t_i), \quad i = 0, \dots, N \\ u''(t_{N-1} - 0) &= u''(t_{N-1} + 0). \end{aligned} \tag{3.10}$$

We write (3.10) in the form of a linear system to determine the coefficients c_j as follows:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ 0 & & \cdots & 0 & 0 & 1 & 1 \\ 0 & & \cdots & -1 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \\ \vdots \\ \\ c_{N-1} \\ c_N \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ \\ f_N \\ 0 \end{pmatrix} \tag{3.11}$$

with $f_i = f(t_i), \quad i = 0, \dots, N$. The system (3.11) has the unique solution because the determinant of its matrix is different from zero. Actually, the solution of (3.11) is

$$\left.\begin{aligned} c_N &= \frac{1}{8}(f_{N-2} - 4f_{N-1} + 7f_N) \\ c_{N-1} &= \frac{1}{8}(-f_{N-2} + 4f_{N-1} + f_N) \\ c_{N-2} &= \frac{1}{8}(f_{N-2} + 4f_{N-1} - f_N) \\ c_{N-3} &= \frac{1}{8}(7f_{N-2} - 4f_{N-1} + f_N) \\ c_{N-4} &= f_{N-3} - c_{N-3} \\ c_i &= f_{i+1} - f_{i+2} + \ldots + (-1)^{N-i} f_{N-3} + (-1)^{N-i+1} c_{N-3} \\ & \quad i = N-5, \dots, -1. \end{aligned}\right\} \tag{3.12}$$

This allows to get

$$\|P_N f\|_{C[0,T]} \le 2 \max_{-1\le i\le N} |c_i| \le 2N\|f\|_{C[0,T]}.$$

Consider the function $f \in C[0,T]$ such that $f(t_i) = (-1)^i$, $i = 0,\dots,N$, being linear between the knots t_i. Then, for example, for $i = 2$ with $N \to \infty$ we have

$$\|P_N f\|_C \ge |(P_N f)(t_i)| = |c_{i-1}B_{i-1}(t_i) + c_i B_i(t_i) + c_{i+1}B_{i+1}(t_i)| \approx N\|f\|_C.$$

Thus, $\|P_N\| \ge \text{const}\cdot N$. It is established that the sequence $\|P_N\|$ has order N as $N \to \infty$.

In [15] the norm of the quadratic spline interpolation operators is explicitly calculated for the case of interpolation conditions which are actually the same as in the classical step-by-step collocation. This formula implies the order N of these projections. Let us mention that the results in [20] do not yield this asymptotics of the projection norms.

4. General convergence theorems

In this section we present some general convergence theorems for operator equations which will be applied to the equation (2.1) in the linear case.

Let E and F be Banach spaces, $\mathcal{L}(E,F)$ and $\mathcal{K}(E,F)$ spaces of linear continuous and linear compact operators. Suppose we have an equation

$$u = Ku + f\ , \tag{4.1}$$

where $K \in \mathcal{K}(E,E)$ and $f \in E$. Let be given a sequence of approximating operators $P_N \in \mathcal{L}(E,E)$, $N = 1,2,\dots$. Consider also equations

$$u_N = P_N K u_N + P_N f. \tag{4.2}$$

The following theorem for second kind equations may be called classical because it is one of the most important tools in the theory of approximate methods for integral equations (see [1, 8]).

Theorem 4.1. *Suppose $u = Ku$ only if $u = 0$ and $P_N u \to u$ for all $u \in E$ as $N \to \infty$. Then:*

1) *Equation (4.1) has the unique solution u^*.*
2) *There exists a number N_0 such that for $N \ge N_0$, equation (4.2) has the unique solution u_N^*.*
3) *$u_N^* \to u^*$ as $N \to \infty$.*

4) *There are constants* $C_1, C_2, C_3 > 0$ *such that*

$$C_1\|P_N u^* - u^*\| \le \|u_N^* - u^*\| \le C_2\|P_N u^* - u^*\| \tag{4.3}$$

$$\|u_N^* - P_N u^*\| \le C_3\|K(P_N u^* - u^*)\|\,. \tag{4.4}$$

Note that this theorem can be deduced from more general ones [13, 22]. The reader can find the following notions and results, for instance, in [22].

The sequence of operators $A_N \in \mathcal{L}(E,F)$ is said to be *stably convergent* to the operator $A \in \mathcal{L}(E,F)$ if A_N converges to A pointwise (i.e. $A_N x \to Ax$ for all $x \in E$) and there is N_0 such that for $N \ge N_0$, $A_N^{-1} \in \mathcal{L}(F,E)$ and $\|A_N^{-1}\| \le$ const. The sequence A_N is said to be *regularly convergent* to A if A_N converges to A pointwise and if x_N is bounded and $A_N x_N$ compact, then x_N is compact itself. The sequence A_N is said to be *compactly convergent* to A if A_N converges to A pointwise and if x_N is bounded, then $A_N x_N$ is compact.

Theorem 4.2. *Having $P_N f \to f$ and compact convergence of $P_N K$ to K instead of $P_N u \to u$ for all $u \in E$, the assertions of Theorem* 4.1 *hold.*

Consider the equations

$$Au = f \tag{4.5}$$

$$A_N u_N = f_N \tag{4.6}$$

with $A, A_N \in \mathcal{L}(E,F)$ and $f, f_N \in F$.

Theorem 4.3. *The following two conditions are equivalent:*

1) $\mathrm{Im}A = F$, *A_N converges to A stably.*

2) $\mathrm{Ker}A = \{0\}$, *A_N are Fredholm operators of index 0 for $N \ge N_0$ with some N_0, and A_N converges to A regularly.*

If one of them is satisfied, then equation (4.5) *has the unique solution u^*. There exists a number N_0 such that for $N \ge N_0$, the equations* (4.6) *are uniquely solvable. If f_N converges to f, then u_N converges to u with the estimate*

$$C_1\|A_N u^* - f_N\| \le \|u_N^* - u^*\| \le C_2\|A_N u^* - f_N\|.$$

Remark 4.4. Without presenting the details let us mention that, for the general equation (4.1) with a nonlinear operator K, it holds a counterpart of Theorem 4.1 ensuring the two-sided error estimate (4.3) provided the projections P_N converge pointwise to the identity operator (see [14], Section 50.2). This theorem needs the complete continuity, i.e. continuity and compactness, of the nonlinear operator K which is guaranteed for the operator (2.5) in the space $C[0,T]$ by the continuity of the kernel $\mathcal{K}(t,s,u)$ (see [12], Chapter 1, Section 3). We prove the convergence of P_N in the next section (see Lemma 5.1).

5. Application of the classical convergence theorem

In this section we show that, for $c \in (0,1)$, Theorem 4.1 is applicable to the equation (2.1). Suppose that the sequence of meshes Δ_N is quasi-uniform.

Lemma 5.1. *For $c \in (0,1)$, the projection operators P_N defined by* (2.4) *converge pointwise to the identity, i.e. $P_N f \to f$ in $C[0,T]$ for all $f \in C[0,T]$ as $N \to \infty$.*

Proof. Choose any $c \in (0,1)$. For given $f \in C^1[0,T]$, let S and z be quadratic splines satisfying (3.1) for the chosen c and for $c = 1/2$ respectively. Taking into account $S = P_N f$, $\|P_N\| \le \text{const}$ and $\|z - f\|_C \to 0$ (see [10]), we get

$$\begin{aligned}\|S - f\|_C &\le \|S - z\|_C + \|z - f\|_C \\ &= \|P_N(f - z)\|_C + \|z - f\|_C \\ &\le \text{const}\|f - z\|_C + \|z - f\|_C \to 0.\end{aligned}$$

This means that $\|P_N f - f\| \to 0$ for all $f \in C^1[0,T]$. Using the Banach-Steinhaus theorem, we get the convergence of the sequence of operators P_N to the identity operator everywhere in the space $C[0,T]$, since $C^1[0,T]$ is dense in $C[0,T]$. The proof is completed. ∎

Taking $E = C[0,T]$, the integral operator

$$(Ku)(t) = \int_0^t \mathcal{K}(t,s)u(s)ds, \quad u \in C[0,T], \tag{5.1}$$

and using Lemma 5.1, Theorem 4.1 directly yields

Theorem 5.2. *Suppose the kernel $\mathcal{K}$ is such that K is compact, $u = Ku$ holds only for $u = 0$, and $c \in (0,1)$. Then the method* (2.2), (2.3) *is convergent in $C[0,T]$ and the estimates* (4.3) *and* (4.4) *hold.*

6. Compact convergence

We have already shown that our method is stable for $c \in (0,1)$. In the case $c = 1$ the sequence of operators P_N is unbounded. So we cannot apply the classical convergence theorem. Taking into consideration Theorem 4.2, it is justified to ask whether there is the compact convergence of $P_N K$ to K. First we state

Proposition 6.1. *Suppose the operator K is given by* (5.1)*, where $\mathcal{K}(t,s)$ is continuous in $\{(t,s)|\ 0 \le s \le t \le T\}$ and continuously differentiable with respect to t. Then the sequence $P_N K$ converges strongly to K in $C[0,T]$.*

Proof. For given $u \in C[0,T]$, denote $f = Ku$. Then $f \in C^1[0,T]$. Let S be the quadratic spline interpolant determined by (3.1) in the case $c = 1/2$. It is known that $\|S - f\|_{C[0,T]} \le \text{const} \cdot h\, \omega(f')$ (see [10]), where $\omega(\cdot)$ is the modulus of continuity. We obtain

$$\|P_N f - f\|_C \le \|P_N\| \|(f - S)\| + \|S - f\|.$$

The last norm converges to 0, and

$$\|P_N\| \|f - S\| \le (\text{const} \cdot N)(\text{const} \cdot h\, \omega(f')) \le \text{const}\, \omega(f') \to 0.$$

Hence, $\|P_N Ku - Ku\| \to 0$ for all $u \in C[0,T]$, which completes the proof. ∎

Let us focus our attention on the operator $(Ku)(t) = \int_0^t u(s)ds$. Consider in the rest of this section (and in the following section, too) the uniform mesh, i.e. suppose that $h_i = h = T/N, \quad i = 1, \dots, N$.

Choose the sequence of functions $u_N \in C[0,T]$ such that, for $i = 1, \dots, N$ and sufficiently small $\delta = \delta(N) > 0$,

$$u_N(t) = \begin{cases} 1 & \text{for all } t \in [t_{i-1} + \delta, t_i - \delta],\ i \text{ even} \\ -1 & \text{for all } t \in [t_{i-1} + \delta, t_i - \delta],\ i \text{ odd}, \end{cases}$$

u_N being linear for $t \in [t_i - \delta, t_i + \delta]$, $i = 1, \dots, N-1$, and constant in $[t_0, t_0 + \delta]$ and $[t_N - \delta, t_N]$. Obviously, $\|u_N\| = 1$. Defining $f_i = f(t_i) = (Ku_N)(t_i)$, we have $f_0 = 0$, $f_i = -h + \delta/2$ for $i = 1, 3, \dots$, $f_i = -\delta/2$ for $i = 2, 4, \dots$, $f_N = 0$ for N even and $f_N = -h$ for N odd. Taking, for example, $\delta = O(h^2)$, and using the equalities (3.12), we calculate for large N and relatively small i the coefficients $c_i = T/2 + O(h)$, $i = -1, 1, \dots$, and $c_i = -T/2 + O(h)$, $i = 0, 2, \dots$. The values of B-splines $B_i(t_i + h/2) = 3/2$ and $B_{i-1}(t_i + h/2) = B_{i+1}(t_i + h/2) = 1/4$ allow to get for i odd

$$2\big(P_N K u_N\big)\big(t_i + \frac{h}{2}\big) = \sum_{j=i-1}^{i+1} c_j B_j\big(t_i + \frac{h}{2}\big) = \frac{T}{2} + O(h)$$

$$\big(P_N K u_N\big)\big(t_{i-1} + \frac{h}{2}\big) = \sum_{j=i-2}^{i} c_j B_j\big(t_{i-1} + \frac{h}{2}\big) = -\frac{T}{2} + O(h).$$

Hence,

$$|\big(P_N K u_N\big)\big(t_i + \frac{h}{2}\big) - \big(P_N K u_N\big)\big(t_{i-1} + \frac{h}{2}\big)| = T + O(h),$$

which means that the functions $P_N K u_N$, as $N \to \infty$ or $h \to 0$, are not equicontinuous and, therefore, the sequence $P_N K u_N$ is not compact. We have proved

Proposition 6.2. *For $(Ku)(t) = \int_0^t u(s)ds$, the sequence $P_N K$ does not converge compactly to K in the case $c = 1$ as $N \to \infty$.*

7. Regular convergence

Our purpose in this section is to prove the regular convergence of operators $I - P_N K$ to $I - K$ in the case $c = 1$ by using a new representation of quadratic splines. Recall that the mesh is assumed to be uniform.

Given a function $f \in C[0,T]$, let $S = P_N f \in S_2(\Delta_N)$ be such that

$$S(t_i) = f(t_i), \quad i = 0, \dots, N$$
$$S''(t_{N-1} - 0) = S''(t_{N-1} + 0).$$

Denote $S_i = S(t_i)$ and $S_{i-1/2} = S(t_{i-1} + h/2)$. Using $t = t_{i-1} + \tau h$, we get the representation of S for $t \in [t_{i-1}, t_i]$

$$S(t) = (1-\tau)(1-2\tau)S_{i-1} + 4\tau(1-\tau)S_{i-1/2} + \tau(2\tau - 1)S_i. \qquad (7.1)$$

The continuity of S' in the knots t_i, i.e. $S'(t_i - 0) = S'(t_i + 0)$, leads to the equations

$$S_{i-1} + 6S_i + S_{i+1} = 4(S_{i-1/2} + S_{i+1/2}), \quad i = 1, \dots, N-1.$$

The not-a-knot boundary condition gives

$$S_N - S_{N-2} = 2(S_{N-1/2} - S_{N-3/2}).$$

Considering the values $S_i = f_i = f(t_i)$, $i = 0, \dots, N$, as known data we have the system

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \\ & & & & & \\ 0 & & \cdots & 0 & 1 & 1 \\ 0 & & \cdots & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} S_{1/2} \\ S_{3/2} \\ \vdots \\ \\ S_{N-3/2} \\ S_{N-1/2} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ \\ d_{N-1} \\ d_N \end{pmatrix}, \qquad (7.2)$$

where $d_i = (f_{i-1} + 6f_i + f_{i+1})/4$, $i = 1, \dots, N-1$, and $d_N = (f_N - f_{N-2})/2$. However, the matrix of (7.2) is regular because its determinant is equal to 2. By direct calculation we obtain

$$\left.\begin{aligned} S_{N-1/2} &= \frac{1}{8}\big(-f_{N-2} + 6f_{N-1} + 3f_N\big) \\ S_{N-3/2} &= \frac{1}{8}\big(3f_{N-2} + 6f_{N-1} - f_N\big) \\ S_{N-5/2} &= \frac{1}{8}\big(2f_{N-3} + 9f_{N-2} - 4f_{N-1} + f_N\big) \\ S_{k-1/2} &= \frac{1}{4}\big(f_{k-1} + 5f_k\big) - f_{k+1} + f_{k+2} - \dots \\ &\quad + \frac{(-1)^{N-k}}{8}\big(7f_{N-2} - 4f_{N-1} + f_N\big), \quad k = N-3, \dots, 1. \end{aligned}\right\} \qquad (7.3)$$

Now, having S_i, $i = 0, \dots, N$, and $S_{i-1/2}$, $i = 1, \dots, N$, the spline S is determined by (7.1).

In the assumptions of Proposition 6.1 we have the strong convergence of $I - P_N K$ to $I - K$.

Theorem 7.1. *Suppose that the functions* $\mathcal{K}$ *and* $\partial\mathcal{K}/\partial t$ *are continuous in* $\{(t,s)|\ 0 \le s \le t \le T\}$. *Then, for* $c = 1$ *and for* K *defined by* (5.1), *the convergence of* $I - P_N K$ *to* $I - K$ *is regular.*

Proof. Choose a sequence $g_N \in C[0,T]$ such that $\|g_N\| \le 1$. Assume that the sequence $(I - P_N K)g_N$ is compact. We have to show the compactness of g_N. Denote here $S = P_N K g_N$ and use the values S_i and $S_{i-1/2}$ of S as before. However, we have to keep in mind that they depend on N.

The continuity of $\mathcal{K}$ and $\partial\mathcal{K}/\partial t$ ensures also the uniform continuity and boundedness, thus, there are numbers M and M_1 such that $|\mathcal{K}(t,s)| \le M$ and $|(\partial\mathcal{K}/\partial t)(t,s)| \le M_1$ on $\{(t,s)|\ 0 \le s \le t \le T\}$.

For $t \in [t_{k-1}, t_k]$, $k = 1, \dots, N$, we have

$$\begin{aligned} g_N(t) - (P_N K g_N)(t) &= g_N(t) - S(t) \\ &= g_N(t) - \big((1-2\tau)(1-\tau)S_{k-1} \\ &\quad + 4\tau(1-\tau)S_{k-1/2} + \tau(2\tau-1)S_k\big). \end{aligned}$$

The difference

$$\begin{aligned} S_{k-1} - (K g_N)(t) &= \int_0^{t_{k-1}} \mathcal{K}(t_{k-1}, s) g_N(s) ds - \int_0^t \mathcal{K}(t,s) g_N(s) ds \\ &= \int_0^{t_{k-1}} \big(\mathcal{K}(t_{k-1}, s) - \mathcal{K}(t,s)\big) g_N(s)\, ds - \int_{t_{k-1}}^t \mathcal{K}(t,s) g_N(s) ds \end{aligned}$$

goes to zero uniformly on $[0,T]$ as $h \to 0$ because of the uniform continuity and boundedness of $\mathcal{K}$. Similarly, $S_k - (K g_N)(t) \to 0$ uniformly on $[0,T]$ as $h \to 0$. Using $(2\tau-1)^2 + 4\tau(1-\tau) = 1$, we obtain

$$g_N(t) - S(t) = g_N(t) - (K g_N)(t) - 4\tau(1-\tau)\big(S_{k-1/2} - (K g_N)(t)\big) + (G_N^1 g_N)(t),$$

where $G_N^1 g_N \to 0$ in $C[0,T]$. Here, the sequence $K g_N$ is compact. To establish the compactness of g_N, we will study the term $S_{k-1/2} - (K g_N)(t)$.

Taking $f_i = (K g_N)(t_i)$ and let us write $S_{k-1/2}$ from (7.3) in the form

$$S_{k-1/2} = f_k + \frac{1}{2} \sum_{i=k}^{N} (-1)^{i-k+1} (f_{i-1} - f_i) + O(h).$$

Using again $S_k - (Kg_N)(t) \to 0$ as $h \to 0$ uniformly on $[0, T]$, we get

$$S_{k-1/2} - (Kg_N)(t) = \frac{1}{2}\sum_{i=k}^{N}(-1)^{i-k}(f_i - f_{i-1}) + (G_N^2 g_N)(t)$$

with $G_N^2 g_N \to 0$ in $C[0, T]$. Denote

$$I_k(g_N) = \int_{t_{k-1}}^{t_k} \mathcal{K}(t_k, s) g_N(s) ds.$$

Then

$$\begin{aligned} f_k - f_{k-1} &= \int_{t_{k-1}}^{t_k} \mathcal{K}(t_k, s) g_N(s) ds + \int_0^{t_{k-1}} \big(\mathcal{K}(t_k, s) - \mathcal{K}(t_{k-1}, s)\big) g_N(s) ds \\ &= I_k(g_N) + h \int_0^{t_{k-1}} \frac{\partial \mathcal{K}}{\partial t}(\xi_k, s) g_N(s) ds \end{aligned}$$

where $\xi_k \in [t_{k-1}, t_k]$. Again, we have

$$\begin{aligned} (f_k - f_{k-1}) - (f_{k+1} - f_k) = I_k(g_N) - I_{k+1}(g_N) \\ - h \int_0^{t_{k-1}} \Big(\frac{\partial \mathcal{K}}{\partial t}(\xi_{k+1}, s) - \frac{\partial \mathcal{K}}{\partial t}(\xi_k, s)\Big) g_N(s)\, ds \\ - h \int_{t_{k-1}}^{t_k} \frac{\partial \mathcal{K}}{\partial t}(\xi_{k+1}, s) g_N(s)\, ds. \end{aligned}$$

In this expression, the uniform continuity of $\partial\mathcal{K}/\partial t$ allows to estimate the first integral by $\varepsilon_N h$ with $\varepsilon_N \to 0$ as $h \to 0$ and the second one by $M_1 h^2$. Summing up all the differences $(f_k - f_{k-1}) - (f_{k+1} - f_k)$ (if there are an odd number of terms $f_k - f_{k-1}$ it suffices to observe that, in fact, $f_k - f_{k-1} = I_k(g_N) + O(h)$), we get

$$\sum_{i=k}^{N}(-1)^{i-k} I_i(g_N) + r_N$$

with $r_N \to 0$ as $h \to 0$. We arrived at

$$\begin{aligned} g_N(t) - (P_N K g_N)(t) = g_N(t) - (K g_N)(t) \\ - 2\tau(1-\tau)\sum_{i=k}^{N}(-1)^{i-k} I_i(g_N) + (G_N^3 g_N)(t)\ , \end{aligned} \tag{7.4}$$

where $G_N^3 g_N \to 0$ in $C[0, T]$.

Denoting $\alpha_N(t) = 2\tau(1-\tau)$, define the operators $Q_N : C[0,T] \to C[0,T]$ and the functions $\varphi_N \in C[0,T]$ by

$$\begin{aligned}(Q_N g_N)(t) &= \alpha_N(t) \sum_{i=k}^{N} (-1)^{i-k} I_i(g_N) \\ \varphi_N &= g_N - Q_N g_N.\end{aligned} \tag{7.5}$$

Clearly, (7.4) yields the compactness of φ_N. For $t \in [t_{N-1}, t_N]$, from $\varphi_N(t) = g_N(t) - \alpha_N(t) I_N(g_N)$ we get $I_N(\varphi_N) = I_N(g_N) - I_N(\alpha_N) I_N(g_N)$. Denoting $\lambda_k = I_k(\alpha_N)$, $k = 1, \ldots, N$, we see that $|\lambda_k| \le Mh/2$. Thus, for sufficiently small h, taking $\mu_k = 1/(1-\lambda_k)$, we have $I_N(g_N) = \mu_N I_N(\varphi_N)$. Now, induction leads to

$$\sum_{i=k}^{N} (-1)^{i-k} I_i(g_N) = \sum_{i=k}^{N} (-1)^{i-k} \Big(\prod_{j=k}^{i} \mu_j \Big) I_i(\varphi_N).$$

Let $R_N : C[0,T] \to C[0,T]$ be defined by

$$(R_N \varphi_N)(t) = \alpha_N(t) \sum_{i=k}^{N} (-1)^{i-k} \Big(\prod_{j=k}^{i} \mu_j \Big) I_i(\varphi_N). \tag{7.6}$$

Write (7.5) in the form

$$g_N = \varphi_N + R_N \varphi_N. \tag{7.7}$$

Hence,

$$Q_N g_N = Q_N \varphi_N + Q_N R_N \varphi_N.$$

Replacing $Q_N g_N$ in (7.5) by the last formula, we obtain

$$g_N = \varphi_N + Q_N \varphi_N + Q_N R_N \varphi_N. \tag{7.8}$$

Now, we establish three lemmas to complete the proof of Theorem 7.1.

Lemma 7.2. *The convergence $\varphi_N \to \psi$ in the space $C[0,T]$ implies $Q_N \varphi_N \to 0$ in $C[0,T]$.*

Proof. Based on (7.4) we have

$$Q_N \varphi_N = P_N K \varphi_N - K \varphi_N + G_N^3 \varphi_N.$$

By Proposition 6.1, $P_N K \varphi_N - K \varphi_N \to 0$. Since φ_N is bounded, we get also $G_N^3 \varphi_N \to 0$ which completes the proof. ∎

Lemma 7.3. *The operators R_N are uniformly bounded.*

Proof. For a function $\varphi \in C[0,T]$, consider

$$(R_N\varphi)(t) = \alpha_N(t) \sum_{i=k}^{N} (-1)^{i-k} \Big(\prod_{j=k}^{i} \mu_j \Big) I_i(\varphi).$$

Taking into account the estimates $|\alpha_N(t)| \leq 1/2$, $|I_k(\varphi)| \leq Mh\|\varphi\|$ and (for small h) $\mu_k = 1/(1-\lambda_k) \leq 1/(1-Mh/2) = \mu$ (denote so), we have

$$\|R_N\varphi\| \leq \frac{1}{2} Mh\|\varphi\| \Big(\sum_{i=1}^{N-k+1} \mu^i \Big) = \frac{h}{2} M \frac{\mu(\mu^{N-k+1}-1)}{\mu-1} \|\varphi\| \leq \text{const}\|\varphi\|,$$

which completes the proof of this lemma. ∎

Lemma 7.4. *It holds* $Q_N R_N = R_N Q_N$.

Proof. From (7.5) and (7.7) we get $(I+R_N)(I-Q_N) = I$. To prove the lemma, it is sufficient to check that $(I-Q_N)(I+R_N) = I$.

Choose an arbitrary $\varphi_N \in C[0,T]$ and determine $g_N = (I+R_N)\varphi_N$. We will prove that $(I-Q_N)g_N = \varphi_N$. Using (7.6), we calculate

$$I_i(g_N) = I_i(\varphi_N) + \lambda_i \sum_{j=i}^{N} (-1)^{j-i} \Big(\prod_{l=i}^{j} \mu_l \Big) I_j(\varphi_N), \quad i = k, \ldots, N.$$

Since $\varphi_N = g_N - R_N\varphi_N$ and, on the other hand, $\varphi_N = g_N - Q_N g_N$, we show that $R_N\varphi_N = Q_N g_N$ or, taking into account the definitions of R_N and Q_N,

$$\sum_{i=k}^{N} (-1)^{i-k} I_i(g_N) = \sum_{i=k}^{N} (-1)^{i-k} \Big(\prod_{j=k}^{i} \mu_j \Big) I_i(\varphi_N). \tag{7.9}$$

However,

$$\begin{aligned} \sum_{i=k}^{N} (-1)^{i-k} I_i(g_N) &= \sum_{i=k}^{N} (-1)^{i-k} I_i(\varphi_N) \\ &\quad + \sum_{i=k}^{N} (-1)^{i-k} \lambda_i \sum_{j=i}^{N} (-1)^{j-i} \Big(\prod_{l=i}^{j} \mu_l \Big) I_j(\varphi_N)\ , \end{aligned}$$

and it is straightforward to check that the coefficients of $I_i(\varphi_N)$ coincide with those in the right hand side of (7.9). This proves the lemma. ∎

Finally, taking into account Lemma 7.4, we may write (7.8) in the form

$$g_N = \varphi_N + Q_N\varphi_N + R_N Q_N \varphi_N.$$

Remembering that φ_N was compact and, using the Lemmas 7.2 and 7.3, we establish the compactness of g_N. The proof of Theorem 7.1 is complete. ∎

Now, we are ready to close this section with the following

Theorem 7.5. *Let $\mathcal{K}$ and $\partial\mathcal{K}/\partial t$ be continuous in* $\{(t,s)|\ 0 \le s \le t \le T\}$. *Then, for $c = 1$ and for the uniform mesh, there is a number N_0 such that for $N \ge N_0$, the problem* (2.2), (2.3) *has the unique solution, and the estimate* (4.3) *holds. If $P_N f$ converges to f, then the solutions of* (2.2), (2.3) *converge to the solution of* (2.1) *in* $C[0,T]$.

Proof. Note that, in the assumptions about $\mathcal{K}$, the operator K defined by (5.1) is such that $u = Ku$ only for $u = 0$. Making use of Theorem 4.3, take $E = F = C[0,T]$, $A = I - K$, $A_N = I - P_N K$ and refer to Theorem 7.1, the assertion is proved. ∎

Remark 7.6. The rate of convergence of the method (2.2), (2.3) for linear equations, as stated in Theorems 5.2 and 7.5, is determined by the two-sided estimate (4.3). It is well known that quadratic spline interpolation projections P_N have the property $\|P_N u - u\| = O(h^3)$ for smooth functions u. The rate $O(h^3)$ is confirmed by the numerical tests presented in Section 9.

8. The method in the space of continuously differentiable functions

We will focus our attention to the study of the method (2.2), (2.3) in the space $C^1[0,T]$. Similarly to the beginning of Section 3, fix $c \in (0,1)$ and define the projections P_N by (3.1). Without any additional assumption we may establish the estimates (3.8) and (3.9). Suppose now $f \in C^1[0,T]$. Then, for $i = 1,\dots,N-1$, and for some $\xi_i \in (\tau_i, \tau_{i+1})$, we have

$$\begin{aligned}|g_i| &= \frac{|f'(\xi_i)|(\tau_{i+1}-\tau_i)}{c(1-c)(h_i+h_{i+1})}\\ &= |f'(\xi_i)|\Big(\frac{1}{c}\ \frac{h_i}{h_i+h_{i+1}} + \frac{1}{1-c}\ \frac{h_{i+1}}{h_i+h_{i+1}}\Big)\\ &\le \max\Big\{\frac{1}{c}\ ,\frac{1}{1-c}\Big\}\|f'\|_{C[0,T]}\end{aligned}$$

and

$$|g_0| \le \frac{1}{1-c}\ \|f'\|_{C[0,T]}.$$

Taking into account (3.8), (3.9) and basing on the representation (3.2) we obtain $\|P_N\|_{C[0,T]} \le \text{const}\ \|f\|_{C^1[0,T]}$ only in the assumption that $h_N^2/h_{N-1} = O(1)$. The derivative of (3.2) allows to get again with the help of (3.8) and (3.9) the estimate

$$\|(P_N f)'\|_{C[0,T]} \le \max_{0\le i\le N} |m_i| \le \text{const}\ \|f\|_{C^1[0,T]}$$

provided $h_N/h_{N-1} = O(1)$. We have proved the following

Proposition 8.1. *For $c \in (0,1)$ assume that $h_N/h_{N-1} = O(1)$, then the projections P_N are uniformly bounded in the space $C^1[0,T]$.*

It holds also

Lemma 8.2. *For $c \in (0,1)$ and $h_N/h_{N-1} = O(1)$, the projections P_N converge pointwise to the identity in the space $C^1[0,T]$, i.e. $P_N f \to f$ in $C^1[0,T]$ for all $f \in C^1[0,T]$ as $h_{\max} \to 0$.*

Proof. Similarly to the proof of Lemma 5.1, for given $f \in C^2[0,T]$ construct the splines $P_N f$ and z. Then $\|z - f\|_{C^1} \to 0$ (see [10]) and

$$\|P_N f - f\|_{C^1} \le \|P_N\|_{C^1 \to C^1} \|f - z\|_{C^1} + \|z - f\|_{C^1} \to 0.$$

As $C^2[0,T]$ is dense in $C^1[0,T]$ it remains to use the Banach-Steinhaus theorem which completes the proof. ∎

Lemma 8.2 and Theorem 4.1 yield

Theorem 8.3. *Suppose the kernel $\mathcal{K}(t,s)$ is such that the operator K defined by* (5.1) *is compact in $C^1[0,T]$, $u = Ku$ only for $u = 0$. Then the method* (2.2), (2.3) *with $c \in (0,1)$ and $h_N/h_{N-1} = O(1)$ is convergent in $C^1[0,T]$, and the estimates* (4.3), (4.4) *hold.*

Let us add that the compactness of K in $C^1[0,T]$ takes place, for example, in the assumptions of Proposition 6.1, but they could be weakened so that weakly singular equations could be also included.

Next, we will study the method in $C^1[0,T]$ for $c = 1$. In the sequel, consider only the uniform mesh. As well as in Section 3, represent $P_N f$ by B-splines. The coefficients of the representation could be estimated from the system (3.12) as $|c_i| \le \text{const}\ \|f\|_{C^1[0,T]}$. Having also $\|B_i'\|_{C[0,T]} = O(N)$, we get that $\|(P_N f)'\|_{C[0,T]} \le \text{const} \cdot N\ \|f\|_{C^1[0,T]}$ which cannot be improved as we will see later. Thus, $\|P_N\|_{C^1 \to C^1} = O(N)$.

Wishing to apply the Theorems 4.2 and 4.3, we have to prove that $P_N K$ converges strongly to K in $C^1[0,T]$. Assume that the kernel $\mathcal{K}$ in (5.1) is continuous and twice continuously differentiable with respect to t. Take $u \in C^1[0,T]$, then $f = Ku \in C^2[0,T]$. Likewise in the proof of Proposition 6.1, concerning the spline S, it is known (see [10]) that $\|S - f\|_{C^1} \le \text{const} \cdot h\ \omega(f'')$. Hence,

$$\|P_N f - f\|_{C^1} \le \|P_N\|_{C^1 \to C^1} \|f - S\|_{C^1} + \|S - f\|_{C^1} \to 0\ ,$$

and we have the pointwise convergence $P_N K \to K$ in the space $C^1[0,T]$.

Let us show that there is no compact convergence $P_N K \to K$ in the space $C^1[0,T]$ even for the operator $(Ku)(t) = \int_0^t u(s)ds$. Take the functions $u_N \in C^1[0,T]$ such that, for sufficiently small $\delta = \delta(N) > 0$,

$$u_N(t) = \begin{cases} t & \text{for } t \in [0, \frac{h}{2} - \delta] \\ (-1)^i (t - t_i) & \text{for } t \in [t_i - \frac{h}{2} + \delta, t_i + \frac{h}{2} - \delta] \\ & \quad i = 1, \dots, N-1 \\ (-1)^N (t - t_N) & \text{for } t \in [t_N - \frac{h}{2} + \delta, t_N], \end{cases}$$

u_N being (uniquely determined by the continuity of u_N') quadratic polynomial for $t \in [t_i - h/2 - \delta, t_i - h/2 + \delta]$, $i = 1, \dots, N$. Clearly $\|u_N\|_C \le h/2$ and $\|u_N'\|_C = 1$ which means that u_N is bounded in $C^1[0,T]$. Taking $\delta = O(h^2)$ and defining $f_i = (Ku_N)(t_i)$, we calculate $f_i = h^2/4 + O(h^3)$ for i odd and $f_i = 0$ for i even. Considering relatively small i and large N, we get $c_i = -Th/8 + O(h^2)$ for $i = -1, 1, \dots$, and $c_i = Th/8 + O(h^2)$ for $i = 0, 2, \dots$. As it holds $B_i'(t_i) = 2/h$, $B_{i-1}'(t_i) = -2/h$ and $B_j'(t_i) = 0$ for $j > i$ and $j < i - 1$, we have for i odd

$$(P_N K u_N)'(t_i) = c_{i-1} B_{i-1}'(t_i) + c_i B_i'(t_i) = -\frac{T}{2} + O(h)$$

$$(P_N K u_N)'(t_{i+1}) = c_i B_i'(t_{i+1}) + c_{i+1} B_{i+1}'(t_{i+1}) = \frac{T}{2} + O(h),$$

consequently, $(P_N K u_N)'$ are not equicontinuous and the sequence $P_N K u_N$ is not compact in $C^1[0,T]$. We have proved

Proposition 8.4. *For $(Ku)(t) = \int_0^t u(s)ds$, the sequence $P_N K$ does not converge compactly to K in the space $C^1[0,T]$ in the case $c = 1$ as $N \to \infty$.*

Note that, actually, in the proof of this proposition we established the inequality $\|P_N\|_{C^1 \to C^1} \ge \text{const} \cdot N$.

We state as an open problem, for $c = 1$, the regular convergence of $I - P_N K$ to $I - K$ in the space $C^1[0,T]$. Numerical results presented in the next section suggest the positive solution of this problem.

9. Numerical tests

In numerical tests, in order to take advantage of the complexity $O(N)$, we chose the test equation

$$y(t) = \lambda \int_0^t y(s)ds + f(t), \qquad t \in [0,T]$$

on the interval $[0,1]$, having the exact solution $y(t) = (\sin t + \cos t + e^t)/2$. We implemented also the method for the equation (2.1) in the linear case with the kernel $\mathcal{K}(t,s) = t - s$ and the function $f(t) = \sin t$, having the exact solution $y(t) = (2\sin t + e^t - e^{-t})/4$ on the interval $[0,1]$. This equation is used in [2, 5, 18]. Actually, we calculated the error $\|u - y\|_C$ approximately as

$$\max_{1\le n\le N}\ \max_{0\le k\le 10}\left|(u-y)\Big(t_{n-1} + \frac{kh}{10}\Big)\right|$$

and similarly the approximate value of $\|u' - y'\|_C$. They are presented in following tables for particular values of N and c, $\|u - y\|_C$ in the upper row and $\|u' - y'\|_C$ in the lower row. The results confirm the rate $\|u' - y'\|_C = O(h^2)$ for smooth solutions predicted by the theory.

`Numerical results for` $y(t) = \lambda \int_0^t y(s)ds + f(t)$:

$\lambda = -2, \quad f(t) = \frac{1}{2}(3\sin t - \cos t + 3e^t)$

N	4	16	64	256
$c = 1$	$1.12 \cdot 10^{-3}$	$2.21 \cdot 10^{-5}$	$3.63 \cdot 10^{-7}$	$5.74 \cdot 10^{-9}$
	$2.68 \cdot 10^{-2}$	$1.90 \cdot 10^{-3}$	$1.22 \cdot 10^{-4}$	$7.66 \cdot 10^{-6}$
$c = 0.7$	$2.66 \cdot 10^{-3}$	$4.79 \cdot 10^{-5}$	$7.66 \cdot 10^{-7}$	$1.20 \cdot 10^{-8}$
	$4.60 \cdot 10^{-2}$	$3.51 \cdot 10^{-3}$	$2.29 \cdot 10^{-4}$	$1.45 \cdot 10^{-5}$
$c = 0.5$	$4.59 \cdot 10^{-3}$	$8.94 \cdot 10^{-5}$	$1.46 \cdot 10^{-6}$	$2.31 \cdot 10^{-8}$
	$5.69 \cdot 10^{-2}$	$4.48 \cdot 10^{-3}$	$2.95 \cdot 10^{-4}$	$1.86 \cdot 10^{-5}$

$\lambda = -1, \quad f(t) = \sin t + e^t$

N	4	16	64	256
$c = 0.1$	$7.71 \cdot 10^{-3}$	$1.62 \cdot 10^{-4}$	$2.70 \cdot 10^{-6}$	$4.29 \cdot 10^{-8}$
	$7.11 \cdot 10^{-2}$	$5.81 \cdot 10^{-3}$	$3.85 \cdot 10^{-4}$	$2.44 \cdot 10^{-5}$
$c = 10^{-3}$	$8.0866 \cdot 10^{-3}$	$1.6899 \cdot 10^{-4}$	$2.8187 \cdot 10^{-6}$	$4.4751 \cdot 10^{-8}$
	$7.2391 \cdot 10^{-2}$	$5.8971 \cdot 10^{-3}$	$3.9151 \cdot 10^{-4}$	$2.4834 \cdot 10^{-5}$
$c = 10^{-6}$	$8.0891 \cdot 10^{-3}$	$1.6902 \cdot 10^{-4}$	$2.8184 \cdot 10^{-6}$	$4.4618 \cdot 10^{-8}$
	$7.2399 \cdot 10^{-2}$	$5.8972 \cdot 10^{-3}$	$3.9148 \cdot 10^{-4}$	$2.4900 \cdot 10^{-5}$

$\lambda = 1, \quad f(t) = \cos t$

N	4	16	64	256
$c = 1$	$1.32 \cdot 10^{-3}$	$2.90 \cdot 10^{-5}$	$4.90 \cdot 10^{-7}$	$7.80 \cdot 10^{-9}$
	$2.29 \cdot 10^{-2}$	$1.85 \cdot 10^{-3}$	$1.25 \cdot 10^{-4}$	$7.98 \cdot 10^{-6}$
$c = 0.7$	$2.19 \cdot 10^{-3}$	$4.73 \cdot 10^{-5}$	$7.98 \cdot 10^{-7}$	$1.27 \cdot 10^{-8}$
	$4.20 \cdot 10^{-2}$	$3.42 \cdot 10^{-3}$	$2.27 \cdot 10^{-4}$	$1.44 \cdot 10^{-5}$
$c = 0.5$	$4.03 \cdot 10^{-3}$	$8.63 \cdot 10^{-5}$	$1.45 \cdot 10^{-6}$	$2.30 \cdot 10^{-8}$
	$5.38 \cdot 10^{-2}$	$4.41 \cdot 10^{-3}$	$2.93 \cdot 10^{-4}$	$1.86 \cdot 10^{-5}$

$\lambda = 2, \quad f(t) = \frac{1}{2}(-\sin t + 3\cos t - e^t)$

N	4	16	64	256
$c = 0.1$	$7.24 \cdot 10^{-3}$	$1.56 \cdot 10^{-4}$	$2.62 \cdot 10^{-6}$	$4.17 \cdot 10^{-8}$
	$6.99 \cdot 10^{-2}$	$5.79 \cdot 10^{-3}$	$3.85 \cdot 10^{-4}$	$2.44 \cdot 10^{-5}$
$c = 10^{-3}$	$7.8735 \cdot 10^{-3}$	$1.6810 \cdot 10^{-4}$	$2.8145 \cdot 10^{-6}$	$4.4722 \cdot 10^{-8}$
	$7.1893 \cdot 10^{-2}$	$5.8948 \cdot 10^{-3}$	$3.9150 \cdot 10^{-4}$	$2.4834 \cdot 10^{-5}$
$c = 10^{-6}$	$7.8789 \cdot 10^{-3}$	$1.6819 \cdot 10^{-4}$	$2.8158 \cdot 10^{-6}$	$4.4719 \cdot 10^{-8}$
	$7.1906 \cdot 10^{-2}$	$5.8950 \cdot 10^{-3}$	$3.9150 \cdot 10^{-4}$	$2.4847 \cdot 10^{-5}$

Numerical results for $y(t) = \int_0^t (t-s)y(s)ds + f(t)$:

N	8	32	128	512
$c = 1$	$4.81 \cdot 10^{-5}$	$9.20 \cdot 10^{-7}$	$1.51 \cdot 10^{-8}$	$2.38 \cdot 10^{-10}$
	$2.15 \cdot 10^{-3}$	$1.56 \cdot 10^{-4}$	$1.01 \cdot 10^{-5}$	$6.36 \cdot 10^{-7}$
$c = 0.5$	$1.95 \cdot 10^{-4}$	$3.71 \cdot 10^{-6}$	$6.07 \cdot 10^{-8}$	$9.59 \cdot 10^{-10}$
	$5.02 \cdot 10^{-3}$	$3.75 \cdot 10^{-4}$	$2.45 \cdot 10^{-5}$	$1.55 \cdot 10^{-6}$
$c = 0.1$	$3.56 \cdot 10^{-4}$	$6.84 \cdot 10^{-6}$	$1.12 \cdot 10^{-7}$	$1.77 \cdot 10^{-9}$
	$6.53 \cdot 10^{-3}$	$4.92 \cdot 10^{-4}$	$3.22 \cdot 10^{-5}$	$2.03 \cdot 10^{-6}$

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