The 100th Jubilee of Riesz Theory

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The *Riesz spectral theory of compact operators* was created just 100 years ago. This jubilee is a very welcome occasion for writing a detailed appreciation. Surely, the Riesz paper [24] and Banach's monograph [2], which grew out of his thesis [3], are the most important publications of classical Banach space theory; see also [21]. The presentation by both these authors is so convincing that most of their proofs can be used in today's lectures.

The Riesz paper was finished in Győr (Hungary) on January 19, 1916, and printed on December 3–5 1916. However, in most bibliographies it is dated to the year 1918, when volume 41 of *Acta Mathematica* was completed; a consequence of World War I. By the way, there is also a Hungarian version from February 14, 1916 with the title "Lineáris függvényegyenletekről"; see [25].

We stress the fact that F. Riesz wrote his contribution at a time when the concept of an abstract Banach space did not exist. Indeed, claiming in [24, p. 71]

> *Der in den neueren Untersuchungen über diverse Funktionalräume bewanderte Leser wird die allgemeinere Verwendbarkeit der Methode sofort erkennen,*

he exclusively used the space $C[a,b]$ of continuous functions on an interval $[a,b]$, equipped with the sup-norm. In other words, he stated that almost all of his results remain true in abstract Banach spaces.

Around the year 1905, D. Hilbert and his pupil E. Schmidt had developed a determinant-free approach to Fredholm's theory of integral equations of the second kind, which is based on ℓ_2 and $L_2[a,b]$ (in a hidden form); see [11]. Subsequently, F. Riesz introduced the spaces $L_p[a,b]$ and ℓ_p with $1 \leq p < \infty$; see [26, 27]. Since orthogonality was no longer available for $p \neq 2$, he tried to overcome this trouble in the most simple case, namely, $C[a,b]$.

To understand the situation in which F. Riesz started his investigations and to see their most important applications, the reader may consult the beautiful survey on *'Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten'* [10] written by E. Hellinger and O. Toeplitz.

D. Hilbert treated completely continuous bilinear forms on $\ell_2 \times \ell_2$, and F. Riesz observed that the corresponding operators are characterised by the property that weakly convergent sequences are mapped to norm convergent sequences; see [27, p. 96]. On the other hand, in the paper under review, he refers to an operator T on $C[a,b]$ as *completely continuous* if every bounded sequence (f_n) contains a subsequence whose image (Tf_{n_i}) is norm convergent. For the moment, we will distinguish these concepts by saying that an operator is completely continuous 'in the sense of Hilbert' or 'in the sense of Riesz'. Naturally, F. Riesz knew that both kinds of complete continuity coincide for operators on ℓ_2. For general spaces, completely continuous operators in the sense of Riesz are also completely continuous in the sense of Hilbert, whereas the converse implication fails. However, the following counterexamples were available only later. J. Schur [30, § 4] showed that the identity map of ℓ_1 is completely continuous in the sense of Hilbert and, of course, it fails to be completely continuous in the sense of Riesz. Moreover, let $C(\mathbb{T})$ be the space of all continuous 2π-periodic functions. Then the rule

$$P : f(t) = \sum_{n=-\infty}^{\infty} \gamma_n \mathrm{e}^{\mathrm{i}nt} \;\mapsto\; (\gamma_{2^k})_{k=0}^{\infty}$$

defines a 2-summing operator from $C(\mathbb{T})$ onto ℓ_2, which is completely continuous in the sense of Hilbert but not in the sense of Riesz; see [33, Sec. III.F].

In Hille's monograph *'Functional analysis and semigroups'* [13, p. 49] the term 'compact' was used instead of completely continuous in the sense of Riesz. Luckily, this proposal has prevailed and our temporary suffix 'in the sense of Hilbert' becomes unnecessary. From now on, we will employ the attributes 'compact' and 'completely continuous' in this way, which has become standard.

Let $\mathfrak{L}(X,Y)$ denote the Banach spaces of all (bounded, linear) operators from the Banach space X into the Banach space Y. If $X = Y$, then we simply write $\mathfrak{L}(X)$ instead of $\mathfrak{L}(X,X)$. The identity map of X is denoted by I or, more precisely, by I_X. Every $A \in \mathfrak{L}(X,Y)$ has the range $\mathcal{M}(A) := \{Ax : x \in X\}$ and the null space $\mathcal{N}(A) := \{x \in X : Ax = \mathsf{o}\}$.

The main results of F. Riesz say that the following properties hold for any compact operator $T \in \mathfrak{L}(X)$:

If

$$\mathcal{M}((I-T)^m) = \mathcal{M}((I-T)^{m+1}),$$

then

$$\mathcal{M}((I-T)^{m+1}) = \mathcal{M}((I-T)^{m+2}).$$

The ranges $\mathcal{M}((I-T)^m)$ are closed and form a non-increasing sequence, which stabilizes for some index m_0.

If

$$\mathcal{N}((I-T)^n) = \mathcal{N}((I-T)^{n+1}),$$

then

$$\mathcal{N}((I-T)^{n+1}) = \mathcal{N}((I-T)^{n+2}).$$

The null spaces $\mathcal{N}((I-T)^n)$ are finite-dimensional and form a non-decreasing sequence, which stabilizes for some index n_0.

The indices m_0 and n_0 coincide when they are chosen as small as possible; their joint value is denoted by p. Then X is the direct sum of the T-invariant subspaces $\mathcal{M}((I-T)^p)$ and $\mathcal{N}((I-T)^p)$.

In the *regular* case $p = 0$, the operator $I - T$ is an isomorphism. In other words, the equation $x - Tx = a$ admits a unique solution $x \in X$ for every $a \in X$. In the *singular* case $p > 0$, the restriction of $I - T$ to $\mathcal{M}((I-T)^p)$ is an isomorphism and the restriction to the finite-dimensional space $\mathcal{N}((I-T)^p)$ is nilpotent. This means that solving the equation $x - Tx = a$ is reduced to a problem of classical linear algebra, at least in principle.

The general concept of a *dual* (adjoint, conjugate) operator, which is based on the Hahn–Banach extension theorem, was introduced only at the end of the 1920s; see [4, Théorème 1]. Hence F. Riesz had to restrict his considerations to the very special case of *transposed* integral equations, which are generated by continuous kernels $K(x, y)$ and $K(y, x)$. The missing keystone was laid by J. Schauder [29]. His main result says that the dual operator $T^* : Y^* \to X^*$ is compact if and only if so is the original operator $T : X \to Y$. Moreover, $\dim(\mathcal{N}(I^* - T^*)) = \dim(\mathcal{N}(I - T))$. In view of this important contribution, the joint outcome is often called the *Riesz–Schauder theory*.

An intermediate result is due to T. H. Hildebrandt [12], who used – in a hidden form – the codimension of $\mathcal{M}(I - T)$. Indeed, in view of the fact that $\mathcal{N}(I^* - T^*)$ and $[X/\mathcal{M}(I - T)]^*$ are isometric, we get $\dim[\mathcal{N}(I^* - T^*)] = \operatorname{cod}[\mathcal{M}(I - T)]$. Therefore $\dim[\mathcal{N}(I^* - T^*)] = \dim[\mathcal{N}(I - T)]$ is equivalent to the formula $\operatorname{cod}[\mathcal{M}(I - T)] = \dim[\mathcal{N}(I - T)]$, which does not require the knowledge of any dual operator.

Let $A \in \mathfrak{L}(X, Y)$. Following [9, pp. 307–308], the equation $Ax = b$ is said to be *normally solvable* provided that, for given $b \in Y$, there exists a solution $x \in X$ if and only if $\langle b, y^* \rangle = 0$ whenever $A^* y^* = o$. Remarkably, this happens just in the case when the range $\mathcal{M}(A)$ is closed. Hence all operators $I - T$ with compact $T \in \mathfrak{L}(X)$ are normally solvable.

The preceding results remain true when $I - T$ is replaced by $I - \zeta T$ with any complex parameter ζ. If $I - \zeta T$ is singular, then we call ζ a *characteristic value*. F. Riesz proved that the characteristic values have no finite accumulation point. Note that he referred to those numbers as *eigenvalues*. This term is now commonly used for $\lambda \in \mathbb{C}$ when working with the scale $\lambda I - T$.

The Riesz paper has stimulated many remarkable developments. Some of them will be sketched in the rest of this review. For more detailed information the reader is referred to [19, Sect. 2.6, Subsect. 5.2.2, 5.2.3, and 8.3.1 (short biography)].

Already F. Riesz [24, p. 74] has observed that the class of compact operators is an ideal, now denoted by $\mathfrak{K}$. Further related ideals are $\mathfrak{F}$, the class of finite rank operators, and $\mathfrak{V}$, the class of completely continuous operators. Note that $\mathfrak{F} \subset \mathfrak{K} \subset \mathfrak{V}$. From Banach's monograph [2, Chap. VI, Théorème 2], we know that $\mathfrak{K}$ is closed in the norm topology of $\mathfrak{L}$. Therefore the closure $\overline{\mathfrak{F}}$, whose members are the *approximable* operators, is contained in $\mathfrak{K}$. A long-standing open problem asked whether even equality holds. The famous negative answer was finally given by P. Enflo [6] when he constructed a Banach space without the *approximation property*.

According to the Russian terminology, $A \in \mathfrak{L}(X, Y)$ is referred to as a Φ-*operator* (Φ stands for Fredholm) if there are operators $U, V \in \mathfrak{L}(Y, X)$, $S \in \mathfrak{K}(X)$, and $T \in \mathfrak{K}(Y)$ such that $UA = I_X - S$ and $AV = I_Y - T$; see [7, p. 195]. The preceding definition means that A is invertible modulo the ideal $\mathfrak{K}$, which can even be replaced by $\mathfrak{F}$. We know from F. V. Atkinson [1, Theorem 1] that Φ-operators are characterised by the property of having finite-dimensional null spaces and finite-codimensional closed ranges. By the way, a famous lemma of T. Kato [14, p. 275] says that $\operatorname{cod}[\mathcal{M}(A)] < \infty$ automatically implies that $\mathcal{M}(A)$ is closed.

F. Riesz has shown that, for compact T, all operators $I - \zeta T$ with $\zeta \in \mathbb{C}$ have very nice properties. Therefore the question arose whether his results hold for more general operators. In a first step, S. M. Nikolskij [16] confirmed this expectation for operators that admit a compact power. To formulate a complete answer, we refer to $T \in \mathfrak{L}(X)$ as a *Riesz operator* if every $I - \zeta T$ with $\zeta \in \mathbb{C}$ behaves in the desired way. To treat real operators, one must pass to their complexifications. Riesz operators can be characterised by various conditions of a quite different flavour.

(1) Every $I - \zeta T$ with $\zeta \in \mathbb{C}$ is a Φ-operator.
(2) According to A. F. Ruston [28, Theorem 3.1], T is *quasinilpotent* with respect to the quotient norm

$$|||T^n||| = \inf\{\|T^n - K\| : K \in \mathfrak{K}(X)\}.$$

This means that

$$\lim_{n\to\infty} |||T^n|||^{1/n} = 0.$$

(3) The resolvent $(I - \zeta T)^{-1}$ is a meromorphic $\mathfrak{L}(X)$-valued function on the complex plane such that the singular part of the Laurent expansion at every pole has finite rank coefficients; see [24, p. 90], [27, pp. 113–121], [29, Footnote 18 on p. 193], [5, p. 198], and [31, p. 660]. Note that the characteristic values coincide with the poles, whose order is just the index p at which the sequences $\{\mathcal{M}((I - \zeta T)^m)\}$ and $\{\mathcal{N}((I - \zeta T)^n)\}$ stabilize.
(4) For every $\varepsilon > 0$ there exists some n such that $T^n(B_X)$ can be covered by a finite number of balls $y + \varepsilon^n B_X$. Here B_X denotes the closed unit ball of X. A presentation of the Riesz theory based on a slightly modified geometric property is given in [18, Sec. 3.2 and 7.4.1].

T. T. West [32, Counterexamples] observed that the set of all Riesz operators on some Banach space may fail to be closed under addition, multiplication, and passing to the limit with respect to the operator norm. So it makes sense to look for closed ideals $\mathfrak{A}$ such that all components $\mathfrak{A}(X)$ consist of Riesz operators. The classical examples are $\overline{\mathfrak{F}}$ and $\mathfrak{K}$.

A much larger ideal $\mathfrak{S}$, introduced by T. Kato [14, pp. 284–288], consists of the *strictly singular* (semicompact) operators $T \in \mathfrak{L}(X, Y)$ defined by the following property: If there exists a constant $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all x in a closed subspace M, then M is finite-dimensional. Dualisation yields the closed ideal of *stricly cosingular* (co-semicompact) operators. Both ideals are extensively treated in a monograph [22, pp. 252–263, 315–317] written by D. Przeworska-Rolewicz and S. Rolewicz. An increasing 1-parameter scale of closed ideals lying between $\mathfrak{K}$ and $\mathfrak{S}$ was constructed in [20].

The largest ideal of this kind, here abbreviated by $\mathfrak{R}$, was introduced in [17, p. 57]. Its components $\mathfrak{R}(X, Y)$ are formed by all $T \in \mathfrak{L}(X, Y)$ such that $I_X + AT$, or equivalently $I_Y + TA$, is a Φ-operator for every $A \in \mathfrak{A}(Y, X)$. Since this definition is based on earlier results of I. Ts. Gokhberg, A. S. Markus, and I. A. Feldman [8] as well as of D. Kleinecke [15], the members of $\mathfrak{R}$ were called *Gochberg operators* or *inessential* (which does not mean that *Gochberg* is *inessential*).

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Up to a technical change concerning references, this article coincides with the 'Looking back' review ([21], 1916; JFM 46.0635.01).