

Homogenization Structures and Applications I

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To the memory of Jacques-Louis Lions 1928 - 2001

Abstract. We lay the foundations of a mathematical theory of homogenization structures and show how the latter arises in the homogenization of partial differential equations. We find out that the concept of a homogenization structure turns out to be exactly the right tool that is needed to systematically extend homogenization theory beyond the classical periodic setting. This permits to work out various outstanding nonperiodic homogenization problems that were out of reach till then for lack of an appropriate mathematical framework. The classical Gelfand representation theory is one of our main tools and our basic approach is an adaptation of the two-scale convergence method.

Keywords: *Homogenization, homogenization algebra, homogenization structure, Gelfand transformation* ■

AMS subject classification: Primary 46J10, 35B40, secondary 46N20, 54E15

1. Introduction

The behaviour of a medium depends essentially on the manner in which the latter is structured. This is true of inhomogeneous media in physics, continuum mechanics, chemistry, biology, etc, and even of human societies from various points of view such as health, economics, etc. One branch of mathematics that specializes in studying behaviours is *homogenization theory* in its broadest interpretation (see, e.g., [3, 5, 12, 20, 21, 34 - 37]). Generally speaking, homogenization theory deals with inhomogeneous media governed by partial differential equations whose coefficients are of the form $a_\alpha(x, \frac{x}{\varepsilon})$ ($x \in \Omega$) where $\varepsilon > 0$, Ω is a bounded open set in \mathbb{R}^N representing a sample of the medium under consideration, and a_α is a real or complex function $(x, y) \rightarrow a_\alpha(x, y)$ on $\Omega \times \mathbb{R}^N$, the properties of the medium being described by means of the finite family $\{a_\alpha\}$ (for example, a_α are the elasticity coefficients of an inhomogeneous elastic solid). The aim is to pass to the limit in the equation (or

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system of equations) when $\varepsilon \rightarrow 0$, the main result sought being the so-called homogenized equation (or system) which gives the effective (or macroscopic) behaviour of the medium. However, in general such an undertaking is hopeless without further information on the behaviour, in $y = (y_1, \dots, y_N) \in \mathbb{R}^N$ (x arbitrarily fixed in Ω) of the family $\{a_\alpha\}$. In other words, we have the need to require the family $\{a_\alpha\}$ to be suitably structured in $y \in \mathbb{R}^N$. The classical case is when the medium is assumed to have a periodic structure, that is, when for any fixed $x \in \Omega$ the functions $a_\alpha(x, \cdot)$ (α ranging over a finite set) are periodic (with the same group of periods). Such an assumption, still termed a *periodicity hypothesis*, sends back to a well known powerful theory, namely classical periodic homogenization theory for which there is a huge bibliography (see, e.g., [4, 5, 14, 36, 37], and especially the bibliography of [14]). However, there seems to be no doubt that for the greater part of inhomogeneous media the right *structure hypothesis* is far from being the periodicity hypothesis (see Subsection 5.3). Unfortunately, except the case of almost periodic homogenization problems (see, e.g., [13, 22, 23, 27, 34]), a great number of non-stochastic homogenization problems beyond the periodic setting remains unsolved. The real reason for it is that there is lack of an appropriate mathematical framework for the study of non-stochastic non-periodic homogenization problems.

Though the use of the word *structure* in the literature of homogenization goes back to the seventies of last century [5], the common understanding of terms such as *periodic structure*, *almost periodic structure*, etc., has remained strictly intuitive till then. This is a serious gap in homogenization theory in so far as, by all appearance, the development of the said theory beyond the beaten track of the periodic setting depends essentially on the way the word *structure* is understood from the mathematical point of view.

In this paper we assign a self-contained mathematical meaning to the word *structure* in the context of homogenization. We call a *homogenization structure* the mathematical tool thus constructed. We lay the foundations of a mathematical theory of homogenization structures and show how it arises in the homogenization of partial differential equations. Of course, all that requires new tools among which are the notion of a homogenization algebra and the underlying concept of mean value.

It should be mentioned that the use of so-called homogenization algebras goes back to [20] and [38] under the names of *algebras with mean values* and *ergodic algebras*. However, the present paper seems to be the first work in which systematic utilization of such algebras leads, by means of the Gelfand representation theory and two-scale convergence, to a general mathematical framework on the model of the classical periodic homogenization theory. Indeed, the new homogenization context proposed in this work is framed in such a way that there is a total analogy between the results obtained here and those

provided by periodic homogenization theory, at least as far as elliptic partial differential equations are concerned. Also, attention is drawn to the paper of Bourgeat et al. [12] in which stochastic two-scale mean convergence is introduced and used for the first time. The general approach presented here is quite likely to inspire a new limiting process generalizing stochastic two-scale mean convergence.

The remaining sections of this paper is framed as follows. Section 2 is devoted to the study of homogenization algebras. The study of homogenization structures proper begins with Section 3, where among many other things a bijective correspondence between homogenization algebras and homogenization structures is established. The basic theory of homogenization structures continues in Section 4 with the study of Σ -convergence following the direct line of two-scale convergence [2]. Section 5 is devoted to drawing attention to the close connection between the theory of homogenization structures and that of partial differential equations. Specifically, in Section 5 we apply some of the preceding results to the homogenization of a second order linear elliptic equation beyond the classical periodic setting. The usual periodicity hypothesis is here replaced by an abstract hypothesis covering various concrete structure hypotheses. Finally, by way of illustration, a few concrete cases are studied.

Except where otherwise stated, we will be concerned with vector spaces over \mathbb{C} (the set of complex numbers) and with scalar functions assuming values in \mathbb{C} . For basic concepts and notation concerning integration theory we refer to [7, 8] (see also [17]). If X is a locally compact space and F is a Banach space, then we use $\mathcal{C}(X; F)$ and $\mathcal{B}(X; F)$ to denote the space of all continuous mappings of X into F and the space of those functions in $\mathcal{C}(X; F)$ that are bounded, respectively. It will always be assumed that $\mathcal{B}(X; F)$ is equipped with the supremum norm $\|u\|_\infty = \sup_{x \in X} \|u(x)\|$ ($u \in \mathcal{B}(X; F)$), where $\|\cdot\|$ denotes the norm on F . For shortness we will write $\mathcal{C}(X) = \mathcal{C}(X; \mathbb{C})$ and $\mathcal{B}(X) = \mathcal{B}(X; \mathbb{C})$. Likewise $L^p(X) = L^p(X; \mathbb{C})$ and $L^p_{loc}(X) = L^p_{loc}(X; \mathbb{C})$ ($1 \leq p \leq +\infty$) where X is assumed to be provided with a positive Radon measure. In this connection \mathbb{R}^N (the N -dimensional numerical space) and its open sets are each provided with the Lebesgue measure $dx = dx_1 \cdots dx_N$. Points in \mathbb{R}^N are denoted by $x = (x_1, \dots, x_N)$ or $y = (y_1, \dots, y_N)$ which we sometimes express by writing \mathbb{R}^N_x or \mathbb{R}^N_y in place of \mathbb{R}^N .

The concept of a homogenization structure turns out to be exactly the right tool that is needed to systematically extend homogenization theory beyond the classical periodic setting. The author hopes that the present paper will encourage the study, in the framework of homogenization, of those physical problems whose natural setting is reasonably excluded from the usual scope of periodic homogenization.

2. Homogenization algebras

We begin by introducing an appropriate notion of mean value on \mathbb{R}_y^N ($1 \leq N \in \mathbb{N}$).

2.1 Mean value on \mathbb{R}_y^N . First of all, a function $u \in \mathcal{B}(\mathbb{R}_y^N)$ is said to be *ponderable* if there exists a complex number $M(u)$ such that $u^\varepsilon \rightarrow M(u)$ in $L^\infty(\mathbb{R}_x^N)$ -weak* as $\varepsilon \rightarrow 0$, where

$$u^\varepsilon(x) = u\left(\frac{x}{\varepsilon}\right) \quad (x \in \mathbb{R}^N). \quad (2.1)$$

We denote by $\Pi^\infty(\mathbb{R}_y^N)$ or simply Π^∞ the set of all functions $u \in \mathcal{B}(\mathbb{R}_y^N)$ that are ponderable. It is an easy exercise to verify that Π^∞ is a Banach space under the supremum norm. Furthermore, the notion of ponderable functions yields a mapping $M : \Pi^\infty \rightarrow \mathbb{C}$ whose main properties are summarized below (see [28]):

- (1) M is a positive continuous linear form on Π^∞ with $M(1) = 1$.
- (2) M is translation invariant, i.e. if $u \in \Pi^\infty$, then $\tau_h u \in \Pi^\infty$ and $M(\tau_h u) = M(u)$ for all $h \in \mathbb{R}^N$ where $\tau_h u(y) = u(y - h)$ ($y \in \mathbb{R}^N$).

This leads to the following

Definition 2.1. The linear form $M : \Pi^\infty \rightarrow \mathbb{C}$ is called the *mean value* on \mathbb{R}_y^N , and the complex number $M(u)$ is called the *mean* of $u \in \Pi^\infty$.

We are now in a position to frame the notion of a homogenization algebra on \mathbb{R}_y^N .

2.2 Generalities on homogenization algebras. We begin with the following definition.

Definition 2.2. By a *homogenization algebra* on \mathbb{R}_y^N is meant any closed subalgebra A of $\mathcal{B}(\mathbb{R}_y^N)$ with the following properties:

- (HA)₁ A with the supremum norm is separable.
- (HA)₂ A contains the constants.
- (HA)₃ If $u \in A$, then $\bar{u} \in A$ (\bar{u} the conjugate of u).
- (HA)₄ $A \subset \Pi^\infty$.

For brevity we will often write *H-algebra* in place of *homogenization algebra*. We shall always assume that an *H-algebra* A on \mathbb{R}_y^N is equipped with the supremum norm. It is clear that A is then a commutative C^* -algebra with identity. The spectrum of A is denoted by $\Delta(A)$, $\Delta(A)$ being provided with the Gelfand topology, i.e. the relative weak * topology on A' (topological dual of A). Thus topologized, $\Delta(A)$ is a metrizable compact space (the compacity

is a classical result, see, e.g., [24: p. 71], and the metrizable follows by property (HA)₁). We denote by \mathcal{G} the Gelfand transformation on A , that is the mapping $\mathcal{G} : A \rightarrow \mathcal{C}(\Delta(A))$ such that if $u \in A$, then $\mathcal{G}(u)(s) = \langle s, u \rangle$ for $s \in \Delta(A)$, where the brackets denote the duality between A' and A . As is classically known (see, e.g., [24: p. 277] and [15: p. 482]), \mathcal{G} is an isometric isomorphism of the C^* -algebra A onto the C^* -algebra $\mathcal{C}(\Delta(A))$.

The appropriate Radon measure on $\Delta(A)$ will be the so-called M -measure for A , denoted below by β .

Proposition 2.1. *There exists a unique Radon measure β on $\Delta(A)$ such that*

$$M(u) = \int_{\Delta(A)} \mathcal{G}(u)(s) d\beta(s) \tag{2.2}$$

for all $u \in A$. Furthermore, β is positive and of total mass 1.

Proof. The mapping $\varphi \rightarrow M(\mathcal{G}^{-1}(\varphi)), \varphi \in \mathcal{C}(\Delta(A))$, is a continuous linear form on $\mathcal{C}(\Delta(A))$ and so there is a Radon measure β on $\Delta(A)$ satisfying (2.2) for $u \in A$. Furthermore, β is positive (since M is positive and \mathcal{G}^{-1} is order preserving) and of total mass 1. The unicity of β being evident, the proof is complete ■

There is no serious difficulty in proving the next useful

Proposition 2.2. *Let $0 < p < +\infty$. For $u \in A$, we have $|u|^p \in A$ with*

$$\mathcal{G}(|u|^p) = |\mathcal{G}(u)|^p \quad \text{and} \quad M(|u|^p) = \int_{\Delta(A)} |\mathcal{G}(u)(s)|^p d\beta(s).$$

We present next a few examples of H -algebras.

Example 2.1. Let S be a network in \mathbb{R}^N , i.e. S is a discrete subgroup of \mathbb{R}^N of rank N (such an S is still called a *réseau*, see [16: p. 111] and [9: Chapter VII/§1]). Let $P_S(\mathbb{R}_y^N) = \{\Psi \in \mathcal{C}(\mathbb{R}_y^N) : \Psi \text{ is } S\text{-periodic}\}$, where by Ψ to be S -periodic we mean that for each $k \in S$ we have $\Psi(y+k) = \Psi(y)$ for all $y \in \mathbb{R}^N$. It is easy to check that $P_S(\mathbb{R}_y^N)$ is an H -algebra on \mathbb{R}_y^N . Here we have $M(u) = \frac{1}{\text{meas}(Y)} \int_Y u(y) dy$ ($u \in P_S(\mathbb{R}_y^N)$), where Y is a closed parallelepiped in \mathbb{R}_y^N centered at the origin of \mathbb{R}^N (see [27]).

Remark 2.1. It is customary to say “ u is Y -periodic” in place of “ u is S -periodic”. But that is a detail.

Example 2.2. Let $AP(\mathbb{R}_y^N)$ be the space of all almost periodic continuous complex functions on \mathbb{R}_y^N (see [19: Chapter 5] and [24: Chapter 10]). Let \mathcal{R} be a countable subgroup of \mathbb{R}^N (being not necessarily a discrete subgroup of \mathbb{R}^N). Let $AP_{\mathcal{R}}(\mathbb{R}_y^N) = \{\psi \in AP(\mathbb{R}_y^N) : Sp(\psi) \subset \mathcal{R}\}$ with $Sp(\psi) = \{k \in \mathbb{R}^N : m(\bar{\gamma}_k \psi) \neq 0\}$, where $\gamma_k(y) = \exp(2i\pi k \cdot y)$, m is the mean value for $AP(\mathbb{R}^N)$.

The space $AP_{\mathcal{R}}(\mathbb{R}_y^N)$ is an H -algebra on \mathbb{R}_y^N (see [27]). In the particular case when \mathcal{R} is a network in \mathbb{R}^N , we have $AP_{\mathcal{R}}(\mathbb{R}_y^N) = P_{\mathcal{R}^*}(\mathbb{R}_y^N)$ (Example 2.1), where $\mathcal{R}^* = \{k \in \mathbb{R}^N : k \cdot y \in \mathbb{Z} \text{ for any } y \in \mathcal{R}\}$ [9: Chapter VIII/p. 7].

Remark 2.2. We have $AP(\mathbb{R}^N) \subset \Pi^\infty$ with $M(u) = m(u)$ for $u \in AP(\mathbb{R}^N)$ (see [28]).

Example 2.3. Let $\mathcal{B}_\infty(\mathbb{R}_y^N)$ be the space of all $u \in \mathcal{C}(\mathbb{R}_y^N)$ with $\lim_{|y| \rightarrow \infty} u(y) = \zeta \in \mathbb{C}$ (ζ depending on u), where $|y|$ denotes the Euclidean norm of y in \mathbb{R}^N . This is an H -algebra. Indeed, properties (HA)₂ and (HA)₃ are evident, property (HA)₁ follows by classical arguments (see [9: Chapter X/p.25] and [9: Chapter IX/p.18]) and property (HA)₄ follows by [28: Proposition 3.3 and Theorem 4.2].

Example 2.4. Let \mathcal{R} be as in Example 2.2. We define $\mathcal{B}_{\infty, \mathcal{R}}(\mathbb{R}_y^N)$ to be the closure in $\mathcal{B}(\mathbb{R}_y^N)$ of the space of all finite sums $\sum_{finite} \varphi_i u_i$ with $\varphi_i \in \mathcal{B}_\infty(\mathbb{R}_y^N)$ and $u_i \in AP_{\mathcal{R}}(\mathbb{R}_y^N)$. The space $\mathcal{B}_{\infty, \mathcal{R}}(\mathbb{R}_y^N)$ is an H -algebra on \mathbb{R}_y^N (use [28: Example 3.4 and Theorem 4.2]).

Remark 2.3. $\mathcal{B}_{\infty, \mathcal{R}}(\mathbb{R}_y^N)$ coincides with the closure in $\mathcal{B}(\mathbb{R}_y^N)$ of the space of all finite sums $\sum_{k \in F} \varphi_k \gamma_k$ with $\varphi_k \in \mathcal{B}_\infty(\mathbb{R}_y^N)$, γ_k defined in Example 2.2 and F ranging over the finite subsets of \mathcal{R} .

2.3 Basic spaces attached to an H -algebra. In this subsection we present two basic spaces associated to a given H -algebra. In what follows, A denotes an H -algebra on \mathbb{R}_y^N .

The space $\mathfrak{X}_A^p(\mathbb{R}_y^N)$. Let $1 \leq p < +\infty$. We first introduce the space Ξ^p of all $u \in L_{loc}^p(\mathbb{R}_y^N)$ for which the sequence $(u^\varepsilon)_{0 < \varepsilon \leq 1}$ is bounded in $L_{loc}^p(\mathbb{R}_x^N)$ with u^ε given by (2.1). This is clearly a vector subspace of $L_{loc}^p(\mathbb{R}_y^N)$. By letting

$$\|u\|_{\Xi^p} = \sup_{0 < \varepsilon \leq 1} \left(\int_{B_N} |u(\frac{x}{\varepsilon})|^p dx \right)^{1/p} \quad (u \in \Xi^p)$$

where B_N is the open unit ball of \mathbb{R}_x^N , we define a norm under which Ξ^p is a Banach space. This being so, we define $\mathfrak{X}_A^p(\mathbb{R}_y^N)$ (or simply \mathfrak{X}_A^p , or even \mathfrak{X}^p when there is no danger of confusion) to be the closure of A in Ξ^p . We provide \mathfrak{X}_A^p with the Ξ^p -norm, which makes it a Banach space.

The following propositions and corollaries can be proved exactly as in [27] (see [27: Proposition 2.7, Theorem 2.1 and its corollaries]).

Proposition 2.3. *The mean value M , viewed as defined on A , extends by continuity to a (unique) continuous linear form (still denoted by M) on \mathfrak{X}_A^p . Furthermore, given $u \in \mathfrak{X}_A^p$ and a fixed bounded open set Ω in \mathbb{R}_x^N , we*

have $u^\varepsilon \rightarrow M(u)$ in $L^p(\Omega)$ weak as $\varepsilon \rightarrow 0$, where u^ε is considered as defined on Ω .

Proposition 2.4. *The Gelfand transformation $\mathcal{G} : A \rightarrow \mathcal{C}(\Delta(A))$ extends by continuity to a (unique) continuous linear mapping, still denoted by \mathcal{G} , of \mathfrak{X}_A^p into $L^p(\Delta(A))$.*

The mapping $\mathcal{G} : \mathfrak{X}_A^p \rightarrow L^p(\Delta(A))$ derived from Proposition 2.4 is referred to as the *canonical mapping* of \mathfrak{X}_A^p into $L^p(\Delta(A))$. It is important to note that (2.2) holds for $u \in \mathfrak{X}_A^p$.

Proposition 2.4 has the following two important corollaries (proceed as in [27]).

Corollary 2.1. *Let $1 < p, q < +\infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1$. If $u \in \mathfrak{X}^p = \mathfrak{X}_A^p$ and $v \in \mathfrak{X}^q$, then $uv \in \mathfrak{X}^r$ and $\mathcal{G}(uv) = \mathcal{G}(u)\mathcal{G}(v)$.*

Corollary 2.2. *The following assertions are true:*

- (i) *If $u \in \mathfrak{X}^p$, then $\bar{u} \in \mathfrak{X}^p$ and $\mathcal{G}(\bar{u}) = \overline{\mathcal{G}(u)}$.*
- (ii) *If $u \in \mathfrak{X}^p$, then $|u|^p \in \mathfrak{X}^1$ and $\mathcal{G}(|u|^p) = |\mathcal{G}(u)|^p$.*
- (iii) *If $\Psi \in A$ and $u \in \mathfrak{X}^p$, then $\Psi u \in \mathfrak{X}^p$ and $\mathcal{G}(\Psi)\mathcal{G}(u) = \mathcal{G}(\Psi u)$.*
- (iv) *If $u \in \mathfrak{X}^1$ and $u \geq 0$ a.e., then $\mathcal{G}(u) \geq 0$ a.e.*
- (v) *If $u \in \mathfrak{X}^1 \cap L^\infty$, then $\mathcal{G}(u) \in L^\infty(\Delta(A))$ and $\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \leq \|u\|_{L^\infty}$.*

The space $H_{\#}^1(\Delta(A))$. The aim here is to construct a Sobolev-type space generalizing the space

$$H_{\#}^1(Y) = \left\{ w \in H^1(Y) : w \text{ is } Y\text{-periodic and } \int_Y w(y) dy = 0 \right\}$$

(Y as in Example 2.1) of the periodic homogenization theory. First, given an integer $m \geq 1$, let

$$A^m = \left\{ \Psi \in A \cap C^m(\mathbb{R}_y^N) : D_y^\alpha \Psi \in A \text{ for } |\alpha| \leq m \right\}$$

where $D_y^\alpha \Psi = \frac{\partial^{|\alpha|} \Psi}{\partial y_1^{\alpha_1} \dots \partial y_N^{\alpha_N}}$. Endowed with the norm $\|\Psi\|_m = \sup_{|\alpha| \leq m} \|D_y^\alpha \Psi\|_\infty$, A^m is a Banach space. Furthermore, let $A^\infty = \bigcap_{m \geq 1} A^m$. We provide A^∞ with the locally convex topology defined by the family of norms $\|\cdot\|_m$ ($m \geq 1$), which makes it a Fréchet space (to show this is a routine exercise). Finally, we put

$$\begin{aligned} \mathcal{D}(\Delta(A)) &= \left\{ \varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^\infty \right\} \\ \mathcal{D}^m(\Delta(A)) &= \left\{ \varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^m \right\} \quad (1 \leq m \in \mathbb{N}). \end{aligned}$$

Definition 2.3. The mapping $\partial_i = \mathcal{G} \circ \frac{\partial}{\partial y_i} \circ \mathcal{G}^{-1}$ (usual composition) of $\mathcal{D}^1(\Delta(A))$ into $\mathcal{C}(\Delta(A))$ is called the *partial derivative of index i* ($1 \leq i \leq N$) on $\Delta(A)$. The function $\partial_i \varphi \in \mathcal{C}(\Delta(A))$ is called the *partial derivative of index i of $\varphi \in \mathcal{D}^1(\Delta(A))$* .

More generally, $\partial^\alpha = \mathcal{G} \circ D_y^\alpha \circ \mathcal{G}^{-1}$ is the partial derivative of index $\alpha \in \mathbb{N}^N$ on $\Delta(A)$. On providing $\mathcal{D}^m(\Delta(A))$ ($m \geq 1$) with the norm $\|\varphi\|_m = \sup_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_\infty$ and $\mathcal{D}(\Delta(A))$ with the locally convex topology defined by the family of norms $\|\cdot\|_m$ ($m \geq 1$), we easily see that $\mathcal{D}^m(\Delta(A))$ is a Banach space and $\mathcal{D}(\Delta(A))$ is a Fréchet space and that, furthermore, \mathcal{G} considered as defined on A^m is an isometric isomorphism of A^m onto $\mathcal{D}^m(\Delta(A))$. The topological dual of $\mathcal{D}(\Delta(A))$, denoted by $\mathcal{D}'(\Delta(A))$, is endowed with the strong dual topology. By a distribution on $\Delta(A)$ we shall mean an element of $\mathcal{D}'(\Delta(A))$. The derivative of index i ($1 \leq i \leq N$) of $T \in \mathcal{D}'(\Delta(A))$ is defined to be the distribution $\partial_i T$ on $\Delta(A)$ given by $\langle \partial_i T, \varphi \rangle = -\langle T, \partial_i \varphi \rangle$ ($\varphi \in \mathcal{D}(\Delta(A))$). The transformation $T \rightarrow \partial_i T$ maps continuously and linearly $\mathcal{D}'(\Delta(A))$ into itself.

Before we proceed any further, let us point out the following fundamental

Proposition 2.5. *Let $\varphi \in \mathcal{D}^m(\Delta(A))$ ($m \geq 1$). For any multi-index α with $1 \leq |\alpha| \leq m$, $\int_{\Delta(A)} \partial^\alpha \varphi(s) d\beta(s) = 0$.*

Proof. The proof in [27: Proposition 4.2] carries over mutatis mutandis to the present general context ■

In the sequel we assume that A^∞ is dense in A (this amounts to saying that $\mathcal{D}(\Delta(A))$ is dense in $\mathcal{C}(\Delta(A))$), as will always be the case in practice. Then we see immediately that we may identify a function $u \in L^1(\Delta(A))$ with the distribution

$$T_u \in \mathcal{D}'(\Delta(A)), \quad \langle T_u, \varphi \rangle = \int_{\Delta(A)} u(s) \varphi(s) d\beta(s) \quad (\varphi \in \mathcal{D}(\Delta(A))).$$

Hence $L^p(\Delta(A)) \subset \mathcal{D}'(\Delta(A))$ with continuous embedding, $1 \leq p \leq +\infty$. Consequently, we may define

$$H^1(\Delta(A)) = \left\{ u \in L^2(\Delta(A)) : \partial_i u \in L^2(\Delta(A)) \quad (1 \leq i \leq N) \right\}.$$

This is a Hilbert space with the norm

$$\|u\|_{H^1(\Delta(A))} = \left(\|u\|_{L^2(\Delta(A))}^2 + \sum_{i=1}^N \|\partial_i u\|_{L^2(\Delta(A))}^2 \right)^{1/2}.$$

However, in practice, instead of $H^1(\Delta(A))$ we will consider its closed subspace

$$H^1(\Delta(A))/\mathbb{C} = \left\{ u \in H^1(\Delta(A)) \mid \int_{\Delta(A)} u \, d\beta = 0 \right\}$$

equipped with the seminorm

$$\|u\|_{H^1(\Delta(A))/\mathbb{C}} = \left(\sum_{i=1}^N \|\partial_i u\|_{L^2(\Delta(A))}^2 \right)^{1/2}.$$

So topologized, $H^1(\Delta(A))/\mathbb{C}$ is a pre-Hilbert space which is in general non-separated and non-complete (see [27: Proposition 4.4]).

Definition 2.4. We define $H^1_{\#}(\Delta(A))$ as separated completion of $H^1(\Delta(A))/\mathbb{C}$, and J to be the canonical mapping of $H^1(\Delta(A))/\mathbb{C}$ into $H^1_{\#}(\Delta(A))$ (see [10: Chapter II, §3, n°7], [11: Chapter I, §1, n°4] and [17: pp. 61 - 62]).

Remark 2.4. $H^1_{\#}(\Delta(A))$ is a Hilbert space and the following classical properties hold:

- 1) J is linear
- 2) $J(H^1(\Delta(A))/\mathbb{C})$ is dense in $H^1_{\#}(\Delta(A))$
- 3) $\|J(u)\|_{H^1_{\#}(\Delta(A))} = \|u\|_{H^1(\Delta(A))/\mathbb{C}}$ ($u \in H^1(\Delta(A))/\mathbb{C}$)
- 4) If F is a Banach space and if l is a continuous linear mapping of $H^1(\Delta(A))/\mathbb{C}$ into F , then there exists a unique continuous linear mapping L of $H^1_{\#}(\Delta(A))$ into F such that $l = L \circ J$.

By this remark we get at once the next

Proposition 2.6. For any given index i ($1 \leq i \leq N$), let the distribution derivative ∂_i be viewed as a mapping of $H^1(\Delta(A))/\mathbb{C}$ into $L^2(\Delta(A))$. Then there exists a unique continuous linear mapping, still denoted by ∂_i , of $H^1_{\#}(\Delta(A))$ into $L^2(\Delta(A))$ such that $\partial_i J(v) = \partial_i v$ for $v \in H^1(\Delta(A))/\mathbb{C}$. Furthermore,

$$\|u\|_{H^1_{\#}(\Delta(A))} = \left(\sum_{i=1}^N \|\partial_i u\|_{L^2(\Delta(A))}^2 \right)^{1/2}$$

for $u \in H^1_{\#}(\Delta(A))$.

3. Homogenization structures

3.1 Definitions and connection with H -algebras. We start with the following preliminary notion.

Definition 3.1. By a *structural representation* on \mathbb{R}_y^N is meant any subset Γ of $\mathcal{B}(\mathbb{R}_y^N)$ with the following properties:

- (HS)₁ Γ is a group under multiplication in $\mathcal{B}(\mathbb{R}_y^N)$.
- (HS)₂ Γ is countable.
- (HS)₃ If $\gamma \in \Gamma$, then $\bar{\gamma} \in \Gamma$.
- (HS)₄ $\Gamma \subset \Pi^\infty$.

Now, in the collection of all structural representations on \mathbb{R}_y^N we consider the binary relation \sim defined by

$$\Gamma \sim \Gamma' \iff CLS(\Gamma) = CLS(\Gamma')$$

where $CLS(\Gamma)$ denotes the closed vector subspace of $\mathcal{B}(\mathbb{R}_y^N)$ spanned by Γ . This is evidently an equivalence relation.

Definition 3.2. Every equivalence class modulo \sim is called a *homogenization structure* on \mathbb{R}_y^N .

For brevity we will sometimes write *H-structure* instead of homogenization structure. If Σ is an *H-structure* on \mathbb{R}_y^N , by a *representation* of Σ is understood any equivalence class representative $\Gamma \in \Sigma$. We then say that Σ is represented by Γ . Reciprocally, for any structural representation Γ on \mathbb{R}_y^N there exists one and only one *H-structure* on \mathbb{R}_y^N of which Γ is a representation.

We denote by \mathcal{HS} the collection of all *H-structures* on \mathbb{R}_y^N and by \mathcal{HA} the collection of all *H-algebras* on \mathbb{R}_y^N . Our next purpose is to establish a bijective correspondence between \mathcal{HS} and \mathcal{HA} . Before we can do this, however, we require the following

Lemma 3.1. *Let $A \in \mathcal{HA}$ and denote by $\text{reg}(A)$ the multiplicative group of all regular (or invertible) elements of A . Then $\text{reg}(A)$ is total in A .*

Proof. Since $\text{reg}[\mathcal{C}(\Delta(A))] = \mathcal{G}(\text{reg}(A))$, we see immediately that the lemma is proved if we can check that $\text{reg}[\mathcal{C}(\Delta(A))]$ is total in $\mathcal{C}(\Delta(A))$. But this follows by the Stone-Weierstrass theorem ■

We turn now to the proof of the claimed result.

Theorem 3.1. *For each $\Sigma \in \mathcal{HS}$, let $\mathcal{J}(\Sigma) = CLS(\Gamma)$, where Γ is a representation of Σ . Then:*

- (i) $\mathcal{J}(\Sigma)$ is an H -algebra on \mathbb{R}_y^N depending only on Σ (and not on the chosen representation Γ of Σ).
- (ii) The mapping $\Sigma \rightarrow \mathcal{J}(\Sigma)$ is a bijection of \mathcal{HS} onto \mathcal{HA} .

Proof. Let $\Sigma \in \mathcal{HS}$. Fix freely any $\Gamma \in \Sigma$. It is an easy task to show that the set $\mathcal{J}(\Sigma) = CLS(\Gamma)$ is an H -algebra on \mathbb{R}_y^N that depends only on Σ and not on the chosen $\Gamma \in \Sigma$. This yields a mapping $\Sigma \rightarrow \mathcal{J}(\Sigma)$ of \mathcal{HS} into \mathcal{HA} which is clearly injective.

Only the surjectivity remains to be shown. Fix freely any $A \in \mathcal{HA}$. Consider $\text{reg}(A)$ (see Lemma 3.1) with the metric $\mu(u, v) = \|u - v\|_\infty$ ($u, v \in \text{reg}(A)$). According to property $(HA)_1$ (Definition 2.2), $\text{reg}(A)$ is separable. So, let \mathcal{R} be a dense countable set in $\text{reg}(A)$, and let $\overline{\mathcal{R}} = \{\bar{u} : u \in \mathcal{R}\} \subset \text{reg}(A)$. Finally, define Γ to be the set of all $\gamma \in \text{reg}(A)$ of the form $\gamma = \varphi_1^{\alpha_1} \varphi_2^{\alpha_2} \cdots \varphi_n^{\alpha_n}$ (where the integer $n \geq 1$ depends on γ) with $\alpha_i \in \mathbb{Z}$ and $\varphi_i \in \mathcal{R} \cup \overline{\mathcal{R}}$ for $1 \leq i \leq n$. It is easily checked that Γ is a structural representation on \mathbb{R}_y^N , that Γ is dense in $\text{reg}(A)$ and thus is total in A , according to Lemma 3.1. Consequently, denoting by Σ the (unique) H -structure on \mathbb{R}_y^N represented by Γ , we have $A = \mathcal{J}(\Sigma)$ and so the surjectivity is established ■

Definition 3.3. The H -algebra $\mathcal{J}(\Sigma)$ is called the *image* of the H -structure Σ on \mathbb{R}_y^N . ■

3.2 Some examples of H -structures. In this subsection we present some fundamental examples of H -structures on \mathbb{R}_y^N .

Example 3.1 (*The trivial H -structure Σ_0*). Let Γ be the set of all constant mappings $\gamma : \mathbb{R}_y^N \rightarrow \mathbb{C}$ assuming values in \mathbb{Q}^* (the non-zero rationals). The set Γ is a structural representation on \mathbb{R}_y^N . By the trivial H -structure on \mathbb{R}_y^N is meant the H -structure Σ_0 on \mathbb{R}_y^N represented by Γ . We have $\mathcal{J}(\Sigma_0) = \mathbb{C}$ (the constant complex functions on \mathbb{R}_y^N).

Example 3.2 (*Periodic H -structures*). Let \mathcal{R} be a network in \mathbb{R}_y^N . Let $\Gamma = \{\gamma_k : k \in \mathcal{R}\}$ where $\gamma_k(y) = \exp(2i\pi k \cdot y)$ ($y \in \mathbb{R}^N$). The set Γ is a structural representation on \mathbb{R}_y^N . The H -structure on \mathbb{R}_y^N represented by Γ is denoted by $\Sigma_{\mathcal{R}}$ and referred to as the periodic H -structure represented by \mathcal{R} . We have $\mathcal{J}(\Sigma_{\mathcal{R}}) = P_{\mathcal{R}^*}(\mathbb{R}_y^N)$ (see Examples 2.1 and 2.2).

Example 3.3 (*Almost periodic H -structures*). Let \mathcal{R} be a countable subgroup of \mathbb{R}_y^N . The set $\Gamma = \{\gamma_k : k \in \mathcal{R}\}$ is a structural representation on \mathbb{R}_y^N . We define $\Sigma_{\mathcal{R}}$ to be the (unique) H -structure on \mathbb{R}_y^N of which Γ is one representation. $\Sigma_{\mathcal{R}}$ is referred to as the *almost periodic H -structure* on \mathbb{R}_y^N represented by \mathcal{R} . We have here $\mathcal{J}(\Sigma_{\mathcal{R}}) = AP_{\mathcal{R}}(\mathbb{R}_y^N)$ (Example 2.2).

Remark 3.1. A periodic H -structure on \mathbb{R}_y^N is none other than an almost periodic H -structure represented by a network in \mathbb{R}_y^N .

Example 3.4 (*The H -structure of the convergence at infinity, Σ_∞*). By this is understood the H -structure Σ_∞ on \mathbb{R}_y^N whose image is the H -algebra $\mathcal{B}_\infty(\mathbb{R}_y^N)$ in Example 2.3 (the existence and uniqueness of Σ_∞ follows by Theorem 3.1).

Example 3.5 (*The H -structures $\Sigma_{\infty, \mathcal{R}}$*). Let \mathcal{R} be as in Example 3.3. We define $\Sigma_{\infty, \mathcal{R}}$ to be the H -structure on \mathbb{R}_y^N whose image is the H -algebra $\mathcal{B}_{\infty, \mathcal{R}}(\mathbb{R}_y^N)$ (Example 2.4).

3.3 Comparison of H -structures. In \mathcal{HS} we consider the binary relation \preceq defined by

$$\Sigma \preceq \Sigma' \iff \mathcal{J}(\Sigma) \subset \mathcal{J}(\Sigma').$$

This is an order in \mathcal{HS} (according to Theorem 3.1, we have $\mathcal{J}(\Sigma) = \mathcal{J}(\Sigma')$ if and only if $\Sigma = \Sigma'$). In the sequel we set (see Subsection 2.3)

$$\mathfrak{X}_\Sigma^p = \mathfrak{X}_{\mathcal{J}(\Sigma)}^p \tag{3.1}$$

for $\Sigma \in \mathcal{HS}$ and $1 \leq p < +\infty$. The following simple comparison results are worth mentioning.

- (1) $\Sigma_0 \preceq \Sigma$ for any $\Sigma \in \mathcal{HS}$.
- (2) If $\Sigma \preceq \Sigma'$, then $\mathfrak{X}_\Sigma^p \subset \mathfrak{X}_{\Sigma'}^p$ ($1 \leq p < +\infty$).
- (3) If \mathcal{R}_1 and \mathcal{R}_2 are two countable subgroups of \mathbb{R}_y^N and if $\Sigma_{\mathcal{R}_i}$ ($i = 1, 2$) is as in Example 3.3, then $\Sigma_{\mathcal{R}_1} \preceq \Sigma_{\mathcal{R}_2}$ amounts to $\mathcal{R}_1 \subset \mathcal{R}_2$.

3.4 Product H -structures. Let $m \in \mathbb{N}$, let $\{N_j\}_{1 \leq j \leq m}$ be a finite family of positive integers, and $N = N_1 + \dots + N_m$, so that

$$\mathbb{R}^N = \mathbb{R}^{N_1} \times \dots \times \mathbb{R}^{N_m}. \tag{3.2}$$

In the sequel M denotes the mean value on \mathbb{R}^{N_j} as well.

Remark 3.2. Let $u_j \in \Pi^\infty(\mathbb{R}^{N_j})$ ($1 \leq j \leq m$). The tensor product $\otimes_{j=1}^m u_j$ lies in $\Pi^\infty(\mathbb{R}^N)$ and further $M(\otimes_{j=1}^m u_j) = \prod_{j=1}^m M(u_j)$.

Now we put

$$\bigodot_{j=1}^m G_j = \{g : g = \otimes_{j=1}^m g_j \ (g_j \in G_j)\} \quad (G_j \subset \mathcal{B}(\mathbb{R}^{N_j})).$$

On the other hand, we recall that $\otimes_{j=1}^m G_j$ denotes the vector subspace of $\mathcal{B}(\mathbb{R}^N)$ spanned by $\odot_{j=1}^m G_j$. Before we can introduce the notion of a product of H -structures, we need the following preliminary

Proposition 3.1. *Consider a family $\{\Sigma_j\}_{1 \leq j \leq m}$, where Σ_j is an H -structure on \mathbb{R}^{N_j} . There exists one and only one H -structure Σ on \mathbb{R}^N with the following property*

$$(P) \begin{cases} \text{If } \Gamma_j \text{ is a representation of } \Sigma_j \text{ (} 1 \leq j \leq m \text{),} \\ \text{then } \Gamma = \odot_{j=1}^m \Gamma_j \text{ is such one of } \Sigma. \end{cases}$$

Proof. Let Γ_j be a representation of Σ_j ($1 \leq j \leq m$). It is an easy exercise to check that $\Gamma = \odot_{j=1}^m \Gamma_j$ is a structural representation on \mathbb{R}^N . Let Σ be the H -structure on \mathbb{R}^N of which Γ is one representation. Let us show that Σ is the claimed H -structure. To this end consider any further family $\{\Gamma'_j\}_{1 \leq j \leq m}$ with $\Gamma'_j \in \Sigma_j$. Clearly, the set $\Gamma' = \odot_{j=1}^m \Gamma'_j$ is a structural representation on \mathbb{R}^N and we have $CLS(\Gamma) = CLS(\Gamma')$, since $CLS(\Gamma'_j) = CLS(\Gamma_j)$ for $1 \leq j \leq m$. Hence Σ satisfies property (P), and it is evident that Σ is the only H -structure on \mathbb{R}^N with that property ■

This leads us to the following

Definition 3.4. The H -structure Σ of Proposition 3.1 is referred to as the *product of the H -structures Σ_j ($1 \leq j \leq m$)*, and is denoted by $\prod_{j=1}^m \Sigma_j = \Sigma_1 \times \cdots \times \Sigma_m$.

The next result will play a fundamental role.

Proposition 3.2. *Let $\{\Sigma_j\}_{1 \leq j \leq m}$, where Σ_j is an H -structure on \mathbb{R}^{N_j} . Then $\otimes_{j=1}^m \mathcal{J}(\Sigma_j)$ is dense in $\mathcal{J}(\prod_{j=1}^m \Sigma_j)$.*

Proof. Let $\Gamma = \odot_{j=1}^m \Gamma_j$, where Γ_j is a representation of Σ_j . Set $\Sigma = \prod_{j=1}^m \Sigma_j$. Clearly, $\langle \Gamma \rangle \subset \otimes_{j=1}^m \mathcal{J}(\Sigma_j) \subset \mathcal{J}(\Sigma)$, where $\langle \Gamma \rangle$ stands for the vector subspace of $\mathcal{B}(\mathbb{R}^N)$ spanned by Γ . But according to Proposition 3.1, $\mathcal{J}(\Sigma) = CLS(\Gamma)$. Therefore, the proposition follows ■

Now, let Σ_j ($1 \leq j \leq m$) be as in Proposition 3.2. Set $\Sigma = \Sigma_1 \times \cdots \times \Sigma_m$, $A_j = \mathcal{J}(\Sigma_j)$ and $A = \mathcal{J}(\Sigma)$. Our main purpose in the sequel is to establish the equality

$$\Delta(A) = \Delta(A_1) \times \cdots \times \Delta(A_m) \tag{3.3}$$

where \times denotes the usual Cartesian product. We need the following

Lemma 3.2. *Let $f_j \in \mathcal{B}(\mathbb{R}^{N_j})$ for $1 \leq j \leq m$. There exists a constant $c > 0$ such that*

$$\left| \prod_{j=1}^m f_j(z_j) - \prod_{j=1}^m f_j(y_j) \right| \leq c \sum_{j=1}^m |f_j(z_j) - f_j(y_j)| \tag{3.4}$$

for all $y_j, z_j \in \mathbb{R}^{N_j}$ ($1 \leq j \leq m$).

Proof. This can be shown by induction on $m \geq 1$. The case $m = 1$ is trivial. Next, assume that $m > 1$, and suppose there is some constant $c(m-1) > 0$ such that

$$\left| \prod_{j=1}^{m-1} f_j(z_j) - \prod_{j=1}^{m-1} f_j(y_j) \right| \leq c(m-1) \sum_{j=1}^{m-1} |f_j(z_j) - f_j(y_j)|$$

for $y_j, z_j \in \mathbb{R}^{N_j}$. If $y_j, z_j \in \mathbb{R}^{N_j}$, then by the equality

$$\begin{aligned} & \prod_{j=1}^m f_j(z_j) - \prod_{j=1}^m f_j(y_j) \\ &= \left[\prod_{j=1}^{m-1} f_j(z_j) - \prod_{j=1}^{m-1} f_j(y_j) \right] f_m(z_m) + [f_m(z_m) - f_m(y_m)] \prod_{j=1}^{m-1} f_j(y_j) \end{aligned}$$

we are quickly led to (3.4) with $c = c(m) > \max(\prod_{j=1}^{m-1} \|f_j\|_\infty, c(m-1)\|f_m\|_\infty)$ ■

At the present time, define

$$\hat{y} = (\delta_{y_1}, \dots, \delta_{y_m}) \quad \text{for } y = (y_1, \dots, y_m) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \dots \times \mathbb{R}^{N_m}$$

where δ_{y_j} is the Dirac measure on \mathbb{R}^{N_j} at y_j , and then set

$$D = \left\{ \hat{y} : y = (y_1, \dots, y_m) \in \mathbb{R}^N \text{ (with (3.2))} \right\}.$$

Of course, $D \subset \prod_{j=1}^m \Delta(A_j)$. In the sequel the compact space $\Delta(A)$ is provided with its natural uniform structure [10: Chapter II, p. 27], and the same is true of each compact space $\Delta(A_j)$ ($1 \leq j \leq m$). The product space $\Delta(A_1) \times \dots \times \Delta(A_m)$ is provided with the corresponding product uniform structure, and D is viewed as a uniform subspace of $\Delta(A_1) \times \dots \times \Delta(A_m)$ (see [10: Chapter II]).

We are now in a position to prove (3.3).

Theorem 3.2. *The mapping $h : D \rightarrow \Delta(A)$ defined by $h(\hat{y}) = \delta_y$ for $y \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \dots \times \mathbb{R}^{N_m}$ is uniformly continuous, and it extends by continuity to a homeomorphism H of $\Delta(A_1) \times \dots \times \Delta(A_m)$ onto $\Delta(A)$.*

Proof. The first point is to show the uniform continuity of h . Let V be a vicinity of the uniform structure of $\Delta(A)$. Of course, we may assume without loss of generality that

$$V = \left\{ (r, s) : r, s \in \Delta(A) \text{ with } |\langle r, \varphi \rangle - \langle s, \varphi \rangle| \leq \alpha \text{ for all } \varphi \in B \right\}$$

where B is a non-empty finite subset of A , $\alpha > 0$, and $\langle \cdot, \cdot \rangle$ denotes the duality between A' and A . This being so, fix freely $\varphi \in B$. In view of Proposition 3.2, we have $\|\varphi - f\|_\infty \leq \frac{\alpha}{4}$ for some suitable $f = \sum_{i=1}^n (\otimes_{j=1}^m u_{ij})$ with $u_{ij} \in A_j$. Hence

$$|\varphi(y) - \varphi(z)| \leq |f(y) - f(z)| + \frac{\alpha}{2}$$

for $y = (y_1, \dots, y_m)$ and $z = (z_1, \dots, z_m)$ in \mathbb{R}^N (with decomposition (3.2)). But

$$\begin{aligned} |f(y) - f(z)| &\leq \sum_{i=1}^n \left| \prod_{j=1}^m u_{ij}(y_j) - \prod_{j=1}^m u_{ij}(z_j) \right| \\ &\leq \sum_{i=1}^n c_i \sum_{j=1}^m |u_{ij}(y_j) - u_{ij}(z_j)| \quad (\text{Lemma 3.2}) \end{aligned}$$

where $c_i > 0$. Letting $c = \max_{1 \leq i \leq n} c_i$, we deduce that

$$|\varphi(y) - \varphi(z)| \leq c \sum_{i=1}^n \sum_{j=1}^m |u_{ij}(y_j) - u_{ij}(z_j)| + \frac{\alpha}{2}$$

for any $y = (y_1, \dots, y_m)$ and $z = (z_1, \dots, z_m)$ as above. Hence

$$|\varphi(y) - \varphi(z)| \leq \alpha \tag{3.5}$$

for $y = (y_1, \dots, y_m)$ and $z = (z_1, \dots, z_m)$ in \mathbb{R}^N (with (3.2)) such that $(\delta_{y_j}, \delta_{z_j}) \in W_\varphi^j$ ($1 \leq j \leq m$) where

$$W_\varphi^j = \left\{ (r, s) : r, s \in \Delta(A_j) \text{ with } |\langle r, u_{ij} \rangle - \langle s, u_{ij} \rangle| \leq \frac{\alpha}{2mnc} \quad (1 \leq i \leq n) \right\}.$$

Now, let pr_j denote the natural projection of $\Delta(A_1) \times \dots \times \Delta(A_m)$ onto $\Delta(A_j)$. Let $q_j = pr_j \times pr_j$, i.e. q_j is the mapping

$$\begin{aligned} q_j &: [\Delta(A_1) \times \dots \times \Delta(A_m)] \times [\Delta(A_1) \times \dots \times \Delta(A_m)] \rightarrow \Delta(A_j) \times \Delta(A_j) \\ q_j(r, s) &= (pr_j(r), pr_j(s)) \quad \text{for all } r, s \in \Delta(A_1) \times \dots \times \Delta(A_m). \end{aligned}$$

Finally, let

$$W = \bigcap_{j=1}^m q_j^{-1}(W^j) \quad \text{with } W^j = \bigcap_{\varphi \in B} W_\varphi^j.$$

The set W is a vicinity of the uniform structure of $\prod_{j=1}^m \Delta(A_j)$ and we have, thanks to (3.5), $(h(\hat{y}), h(\hat{z})) \in V$ whenever $(\hat{y}, \hat{z}) \in W$. This shows that h is uniformly continuous.

Consequently, since D is dense in $\Delta(A_1) \times \dots \times \Delta(A_m)$ [indeed, it is well known that the range of the mapping $z \rightarrow \delta_z$ of \mathbb{R}^{N_j} into $\Delta(A_j)$ is dense in

$\Delta(A_j)]$, the mapping h extends by continuity to a continuous mapping H of $\Delta(A_1) \times \cdots \times \Delta(A_m)$ into $\Delta(A)$ (see [10: Chapter II, p. 29]). Thus, thanks to a well-known argument, the theorem is proved if we can verify that H is bijective.

We begin by showing the surjectivity. Let $s \in \Delta(A)$. Observing that $h(D)$ is dense in $\Delta(A)$, we see that there exists a sequence $(\zeta_n)_{n \geq 0} \subset \mathbb{R}^N$ (\mathbb{R}^N factorized as in (3.2)) such that $h(\zeta_n) \rightarrow s$ in $\Delta(A)$ as $n \rightarrow \infty$. Furthermore, since $\Delta(A_1) \times \cdots \times \Delta(A_m)$ is compact and metrizable, we can extract a subsequence, still denoted by (ζ_n) for simplicity, such that $\zeta_n \rightarrow r$ in $\Delta(A_1) \times \cdots \times \Delta(A_m)$ as $n \rightarrow \infty$. Hence $H(r) = s$, which shows that H is surjective.

It remains to check that H is injective. First of all, let $u_j \in A_j$ ($1 \leq j \leq m$). If $y = (y_1, \dots, y_m) \in \mathbb{R}^N$ (with (3.2)), then the classical equality $\delta_y = \otimes_{j=1}^m \delta_{y_j}$ shows that

$$\left\langle H(\overset{\circ}{y}), \bigotimes_{j=1}^m u_j \right\rangle = \sum_{j=1}^m \langle \delta_{y_j}, u_j \rangle.$$

By combining the continuity of H with the fact that D is dense in $\Delta(A_1) \times \cdots \times \Delta(A_m)$, we deduce that

$$\left\langle H(s), \bigotimes_{j=1}^m u_j \right\rangle = \prod_{j=1}^m \langle s_j, u_j \rangle \quad \text{for } s = (s_1, \dots, s_m) \in \prod_{j=1}^m \Delta(A_j). \quad (3.6)$$

Having made this point, we now consider $s = (s_1, \dots, s_m)$ and $r = (r_1, \dots, r_m)$ in $\Delta(A_1) \times \cdots \times \Delta(A_m)$ such that $H(s) = H(r)$. Then

$$\prod_{j=1}^m \langle s_j, u_j \rangle = \prod_{j=1}^m \langle r_j, u_j \rangle \quad \text{for any } u_j \in A_j. \quad (3.7)$$

Fix freely an integer i ($1 \leq i \leq m$) and let $\psi \in A_i$. In (3.7) take $u_i = \psi$ and $u_j = 1$ if $j \neq i$. Since $\langle r_j, 1 \rangle = \langle s_j, 1 \rangle = 1$, it follows $\langle s_i, \psi \rangle = \langle r_i, \psi \rangle$. Hence $s = r$. Therefore H is injective and so the proof is complete ■

Thanks to Theorem 3.2, $\Delta(A)$ may be identified (by means of H) with $\Delta(A_1) \times \cdots \times \Delta(A_m)$. Thus, we are justified in setting equality (3.3), and it is worth noting that the use of the said equality will be systematic throughout.

Theorem 3.2, or rather its concrete version (3.3) has the following two corollaries of considerable interest.

Corollary 3.1. *We have $\mathcal{G}(\otimes_{j=1}^m u_j) = \otimes_{j=1}^m \mathcal{G}(u_j)$ for $u_j \in A_j$ where the same \mathcal{G} denotes the Gelfand transformation on A_j as well.*

Proof. This follows directly by (3.6) ■

Corollary 3.2. *Let β and β_j be the M -measures for A and A_j , respectively. Then $\beta = \otimes_{j=1}^m \beta_j$.*

Proof. By combining Corollary 3.1 with Remark 3.2 and use of (2.2) we are immediately led to

$$\int_{\Delta(A)} \left(\bigotimes_{j=1}^m \varphi_j \right) (s) d\beta(s) = \prod_{j=1}^m \int_{\Delta(A_j)} \varphi_j(s_j) d\beta_j(s_j)$$

for $\varphi_j \in \mathcal{C}(\Delta(A_j))$, thereby proving the corollary (see [7: p. 82/Theorem 1]) ■

The notion of a product H -structure will prove to be particularly adapted for tackling homogenization problems (in time) for evolution partial differential equations beyond the classical periodic setting. But this is outside the scope of the present study.

We wish to describe two examples of product H -structures.

Example 3.6 (*Product of almost periodic H -structures*). Let $\Sigma_{\mathcal{R}_j}$ be the almost periodic H -structure on \mathbb{R}^{N_j} represented by a countable subgroup \mathcal{R}_j of \mathbb{R}^{N_j} (see Example 3.3), $1 \leq j \leq m$, and let $\Sigma_{\mathcal{R}_1 \times \dots \times \mathcal{R}_m}$ be the almost periodic H -structure on $\mathbb{R}^N = \mathbb{R}^{N_1} \times \dots \times \mathbb{R}^{N_m}$ represented by the product $\mathcal{R}_1 \times \dots \times \mathcal{R}_m$. Then $\Sigma_{\mathcal{R}_1 \times \dots \times \mathcal{R}_m} = \Sigma_{\mathcal{R}_1} \times \dots \times \Sigma_{\mathcal{R}_m}$.

Example 3.7. We consider the product H -structure $\Sigma = \Sigma_1 \times \Sigma_\infty$ on $\mathbb{R}^{N_1+1} = \mathbb{R}^{N_1} \times \mathbb{R}$, where Σ_1 is an H -structure on \mathbb{R}^{N_1} and Σ_∞ is the H -structure on \mathbb{R} defined as in Example 3.4. Our main purpose is to identify Σ . Let $A_1 = \mathcal{J}(\Sigma_1)$ and introduce the space $\mathcal{B}_\infty(\mathbb{R}; A_1)$ of all $\psi \in \mathcal{B}(\mathbb{R}; A_1)$ such that

$$\lim_{|\tau| \rightarrow +\infty} \|\psi(\tau) - l_1(\psi)\|_\infty = 0$$

where $l_1(\psi) \in A_1$. $\mathcal{B}_\infty(\mathbb{R}; A_1)$ is a Banach space under the $\mathcal{B}(\mathbb{R}^{N_1+1})$ -norm, and the correspondence $\psi \rightarrow l_1(\psi)$ is a continuous linear mapping of $\mathcal{B}_\infty(\mathbb{R}; A_1)$ into A_1 . We will denote by $\mathcal{B}_0(\mathbb{R}; A_1)$ the space of all $\psi \in \mathcal{B}_\infty(\mathbb{R}; A_1)$ with $l_1(\psi) = 0$. Note that $\mathcal{B}_0(\mathbb{R}; A_1)$ coincides with the closure of $A_1 \otimes \mathcal{K}(\mathbb{R})$ in $\mathcal{B}(\mathbb{R}^{N_1+1})$ (see, e.g., [7: pp. 45 - 46]) where $\mathcal{K}(\mathbb{R})$ is the space of all continuous complex functions on \mathbb{R} with compact supports.

Proposition 3.3. *We have $\mathcal{J}(\Sigma) = \mathcal{B}_\infty(\mathbb{R}; A_1)$.*

Proof. We clearly have $A_1 \otimes \mathcal{B}_\infty(\mathbb{R}) \subset \mathcal{B}_\infty(\mathbb{R}; A_1)$. Therefore, since $A_1 \otimes \mathcal{B}_\infty(\mathbb{R})$ is dense in $A = \mathcal{J}(\Sigma)$ (Proposition 3.2), it follows that $\mathcal{J}(\Sigma) \subset \mathcal{B}_\infty(\mathbb{R}; A_1)$. To show the inverse inclusion, observe that each $\psi \in \mathcal{B}_\infty(\mathbb{R}; A_1)$ is (uniquely) expressible in the form

$$\psi = l_1(\psi) \otimes 1 + \psi_0 \quad (\psi_0 \in \mathcal{B}_0(\mathbb{R}; A_1)). \tag{3.8}$$

Hence $\mathcal{B}_\infty(\mathbb{R}; A_1) \subset \mathcal{J}(\Sigma)$, which completes the proof ■

Having identified Σ , let us prove an isomorphism result we will need later. The same \mathcal{G} will denote the Gelfand transformation on $A_1 = \mathcal{J}(\Sigma_1)$, on $A = \mathcal{J}(\Sigma)$ and on $\mathcal{B}_\infty(\mathbb{R})$, as well.

Proposition 3.4. *The mapping*

$$L_1 : \mathcal{C}(\Delta(A)) \rightarrow \mathcal{C}(\Delta(A_1)), \quad L_1(\mathcal{G}(\psi)) = \mathcal{G}(\ell_1(\psi)) \quad (\psi \in A)$$

extends by continuity to an isometric isomorphism, still denoted by L_1 , of $L^2(\Delta(A))$ onto $L^2(\Delta(A_1))$.

Proof. We begin by observing that

$$\|L_1(\widehat{\psi})\|_{L^2(\Delta(A_1))} = \|\widehat{\psi}\|_{L^2(\Delta(A))} \quad (\psi \in A) \quad (3.9)$$

where $\widehat{\psi} = \mathcal{G}(\psi)$. Indeed, given $\psi \in A$, from (3.8) we have

$$M(|\psi|^2) = M(|\ell_1(\psi)|^2 \otimes 1) + 2\operatorname{Re} M([\ell_1(\psi) \otimes 1]\overline{\psi}_0) + M(|\psi_0|^2)$$

where M denotes the mean value on \mathbb{R}^{N_1+1} , \mathbb{R}^{N_1} and \mathbb{R} , as well. But the last two terms on the right reduce to zero because M vanishes on $\mathcal{B}_0(\mathbb{R}; A_1)$, and on the other hand we have $M(|\ell_1(\psi)|^2 \otimes 1) = M(|\ell_1(\psi)|^2)$ (see Remark 3.2). Therefore $M(|\psi|^2) = M(|\ell_1(\psi)|^2)$. Hence (3.9) follows, according to (2.2). Consequently, thanks to the density of $\mathcal{C}(\Delta(A))$ in $L^2(\Delta(A))$, the mapping L_1 extends by continuity to an isometry, still called L_1 , of $L^2(\Delta(A))$ into $L^2(\Delta(A_1))$.

Thus, it only remains to check that this isometry is surjective. From the obvious equality $\ell_1(f \otimes 1) = \ell_1(f)$ ($f \in A_1$) we see that

$$\mathcal{C}(\Delta(A_1)) = L_1[\mathcal{G}(A_1 \otimes \mathbb{C})] \subset L_1[L^2(\Delta(A))] \subset L^2(\Delta(A_1)).$$

We deduce that $L_1[L^2(\Delta(A))]$ is dense in $L^2(\Delta(A_1))$. Therefore $L_1[L^2(\Delta(A))] = L^2(\Delta(A_1))$, since the space on the left is closed in that of the right (recall that L_1 is an isometry) ■

3.5 Summable families of H -structures. Let $\{\Sigma_i\}_{1 \leq i \leq m}$ be a finite family of H -structures on \mathbb{R}^N . For each i , we set $A_i = \mathcal{J}(\Sigma_i)$ and we denote by $A_1 + \dots + A_m$ the space of all finite sums $\sum_{i=1}^m \psi_i$ with $\psi_i \in A_i$.

Definition 3.5. The family Σ_i ($1 \leq i \leq m$) is said to be *summable* if the vector space $A_1 + \dots + A_m$ is stable under pointwise multiplication.

The following proposition is obvious and the proof is therefore omitted.

Proposition 3.5. *Suppose the family $\{\Sigma_i\}_{1 \leq i \leq m}$ is summable. Let A be the closure of $A_1 + \dots + A_m$ in $\mathcal{B}(\mathbb{R}^N)$. Then A is an H -algebra on \mathbb{R}^N .*

This leads to the following

Definition 3.6. Suppose the family $\{\Sigma_i\}_{1 \leq i \leq m}$ is summable. Then the H -structure Σ on \mathbb{R}^N whose image is the H -algebra A of Proposition 3.5 (see Theorem 3.1) is called the *sum* of the (summable) family $\{\Sigma_i\}_{1 \leq i \leq m}$ and is denoted by $\Sigma_1 + \dots + \Sigma_m$.

Now we consider a pair $\{\Sigma_1, \Sigma_2\}$ of H -structures on \mathbb{R}^N . In the sequel we use the same \mathcal{G} to denote the Gelfand transformation on $A_1 = \mathcal{J}(\Sigma_1)$ and $A_2 = \mathcal{J}(\Sigma_2)$, as well. Let

$$A_1/\mathbb{C} = \{\psi \in A_1 : M(\psi) = 0\}.$$

We assume the following:

$$A_1/\mathbb{C} \text{ is stable under multiplication} \quad (3.10)$$

$$\text{If } \varphi \in A_1/\mathbb{C} \text{ and } \psi \in A_2, \text{ then } \varphi\psi \in A_1/\mathbb{C} \quad (3.11)$$

$$(A_1/\mathbb{C}) \cap A_2 = \{0\}. \quad (3.12)$$

With (3.12) in mind, we let $\mathcal{V} = (A_1/\mathbb{C}) \oplus A_2$ (direct sum) and we define the mapping $\ell_2 : \mathcal{V} \rightarrow \mathcal{V}$ to be the projection (of \mathcal{V}) on A_2 along A_1/\mathbb{C} . Thus, $A_1/\mathbb{C} = \{\psi \in \mathcal{V} : \ell_2\psi = 0\}$ and $A_2 = \{\psi \in \mathcal{V} : \psi = \ell_2\psi\}$ (range of ℓ_2), and each $\psi \in \mathcal{V}$ is uniquely expressible in the form

$$\psi = \psi_0 + \ell_2\psi \quad \text{with } \psi_0 \in A_1/\mathbb{C}. \quad (3.13)$$

Our goal is to establish the following

Proposition 3.6. *Let (3.10)–(3.12) be satisfied. Then the family $\{\Sigma_1, \Sigma_2\}$ is summable. Furthermore, if we set $A = \mathcal{J}(\Sigma_1 + \Sigma_2)$ and if the Gelfand transformation on A is also denoted by \mathcal{G} , then the mapping*

$$L_2 : \mathcal{G}(\mathcal{V}) \rightarrow \mathcal{C}(\Delta(A_2)), \quad L_2(\mathcal{G}(\psi)) = \mathcal{G}(\ell_2(\psi)) \quad (\psi \in \mathcal{V})$$

extends by continuity to an isometric isomorphism L of $L^2(\Delta(A))$ onto $L^2(\Delta(A_2))$. ■

Proof. According to (3.10) - (3.11), the space \mathcal{V} is stable under multiplication. Hence the summability of the family $\{\Sigma_1, \Sigma_2\}$ follows by the trivial equality $A_1 + A_2 = (A_1/\mathbb{C}) + A_2$. For the rest we proceed as in the proof of Proposition 3.4: Let $\psi \in \mathcal{V}$. Starting from (3.13) and using the fact that the functions $|\psi_0|^2, \overline{\psi_0}\ell_2(\psi)$ and $\psi_0\overline{\ell_2(\psi)}$ lie in A_1/\mathbb{C} (thanks to (3.10) - (3.11)), one arrives at $M(|\psi|^2) = M(|\ell_2(\psi)|^2)$. Therefore $\|L_2(\widehat{\psi})\|_{L^2(\Delta(A_2))} = \|\widehat{\psi}\|_{L^2(\Delta(A))}$ for all $\psi \in \mathcal{V}$ where $\widehat{\psi} = \mathcal{G}(\psi)$. Consequently, noting that $\mathcal{G}(\mathcal{V})$ is dense in $L^2(\Delta(A))$ (since \mathcal{V} is dense in A) we see that L_2 extends by continuity to an isometry, denoted by L , of $L^2(\Delta(A))$ into $L^2(\Delta(A_2))$. Furthermore, by simple arguments similar to those we used in proving Proposition 3.4, it is easily checked that L is surjective. This completes the proof ■

This situation is worth illustrating.

Example 3.8. Let $\Sigma_1 = \Sigma_\infty$ (Example 3.4), and let Σ_2 be an H -structure on \mathbb{R}^N with the following property:

$$0 \leq u \in \mathcal{J}(\Sigma_2) \implies \{M(u) = 0 \Rightarrow u = 0\}. \quad (3.14)$$

Then (3.10) - (3.12) are fulfilled. Indeed, we have $A_1/\mathbb{C} = \mathcal{B}_0(\mathbb{R}^N)$, and it is therefore apparent that (3.10) - (3.11) are satisfied. As regards (3.12), let $\psi \in (A_1/\mathbb{C}) \cap A_2$. Then $|\psi|^2 \in (A_1/\mathbb{C}) \cap A_2$ thanks to (3.11) (use also Proposition 2.2). Hence $M(|\psi|^2) = 0$. Therefore $\psi = 0$, according to (3.14). This shows (3.12).

As a consequence of the preceding example, we have the following

Proposition 3.7. *Let $\Sigma_{\mathcal{R}}$ and Σ_∞ be as in Examples 3.3 – 3.4. The pair $\{\Sigma_\infty, \Sigma_{\mathcal{R}}\}$ is summable and we have $\Sigma_\infty + \Sigma_{\mathcal{R}} = \Sigma_{\infty, \mathcal{R}}$ (see Example 3.5).*

Proof. If a function $0 \leq u \in A_2 = \mathcal{J}(\Sigma_{\mathcal{R}})$ satisfies $M(u) = 0$, since β_2 (the M -measure for A_2) is the Haar measure on $\Delta(A_2)$ (thus, the support of β_2 is exactly $\Delta(A_2)$) [27: Proposition 2.6], then $\mathcal{G}(u)(s) = 0$ for all $s \in \Delta(A_2)$ (use (2.2)). Hence $u = 0$. Therefore, the summability of $\{\Sigma_\infty, \Sigma_{\mathcal{R}}\}$ follows by Example 3.8 with $\Sigma_2 = \Sigma_{\mathcal{R}}$. Furthermore, it is easily seen that $\mathcal{B}_\infty(\mathbb{R}^N) + AP_{\mathcal{R}}(\mathbb{R}^N)$ coincides with the space of all finite sums $\sum \varphi_i u_i$ with $\varphi_i \in \mathcal{B}_\infty(\mathbb{R}^N)$ and $u_i \in AP_{\mathcal{R}}(\mathbb{R}^N)$, hence $\Sigma_\infty + \Sigma_{\mathcal{R}} = \Sigma_{\infty, \mathcal{R}}$ ■

However, it would not be out of interest to exhibit a non-summable family of H -structures.

Proposition 3.8. *Let \mathcal{R}_1 and \mathcal{R}_2 be two countable subgroups of \mathbb{R}^N such that the union $\mathcal{R}_1 \cup \mathcal{R}_2$ is not a subgroup of \mathbb{R}^N . Then the pair $\{\Sigma_{\mathcal{R}_1}, \Sigma_{\mathcal{R}_2}\}$ (see Example 3.3) is not summable.*

Proof. If $k_1 \in \mathcal{R}_1$ and $k_2 \in \mathcal{R}_2$ are such that $k_1 + k_2$ lies outside $\mathcal{R}_1 \cup \mathcal{R}_2$, then $\gamma_{k_1} \gamma_{k_2}$ lies off $AP_{\mathcal{R}_1}(\mathbb{R}^N) + AP_{\mathcal{R}_2}(\mathbb{R}^N)$. The proposition follows ■

4. Σ -convergence

Throughout this section, Ω denotes a bounded open set in \mathbb{R}_x^N .

4.1 Preliminaries. For $u \in \mathcal{C}(\Omega \times \mathbb{R}_y^N)$, and in particular for $u \in \mathcal{C}(\Omega; \mathcal{B}(\mathbb{R}_y^N))$, ■
let

$$u^\varepsilon(x) = u(x, \frac{x}{\varepsilon}) \quad (x \in \Omega) \quad (4.1)$$

where $\varepsilon > 0$. This gives a function $u^\varepsilon \in \mathcal{C}(\Omega)$. Of course, we may also take u in $\mathcal{C}(\overline{\Omega}; \mathcal{B}(\mathbb{R}_y^N))$ ($\overline{\Omega}$ the closure of Ω in \mathbb{R}_x^N). Later on we will have the need to

give meaning to the right side of (4.1) for certain functions $u \in L^1_{loc}(\Omega \times \mathbb{R}_y^N)$. Specifically, we wish to define $u|_{\Delta_\varepsilon}$, i.e. *the trace* of u on $\Delta_\varepsilon = \{(x, y) : y = \frac{x}{\varepsilon} \ (x \in \Omega)\}$, in the following two cases:

- 1) $u \in L^p(\Omega; \mathcal{B}(\mathbb{R}_y^N)) \ (1 \leq p \leq +\infty)$
- 2) $u \in \mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}_y^N))$.

This is a delicate question because the set Δ_ε is negligible in $\mathbb{R}_x^N \times \mathbb{R}_y^N$ for Lebesgue measure (see, e.g., [6: p. 33]). Nevertheless, we have the following two trace results (for further details, including the proofs, see [27] or [29]).

Proposition 4.1. *Let $1 \leq p \leq +\infty$. There exists a linear operator $u \rightarrow u^\varepsilon$ of $L^p(\Omega; \mathcal{B}(\mathbb{R}_y^N))$ into $L^p(\Omega)$ with the following properties :*

- (i) $\|u^\varepsilon\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega; \mathcal{B}(\mathbb{R}_y^N))}$ for all $u \in L^p(\Omega; \mathcal{B}(\mathbb{R}_y^N))$.
- (ii) If $u \in \mathcal{C}(\overline{\Omega}; \mathcal{B}(\mathbb{R}_y^N))$, then u^ε is given by (4.1) (as is customary, $\mathcal{C}(\overline{\Omega}; \mathcal{B}(\mathbb{R}_y^N))$ may be viewed as a subspace of $L^p(\Omega; \mathcal{B}(\mathbb{R}_y^N))$).

Proposition 4.2. *For each u of the form $u = \sum_{i \in I} \varphi_i \otimes u_i$ with $\varphi_i \in \mathcal{C}(\overline{\Omega})$ and $u_i \in L^\infty(\mathbb{R}_y^N)$, I being a finite set, let $u^\varepsilon = \sum_{i \in I} \varphi_i u_i^\varepsilon$ (u_i^ε defined as in (2.1)). This gives a mapping $u \rightarrow u^\varepsilon$ of $\mathcal{C}(\overline{\Omega}) \otimes L^\infty(\mathbb{R}_y^N)$ into $L^\infty(\Omega)$ that extends by continuity to a continuous linear mapping, still denoted by $u \rightarrow u^\varepsilon$, of $\mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}_y^N))$ into $L^\infty(\Omega)$ with the property that $\|u^\varepsilon\|_{L^\infty(\Omega)} \leq \sup_{x \in \overline{\Omega}} \|u(x)\|_{L^\infty(\mathbb{R}_y^N)}$ for all $u \in \mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}_y^N))$.*

Having made this point, let Σ be an H -structure on \mathbb{R}^N , and let $A = \mathcal{J}(\Sigma)$. For $u \in L^1_{loc}(\Omega; A)$, let $\tilde{u}(x) = M(u(x)) \ (x \in \Omega)$, where M denotes the mean value on \mathbb{R}_y^N . This defines a function $\tilde{u} \in L^1_{loc}(\Omega)$, hence a linear transformation $u \rightarrow \tilde{u}$ of $L^1_{loc}(\Omega; A)$ into $L^1_{loc}(\Omega)$ that maps continuously $\mathcal{C}(\overline{\Omega}; A)$ into $\mathcal{B}(\Omega)$ and $L^p(\Omega; A)$ into $L^p(\Omega)$ ($1 \leq p \leq +\infty$). Furthermore, with (4.1) and Proposition 4.1 in mind, we have

Proposition 4.3. *As $\varepsilon \rightarrow 0$, we have $u^\varepsilon \rightarrow \tilde{u}$ in $L^\infty(\Omega)$ -weak * for $u \in \mathcal{C}(\overline{\Omega}; A)$, and $u^\varepsilon \rightarrow \tilde{u}$ in $L^p(\Omega)$ -weak for $u \in L^p(\Omega; A)$ ($1 \leq p < +\infty$).*

Proof. This follows by a simple adaptation of [27: Proofs of Propositions 1.9 and 1.10] ■

For $u \in L^p(\Omega; \mathfrak{X}_\Sigma^p)$ (see (3.1)) with $1 \leq p < +\infty$ we set $\mathcal{G}_\Omega(u) = \mathcal{G} \circ u$ (usual composition), where we recall that \mathcal{G} denotes the Gelfand transformation on A as well as the canonical mapping of \mathfrak{X}^p into $L^p(\Delta(A))$ (Subsection 2.3).

This defines a linear transformation \mathcal{G}_Ω that maps continuously

$$\begin{aligned} L^p(\Omega; \mathfrak{X}_\Sigma^p) &\rightarrow L^p(\Omega \times \Delta(A)) \\ L^p(\Omega; A) &\rightarrow L^p(\Omega \times \Delta(A)) \\ \mathcal{C}(\overline{\Omega}; \mathfrak{X}_\Sigma^p) &\rightarrow \mathcal{C}(\overline{\Omega}; L^p(\Delta(A))) \\ \mathcal{C}(\overline{\Omega}; A) &\rightarrow \mathcal{C}(\overline{\Omega} \times \Delta(A)). \end{aligned}$$

For convenience we will most of the time write

$$\widehat{u} = \mathcal{G}_\Omega(u). \tag{4.2}$$

Observe that we may therefore still put $\widehat{u} = \mathcal{G}(u)$ if $u \in \mathfrak{X}_\Sigma^p$.

Finally, we will need the following definition: By a *fundamental sequence* we will mean any ordinary sequence of real numbers $0 < \varepsilon_n \leq 1$ ($n \in \mathbb{N}$) with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

Remark 4.1. Given $\zeta \in \mathbb{C}$ and a sequence of complex numbers $(\zeta_\varepsilon)_{\varepsilon>0}$, $\zeta_\varepsilon \rightarrow \zeta$ as $\varepsilon \rightarrow 0$ if and only if $\zeta_{\varepsilon_n} \rightarrow \zeta$ as $n \rightarrow \infty$ for any fundamental sequence (ε_n) .

4.2 The Σ -convergence in $L^p(\Omega)$. Let $p \in \mathbb{R}$ with $p \geq 1$, and let $\frac{1}{p'} = 1 - \frac{1}{p}$. Let E be a subset of $\mathbb{R}_+^* = (0, +\infty)$ whose closure in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ contains 0. For example, E may be the whole \mathbb{R}_+^* , or a fundamental sequence. There are many other examples of such an E .

Finally, let Σ be an H -structure on \mathbb{R}_y^N , and let $A = \mathcal{J}(\Sigma)$.

Definition 4.1. A sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega)$ is said to be *weakly Σ -convergent* in $L^p(\Omega)$ if there exists $u_0 \in L^p(\Omega \times \Delta(A))$ such that, as $E \ni \varepsilon \rightarrow 0$, we have

$$\int_\Omega u_\varepsilon(x) f^\varepsilon(x) dx \rightarrow \int \int_{\Omega \times \Delta(A)} u_0(x, s) \widehat{f}(x, s) dx d\beta(s) \tag{4.3}$$

for every $f \in L^{p'}(\Omega; A)$, where $\widehat{f} = \mathcal{G}_\Omega(f)$ (see (4.2)).

We express this by writing $u_\varepsilon \rightarrow u_0$ in $L^p(\Omega)$ -weak Σ , and we refer to u_0 as the *weak Σ -limit* of the sequence $(u_\varepsilon)_{\varepsilon \in E}$ (the unicity of u_0 is evident).

Example 4.1. If $u \in L^p(\Omega; A)$, then according to Proposition 4.3 we have $u^\varepsilon \rightarrow \widehat{u}$ in $L^p(\Omega)$ -weak Σ . This is true in particular for $u \in A$, hence also for $u \in \mathfrak{X}^p = \mathfrak{X}_\Sigma^p$ by density.

Now, we wish to record the main results related to weak Σ -convergence. The proofs of Propositions 4.4 and 4.5 and of Theorem 4.1 below are quite the same as in the case of an almost periodic H -structure [27] and are therefore not worth repeating.

Proposition 4.4. *Suppose a sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega)$ is weakly Σ -convergent in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A))$. Then as $E \ni \varepsilon \rightarrow 0$, $u_\varepsilon \rightarrow \tilde{u}_0$ in $L^p(\Omega)$ -weak where $\tilde{u}_0(x) = \int_{\Delta(A)} u_0(x, s) d\beta(s)$ ($x \in \Omega$).*

Theorem 4.1. *Assume that $1 < p < +\infty$. Then the space $L^p(\Omega)$ is Σ -reflexive in the following sense: Given a fundamental sequence E and a sequence $(u_\varepsilon)_{\varepsilon \in E}$ which is bounded in $L^p(\Omega)$, a subsequence E' can be extracted from E such that the sequence $(u_\varepsilon)_{\varepsilon \in E'}$ is weakly Σ -convergent in $L^p(\Omega)$.*

In the sequel we set $\mathfrak{X}^{r,\infty} = \mathfrak{X}^r \cap L^\infty(\mathbb{R}_y^N)$ ($1 \leq r < +\infty$), where $\mathfrak{X}^r = \mathfrak{X}_\Sigma^r$. We equip $\mathfrak{X}^{r,\infty}$ with the L^∞ -norm, which makes it a Banach space.

Proposition 4.5. *Let $1 < p < +\infty$. Suppose a sequence $u_\varepsilon \in L^p(\Omega)$ ($\varepsilon \in E$) is weakly Σ -convergent in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A))$. Then, as $\varepsilon \rightarrow 0$ ($\varepsilon \in E$), (4.3) holds for $f \in \mathcal{C}(\bar{\Omega}; \mathfrak{X}^{p',\infty})$.*

Combining this with Example 4.1, we quickly deduce the following

Corollary 4.1. *For $u \in \mathcal{C}(\bar{\Omega}; \mathfrak{X}^{p,\infty})$ with $1 < p < +\infty$ the sequence $(u^\varepsilon)_{\varepsilon > 0}$ is weakly Σ -convergent in $L^p(\Omega)$ to \hat{u} .*

Proposition 4.4 points out a close connection between the weak Σ -convergence and usual weak convergence in $L^p(\Omega)$. Besides, observe that the latter is none other than the weak Σ_0 -convergence in $L^p(\Omega)$, Σ_0 being defined in Example 3.1. Theorem 4.1 provides a justification of the concept of weak Σ -convergence. Note that this theorem was already available in the particular case where Σ is a periodic H -structure (see [2, 30]).

Let us turn now to the concept of strong Σ -convergence. Based on the density of $\mathcal{G}_\Omega(L^p(\Omega; A)) = L^p(\Omega; \mathcal{C}(\Delta(A)))$ in $L^p(\Omega \times \Delta(A))$, we can frame the following

Definition 4.2. A sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega)$ is said to be *strongly Σ -convergent* in $L^p(\Omega)$ if there exists $u_0 \in L^p(\Omega \times \Delta(A))$ such that the following holds true:

$$(\text{SSC}) \left\{ \begin{array}{l} \text{Given } \eta > 0 \text{ and } f \in L^p(\Omega; A) \text{ with } \|u_0 - \hat{f}\|_{L^p(\Omega \times \Delta(A))} \leq \frac{\eta}{2} \\ \text{there is some } \alpha > 0 \text{ such that } \|u_\varepsilon - f^\varepsilon\|_{L^p(\Omega)} \leq \eta \text{ provided } E \ni \varepsilon \leq \alpha. \end{array} \right.$$

We express this by writing $u_\varepsilon \rightarrow u_0$ in $L^p(\Omega)$ -strong Σ . The unicity of u_0 is obtained exactly as in [27: Proposition 3.4], and we call u_0 the *strong Σ -limit* of $(u_\varepsilon)_{\varepsilon \in E}$.

Remark 4.2. Suppose $u_0 = \hat{v}_0$ with $v_0 \in L^p(\Omega; A)$. Then $u_\varepsilon \rightarrow u_0$ in $L^p(\Omega)$ -strong Σ if and only if $\|u_\varepsilon - v_0^\varepsilon\|_{L^p(\Omega)} \rightarrow 0$ as $E \ni \varepsilon \rightarrow 0$.

Now by Proposition 4.3 we have $\lim_{\varepsilon \rightarrow 0} \|\psi^\varepsilon\|_{L^p(\Omega)} = \|\hat{\psi}\|_{L^p(\Omega \times \Delta(A))}$ for $\psi \in L^p(\Omega; A)$. Proceeding as in [27: Examples 3.2 and 3.3], we deduce immediately the following two basic examples.

Example 4.2.

(1) For $u \in L^p(\Omega; A)$ we have $u^\varepsilon \rightarrow \widehat{u}$ in $L^p(\Omega)$ -strong Σ as $\varepsilon \rightarrow 0$.

(2) Let $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega)$. If $u_\varepsilon \rightarrow u$ in $L^p(\Omega)$ (strong) as $E \ni \varepsilon \rightarrow 0$, then $u_\varepsilon \rightarrow u$ in $L^p(\Omega)$ -strong Σ .

However, in general a strongly Σ -convergent sequence in $L^p(\Omega)$ is not necessarily convergent in $L^p(\Omega)$ (see [27: Remark 3.4]).

The next proposition provides an illustration of the concept of strong Σ -convergence.

Proposition 4.6. *Suppose a sequence $(u_\varepsilon)_{\varepsilon \in E}$ is strongly Σ -convergent in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A))$. Then, as $E \ni \varepsilon \rightarrow 0$, we have:*

(i) $u_\varepsilon \rightarrow u_0$ in $L^p(\Omega)$ -weak Σ .

(ii) $\|u_\varepsilon\|_{L^p(\Omega)} \rightarrow \|u_0\|_{L^p(\Omega \times \Delta(A))}$.

Reciprocally, if $p = 2$ and assertions (i) - (ii) hold, then $u_\varepsilon \rightarrow u_0$ in $L^p(\Omega)$ -strong Σ .

Proof. The procedure is exactly that which leads to [27: Propositions 3.5 - 3.6] ■

We conclude this subsection with the following

Proposition 4.7. *Suppose the two real numbers $p, q \geq 1$ are such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \leq 1$. Let $u_0 \in L^p(\Omega \times \Delta(A))$ and $v_0 \in L^q(\Omega \times \Delta(A))$, and let $u_\varepsilon \in L^p(\Omega)$ and $v_\varepsilon \in L^q(\Omega)$ for $\varepsilon \in E$. Finally, assume that $u_\varepsilon \rightarrow u_0$ in $L^p(\Omega)$ -strong Σ and $v_\varepsilon \rightarrow v_0$ in $L^q(\Omega)$ -weak Σ . Then $u_\varepsilon v_\varepsilon \rightarrow u_0 v_0$ in $L^r(\Omega)$ -weak Σ .*

Proof. This is a reproduction of the proof of [27: Proposition 3.7] ■

4.3 Remarks. Let us state the following:

(1) Subsections 4.1 - 4.2 are still valid if Ω is not bounded provided some slight modifications are made.

(2) Suppose E is a fundamental sequence and let $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega)$ ($1 < p < +\infty$) with $u_\varepsilon \rightarrow u$ in $L^p(\Omega)$ -weak as $E \ni \varepsilon \rightarrow 0$. Let $f \in L^{p'}(\Omega; A)$. According to Proposition 4.3, $f^\varepsilon \rightarrow \widehat{f}$ in $L^{p'}(\Omega)$ -weak as $\varepsilon \rightarrow 0$. By confining ourselves to the mere resources of classical weak convergence, it is not in general possible to state whether or not the sequence $(u_\varepsilon f^\varepsilon)_{\varepsilon \in E}$ is weakly convergent in $L^1(\Omega)$ when $E \ni \varepsilon \rightarrow 0$. However, by appealing to the concept of weak Σ -convergence we know at least that by Theorem 4.1 we can extract a subsequence E' from E such that, as $E' \ni \varepsilon \rightarrow 0$, $u_\varepsilon f^\varepsilon \rightarrow v$ in $L^1(\Omega)$ -weak, where

$$v(x) = \int_{\Delta(A)} u_0(x, s) \widehat{f}(x, s) dx d\beta(s) \quad (x \in \Omega)$$

with $u_0 \in L^p(\Omega \times \Delta(A))$. The lesson drawn from this is that in general the sequence $(u_\varepsilon f^\varepsilon)_{\varepsilon \in E}$ does not converge weakly in $L^1(\Omega)$. Moreover, in the case this should happen the corresponding limit would not be $u\tilde{f}$. One of the main purposes of the weak Σ -convergence is precisely to supply this deficiency. Indeed, if instead of the preceding weak convergence hypothesis on $(u_\varepsilon)_{\varepsilon \in E}$ we assume that $(u_\varepsilon)_{\varepsilon \in E}$ is weakly Σ -convergent in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A))$, then $u_\varepsilon f^\varepsilon \rightarrow u_0 \hat{f}$ in $L^1(\Omega)$ -weak Σ (use Proposition 4.7 and Example 4.2), hence $u_\varepsilon f^\varepsilon \rightarrow v$ in $L^1(\Omega)$ -weak as $E \ni \varepsilon \rightarrow 0$, where v is as above. On the other hand, as we mentioned in [27: Subsection 1.1], the stiffness of the usual strong convergence in $L^p(\Omega)$ needed to be tempered with the new concept of strong Σ -convergence (see part (2) of Example 4.2 and the subsequent comment). Thus, if we write W and S for usual Weak Convergence and Strong Convergence, respectively, and $W\Sigma$ and $S\Sigma$ for the Weak Σ -Convergence and Strong Σ -Convergence, respectively, then $S \Rightarrow S\Sigma \Rightarrow W\Sigma \Rightarrow W$.

4.4 Proper H -structures. As will be seen later, our capability of applying the theory of H -structures to partial differential equations is based on the forthcoming notion of a proper H -structure. We begin with a few preliminaries. The basic notation and hypotheses are as before, in particular Σ denotes an H -structure on \mathbb{R}^N with $A = \mathcal{J}(\Sigma)$, and β is the M -measure for A .

Definition 4.3. Σ is said to be

- (i) of class C^∞ if A^∞ is dense in A
- (ii) total if $\mathcal{D}(\Delta(A))$ is dense in $H^1(\Delta(A))$.

It is worth recalling that the construction of the space $H^1(\Delta(A))$ in Subsection 2.3 is based on the hypothesis that Σ is of class C^∞ . The following results can easily be established:

- (1) Suppose $\Sigma = \Sigma_1 \times \Sigma_2$, where Σ_i is an H -structure of class C^∞ on \mathbb{R}^{N_i} ($i = 1, 2$) with $N = N_1 + N_2$. Then Σ is of class C^∞ .
- (2) Suppose $\Sigma = \Sigma_1 + \Sigma_2$, where Σ_i is an H -structure of class C^∞ on \mathbb{R}^N . Then Σ is of class C^∞ .
- (3) The H -structures in Examples 3.1 - 3.5 are each of class C^∞ .

Throughout the rest of this subsection Σ is assumed to be of class C^∞ .

Proposition 4.8. *Suppose Σ is total. Then the following assertions are true:*

- (i) $J(\mathcal{D}(\Delta(A))/\mathbb{C})$ is dense in $H^1_{\#}(\Delta(A))$, where $\mathcal{D}(\Delta(A))/\mathbb{C} = \{\varphi \in \mathcal{D}(\Delta(A)) : \int_{\Delta(A)} \varphi d\beta = 0\}$.
- (ii) $\int_{\Delta(A)} \partial_i u d\beta = 0$ ($i = 1, \dots, N$) for $u \in H^1_{\#}(\Delta(A))$.

Proof. Assuming that Σ is total implies immediately that $\mathcal{D}(\Delta(A))/\mathbb{C}$ is dense in the pre-Hilbert space $H^1(\Delta(A))/\mathbb{C}$. Hence assertion (i) follows by points 2) and 3) of Remark 2.4. Combining this with Propositions 2.5 - 2.6 we finally arrive at assertion (ii) ■

We turn now to the next

Definition 4.4. We say the Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$ [1, 25, 26] is Σ -*reflexive* if the following holds: Given a fundamental sequence E and a sequence $(u_\varepsilon)_{\varepsilon \in E}$ which is bounded in $H^1(\Omega)$, a subsequence E' can be extracted from E such that, as $E' \ni \varepsilon \rightarrow 0$, $u_\varepsilon \rightarrow u_0$ in $H^1(\Omega)$ -weak and $\frac{\partial u_\varepsilon}{\partial x_j} \rightarrow \frac{\partial u_0}{\partial x_j} + \partial_j u_1$ in $L^2(\Omega)$ -weak Σ ($1 \leq j \leq N$), where $u_1 \in L^2(\Omega; H^1_\#(\Delta(A)))$.

Remark 4.3. By considering ∂_j as a mapping of $H^1_\#(\Delta(A))$ into $L^2(\Delta(A))$ ■ by Proposition 2.6, $\partial_j \circ u_1 \in L^2(\Omega \times \Delta(A))$ for $u_1 \in L^2(\Omega; H^1_\#(\Delta(A)))$, and the transformation $u_1 \rightarrow \partial_j \circ u_1$ is a continuous linear mapping of $L^2(\Omega; H^1_\#(\Delta(A)))$ into $L^2(\Omega \times \Delta(A))$ (this follows by [7: p. 132/Theorem 4]). We shall set $\partial_j u_1 = \partial_j \circ u_1$.

We are now able to define a so-called proper H -structure.

Definition 4.5. Σ is said to be *proper* if the following conditions are fulfilled :

(PR)₁ Σ is of class C^∞ .

(PR)₂ Σ is total.

(PR)₃ For any bounded open set $\Omega \subset \mathbb{R}^N$, $H^1(\Omega)$ is Σ -reflexive.

Example 4.3. Any almost periodic H -structure is proper (see [27]). Of course, this includes the particular case of periodic H -structures.

Because of the important role the proper H -structures are destined to play in the homogenization of partial differential equations, we wish to prove a fundamental theorem that will later allow us to establish the properness of some specific H -structures. Let us first state the hypotheses.

Let Σ_2 be a further H -structure on \mathbb{R}^N with $A_2 = \mathcal{J}(\Sigma_2)$. We assume that Σ_2 is of class C^∞ . On the other hand, we denote by β_2 the M -measure for A_2 . Now, we assume that hypothesis (H) below is satisfied:

(H) There exist an isometric isomorphism L of $L^2(\Delta(A))$ onto $L^2(\Delta(A_2))$, a dense vector subspace \mathcal{V} of A , a surjective linear mapping $l : \mathcal{V} \rightarrow A_2$, and a vector subspace \mathcal{V}^∞ of $A^\infty \cap \mathcal{V}$ such that the following conditions hold:

$$L(\mathcal{G}(v)) = \mathcal{G}(lv) \quad (v \in \mathcal{V}) \tag{4.4}$$

$$L(\widehat{v}u) = L(\widehat{v})L(u) \quad (v \in \mathcal{V}, u \in L^2(\Delta(A))) \text{ where } \widehat{v} = \mathcal{G}(v) \tag{4.5}$$

$$(v - lv)^\varepsilon \rightarrow 0 \text{ in } L^2_{loc}(\mathbb{R}_x^N) \text{ as } \varepsilon \rightarrow 0 \quad (v \in \mathcal{V}) \quad (4.6)$$

$$\text{If } v \in \mathcal{V}^\infty, \text{ then } D_y^\alpha v \in \mathcal{V} \quad (\alpha \in \mathbb{N}^N) \quad (4.7)$$

$$\text{The restriction of } l \text{ to } \mathcal{V}^\infty \text{ maps } \mathcal{V}^\infty \text{ onto } A_2^\infty \quad (4.8)$$

$$D_y^\alpha(lv) = l(D_y^\alpha v) \quad (v \in \mathcal{V}^\infty, \alpha \in \mathbb{N}^N). \quad (4.9)$$

Under the preceding hypotheses, we wish to prove that if Σ_2 is proper, then so also is Σ . However, before we can do this, some preliminary lemmas are necessary.

Lemma 4.1. *Suppose Σ_2 is total. Then the following assertions are true:*

(i) *If $u \in H^1(\Delta(A))$, then $Lu \in H^1(\Delta(A_2))$ and further*

$$\partial_i(Lu) = L(\partial_i u) \quad (1 \leq i \leq N). \quad (4.10)$$

(ii) *The restriction of L to $H^1(\Delta(A))$ is an isometric isomorphism of $H^1(\Delta(A))$ onto $H^1(\Delta(A_2))$.*

(iii) Σ is total.

Proof. First, it is not difficult to show that

$$\int_{\Delta(A_2)} Lu d\beta_2 = \int_{\Delta(A)} u d\beta \quad (u \in L^2(\Delta(A))). \quad (4.11)$$

With this in mind, let now $u \in H^1(\Delta(A))$. Fix freely $\psi \in A_2^\infty$. Then, in the distribution sense on $\Delta(A_2)$, we have

$$\langle \partial_i Lu, \widehat{\psi} \rangle = - \int_{\Delta(A_2)} Lu \partial_i \widehat{\psi} d\beta_2.$$

But according to (4.8), $\psi = lv$ with $v \in \mathcal{V}^\infty$, and further $\partial_i \widehat{\psi} = L(\partial_i \widehat{v}) = L(\widehat{\frac{\partial v}{\partial y_i}})$ thanks to (4.4), (4.7) and (4.9). Hence

$$\begin{aligned} \langle \partial_i Lu, \widehat{\psi} \rangle &= - \int_{\Delta(A_2)} L(u \partial_i \widehat{v}) d\beta_2 \quad (\text{according to (4.5)}) \\ &= - \int_{\Delta(A)} u \partial_i \widehat{v} d\beta \quad (\text{use (4.11)}) \\ &= \int_{\Delta(A)} \widehat{v} \partial_i u d\beta \quad (\text{since } u \in H^1(\Delta(A))) \\ &= \int_{\Delta(A_2)} L(\widehat{v} \partial_i u) d\beta_2 \quad (\text{use (4.11), once more}) \\ &= \int_{\Delta(A_2)} \widehat{\psi} L(\partial_i u) d\beta_2 \quad (\text{thanks to (4.5), once more}). \end{aligned}$$

Therefore assertion (i) follows by the arbitrariness of ψ . We next show assertion (ii). According to (i), L maps isometrically $H^1(\Delta(A))$ into $H^1(\Delta(A_2))$. Thus, it only remains to establish that $H^1(\Delta(A_2)) = L[H^1(\Delta(A))]$, which reduces to showing that $L[H^1(\Delta(A))]$ is dense in $H^1(\Delta(A_2))$, since L is isometric. But this follows by

$$\mathcal{D}(\Delta(A_2)) = \mathcal{G}(A_2^\infty) = L(\mathcal{G}(\mathcal{V}^\infty)) \subset L[H^1(\Delta(A))] \subset H^1(\Delta(A_2))$$

and use of the fact that Σ_2 is total. This shows assertion (ii). Finally, reference to what precedes reveals that $L(\mathcal{G}(\mathcal{V}^\infty))$ is dense in $H^1(\Delta(A_2))$. Hence assertion (iii) follows by (ii) ■

In the sequel J denotes the canonical mapping of $H^1(\Delta(A))/\mathbb{C}$ into $H^1_\#(\Delta(A))$ ■ whereas J_2 denotes the canonical mapping of $H^1(\Delta(A_2))/\mathbb{C}$ into $H^1_\#(\Delta(A_2))$.

Lemma 4.2. *Suppose Σ_2 is total. Then there exists an isometric isomorphism $L_\# : H^1_\#(\Delta(A)) \rightarrow H^1_\#(\Delta(A_2))$ such that*

$$L_\#(Jf) = J_2(Lf) \quad (f \in H^1(\Delta(A))/\mathbb{C}) \quad (4.12)$$

$$\partial_i L_\#(u) = L(\partial_i u) \quad (u \in H^1_\#(\Delta(A)), 1 \leq i \leq N). \quad (4.13)$$

Proof. According to Lemma 4.1/(i) - (ii), L is an isometric isomorphism of $H^1(\Delta(A))/\mathbb{C}$ onto $H^1(\Delta(A_2))/\mathbb{C}$. With this in mind, observe that

$$\|Lf\|_{H^1(\Delta(A_2))/\mathbb{C}} = \|J_2(Lf)\|_{H^1_\#(\Delta(A_2))} \quad (f \in H^1(\Delta(A))/\mathbb{C}).$$

Therefore, $J_2 \circ L$ is an isometry of $H^1(\Delta(A))/\mathbb{C}$ into $H^1_\#(\Delta(A_2))$. Hence, in view of Remark 2.4/4), there is a unique continuous linear mapping $L_\# : H^1_\#(\Delta(A)) \rightarrow H^1_\#(\Delta(A_2))$ such that (4.12) holds, and it is clear on the other hand that $L_\#$ is an isometric mapping. Furthermore, the surjectivity of $L_\#$ follows by noting that

$$J_2[H^1(\Delta(A_2))/\mathbb{C}] = J_2[L(H^1(\Delta(A))/\mathbb{C})] \subset L_\#[H^1_\#(\Delta(A))] \subset H^1_\#(\Delta(A_2)),$$

and following the same line of argument as in part (ii) of the proof of Lemma 4.1. Thus, only (4.13) remains to be shown. Of course, it suffices to show (4.13) for $u = Jf$ with $f \in H^1(\Delta(A))/\mathbb{C}$. But this results immediately by (4.10) and (4.12) ■

Before we can prove the main result in this subsection, we need one further lemma and a few notations and remarks. To begin with, if $u \in L^2(\Omega; L^2(\Delta(A)))$, we set $(L_\Omega u)(x) = L(u(x)) \quad (x \in \Omega)$ which defines a mapping

$$L_\Omega : L^2(\Omega; L^2(\Delta(A))) \rightarrow L^2(\Omega; L^2(\Delta(A_2)))$$

(use [7: p. 132]). Furthermore, according to hypothesis (H), L_Ω is an isometric isomorphism of $L^2(\Omega; L^2(\Delta(A)))$ onto $L^2(\Omega; L^2(\Delta(A_2)))$. Likewise, provided Σ_2 is total, Lemma 4.2 reveals that the mapping

$$L_{\#\Omega} : L^2(\Omega; H_{\#}^1(\Delta(A))) \rightarrow L^2(\Omega; H_{\#}^1(\Delta(A_2)))$$

defined by $L_{\#\Omega}u = L_{\#} \circ u$ is an isometric isomorphism with the further property

$$\partial_i(L_{\#\Omega}u) = L_\Omega(\partial_i u) \quad (4.14)$$

for $u \in L^2(\Omega; H_{\#}^1(\Delta(A)))$ and $1 \leq i \leq N$.

Finally, for $\psi \in \mathcal{K}(\Omega) \otimes \mathcal{V}$, we set $(l_\Omega \psi)(x) = l(\psi(x))$ ($x \in \Omega$) which gives a function $l_\Omega \psi \in \mathcal{K}(\Omega) \otimes A_2$. By (4.4) and (4.6) we have

$$L_\Omega(\mathcal{G}_\Omega(\psi)) = \mathcal{G}_\Omega(l_\Omega \psi) \quad (4.15)$$

$$(\psi - l_\Omega \psi)^\varepsilon \rightarrow 0 \text{ in } L^2(\Omega) \text{ as } \varepsilon \rightarrow 0. \quad (4.16)$$

Lemma 4.3. *Suppose E is a fundamental sequence, and let $(u_\varepsilon)_{\varepsilon \in E}$ be a sequence in $L^2(\Omega)$ such that*

$$u_\varepsilon \rightarrow v_0 \quad \text{in } L^2(\Omega)\text{-weak } \Sigma_2 \text{ as } \varepsilon \rightarrow 0 \quad (4.17)$$

where $v_0 \in L^2(\Omega \times \Delta(A_2))$. Then $u_\varepsilon \rightarrow L_\Omega^{-1}v_0$ in $L^2(\Omega)$ -weak Σ .

Proof. In view of (4.17), the sequence $(u_\varepsilon)_{\varepsilon \in E}$ is bounded in $L^2(\Omega)$ (use Proposition 4.4). On the other hand, $\mathcal{K}(\Omega) \otimes \mathcal{V}$ is dense in $L^2(\Omega; A)$. Hence, by a routine technique (see, e.g., the proof of [27: Proposition 3.2]) our objective reduces to showing that, as $E \ni \varepsilon \rightarrow 0$, we have

$$\int_\Omega u_\varepsilon \psi^\varepsilon dx \rightarrow \iint_{\Omega \times \Delta(A)} u_0(x, s) \widehat{\psi}(x, s) dx d\beta(s)$$

for all $\psi \in \mathcal{K}(\Omega) \otimes \mathcal{V}$, where $u_0 = L_\Omega^{-1}v_0$. But according to (4.16), it amounts to verifying that, as $E \ni \varepsilon \rightarrow 0$,

$$\int_\Omega u_\varepsilon (l_\Omega \psi)^\varepsilon dx \rightarrow \iint_{\Omega \times \Delta(A)} u_0(x, s) \widehat{\psi}(x, s) dx d\beta(s) \quad (4.18)$$

for $\psi \in \mathcal{K}(\Omega) \otimes \mathcal{V}$. So let ψ be as stated. By (4.17) we have

$$\int_\Omega u_\varepsilon (l_\Omega \psi)^\varepsilon dx \rightarrow \iint_{\Omega \times \Delta(A_2)} v_0(x, s) \mathcal{G}_\Omega(l_\Omega \psi)(x, s) dx d\beta(s)$$

as $E \ni \varepsilon \rightarrow 0$. But by using (4.15), (4.5) and (4.11), we see that

$$\iint_{\Omega \times \Delta(A_2)} v_0(x, s) \mathcal{G}_\Omega(l_\Omega \psi)(x, s) dx d\beta_2(s) = \int_\Omega \int_{\Delta(A)} u_0(x, s) \widehat{\psi}(x, s) d\beta(s) dx.$$

Hence (4.18) follows and so the lemma is proved \blacksquare

We are now able to prove the desired result.

Theorem 4.2. *Suppose Σ_2 is proper. Then Σ is proper.*

Proof. Since Σ is of class C^∞ (by hypothesis) and total (Lemma 4.1), it only remains to show that $H^1(\Omega)$ is Σ -reflexive. So let $(u_\varepsilon)_{\varepsilon \in E}$ be a bounded sequence in $H^1(\Omega)$, E being fundamental. Using the Σ_2 -reflexivity of $H^1(\Omega)$, we get a subsequence E' from E and two functions $u_0 \in H^1(\Omega)$ and $v_1 \in L^2(\Omega, H^1_{\#}(\Delta(A_2)))$ such that, as $E' \ni \varepsilon \rightarrow 0$, $u_\varepsilon \rightarrow u_0$ in $H^1(\Omega)$ -weak and $\frac{\partial u_\varepsilon}{\partial x_j} \rightarrow \frac{\partial u_0}{\partial x_j} + \partial_j v_1$ in $L^2(\Omega)$ -weak Σ_2 ($1 \leq j \leq N$). Furthermore, thanks to Lemma 4.3, $\frac{\partial u_\varepsilon}{\partial x_j} \rightarrow L_\Omega^{-1}(\frac{\partial u_0}{\partial x_j}) + L_\Omega^{-1}(\partial_j v_1)$ in $L^2(\Omega)$ -weak Σ ($1 \leq j \leq N$). But $L_\Omega^{-1}(\frac{\partial u_0}{\partial x_j}) = \frac{\partial u_0}{\partial x_j}$ (because $\frac{\partial u_0}{\partial x_j}$ does not depend on the variable $s \in \Delta(A_2)$) and $L_\Omega^{-1} \partial_j v_1 = \partial_j u_1$ (according to (4.14)), where $u_1 = L_{\# \Omega}^{-1} v_1 \in L^2(\Omega, H^1_{\#}(\Delta(A)))$. Therefore the theorem follows ■

As mentionned above, the real object of Theorem 4.2 is to allow us to establish the properness of some specific H -structures. So, by way of illustration, we have the following three corollaries.

Corollary 4.2. *The H -structure $\Sigma_{\infty, \mathcal{R}}$ on \mathbb{R}^N (Example 3.5) is proper.*

Proof. Let $\Sigma_1 = \Sigma_\infty$, $\Sigma_2 = \Sigma_{\mathcal{R}}$, and $\Sigma = \Sigma_{\infty, \mathcal{R}}$. As usual, we will set $A_i = \mathcal{J}(\Sigma_i)$ ($i = 1, 2$) and $A = \mathcal{J}(\Sigma)$. We recall that $\Sigma = \Sigma_1 + \Sigma_2$ (Proposition 3.7) and, further, Σ is of class C^∞ . Now, let L be the isometric isomorphism constructed in Proposition 3.6, put $\mathcal{V} = (A_1/\mathbb{C}) \oplus A_2$, and let l be the mapping of \mathcal{V} into A_2 defined by $l\psi = l_2\psi$ ($\psi \in \mathcal{V}$), where $l_2 : \mathcal{V} \rightarrow \mathcal{V}$ denotes the projection on A_2 along A_1/\mathbb{C} (see Subsection 3.5). Finally, let $\mathcal{V}^\infty = \mathcal{D}(\mathbb{R}^N) + A_2^\infty$ where $\mathcal{D}(\mathbb{R}^N)$ is the space of all complex functions on \mathbb{R}^N that are of class C^∞ and of compact supports. This is clearly a vector subspace of $A^\infty \cap \mathcal{V}$. On the other hand, it is clear that \mathcal{V} is dense in A , and l is surjective and linear. Thus, the corollary is proved if we can verify that conditions (4.4) - (4.9) are satisfied. Condition (4.4) is immediate by Proposition 3.6, condition (4.6) follows by the fact that $v - lv \in \mathcal{B}_0(\mathbb{R}_y^N) = A_1/\mathbb{C}$ for $v \in \mathcal{V}$ (use (3.13)). As regards condition (4.5), decomposition (3.13) and use of (3.12) reveal that $l(v\psi) = l(v)l(\psi)$ for $v, \psi \in \mathcal{V}$. Hence (4.5) follows by (4.4) and use of the fact that $\mathcal{G}(\mathcal{V})$ is dense in $L^2(\Delta(A))$. Finally, the verification of (4.7) - (4.9) is an easy matter. Therefore, since $\Sigma_{\mathcal{R}}$ is proper (Example 4.3), the corollary follows by Theorem 4.2 ■

Corollary 4.3. *The H -structure Σ_∞ on \mathbb{R}^N (Example 3.4) is proper.*

Proof. Note that $\Sigma_\infty = \Sigma_\infty + \Sigma_0$ with $\Sigma_0 = \Sigma_{\mathcal{R}=\{O\}}$ (O the origin of \mathbb{R}^N) and apply Corollary 4.2 ■

Corollary 4.4. *Let $\Sigma = \Sigma_1 \times \Sigma_\infty$ on $\mathbb{R}^N = \mathbb{R}^{N-1} \times \mathbb{R}$ ($N \geq 2$) where Σ_1 is an H -structure of class C^∞ on \mathbb{R}^{N-1} and Σ_∞ is the H -structure of the*

convergence at infinity on \mathbb{R} . Suppose the product H -structure $\Sigma_2 = \Sigma_1 \times \Sigma_0$ on $\mathbb{R}^N = \mathbb{R}^{N-1} \times \mathbb{R}$ (Σ_0 is the trivial H -structure on \mathbb{R}) is proper. Then Σ is proper, too.

Proof. The H -structures Σ and Σ_2 are of class C^∞ so that, thanks to Theorem 4.2, the corollary is proved once we have shown that the pair $\{\Sigma, \Sigma_2\}$ satisfies hypothesis (H). For this purpose, let $\mathcal{V} = A = \mathcal{J}(\Sigma)$ and $A_2 = \mathcal{J}(\Sigma_2)$, and consider the mapping $l : \mathcal{V} \rightarrow A_2$ given by $l(\psi) = l_1(\psi) \otimes 1$ ($\psi \in \mathcal{V}$) where 1 denotes the identity element of the algebra $\mathcal{B}(\mathbb{R})$, and l_1 is the mapping of $A = \mathcal{B}_\infty(\mathbb{R}; A_1)$ into A_1 defined in Example 3.7. It is clear that l maps continuously \mathcal{V} into A_2 , and that $l(\psi) = \psi$ for $\psi \in A_2$. Next, let L be the mapping of $\mathcal{C}(\Delta(A))$ into $\mathcal{C}(\Delta(A_2))$ such that $L(\mathcal{G}(\psi)) = \mathcal{G}(l(\psi))$ ($\psi \in A$), where \mathcal{G} is the Gelfand transformation on A and A_2 , as well. By using Corollary 3.1 we see that $L(\widehat{\psi}) = L_1(\widehat{\psi}) \otimes 1$ ($\psi \in A$) where 1 is this time the identity element of $\mathcal{C}(\Delta(\mathbb{C}))$ (recall that $\mathcal{J}(\Sigma_0) = \mathbb{C}$) and where L_1 is defined in Proposition 3.4. By the said proposition and use of Corollary 3.2 we deduce that $\|L(\widehat{\psi})\|_{L^2(\Delta(A_2))} = \|\widehat{\psi}\|_{L^2(\Delta(A))}$ for $\psi \in A$. Hence an obvious argument reveals that L extends to an isometric mapping, still called L , of $L^2(\Delta(A))$ into $L^2(\Delta(A_2))$. Furthermore, by the property $L(\widehat{\psi}) = \widehat{\psi}$ ($\psi \in A_2$) we have that

$$\mathcal{C}(\Delta(A_2)) = \mathcal{G}(A_2) \subset L[L^2(\Delta(A))] \subset L^2(\Delta(A_2)).$$

Therefore the same routine argument as used in the proof of Proposition 3.4 reveals that L is an isometric isomorphism of $L^2(\Delta(A))$ onto $L^2(\Delta(A_2))$.

Finally, we set $\mathcal{V}^\infty = A_2^\infty + (A_1^\infty \otimes \mathcal{D}(\mathbb{R}))$, and we note that this is a vector subspace of $A^\infty \cap \mathcal{V} = A^\infty$. Now, it is an easy exercise to check that conditions (4.4) - (4.9) are satisfied and so Σ is proper ■

Example 4.4. Let Σ_∞ be as above, and let $\Sigma_{\mathcal{R}'}$ be the almost periodic H -structure on \mathbb{R}^{N-1} represented by a countable subgroup \mathcal{R}' of \mathbb{R}^{N-1} . Then the H -structure $\Sigma = \Sigma_{\mathcal{R}'} \times \Sigma_\infty$ on \mathbb{R}^N is proper. Indeed, this follows by Corollary 4.4 (with $\Sigma_1 = \Sigma_{\mathcal{R}'}$) owing to the fact that $\Sigma_2 = \Sigma_1 \times \Sigma_0$ is precisely the almost periodic H -structure $\Sigma_{\mathcal{R}' \times \{0\}}$ on \mathbb{R}^N (see Example 4.3).

5. Application to the homogenization of a linear elliptic partial differential equation

5.1 Statement of the abstract model problem. Preliminaries. Let

$$-\sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j} \right) = f \quad \text{in } \Omega, u_\varepsilon \in H_0^1(\Omega) \tag{5.1}$$

where $\varepsilon > 0$, Ω is a fixed bounded open set in \mathbb{R}_x^N , $f \in H^{-1}(\Omega)$, $a_{ij}^\varepsilon(x) = a_{ij}(x, \frac{x}{\varepsilon})$ for $x \in \Omega$ (in the sense of Proposition 4.2) with $a_{ij} \in C(\bar{\Omega}; L^\infty(\mathbb{R}_y^N))$, and with the classical ellipticity condition: there exists $\alpha > 0$ such that for any $x \in \bar{\Omega}$

$$\operatorname{Re} \sum_{i,j=1}^N a_{ij}(x, y) \xi_j \bar{\xi}_i \geq \alpha |\xi|^2 \quad (\xi \in \mathbb{C}^N, \text{ a.e. in } y \in \mathbb{R}^N). \quad (5.2)$$

Under these hypotheses, u_ε (for each fixed $\varepsilon > 0$) is uniquely determined by (5.1) (see [27: Proposition 1.6]). Now, let Σ be a proper H -structure on \mathbb{R}^N . We assume that

$$a_{ij}(x, \cdot) \in \mathfrak{X}_\Sigma^2 \quad \text{for all } x \in \bar{\Omega} \quad (1 \leq i, j \leq N) \quad (5.3)$$

and we wish to investigate under this abstract structure hypothesis the behavior of u_ε when $\varepsilon \rightarrow 0$. For convenience we will require the family $\{a_{ij}\}$ to satisfy the symmetry condition $a_{ji} = \bar{a}_{ij}$ ($1 \leq i, j \leq N$), but this is not essential.

We now collect the basic tools and preliminary results we need. First, as usual, the Hilbert space $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ is considered with the norm

$$\|v\|_{H_0^1(\Omega)} = \left(\sum_{i=1}^N \left\| \frac{\partial v}{\partial x_i} \right\|_{L^2(\Omega)}^2 \right)^{1/2},$$

equivalent to the $H^1(\Omega)$ -norm. Next, we let

$$\mathbb{F}_0^1 = H_0^1(\Omega) \times L^2(\Omega; H_{\#}^1(\Delta(A))) \quad \text{with } A = \mathcal{J}(\Sigma),$$

defining thus a Hilbert space under the norm

$$\|V\|_{\mathbb{F}_0^1} = \left[\|v_0\|_{H_0^1(\Omega)}^2 + \|v_1\|_{L^2(\Omega; H_{\#}^1(\Delta(A)))}^2 \right]^{1/2}, \quad V = (v_0, v_1) \in \mathbb{F}_0^1.$$

Furthermore, by Proposition 4.8/(i),

$$F_0^\infty = \mathcal{D}(\Omega) \times [\mathcal{D}(\Omega) \otimes J(\mathcal{D}(\Delta(A)))/\mathbb{C}] \quad \text{is dense in } \mathbb{F}_0^1. \quad (5.4)$$

By letting

$$\mathbb{D}_i V = \frac{\partial v_0}{\partial x_i} + \partial_i v_1, \quad V = (v_0, v_1) \in \mathbb{F}_0^1, 1 \leq i \leq N$$

and using Proposition 4.8/(ii) we are quickly led to

$$\|V\|_{\mathbb{F}_0^1} = \left[\sum_{i=1}^N \|\mathbb{D}_i V\|_{L^2(\Omega \times \Delta(A))}^2 \right]^{1/2} \quad (V \in \mathbb{F}_0^1).$$

This expression of the \mathbb{F}_0^1 -norm will prove very useful.

Now, according to (5.3), we have $a_{ij} \in \mathcal{C}(\bar{\Omega}, \mathfrak{X}^{2,\infty})$ ($1 \leq i, j \leq N$) where $\mathfrak{X}^{2,\infty} = \mathfrak{X}_\Sigma^2 \cap L^\infty$ is equipped with the L^∞ -norm. Thanks to Corollary 2.2 it follows that $\hat{a}_{ij} = \mathcal{G}_\Omega(a_{ij}) \in \mathcal{C}(\bar{\Omega}, L^\infty \Delta(A))$ with $\hat{a}_{ji} = \overline{\hat{a}_{ij}}$ and

$$\operatorname{Re} \sum_{i,j=1}^N \hat{a}_{ij}(x, s) \xi_j \bar{\xi}_i \geq \alpha |\xi|^2 \quad (\xi \in \mathbb{C}^N) \quad (5.5)$$

for all $x \in \bar{\Omega}$ and almost all $s \in \Delta(A)$ (i.e. given $x \in \bar{\Omega}$, we have (5.5) for any s lying outside some β -negligible set $R_x \subset \Delta(A)$). The preceding ellipticity condition is deduced from (5.2) exactly as in [27: Proposition 5.2].

Consequently, the sesquilinear form $\hat{a}_\Omega(\cdot, \cdot)$ on $\mathbb{F}_0^1 \times \mathbb{F}_0^1$ defined by

$$\hat{a}_\Omega(U, V) = \sum_{i,j=1}^N \int \int_{\Omega \times \Delta(A)} \hat{a}_{ij}(x, s) \mathbb{D}_i U(x, s) \overline{\mathbb{D}_j V(x, s)} \, dx d\beta(s)$$

is Hermitian, continuous, and coercive in the sense that $\operatorname{Re} \hat{a}_\Omega(V, V) \geq \alpha \|V\|_{\mathbb{F}_0^1}^2$ for all $V \in \mathbb{F}_0^1$. We deduce in passing that if L denotes the continuous anti-linear form on \mathbb{F}_0^1 given by $L(V) = \langle f, \bar{v}_0 \rangle$ with $V = (v_0, v_1) \in \mathbb{F}_0^1$, then the variational problem

$$\left. \begin{aligned} U &= (u_0, u_1) \in \mathbb{F}_0^1 \\ \hat{a}_\Omega(U, V) &= L(V) \quad (V \in \mathbb{F}_0^1) \end{aligned} \right\} \quad (5.6)$$

admits a unique solution.

We will need a few basic convergence results. Let us begin by noting that F_0^∞ (in (5.4)) is precisely the space of all Φ of the form

$$\Phi = (\psi_0, J_\Omega(\hat{\psi}_1)) \quad (\psi_0 \in \mathcal{D}(\Omega), \psi_1 \in \mathcal{D}(\Omega) \otimes (A^\infty/\mathbb{C})) \quad (5.7)$$

where $A^\infty/\mathbb{C} = \{\psi \in A^\infty : M(\psi) = 0\}$, $\hat{\psi}_1 = \mathcal{G}_\Omega(\psi_1)$ and $J_\Omega(\hat{\psi}_1) = J \circ \hat{\psi}_1$, $\hat{\psi}_1$ being viewed as a mapping of Ω into $\mathcal{D}(\Delta(A))/\mathbb{C}$.

This being so, let Φ be given by (5.7). For $\varepsilon > 0$, let

$$\Phi_\varepsilon = \psi_0 + \varepsilon \psi_1^\varepsilon, \quad (5.8)$$

i.e. $\Phi_\varepsilon(x) = \psi_0(x) + \varepsilon \psi_1(x, \frac{x}{\varepsilon})$ for $x \in \Omega$. We have $\Phi_\varepsilon \in \mathcal{D}(\Omega)$, and it is an easy exercise to verify that, as $\varepsilon \rightarrow 0$,

$$\Phi_\varepsilon \rightarrow \psi_0 \quad \text{in } H_0^1(\Omega)\text{-weak} \quad (5.9)$$

$$\frac{\partial \Phi_\varepsilon}{\partial x_i} \rightarrow \mathbb{D}_i \Phi = \frac{\partial \psi_0}{\partial x_i} + \partial_i \hat{\psi}_1 \quad \text{in } L^p(\Omega)\text{-strong } \Sigma, 1 \leq i \leq N \quad (5.10)$$

where $1 \leq p < +\infty$. This leads to the following fundamental

Lemma 5.1. *Let $(u_\varepsilon)_{\varepsilon \in E'} \subset H_0^1(\Omega)$ where E' is a fundamental sequence. Let $\Phi \in F_0^\infty$ and $\Phi_\varepsilon \in \mathcal{D}(\Omega)$ with (5.7) – (5.8). Suppose that, as $E' \ni \varepsilon \rightarrow 0$, we have $\frac{\partial u_\varepsilon}{\partial x_j} \rightarrow \mathbb{D}_j U$ in $L^2(\Omega)$ -weak Σ ($1 \leq j \leq N$) where $U \in \mathbb{F}_0^1$. Then $a^\varepsilon(u_\varepsilon, \Phi_\varepsilon) \rightarrow \widehat{a}_\Omega(U, \Phi)$ as $E' \ni \varepsilon \rightarrow 0$.*

Proof. Based on (5.10) and on Propositions 4.5 and 4.7, we see immediately that the lemma follows by the same line of reasoning as in [27: Lemma 5.2] ■

Finally, given $x \in \overline{\Omega}$, let

$$\widehat{a}(x; u, v) = \sum_{i,j=1}^N \int_{\Delta(A)} \widehat{a}_{ij}(x, s) \partial_j u(s) \overline{\partial_i v(s)} d\beta(s)$$

for $u, v \in H_{\#}^1(\Delta(A))$. This defines a coercive (see (5.5)), continuous, Hermitian sesquilinear form $\widehat{a}(x; \cdot)$ on $H_{\#}^1(\Delta(A)) \times H_{\#}^1(\Delta(A))$. Thus, given $1 \leq j \leq N$, to each $x \in \overline{\Omega}$ there is attached a unique $\chi^j(x) \in H_{\#}^1(\Delta(A))$ such that

$$\widehat{a}(x; \chi^j(x), v) = \sum_{k=1}^N \int_{\Delta(A)} \widehat{a}_{kj}(x, s) \overline{\partial_k v(s)} d\beta(s) \quad (5.11)$$

for all $v \in H_{\#}^1(\Delta(A))$. This gives a family of mappings $\chi^j : \overline{\Omega} \rightarrow H_{\#}^1(\Delta(A))$ ($1 \leq j \leq N$), hence a family of functions $q_{ij} : \overline{\Omega} \rightarrow \mathbb{C}$ ($1 \leq i, j \leq N$) with

$$q_{ij}(x) = \int_{\Delta(A)} \widehat{a}_{ij}(x, s) d\beta(s) - \sum_{l=1}^N \int_{\Delta(A)} \widehat{a}_{il}(x, s) \partial_l \chi^j(x, s) d\beta(s) \quad (5.12)$$

for $x \in \overline{\Omega}$, where $s \rightarrow \partial_l \chi^j(x, s)$ (for fixed x) actually denotes the function $\partial_l(\chi^j(x))$.

Lemma 5.2. *The following assertions are true:*

- (i) $\chi^j \in \mathcal{C}(\overline{\Omega}; H_{\#}^1(\Delta(A)))$.
- (ii) $q_{ij} \in \mathcal{C}(\overline{\Omega})$, $q_{ji} = \overline{q_{ij}}$.
- (iii) *There exists a constant $\alpha_0 > 0$ such that $\operatorname{Re} \sum_{i,j=1}^N q_{ij}(x) \xi_j \overline{\xi_i} \geq \alpha_0 |\xi|^2$ ($\xi \in \mathbb{C}^N$) for any $x \in \overline{\Omega}$.*

Proof. This is an exact reproduction of the proof of [27: Lemma 5.3] ■

5.2 Homogenization of the abstract model problem. The first point is to prove the following

Theorem 5.1. *Suppose (5.3) with, moreover, Σ proper. Let $U = (u_0, u_1)$ be (uniquely) defined by (5.6), and for each real $\varepsilon > 0$ let u_ε be the unique solution of (5.1). Then, as $\varepsilon \rightarrow 0$,*

$$u_\varepsilon \rightarrow u_0 \quad \text{in } H_0^1(\Omega)\text{-weak} \tag{5.13}$$

$$u_\varepsilon \rightarrow u_0 \quad \text{in } L^2(\Omega) \tag{5.14}$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \rightarrow \mathbb{D}_j U = \frac{\partial u_0}{\partial x_j} + \partial_j u_1 \quad \text{in } L^2(\Omega)\text{-weak } \Sigma \quad (1 \leq j \leq N). \tag{5.15}$$

Proof. First, for fixed $\varepsilon > 0$, we have

$$\sum_{i,j=1}^N \int_{\Omega} a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j} \frac{\partial \bar{v}}{\partial x_i} dx = \langle f, \bar{v} \rangle \quad (v \in H_0^1(\Omega)).$$

By a routine trick we see immediately that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $H_0^1(\Omega)$. Therefore, given an arbitrary fundamental sequence E , the Σ -reflexivity of $H^1(\Omega)$ guarantees the existence of a subsequence E' from E and of some $U = (u_0, u_1) \in \mathbb{F}_0^1$ such that, as $E' \ni \varepsilon \rightarrow 0$, we have (5.13) - (5.14) (as a direct consequence of (5.13) by reason of the Rellich theorem), and (5.15). Thus, if we succeed in proving that U is the solution of (5.6), then according to the unicity we will be justified in claiming that (5.13) - (5.15) actually hold when $E \ni \varepsilon \rightarrow 0$ ($\varepsilon \in E$ instead of $\varepsilon \in E'$). Hence the theorem will follow by Remark 4.1. To this end, take in the preceding equation (for fixed $\varepsilon \in E'$) the particular test function $v = \Phi_\varepsilon$ with (5.7) - (5.8), then use (5.9) and Lemma 5.1 to see that $\widehat{a}_\Omega(U, \Phi) = L(\Phi)$ holds for all $\Phi \in F_0^\infty$. By (5.4) we deduce that U satisfies (5.6). Hence the theorem follows ■

This leads us to the next point of the present section.

Corollary 5.1. *The following assertions are true:*

(i) *For almost every $x \in \Omega$, $u_1(x)$ is the (unique) solution of the coercive variational problem*

$$\left. \begin{aligned} u_1(x) &\in H_{\#}^1(\Delta(A)) \\ \widehat{a}(x; u_1(x), v) &= - \sum_{j,k=1}^N \frac{\partial u_0}{\partial x_j}(x) \int_{\Delta(A)} \widehat{a}_{kj}(x, s) \overline{\partial_k v}(s) d\beta(s) \\ \forall v &\in H_{\#}^1(\Delta(A)). \end{aligned} \right\} \tag{5.16}$$

(ii) *For almost every $x \in \Omega$, we have*

$$u_1(x) = - \sum_{j=1}^N \frac{\partial u_0}{\partial x_j}(x) \chi^j(x)$$

with $\chi^j(x) \in H_{\#}^1(\Delta(A))$ given by (5.11).

(iii) The function u_0 is the weak solution of

$$-\sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(q_{ij} \frac{\partial u_0}{\partial x_j} \right) = f \quad \text{in } \Omega, u_0 \in H_0^1(\Omega) \quad (5.17)$$

where q_{ij} is given by (5.12) (see also Lemma 5.2).

Proof. Proceed exactly as in the proofs of [27: Propositions 5.4 and 5.5] ■

Remark 5.1. It is not difficult to check that if u_0 is the solution of (5.17), and if a measurable function $u_1 : \Omega \rightarrow H_{\#}^1(\Delta(A))$ satisfies (5.16) for almost every $x \in \Omega$, then $U = (u_0, u_1)$ satisfies (5.6).

5.3 A few concrete examples. Thus, the homogenization of (5.1) under both the properness hypothesis on Σ and the abstract hypothesis (5.3) leads precisely to explicit results that are in every respect similar to those provided by the usual periodic theory (see [2, 5, 30]). Now, with a view to illustrating the wide scope of this general homogenization approach, let us exhibit various concrete examples of structure hypotheses that are reducible to (5.3).

Example 5.1 (Periodicity hypothesis). Suppose there exists a parallelepiped Y in \mathbb{R}_y^N , e.g. $Y = (0, 1)^N$, such that $a_{ij}(x, \cdot)$ is Y -periodic for each fixed $x \in \bar{\Omega}$ ($1 \leq i, j \leq N$). This amounts to saying that there exists a network \mathcal{R} in \mathbb{R}^N such that, given $x \in \bar{\Omega}$, we have $a_{ij}(x, y + k) = a_{ij}(x, y)$ for all $k \in \mathcal{R}$ and for almost all $y \in \mathbb{R}^N$. This leads us immediately to (5.3) with $\Sigma = \Sigma_{\mathcal{R}}$, $\Sigma_{\mathcal{R}}$ being the periodic H -structure on \mathbb{R}^N represented by \mathcal{R} (Example 3.2). It is essential to recall that $\Sigma_{\mathcal{R}}$ is a proper H -structure (see Example 4.3). Also, note that we have here

$$\begin{aligned} \mathfrak{X}_{\Sigma}^2 &= L_{per}^2(Y) \equiv \{w \in L_{loc}^2(\mathbb{R}_y^N) : w \text{ is } Y\text{-periodic}\} = L^2(\Delta(A)) \\ H_{\#}^1(\Delta(A)) &= \{w \in H_{loc}^1(\mathbb{R}_y^N) : w \text{ is } Y\text{-periodic and } \int_Y w \, dy = 0\} \end{aligned}$$

(see [27] for more details), and so we are right in the classical periodic setting.

Example 5.2 (Almost periodicity hypothesis). We assume that

$$a_{ij}(x, \cdot) \in L_{AP}^2(\mathbb{R}_y^N) \quad \text{for all } x \in \bar{\Omega}, 1 \leq i, j \leq N \quad (5.18)$$

where $L_{AP}^2(\mathbb{R}_y^N)$ denotes the space of all functions $w \in L_{loc}^2(\mathbb{R}_y^N)$ that are almost periodic in the sense of Stepanoff [27]. Then (5.3) holds with $\Sigma = \Sigma_{\mathcal{R}}$, where $\Sigma_{\mathcal{R}}$ is the almost periodic H -structure on \mathbb{R}^N represented by a suitable

countable subgroup \mathcal{R} of \mathbb{R}^N (see Examples 3.3 and 4.3). Here also, it should be stressed that the properness of $\Sigma_{\mathcal{R}}$ is a fundamental property.

The present example is worth illustrating:

(1) Suppose there is a family of parallelepipeds Y_{ij} in \mathbb{R}_y^N ($1 \leq i, j \leq N$) such that $a_{ij}(x, \cdot)$ is Y_{ij} -periodic for each $x \in \bar{\Omega}$. Then (5.18) is satisfied.

(2) Suppose that to each $x \in \bar{\Omega}$ there is attached a parallelepiped Y_x in \mathbb{R}^N such that $a_{ij}(x, \cdot)$ is Y_x -periodic ($1 \leq i, j \leq N$). Then (5.18) is satisfied.

Example 5.3. Let $Y' = (0, 1)^{N-1}$ with $N \geq 2$. We suppose here that

$$a_{ij}(x, \cdot) \in \mathcal{B}_{\infty}(\mathbb{R}; L^2_{per}(Y')) \quad \text{for all } x \in \bar{\Omega}, 1 \leq i, j \leq N \quad (5.19)$$

where $L^2_{per}(Y')$ denotes the usual Hilbert space of all functions in $L^2_{loc}(\mathbb{R}^{N-1})$ that are Y' -periodic and $\mathcal{B}_{\infty}(\mathbb{R}; L^2_{per}(Y'))$ denotes the space of all $u \in \mathcal{B}(\mathbb{R}; L^2_{per}(Y'))$ such that $u(y_N)$ converges in $L^2_{per}(Y')$ as $|y_N| \rightarrow \infty$. ■

Proposition 5.1. *Suppose (5.19) holds. Then we have (5.3) with $\Sigma = \Sigma_{\mathcal{R}'} \times \Sigma_{\infty}$ where $\Sigma_{\mathcal{R}'}$ is the periodic H -structure on $\mathbb{R}_{y'}^{N-1}$ (the space \mathbb{R}^{N-1} of the variable $y' = (y_1, \dots, y_{N-1})$) represented by $\mathcal{R}' = \mathbb{Z}^{N-1}$, and Σ_{∞} is the H -structure on \mathbb{R} defined in Example 3.4. Furthermore, Σ is proper.*

Proof. Let $\Sigma = \Sigma_{\mathcal{R}'} \times \Sigma_{\infty}$ be as stated above. According to Proposition 3.3, we have $A = \mathcal{J}(\Sigma) = \mathcal{B}_{\infty}(\mathbb{R}; C_{per}(Y'))$, where $C_{per}(Y') = \mathcal{J}(\Sigma_{\mathcal{R}'})$ is the space of all Y' -periodic continuous complex functions on \mathbb{R}^{N-1} . Hence (5.3) follows by the fact that $\mathcal{B}_{\infty}(\mathbb{R}; C_{per}(Y'))$ is dense in $\mathcal{B}_{\infty}(\mathbb{R}; L^2_{per}(Y'))$ (provided with the $\mathcal{B}(\mathbb{R}; L^2_{per}(Y'))$ -norm) and the latter is continuously embedded into Ξ^2 . Finally, the properness of Σ was established in Example 4.4 ■

Remark 5.2. If in (5.19) we consider $L^2_{AP}(\mathbb{R}^{N-1})$ instead of $L^2_{per}(Y')$, then it can be shown that Proposition 5.1 remains valid, $\Sigma_{\mathcal{R}'}$ being this time a suitable almost periodic H -structure on \mathbb{R}^{N-1} .

Example 5.4. Let (L^2, l^{∞}) be the amalgam of L^2 and l^{∞} on \mathbb{R}^N [18] (see also [27]), i.e. (L^2, l^{∞}) is the space of all $u \in L^2_{loc}(\mathbb{R}^N)$ such that

$$\|u\|_{2,\infty} = \sup_{k \in \mathbb{Z}^N} \left[\int_{k+Y} |u(y)|^2 dy \right]^{1/2} < +\infty \quad (Y = (0, 1)^N). \quad (5.20)$$

This is a Banach space under the norm $\|\cdot\|_{2,\infty}$. This being so, we denote by $L^2_{\infty,per}(Y)$ the closure in (L^2, l^{∞}) of the space of all finite sums

$$\sum_{finite} \varphi_i u_i \quad (\varphi_i \in \mathcal{B}_{\infty}(\mathbb{R}_y^N), u_i \in C_{per}(Y))$$

where $C_{per}(Y)$ is the space of all $u \in \mathcal{C}(\mathbb{R}_y^N)$ such that $u(y+k) = u(y)$ for all $k \in \mathbb{Z}^N$ and all $y \in \mathbb{R}^N$ (such a u is said to be Y -periodic, $Y = (0, 1)^N$). In the present example we assume that

$$a_{ij}(x, \cdot) \in L_{\infty, per}^2(Y) \quad \text{for all } x \in \overline{\Omega}, 1 \leq i, j \leq N. \quad (5.21)$$

Proposition 5.2. *Suppose (5.21). Then (5.3) holds with $\Sigma = \Sigma_{\infty, \mathcal{R}}$ (Example 3.5), where $\mathcal{R} = \mathbb{Z}^N$. Moreover, Σ is proper.*

Proof. $L_{\infty, per}^2(Y)$ is exactly the closure of $\mathcal{J}(\Sigma_{\infty, \mathcal{R}}) = \mathcal{B}_{\infty, \mathcal{R}}(\mathbb{R}^N)$ in (L^2, l^∞) . Therefore, since (L^2, l^∞) is continuously embedded into Ξ^2 (see [27: Lemma 1.3]), (5.3) follows with $\Sigma = \Sigma_{\infty, \mathcal{R}}$ and $\mathcal{R} = \mathbb{Z}^N$, the latter H -structure being proper (Corollary 4.2) ■

As a direct consequence of this, we have the following

Corollary 5.2. *Suppose we have $a_{ij}(x, \cdot) \in L^2(\mathbb{R}_y^N) + L_{per}^2(Y)$ for all $x \in \overline{\Omega}$, $1 \leq i, j \leq N$, where $Y = (0, 1)^N$. Then (5.3) holds with $\Sigma = \Sigma_{\infty, \mathcal{R}}$ as above ■*

Proof. Indeed, both $L^2(\mathbb{R}^N)$ and $L_{per}^2(Y)$ are contained in $L_{\infty, per}^2(Y)$, so that (5.21) is satisfied. Therefore, the corollary follows by Proposition 5.2 ■

Example 5.5. For the sake of convenience we assume here that the coefficients a_{ij} ($1 \leq i, j \leq N$) do not depend on $x \in \overline{\Omega}$. Suppose that for each $k \in \mathbb{Z}^N$ we have $a_{ij}(y) = r_{ij}(k)$ for $y \in k + Y$ (Y as in Example 5.4), where the family $r_{ij} \in l^\infty(\mathbb{Z}^N)$ ($1 \leq i, j \leq N$) is given. In other words, the function a_{ij} is constant in each cell $k + Y$, the corresponding constant being $r_{ij}(k)$. Then it can be checked that (5.3) (where x must be disregarded, of course) is satisfied in each of the following three cases:

- (1) $r_{ij} \in AP(\mathbb{Z}^N)$ [24: p. 323].
- (2) $r_{ij} = c_{ij} + \gamma_{ij}$ with $c_{ij} \in \mathbb{C}$ and $\gamma_{ij} \in l^1(\mathbb{Z}^N)$.
- (3) $r_{ij} \in \mathcal{B}_\infty(\mathbb{Z}^N)$

where $\mathcal{B}_\infty(\mathbb{Z}^N)$ denotes the space of all complex mappings on \mathbb{Z}^N that converge at infinity. These results are established in [27] (for the case (1)) and [31] (for the cases (2) and (3)).

5.4 Concluding remarks. Thus, with the aid of the theory of H -structures, problem (5.1) can be homogenized under various concrete structure hypotheses (such as, e.g., (5.18), (5.19), (5.21), and the hypotheses of Example 5.5) beyond the classical periodic framework. In each case, the operating procedure consists in reducing the concrete homogenization problem to the abstract statement of Subsection 5.1 so as to be able to apply Theorem 5.1. This is a

significant progress in homogenization theory: Not only it is from now on possible to tackle outstanding nonperiodic homogenization problems, but also the results achieved are in every respect similar to those of the periodic homogenization theory, with the same explicitness and the same degree of accuracy. The differential equation (see (5.1)) taken to illustrate our approach was purposely classical and simple. In this connection it is of interest to anticipate that one can obtain results of equal exactness with more complex partial differential equations, for example nonlinear equations and evolution (linear or nonlinear) equations. But that is quite another matter [32, 33].

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