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Expansions of Certain del bar Closed Forms via Fourier-Laplace Transform

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Abstract. We derive Laurent-type expansions of $\overline{\partial}$-closed $(0, n-1)$-forms in certain domains in $\mathbb{C}^n$. These expansions involve the Bochner-Martinelli kernel and its derivatives and are based on the Fourier-Laplace transform.

Keywords: *$\overline{\partial}$-closed $(0, n-1)$-forms, Fourier-Laplace tranform, Laurent-type expansions, derivatives of the Bochner-Martinelli kernel*

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1. Introduction

If D is an open set in $\mathbb{C}^n$ and f is a C^∞-function in D, one sets $\overline{\partial} f$ to be the $(0,1)$-form

$$\overline{\partial} f = \sum \frac{\partial f}{\partial \overline{z}_j}\, d\overline{z}_j$$

where

$$\frac{\partial f}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right) \qquad (z_j = x_j + iy_j;\ x_j, y_j \in \mathbb{R},\ j = 1, \ldots, n)$$

and, in general, if

$$u = \sum f_{j_1 \ldots j_q} d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q}$$

is a $(0,q)$-form with C^∞-coefficients in D, then

$$\overline{\partial} u = \sum \overline{\partial} f_{j_1 \ldots j_q} \wedge d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q}.$$

A $(0,q)$-form u is called *$\overline{\partial}$-closed* if it satisfies the differential equation $\overline{\partial} u = 0$. The set of $\overline{\partial}$-closed $(0,q)$-forms with C^∞-coefficients in D will be denoted by $Z^{(0,q)}_{\overline{\partial}}(D)$. In particular, $Z^{(0,0)}_{\overline{\partial}}(D)$ is the set $\mathcal{O}(D)$ of holomorphic functions in D.

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A $(0,q)$-form u is called $\overline{\partial}$-*exact* in D, if there exists a $(0,q-1)$-form v with C^∞-coefficients in D such that $\overline{\partial}v = u$. The set of $\overline{\partial}$-exact $(0,q)$-forms in D will be denoted by $B_{\overline{\partial}}^{(0,q)}(D)$. Recall also the definition of the $\overline{\partial}$-*cohomology groups*

$$H_{\overline{\partial}}^{(0,q)}(D) = Z_{\overline{\partial}}^{(0,q)}(D)/B_{\overline{\partial}}^{(0,q)}(D).$$

In certain cases the sets $Z_{\overline{\partial}}^{(0,n-1)}(D)$ and $H_{\overline{\partial}}^{(0,n-1)}(D)$ play, in a sense, the role of the set of holomorphic functions in D. These phenomena in several complex variables have attracted the attention of many mathematicians including Andreotti, Grauert, Griffiths, Henkin and Norguet. More relevant for what we do here is [2], where we showed that the space $H_{\overline{\partial}}^{(0,n-1)}(\mathbb{C}^n - \{0\})$ is infinite-dimensional by constructing explicitly an infinite set of $\overline{\partial}$-closed forms whose classes in $H_{\overline{\partial}}^{(0,n-1)}(\mathbb{C}^n - \{0\})$ are linearly independent. These forms were constructed by differentiating appropriately the Bochner-Martinelli kernel.

In this paper we will show that the set $\{\eta_k\}$ of these forms is maximal in the sense that any class $[\theta]$ in $H_{\overline{\partial}}^{(0,n-1)}(\mathbb{C}^n - \{0\})$ has an expansion of the form

$$[\theta] = \sum_k c_k[\eta_k] \qquad (c_k \in \mathbb{C}).$$

To describe this expansion more precisely, let us consider the Bochner-Martinelli kernel

$$M(z,w) = \frac{\beta_n}{|z-w|^{2n}} \sum_{j=1}^{n} (-1)^{j-1}(\overline{z}_j - \overline{w}_j) d\overline{z}_1 \wedge \ldots (j) \ldots \wedge d\overline{z}_n \quad (z \neq w)$$

where $\beta_n = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi i)^n}$.

For each $k = (k_1, \ldots, k_n)$, where k_j are non-negative integers, let us define the $(0,n-1)$-*forms*

$$\eta_k(z) = \left. \frac{\partial^{k_1+\ldots+k_n} M(z,w)}{\partial w_1^{k_1} \cdots \partial w_n^{k_n}} \right|_{w=0}.$$

Since $\overline{\partial}_z M(z,w) = 0$, it follows that $\overline{\partial}\eta_k = 0$, i.e. $\eta_k \in Z_{\overline{\partial}}^{(0,n-1)}(\mathbb{C}^n - \{0\})$. With this notation we will prove the following theorem.

Theorem 1. *Every $\theta \in Z_{\overline{\partial}}^{(0,n-1)}(\mathbb{C}^n - \{0\})$ has an expansion of the form*

$$\theta = \sum_k c_k \eta_k + \overline{\partial}v$$

where $c_k \in \mathbb{C}$ and v is a $(0,n-2)$-form with C^∞-coefficients in $\mathbb{C}^n - \{0\}$.

In the above theorem we assume that $n \geq 2$. The analogous statement in the case $n = 1$ is the Laurent expansion stated as follows: Every holomorphic function in $\mathbb{C} - \{0\}$ has an expansion of the form

$$\sum_{k=1}^{\infty} \frac{c_k}{z^k} + \textit{a function holomorphic in } \mathbb{C}.$$

Moreover, these expansions are unique as far as the coefficients c_k are concerned. Of course, part of what we have to do in the proof of this theorem is to deal with the appropriate convergence of the series which appear in the expansions. This is based on an estimate, which follows from an inequality satisfied by the Fourier-Laplace transform of a $\overline{\partial}$-closed $(0, n-1)$-form in $\mathbb{C}^n - \{0\}$ and the Cauchy inequality, and it is a generalization to several variables of a part of Polya's classical proof of the representation of analytic functionals by their Borel transform. (See [1, 4].)

2. The Fourier-Laplace transform of $\overline{\partial}$-closed $(0, n-1)$-forms in $\mathbb{C}^n - \{0\}$

Let $\theta \in Z^{(0,n-1)}_{\overline{\partial}}(\mathbb{C}^n - \{0\})$. For $\zeta \in \mathbb{C}^n$, define

$$F_\theta(\zeta) = \int_{z\in S} e^{\langle \zeta, z\rangle}\theta(z) \wedge \omega(z)$$

where $\langle \zeta, z\rangle = \sum \zeta_j z_j$, $\omega(z) = dz_1 \wedge \ldots \wedge dz_n$ and S is a simple closed surface surrounding the point $0 \in \mathbb{C}^n$. (By a simple closed surface S, surrounding 0, we mean that $S = \partial\Omega$ where $\Omega \subset \mathbb{C}^n$ is a bounded open set with smooth boundary and with $0 \in \Omega$. For our purposes in this section, however, Ω could be just a ball centered at 0.) Since $\overline{\partial}\theta = 0$, we have $d_z[e^{\langle \zeta, z\rangle}\theta(z) \wedge \omega(z)] = 0$. Therefore, by Stokes's theorem, the above integral is independent of the choice of the surface S and defines an entire function $F_\theta : \mathbb{C}^n \to \mathbb{C}$ which depends only on θ. This function is the Fourier-Laplace transform of θ.

It is also easy to see that the function F_θ satisfies the following estimate:

For every $\delta > 0$ there is a constant $C_\delta > 0$ so that

$$|F_\theta(\zeta)| \leq C_\delta e^{\delta|\zeta|} \qquad (\zeta \in \mathbb{C}^n). \tag{1}$$

Indeed, it suffices to notice that for every $\varepsilon > 0$

$$F_\theta(\zeta) = \int_{|z|=\varepsilon} e^{\langle \zeta, z\rangle}\theta(z) \wedge \omega(z).$$

Next we will use the expansion of the entire function F_θ in order to prove Theorem 1.

3. Proof of Theorem 1

A straightforward computation shows that

$$\begin{aligned}\frac{\partial^{k_1+\ldots+k_n} M(z,w)}{\partial w_1^{k_1}\cdots\partial w_n^{k_n}} &= \beta_n n(n+1)\cdots(n+k_1+\ldots+k_n-1)\\ &\quad\times \frac{(\overline{z}_1-\overline{w}_1)^{k_1}\cdots(\overline{z}_n-\overline{w}_n)^{k_n}}{|z-w|^{2(n+k_1+\ldots+k_n)}}\\ &\quad\times \sum_{j=1}^{n}(-1)^{j-1}(\overline{z}_j-\overline{w}_j)d\overline{z}_1\wedge\ldots(j)\ldots\wedge d\overline{z}_n\end{aligned}$$

and therefore

$$\eta_{k_1,\dots,k_n}(z) = \beta_n n(n+1)\cdots(n+k_1+\ldots+k_n-1)$$
$$\times \frac{\overline{z}_1^{k_1}\cdots\overline{z}_n^{k_n}}{|z|^{2(n+k_1+\ldots+k_n)}}\sum_{j=1}^{n}(-1)^{j-1}\overline{z}_j d\overline{z}_1\wedge\ldots(j)\ldots\wedge d\overline{z}_n.$$

Now we consider the power series expansion of the entire function F_θ:

$$F_\theta(\zeta) = \sum_{k_1,\dots,k_n\geq 0} c_{k_1,\dots,k_n}\zeta_1^{k_1}\cdots\zeta_n^{k_n} \qquad \big((\zeta_1,\dots,\zeta_n)\in\mathbb{C}^n\big).$$

Then, by (1) and the Cauchy inequalities, for $\delta>0$

$$|c_{k_1,\dots,k_n}| \leq C_\delta \frac{e^{\delta(R_1+\ldots+R_n)}}{R_1^{k_1}\cdots R_n^{k_n}} \qquad (R_1,\dots,R_n>0).$$

Applying this inequality with $R_1=\frac{k_1}{\delta},\dots,R_n=\frac{k_n}{\delta}$ we obtain that for every $\delta>0$

$$|c_{k_1,\dots,k_n}| \leq C_\delta \frac{(\delta e)^{k_1+\ldots+k_n}}{k_1^{k_1}\cdots k_n^{k_n}} \qquad \text{for all } k_1,\dots,k_n. \tag{2}$$

We claim that

$$\sum_{k_1,\dots,k_n\geq 0} n(n+1)\cdots(n+k_1+\ldots+k_n-1)|c_{k_1,\dots,k_n}|s_1^{k_1}\cdots s_n^{k_n}<\infty \tag{3}$$

for every $s_1,\dots,s_n>0$. Since (2) holds for every $\delta>0$, to prove (3) it suffices to show that

$$\sum_{k_1,\dots,k_n\geq 0}\frac{n(n+1)\cdots(n+k_1+\ldots+k_n-1)}{k_1^{k_1}\cdots k_n^{k_n}}\,\sigma_1^{k_1}\cdots\sigma_n^{k_n}<\infty \tag{4}$$

for some $\sigma_1,\dots,\sigma_n>0$. Let us first consider the series

$$\sum_{n\leq k_1\leq\ldots\leq k_n}\frac{n(n+1)\cdots(n+k_1+\ldots+k_n-1)}{k_1^{k_1}\cdots k_n^{k_n}}\,\sigma_1^{k_1}\cdots\sigma_n^{k_n}. \tag{5}$$

Writing

$$\frac{n(n+1)\cdots(n+k_1+\ldots+k_n-1)}{k_1^{k_1}\cdots k_n^{k_n}}\,\sigma_1^{k_1}\cdots\sigma_n^{k_n}$$
$$=\frac{n\cdots(n+k_1-1)}{(2k_1)^{k_1}}\cdots$$
$$\frac{(n+k_1+\ldots+k_{n-1})\cdots(n+k_1+\ldots+k_n-1)}{[(n+1)k_n]^{k_n}}(2\sigma_1)^{k_1}\cdots[(n+1)\sigma_n]^{k_n}$$

and observing that $n \le k_1 \le \ldots \le k_n$ implies

$$\left.\begin{array}{r} \dfrac{n\cdots(n+k_1-1)}{(2k_1)^{k_1}} \\ \vdots \\ \dfrac{(n+k_1+\ldots+k_{n-1})\cdots(n+k_1+\ldots+k_n-1)}{[(n+1)k_n]^{k_n}} \end{array}\right\} \le 1$$

we see that the general term of (5) is dominated by

$$(2\sigma_1)^{k_1}\cdots[(n+1)\sigma_n]^{k_n}.$$

Therefore, series (5) converges if $2\sigma_1 < 1, \ldots, (n+1)\sigma_n < 1$, i.e. if $\sigma_1 < \frac{1}{2}, \ldots, \sigma_n < \frac{1}{n+1}$. Since the general term of the series in (4) is symmetric with respect to $k_1, \ldots, k_n$, we conclude that (4) holds provided $\sigma_j < \frac{1}{n+1}$ for $j = 1, 2, \ldots, n$, and this implies (3).

Next, writing the factor

$$\frac{\overline{z}_1^{k_1}\cdots\overline{z}_n^{k_n}}{|z|^{2(n+k_1+\ldots+k_n)}}$$

of $\eta_{k_1,\ldots,k_n}(z)$ in the form

$$\frac{1}{|z|^{2n}}\left(\frac{\overline{z}_1}{|z|^2}\right)^{k_1}\cdots\left(\frac{\overline{z}_n}{|z|^2}\right)^{k_n}$$

we see that (3) implies that the series

$$\eta(z) = \sum_{k_1,\ldots,k_n\ge 0} c_{k_1,\ldots,k_n}\eta_{k_1,\ldots,k_n}(z)$$

converges and defines a $(0, n-1)$-form with C^∞-coefficients in $\mathbb{C}^n - \{0\}$. Moreover, (3) gives

$$\overline{\partial}\eta = \overline{\partial}\Bigg(\sum_{k_1,\ldots,k_n\ge 0} c_{k_1,\ldots,k_n}\eta_{k_1,\ldots,k_n}\Bigg) = \sum_{k_1,\ldots,k_n\ge 0} c_{k_1,\ldots,k_n}\overline{\partial}\eta_{k_1,\ldots,k_n} = 0,$$

i.e. $\eta \in Z_{\overline{\partial}}^{(0,n-1)}(\mathbb{C}^n - \{0\})$.

We claim that

$$\int_{z\in S} e^{\langle\zeta,z\rangle}(\theta(z)-\eta(z))\wedge\omega(z) = 0 \qquad (\zeta\in\mathbb{C}^n). \tag{6}$$

To prove this, we will compute the integrals

$$\int_{z\in S} e^{\langle\zeta,z\rangle}\eta_{k_1,\ldots,k_n}(z)\wedge\omega(z).$$

By the Bochner-Martinelli formula (see [6]), for $f \in \mathcal{O}(\mathbb{C}^n)$,

$$\int_{z\in S} f(z)M(z,w)\wedge\omega(z) = f(w) \qquad (\textit{for } w \textit{ close to } 0\in\mathbb{C}^n).$$

Applying to the above equation the operator

$$\frac{\partial^{k_1+\ldots+k_n}}{\partial w_1^{k_1}\cdots\partial w_n^{k_n}}$$

and evaluating at $w=0$, we obtain

$$\int_{z\in S} f(z)\eta_{k_1,\ldots,k_n}(z)\wedge\omega(z) = \left.\frac{\partial^{k_1+\ldots+k_n} f(w)}{\partial w_1^{k_1}\cdots\partial w_n^{k_n}}\right|_{w=0}.$$

Setting $f(z)=e^{\langle\zeta,z\rangle}$ (with ζ fixed), we find that

$$\int_{z\in S} e^{\langle\zeta,z\rangle}\eta_{k_1,\ldots,k_n}(z)\wedge\omega(z) = \zeta_1^{k_1}\cdots\zeta_n^{k_n}.$$

But by (3), the series

$$\sum_{k_1,\ldots,k_n\geq 0} c_{k_1,\ldots,k_n} e^{\langle\zeta,z\rangle}\eta_{k_1,\ldots,k_n}(z)\wedge\omega(z)$$

converges uniformly for $z\in S$, and therefore

$$\begin{aligned}\int_{z\in S} e^{\langle\zeta,z\rangle}\eta(z)\wedge\omega(z) &= \sum_{k_1,\ldots,k_n\geq 0} c_{k_1,\ldots,k_n}\int_{z\in S} e^{\langle\zeta,z\rangle}\eta_{k_1,\ldots,k_n}(z)\wedge\omega(z)\\ &= \sum_{k_1,\ldots,k_n\geq 0} c_{k_1,\ldots,k_n}\zeta_1^{k_1}\cdots\zeta_n^{k_n}\\ &= F_\theta(\zeta).\end{aligned}$$

Since $F_\theta(\zeta)=\int_{z\in S} e^{\langle\zeta,z\rangle}\theta(z)\wedge\omega(z)$, (6) follows.

According to [3: Lemma 5], a $\overline{\partial}$-closed $(0,n-1)$-form $\chi(z)$ in $\mathbb{C}^n-\{0\}$ is $\overline{\partial}$-exact (in $\mathbb{C}^n-\{0\}$) if and only if

$$\int_{z\in S} e^{\langle\zeta,z\rangle}\chi(z)\wedge\omega(z) = 0 \qquad (\zeta\in\mathbb{C}^n).$$

Therefore (6) implies that $\theta-\eta$ is $\overline{\partial}$-exact in $\mathbb{C}^n-\{0\}$, i.e. there exists a $(0,n-2)$-form v in $\mathbb{C}^n-\{0\}$ so that $\theta-\eta=\overline{\partial}v$. This gives the expansion of Theorem 1 and completes its proof.

4. A calculus of residues

Suppose that $\theta \in Z_{\bar\partial}^{(0,n-1)}(B(0,\varepsilon)-\{0\})$ where $B(0,\varepsilon)=\{z\in\mathbb{C}^n : |z|<\varepsilon\}$ $(\varepsilon>0)$. Then the proof of Theorem 1 can be carried out in this case too and the conclusion is that θ can be written in the form

$$\theta = \sum_k c_k \eta_k + \bar\partial v$$

where v is a $(0,n-2)$-form in $B(0,\varepsilon)-\{0\}$ and

$$\begin{aligned} c_k &= \frac{1}{k_1!\cdots k_n!}\frac{\partial^{k_1+\ldots+k_n}}{\partial\zeta_1^{k_1}\cdots\partial\zeta_n^{k_n}}\bigg|_{\zeta=0}\left(\int_{|z|=r} e^{\langle\zeta,z\rangle}\theta(z)\wedge\omega(z)\right) \\ &= \frac{1}{k_1!\cdots k_n!}\int_{|z|=r} z_1^{k_1}\cdots z_n^{k_n}\theta(z)\wedge\omega(z) \end{aligned}$$

for $0<r<\varepsilon$. Notice that although the expansion $\theta=\sum_k c_k\eta_k+\bar\partial v$ holds in $B(0,\varepsilon)-\{0\}$, the series $\sum_k c_k\eta_k$ converges in $\mathbb{C}^n-\{0\}$ and defines there a $\bar\partial$-closed $(0,n-1)$-form, i.e. $\sum_k c_k\eta_k \in Z_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n-\{0\})$.

More generally, let us associate to each point $\alpha\in\mathbb{C}^n$ the differential forms

$$\begin{aligned} \eta_k(z;\alpha) &= \frac{\partial^{k_1+\ldots+k_n}M(z,w)}{\partial w_1^{k_1}\cdots\partial w_n^{k_n}}\bigg|_{w=\alpha} \\ &= \beta_n n(n+1)\cdots(n+k_1+\ldots+k_n-1) \\ &\quad\times\frac{(\bar z_1-\bar\alpha_1)^{k_1}\ldots(\bar z_n-\bar\alpha_n)^{k_n}}{|z-\alpha|^{2(n+k_1+\ldots+k_n)}} \\ &\quad\times\sum_{j=1}^n(-1)^{j-1}(\bar z_j-\bar\alpha_j)d\bar z_1\wedge\ldots(j)\ldots\wedge d\bar z_n. \end{aligned}$$

Then every $\theta\in Z_{\bar\partial}^{(0,n-1)}(B(\alpha,\varepsilon)-\{\alpha\})$ has an expansion of the form

$$\theta=\sum_k c_k(\alpha)\eta_k(\cdot\,;\alpha)+\bar\partial v$$

where

$$c_k(\alpha)=\frac{1}{k_1!\cdots k_n!}\int_{|z-\alpha|=r}(z_1-\alpha_1)^{k_1}\cdots(z_n-\alpha_n)^{k_n}\theta(z)\wedge\omega(z)\quad(0<r<\varepsilon)$$

and v is a $(0,n-2)$-form in $B(\alpha,\varepsilon)$. Thus the coefficient $c_0(\alpha)=\int_{|z-\alpha|=r}\theta(z)\wedge\omega(z)$ may be thought of as the residue of θ at α, and then the coefficient $c_k(\alpha)$ is the residue, at the point α, of the differential form

$$\frac{1}{k_1!\cdots k_n!}(z_1-\alpha_1)^{k_1}\cdots(z_n-\alpha_n)^{k_n}\theta(z).$$

Notice also that if $f \in \mathcal{O}(B(\alpha, \varepsilon))$, then

$$f(z)\theta(z) \wedge \omega(z) = \sum_k c_k(\alpha) f(z) \eta_k(z;\alpha) \wedge \omega(z) + d\big[f(z)v(z) \wedge \omega(z)\big]$$

and therefore, by Stokes's formula and the uniform convergence of the series on the sphere $|z-\alpha| = r$,

$$\begin{aligned} \int_{|z-\alpha|=r} f(z)\theta(z) \wedge \omega(z) &= \sum_k c_k(\alpha) \int_{|z-\alpha|=r} f(z)\eta_k(z;\alpha) \wedge \omega(z) \\ &= \sum_k c_k(\alpha) \frac{\partial^{k_1+\ldots+k_n} f(z)}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}\bigg|_{z=\alpha}. \end{aligned}$$

Now we can prove the following theorem.

Theorem 2. *Let us consider an open set $\Omega \subset \mathbb{C}^n$ of the form $\Omega = D - (G_1 \cup \ldots \cup G_N)$ where D is a pseudoconvex set and $G_1, \ldots, G_N$ are compact convex sets in $\mathbb{C}^n$ so that $G_l \subset D$ and $G_l \cap G_m = \emptyset$ for $l \neq m$. Let us also consider simple closed surfaces S_l, each one around the set G_l and close to it. Then the following statements hold:*

(I) *A differential form $\chi \in Z_{\overline{\partial}}^{(0,n-1)}(\Omega)$ is $\overline{\partial}$-exact (in Ω) if and only if*

$$\int_{z \in S_l} e^{\langle \zeta, z\rangle} \chi(z) \wedge \omega(z) = 0 \qquad (7)$$

for every $l = 1, \ldots, N$ and $\zeta \in \mathbb{C}^n$.

(II) *Every $\theta \in Z_{\overline{\partial}}^{(0,n-1)}(D - \{\alpha^1, \ldots, \alpha^N\})$, where $\alpha^1, \ldots, a^N \in D$, has an expansion of the form*

$$\theta = \sum_{l=1}^{N} \sum_k c_k(\alpha^l)\eta_k(\cdot\,;\alpha^l) + \overline{\partial} v$$

where v is a $(0, n-2)$-form with C^∞-coefficients in $D - \{\alpha^1, \ldots, \alpha^N\}$.

Proof. `Statement` (I): The one direction follows from Stokes's formula. The other direction is a generalization of [3: Lemma 5] and its proof is similar in this case too, but we will nevertheless outline it. First we exhaust the set Ω with a sequence of compact sets of the form

$$K = \{\lambda \le 0\} - \big(\{\rho_1 < 0\} \cup \ldots \cup \{\rho_N < 0\}\big)$$

so that $\{\lambda < 0\}$ is a bounded strictly pseudoconvex set with smooth boundary and the sets $\{\rho_1 < 0\}, \ldots, \{\rho_N < 0\}$ are strictly convex neighborhoods of the convex sets $G_1, \ldots, G_N$. In other words, the sets $\{\lambda < 0\}$ should exhaust the pseudoconvex set D, while the sets $\{\rho_l < 0\}$ shrink down to the set G_l, for $l = 1, \ldots, N$.

Fixing such a set K, we consider the map $\gamma : (\partial K) \times \mathrm{int}(K) \to \mathbb{C}^n$ as follows: For $(\zeta, z) \in (\partial K) \times \mathrm{int}(K)$, $\{\gamma_j(\zeta, z)\}_{j=1}^n$ is defined to be a Henkin-Ramirez map of the strictly pseudoconvex set $\{\lambda < 0\}$, if $\zeta \in \{\lambda = 0\}$, and

$$\gamma_j(\zeta, z) = \frac{\partial \rho_l}{\partial \zeta_j}(z) \ \text{ if } \zeta \in \{\rho_l = 0\}.$$

(For exhaustions of pseudoconvex sets by strictly pseudoconvex domains and constructions of Henkin-Ramirez maps, see [5, 7].) Then

$$\sum_{j=1}^{n}(\zeta_j - z_j)\gamma_j(\zeta, z) \neq 0 \qquad \text{for } (\zeta, z) \in (\partial K) \times \text{int}(K)$$

and therefore we may write down the Cauchy-Leray formula

$$u = \overline{\partial}_z(T_{q-1}u) + T_q(\overline{\partial}u) + L_q^{\gamma}(u) \tag{8}$$

for $(0, q)$-forms u in a neighborhood of K (notation is as in [3: p. 912]).

Now if $\chi \in Z_{\overline{\partial}}^{(0,n-1)}(\Omega)$ satisfies (7), it follows as in the proof of [3: Lemma 5] that $L_{n-1}^{\gamma}(\chi) = 0$, and therefore (8) gives $\chi = \overline{\partial}_z(T_{n-2}\chi)$ in $\text{int}(K)$. Now the conclusion that χ is $\overline{\partial}$-exact in Ω follows from [3: Lemma 4], and this completes the proof of part (I).

`Statement` (II): Let $\theta \in Z_{\overline{\partial}}^{(0,n-1)}(D - \{\alpha^1, \ldots, \alpha^N\})$. For each l and k, define

$$c_k(\alpha^l) = \frac{1}{k_1! \cdots k_n!} \int_{|z-\alpha^l|=r} (z_1 - \alpha_1^l)^{k_1} \cdots (z_n - \alpha_n^l)^{k_n} \theta(z) \wedge \omega(z)$$

where $r > 0$ is sufficiently small. Then, by what we said in Section 4, the series

$$\sum_k c_k(\alpha^l)\eta_k(\cdot\,; \alpha^l)$$

converges and defines a $\overline{\partial}$-closed $(0, n-1)$-form in $\mathbb{C}^n - \{\alpha^l\}$. Therefore

$$\chi = \theta - \sum_{l=1}^{N}\sum_k c_k(\alpha^l)\eta_k(\cdot\,; \alpha^l) \in Z_{\overline{\partial}}^{(0,n-1)}(D - \{\alpha^1, \ldots, \alpha^N\})$$

and it is easy to check that

$$\int_{|z-\alpha^l|=r} e^{\langle \zeta, z \rangle}\chi(z) \wedge \omega(z) = 0$$

for every $l = 1, \ldots, N$ and $\zeta \in \mathbb{C}^n$. Therefore it follows from part (I) that $\chi = \overline{\partial}v$ for some $(0, n-2)$-form v with C^∞-coefficients in $D - \{\alpha^1, \ldots, \alpha^N\}$. This gives the required expansion of θ and completes the proof of part (II) ∎

5. A residue formula

Recall that if $\mathcal{S} : \mathbb{S}^{2n-1} \to \mathbb{C}^n - \{\alpha\}$ is a closed surface which does not pass through the point α, then the integral

$$\delta_{\mathcal{S}}(\alpha) = \int_{\mathcal{S}} M(z,\alpha) \wedge \omega(z) = \int_{\mathbb{S}^{2n-1}} \mathcal{S}^*[M(z,\alpha) \wedge \omega(z)]$$

is an integer which is the index of the surface $\mathcal{S}$ around the point α. It is easy to see that

$$\int_{\mathcal{S}} \eta_k(z;\alpha) \wedge \omega(z) = \begin{cases} \delta_{\mathcal{S}}(\alpha) & \text{if } k = (0,\dots,0) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, with the notation of Theorem 2,

$$\int_{\mathcal{S}} \theta(z) \wedge \omega(z) = \sum_{l=1}^{N} c_0(\alpha^l)\delta_{\mathcal{S}}(\alpha^l)$$

provided that $\mathcal{S}$ is a surface in D which does not pass through any of the points α^l. More generally, if $f \in \mathcal{O}(D)$, then

$$\int_{\mathcal{S}} f(z)\theta(z) \wedge \omega(z) = \sum_{l=1}^{N} \left(\delta_{\mathcal{S}}(\alpha^l) \sum_k c_k(\alpha^l) \frac{\partial^{k_1+\dots+k_n} f(z)}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \bigg|_{z=\alpha^l} \right).$$

Finally, it is easy to see that the above formulas hold if S is any surface of the more general form $S = \partial\Omega \to \mathbb{C}^n$ (smooth), provided that $\alpha^l \notin S(\partial\Omega)$ for every l.

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