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On a Class of Inclusions in Ordered Spaces

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Abstract. Let W be a non-empty set, X an ordered topological space, $L : W \to X$ a single-valued operator and $N : W \to 2^X \setminus \{\emptyset\}$ a set-valued operator. Under approximate assumptions on monotonicity of L and N we prove existence results for inclusions $Lx \in Nx$. An application of the obtained results to implicit elliptic equations of the form $Lu = f(x, u, Lu)$ is given.

Keywords: *Increasing multi-valued operators, inclusions, ordered spaces, implicit elliptic equations*

AMS subject classification: 35J65, 47H04, 47H07, 47H10

1. Introduction

Fixed point theorems for single-valued increasing operators in ordered spaces are widely investigated and have found various applications to differential equations (see [3, 6] and references therein). Recently, for some operators L and N, the existence of solutions for operator equations of the type $Lx = Nx$ in ordered spaces with applications to implicit differential equations were given in [4]. In the present paper we shall deal with similar results for multi-valued operators.

The notion of monotonicity of multi-valued operators and the existence of fixed points for increasing multi-valued operators were first given by Nishnianidze in [11]. Since the appearance of that paper the study of increasing multi-valued operators has received little attention. In our recent paper [9] we have presented some simple fixed point theorems for increasing multi-valued operators. More interesting fixed point theorems with an application to discontinuous elliptic equations are given in [7, 8]. Further extensions have been obtained in [5].

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In the present paper we shall use a general fixed point theorem of [5] to prove existence results for inclusions of the form $Lx \in Nx$. Then we demonstrate applicability of the obtained results, using them as an alternative way to prove the existence for solutions of implicit elliptic equations.

2. Solvability of Inclusions in ordered Spaces

Let $(X, \leq)$ be an ordered topological space, that is a topological space X in which there is defined an ordering "$\leq$" such that the sets $\{y \in X | \, y \leq x\}$ and $\{y \in X | \, x \leq y\}$ are closed for all $x \in X$. Throughout this section we assume that $(X, \leq)$ has the following property:

- **(C)** Each well-ordered chain C of X whose increasing sequences converge contains an increasing sequence which converges to $\sup C$.

It is proved that each ordered metric space has the property (C) [6: Proposition 1.1.5] and that each ordered normed space equipped with weak topology has property (C) [3: Lemma A.3.1].

Following Nishnianidze [11] we define a pre-ordering in the set $2^X \setminus \{\emptyset\}$ as

$$A < B \quad \text{if and only if} \quad \forall\, a \in A \;\; \exists\, b \in B : \quad a \leq b.$$

A set $A \subset X$ is said to be directed upwards if

$$\forall\, x, y \in A \;\; \exists\, z \in A : \quad x \leq z \;\; \text{and} \;\; y \leq z.$$

A multi-valued operator $F : M \subset X \to 2^X \setminus \{\emptyset\}$ is called increasing if $x, y \in M$ with $x \leq y$ implies $Fx < Fy$. If $x \in Fx$, then x is called fixed point of F.

In the sequence we need the following two general fixed point theorems.

Theorem A [5]. *Let $F : M \subset X \to 2^X \setminus \{\emptyset\}$ satisfy the following hypotheses:*

- **(F1)** *The set $M_0 = \{x \in M | \, x < Fx\}$ is non-empty.*
- **(F2)** *If $\{x_n\}$ and $\{y_n\}$ are increasing sequences in M_0 and if $y_n \in Fx_n$ $(n \in \mathbb{N})$, then $\{y_n\}$ converges in X.*
- **(F3)** *If $x \in M_0$ and $x \leq y \in Fx$, then $y \in M_0$.*
- **(F4)** *Each increasing and convergent sequence of M_0 has an upper bound in M_0.*

Then the set M_0 has a maximal element, and each maximal element of M_0 is also a maximal fixed point of F. Further, if M_0 is directed upwards, then $\max M_0$ *exists and is the greatest fixed point of F.*

Theorem B [6]. *Let X be an ordered metric space and $f : M \subset X \to M$ be a single-valued increasing operator such that:*

1) *There exists an element $u_0 \in M$ satisfying $u_0 \leq f(u_0)$.*

2) *The sequence $\{f(u_n)\}$ converges in M whenever $\{u_n\}$ is an increasing sequence in $M_0 = \{u \in M | \, u_0 \leq u \leq f(u)\}$.*

Then f has a least fixed point in M_0.

Let W be a non-empty set, X an ordered topological space, $L : W \to X$ a single-valued operator and $N : W \to 2^X \setminus \{\emptyset\}$ set-valued. Motivated by the study in [4] we want to establish some existence results for solutions of the inclusion

$$Lu \in Nu. \tag{1}$$

By adaptation of a monotonicity condition in [4] to the multi-valued case we can apply Theorem A to obtain the solvability of inclusion (1).

Theorem 1. *Let L and N be operators satisfying the following hypotheses:*

- **(N1)** *The set $W_0 = \{u \in W | \, Lu < Nu\}$ is non-empty.*
- **(N2)** *If $u_n \in W_0$ and $y_n \in Nu_n$ $(n \in \mathbb{N})$ such that $\{Lu_n\}$ and $\{y_n\}$ are increasing, then $\{y_n\}$ converges.*
- **(N3)** *If $u \in W_0$ and $Lu \leq x \in Nu$, then $x \in L(W_0)$.*
- **(N4)** *Every increasing and convergent sequence of $L(W_0)$ has an upper bound in $L(W_0)$.*

Then inclusion (1) has a solution.

Proof. Let us define two multi-valued operators

$$L^{-1} : L(W_0) \to 2^{W_0} \setminus \{\emptyset\}$$
$$F : L(W_0) \to 2^X \setminus \{\emptyset\}$$

by

$$L^{-1}x = \{u \in W_0 | \, Lu = x\}$$
$$Fx = N \circ L^{-1}x = \bigcup_{u \in L^{-1}x} Nu.$$

Clearly, if x is a fixed point of F and $u \in L^{-1}x$, then u will be a solution of inclusion (1). We shall verify that the defined above operator F satisfies all hypotheses (F1) - (F4) of Theorem A.

First we observe from the definition of F that

$$y \in Fx \quad \Longleftrightarrow \quad \exists u \in W_0 : \; y \in Nu \;\text{ and }\; Lu = x \tag{2}$$

and, consequently,

$$u_0 \in W_0 \quad \Longrightarrow \quad x_0 = Lu_0 \in M_0 = \{x |\, x < Fx\}. \tag{3}$$

Assuming that $x \le y \in Fx$, we shall prove $y \in M_0$. By (2) there exists $u \in W_0$ such that $x = Lu \le y \in Nu$ and so $y = Lv$ for some $v \in W_0$ by Hypothesis (N3). Therefore, $y \in M_0$ by (3). If $\{x_n\}$ and $\{y_n\}$ are two increasing sequences of M_0 such that $y_n \in Fx_n$, then there exists $u_n \in W_0$ satisfying $x_n = Lu_n$ and $y_n \in Nu_n$ $(n \in \mathbb{N})$ and hence $\{y_n\}$ converges by Hypothesis (N2). Thus, Hypothesis (F2) of Theorem A holds.

Finally, if $\{x_n\} \subset M_0$ is an increasing and convergent sequence, then $\{x_n\}$ has an upper bound in M_0. Indeed, we have $x_n = Lu_n$ $(n \in \mathbb{N})$ with $\{u_n\} \subset W_0$ and hence there exists $u \in W_0$ such that $Lu_n \le Lu$ by Hypothesis (N4) or, equivalently, $x_n \le x$ with $x = Lu \in M_0$. The proof is complete ∎

Theorem 2. *Let N and L be operators satisfying Hypotheses* (N1), (N2), (N4) *of Theorem* 1 *and the following hypothesis:*

(N5) *The operator L is surjective, moreover $Lu \le Lv$ implies $Nu < Nv$.*

Then inclusion (1) *has a solution.*

Proof. It is sufficient to show that Hypothesis (N3) of Theorem 1 holds. Assume $Lu \le x \in Nu$ and choose $v \in W$ such that $x = Lv$. We have $Lv < Nu$ and $Lu \le Lv$. This implies $Nu < Nv$ and so $Lv < Nv$. Thus $x \in L(W_0)$ and Hypothesis (N3) holds ∎

Theorem 3. *Let Hypotheses* (N1) *and* (N2) *of Theorem* 1 *and Hypothesis* (N5) *of Theorem* 2 *hold. Assume, in addition, the following:*

(N6) *For each $u \in W$ the set Nu is directed upwards and each increasing sequence of Nu converges to an element of Nu.*

Then inclusion (1) *has a solution.*

Proof. Let us prove that the combination of Hypotheses (N5) and (N6) implies Hypothesis (N4). For this assume that $\{Lu_n\}$ is an increasing sequence in $L(W_0)$ such that $x = \lim Lu_n$ exists in X. Assuming $x = Lu$, we have from Hypothesis (N5) that $Nu_n < Nu$. Since $Lu_n < Nu_n$, then $Lu_n < Nu$ whence there is $y_n \in Nu$ such that $Lu_n \le y_n$ $(n \in \mathbb{N})$. Since Nu is directed upwards, we can assume that $\{y_n\}$ is increasing. Therefore, $y = \lim y_n$ exists and belongs to Nu. Assuming $y = Lv$ we have $Lu \le Lv$, hence $Nu < Nv$. Since $Lu < Nu$ we have $Lv < Nv$ and so $Lu_n \le Lv \in L(W_0)$. Thus, Hypothesis (N4) holds. The theorem is completely proved ∎

3. Applications to implicit elliptic equations

Let us consider the implicit elliptic boundary value problem

$$\left.\begin{aligned} Lu &= f(x,u,Lu) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}\right\} \tag{4}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, where L is a second order elliptic operator in divergence form $Lu = -\mathrm{div}(A(x) \cdot \nabla u)$ with $A(x) = (a_{ij}(x))$ a symmetric matrix and $a_{ij} \in L^\infty(\Omega)$ satisfying for some positive number μ

$$A(x)\xi\xi \geq \mu|\xi|^2 \qquad \text{for a.e } x \in \Omega \text{ and all } \xi \in \mathbb{R}^N.$$

Problem (4) and its variants have been studied under various conditions on the function f by Carl, Heikkilä, Marano and others (see [4, 8] and references therein). We shall consider problem (4) for two cases. In the first case we impose the same monotonicity condition on f as in [4] but the growth conditions are different. In the second case the function $f = f(x,u,v)$ is assumed to be non-decreasing in u and continous in v. To study the first case we apply Theorem 2, while the second case will be considered by Theorem 3.

In this section we make blanked assumption that the function $f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is sup-measurable. Moreover:

- **(H1)** $0 \leq f(x,u,v) \leq a(x)|u|^\alpha + b(x)|v|^\beta$ for some $\alpha, \beta \in (0,1)$, $a \in L^{p_1}(\Omega)$ and $b \in L^{p_2}(\Omega)$ where $p_1 = \frac{2^*}{2^*-1-\alpha}, p_2 = \frac{2^*}{(2^*-1)(1-\beta)}$ and $2^* = \frac{2N}{N-2}$.
- **(H2)** The function $u \mapsto f(x,u,v)$ is non-decreasing for all $v \in \mathbb{R}$ and a.e $x \in \Omega$.
- **(H3)** There are a non-decreasing function $g : \mathbb{R} \to \mathbb{R}$ with $g(0) = 0$ and an open bounded subset $\Omega' \Subset \Omega$ satisfying $f(x,u,v) \geq g(u)$ for all $(x,u,v) \in \Omega' \times \mathbb{R}^+ \times \mathbb{R}$ and $\lim_{u\to 0^+} \frac{g(u)}{u} > \lambda_1$ where λ_1 is the first eigenvalue of the operator L on Ω'.

Lemma. *There exists a function $u_0 \in W_0^{1,2}(\Omega)$ such that $Lu_0 \in L^{(2^*)'}$ and*

$$Lu_0 \leq f(x,u_0,Lu_0) \qquad a.e \text{ on } \Omega. \tag{5}$$

Proof. Let $v_0 = \varepsilon\varphi$, where φ is the first eigenfunction of the operator L on Ω', set to O in $\Omega \setminus \Omega'$ and $\varepsilon > 0$ is a number. Since $v_0 \in L^{2^*}(\Omega)$ and $(2^*)' < 2^*$, we have $v_0 \in L^{(2^*)'} \subset W^{-1,2}$. Therefore, there exists $u_0 \in W_0^{1,2}(\Omega)$ such that $Lu_0 = v_0$. It follows from results of [2] that $Lv_0 \leq \lambda_1 v_0$ in the weak sense, consequently

$$\langle Lv_0 - \lambda_1 Lu_0, w\rangle = \langle Lv_0 - \lambda_1 v_0, w\rangle \leq 0$$

for all $0 \le w \in W_0^{1,2}(\Omega)$ where $\langle \cdot, \cdot \rangle$ stands for the dual pairing between $W_0^{1,2}(\Omega)$ and $W_0^{-1,2}(\Omega)$. Choosing $w = (v_0 - \lambda_1 u_0)^+$ as a test function we conclude $v_0 \le \lambda_1 u_0$ a.e on Ω. Therefore, if ε is small we have by condition (H3) that

$$Lu_0 - f(x, u_0, Lu_0) \le \lambda_1 u_0 - g(u_0) \le 0$$

a.e on Ω ∎

Theorem 4. *Let Hypotheses* (H1) - (H3) *hold. Assume, in addition, the following:*

(H4) *The function $v \mapsto f(x,u,v)$ is non-decreasing for all $u \in \mathbb{R}$ and a.e. $x \in \Omega$.*

Then problem (4) *has a non-trivial solution.*

Proof. Let u_0 and v_0 be as in Lemma and $p = (2^*)' = \frac{2^*}{2^*-1}$. We define the set

$$W = \Big\{ u \in W_0^{1,2}(\Omega) |\, u \ge u_0 \ \text{ and } \ Lu \in L^p(\Omega) \Big\}.$$

For every $u \in W$ let us consider the problem of finding $v \in L^p(\Omega)$ such that

$$v(x) = f\big(x, u(x), v(x)\big) := F_u(v) \qquad \text{for a.e } \ x \in \Omega. \tag{6}$$

The function $F_u(v)$ is measurable and satisfies

$$\begin{aligned}
&\Big(\int_\Omega |F_u(v)|^p dx \Big)^{\frac{1}{p}} \\
&\quad \le \Big(\int_\Omega a^p(x) |u(x)|^{\alpha p} dx \Big)^{\frac{1}{p}} + \Big(\int_\Omega b^p(x) |v(x)|^{\beta p} dx \Big)^{\frac{1}{p}} \\
&\quad \le \|a\|_{p_1} \|u\|_{p'}^{\alpha} + \|b\|_{p_2} \|v\|_p^{\beta}.
\end{aligned} \tag{7}$$

Therefore, F_u is a map from $L^p(\Omega)$ into itself, it is increasing and satisfies $F_u(v_0) \ge v_0$. Moreover, if $\{v_n\}$ is an increasing sequence such that $v_n \le F_u(v_n)$, then it follows from (7) that the sequence $\{F_u(v_n)\}$ is bounded, hence being increasing it converges. Consequently, the solution set of equation (6), which we denote by Nu, is non-empty by Theorem B. Problem (4) is now reduced to the inclusion $Lu \in Nu$, which will be solved by applying Theorem 2 for W defined above and $X = L^p(\Omega)$.

Let us verify conditions (N1), (N2), (N4) and (N5). If $Lu_1 \le Lu_2$, then $u_1 \le u_2$ and we need to show $Nu_1 < Nu_2$. For an element $v_1 \in Nu_1$ one has $v_1 = F_{u_1}(v_1) \le F_{u_2}(v_1)$. Therefore, equation (6) with $u = u_2$ has a solution

$v \geq v_1$ by Theorem B. Thus, there exists $v \in Nu_2$ such that $v \geq v_1$. It follows from the definition of Nu that

$$v_1, v_2 \in Nu \quad \Longrightarrow \quad \max(v_1, v_2) \in Nu. \tag{8}$$

Hence, the set Nu is directed up-wards.

Considering increasing sequences $\{v_n\}$ and $\{Lu_n\}$ satisfying $v_n \in Nu_n$ and $Lu_n < Nu_n$, we shall show that $\{v_n\}$ converges. Actually, from property (8) we can construct a sequence $\{w_n\}$ such that $v_n \leq w_n$ and

$$w_n = F_{u_n}(w_n) \tag{9}$$

$$Lu_n \leq w_n. \tag{10}$$

Applying estimation (7) for $u = u_n$ and $v = w_n$ we get

$$\|w_n\|_p \leq C\big(1 + \|u_n\|_{p'}^{\alpha}\big). \tag{11}$$

Using u_n as a test function in (10), by the inequalities of Hölder and Sobolev we obtain

$$C\bigg(\int_\Omega u_n^{2^*}\,dx\bigg)^{\frac{2}{2^*}} \leq \mu \int_\Omega |\nabla u_n|^2 dx \leq \bigg(\int_\Omega |w_n|^p dx\bigg)^{\frac{1}{p}} \bigg(\int_\Omega u_n^{2^*}\,dx\bigg)^{\frac{1}{2^*}}$$

which implies $\|u_n\|_{2^*} \leq \frac{1}{C}\|w_n\|_p$. This estimation and (11) prove the boundedness of $\{w_n\}$. The sequence $\{v_n\}$ is increasing and bounded, hence it converges.

Finally, we verify condition (N4). If a sequence $\{Lu_n\}$ is increasing and satisfies $Lu_n < Nu_n$, then we can construct an increasing sequence $\{w_n\}$ satisfying (9) - (10). The sequences $\{Lu_n\}$ and $\{w_n\}$ converge in $L^p(\Omega)$ to some functions Lu and w, respectively. Since $Lu_n \leq Lu$ one has $u_n \leq u$, hence $Lu \leq w \leq f(x, u, w)$ by letting $n \to \infty$ in the inequality $Lu_n \leq w_n \leq f(x, u, w)$. Consequently, the set Nu contains an element $v \geq Lu$ and so $Lu < Nu$. Thus, we have proved that Lu is an upper bound of $\{Lu_n\}$ in $L(W_0)$. The theorem is completely proved ∎

Theorem 5. *Let hypotheses* (H1) - (H3) *be satisfied. Assume, in addition, the following:*

(H5) *The function* $v \mapsto f(x, u, v)$ *is continuous for all* $u \in \mathbb{R}$ *and a.e.* $x \in \Omega$.

Then problem (4) *has a solution.*

Proof. Let u_0, v_0 and W, X be defined as in the proof of Theorem 4, and let us define a multi-valued operator N that assigns to each function $u \in W$ the set

$$Nu = \Big\{ v |\, v \text{ is measurable and } v(x) = f(x, u(x), v(x)) \text{ a.e on } \Omega \Big\}.$$

Then problem (4) is reduced to the inclusion $Lu \in Nu$, which will be solved by applying Theorem 3.

First we need to prove $Nu \neq \emptyset$ for every $u \in W$. In fact, since $v_0(x) \leq f(x, u(x), v_0(x))$ and $f(x, u(x), v) \leq v$ for sufficiently large v, Hypothesis (H5) yields that the set

$$H(x) = \Big\{ v \in \mathbb{R} |\, f(x, u(x), v) - v = 0 \Big\}$$

is non-empty and closed. Moreover, the multi-valued function $x \mapsto H(x)$ is measurable by [1: Theorem 8.2.9]. This implies $Nu \neq \emptyset$ by [1: Measurable Selection Theorem 8.1.3]. Furthermore, it follows from Hypotheses (H1) and (H5) and the Dominated Convergence Theorem that the set Nu is closed in $L^p(\Omega)$ and condition (N6) holds. Clearly, property (8) also holds for the defined operator N. Since $Lu_1 \leq Lu_2$ implies $u_1 \leq u_2$, hence to verify condition (N5) it is sufficient to show $Nu_1 < Nu_2$ if $u_1 \leq u_2$. Indeed, for a given function $y \in Nu_1$ we define a multi-valued function $x \mapsto G(x)$ by

$$G(x) = \Big\{ v \in \mathbb{R} |\, y(x) \leq v = f(x, u_2(x), v) \Big\}.$$

Since $y(x) = f(x, u_1(x), y(x)) \leq f(x, u_2(x), y(x))$, hence the equation $v = f(x, u_2(x), v)$ has a solution $v \geq y(x)$, so $G(x) \neq \emptyset$. Again, the Measurable Selection Theorem yields that the set Nu_2 contains an element that is greater than y.

Condition N2 can be verified as in proof of Theorem 4. The proof is complete ∎

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