Solved and Unsolved Problems

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As for me, all the various journeys on which
one by one I found myself engaged,
were leading me to Analysis Situs.
Henri Poincaré (1854–1912)

The present column is devoted to topology. The proposed problems range from tractable to fairly demanding, so that a wide range of our readers could try to tackle them. As always, there is also a proposed open research problem. The open problem in this column, along with the relevant discussion, is contributed by Simon Donaldson.

The word *topology* is derived from the two Greek words τόπος meaning *place* and λόγος meaning *study*. In mathematics, topology is considered to be the study of those properties of geometrical objects that remain invariant under topological transformations. But what is a topological transformation? A transformation of a geometric figure is called *topological* if under this transformation the relations of adjacency of various parts of the figure are not destroyed and also no new ones appear. That is, in such a transformation, the parts of a geometric figure that were in contact will remain in contact, and the parts that were not in contact cannot come into contact. Therefore, under a topological transformation we can stretch, twist, crumple, and bend, but we can neither tear nor glue.

In the above, the notion of continuity plays an integral role and for this reason topology progressed being influenced by the rigorous construction of mathematical analysis. Among the mathematicians who have been involved in the development of topology, the ones who are generally considered to have played the most profound roles are G. Leibniz, L. Euler, F. Gauss, B. Riemann, E. Betti, and most importantly H. Poincaré.

Leibniz coined the term *geometria situs* to describe the field of mathematics that is known today as topology, but it wasn't until Euler that an important topological concept arose with his proof of the now famous Euler polyhedron formula. In this study Euler introduced the concept of what is now called Euler's characteristic. Later, Gauss also made essential contributions to the field. Subsequently, Riemann's work had a profound impact in the development of topology when he introduced the concept of a Riemann surface. He introduced the concept of connectivity of a surface, which helped classify topologically compact orientable surfaces. Inspired by Riemann's concept of connectivity, Betti introduced connectivity numbers of surfaces, now known as Betti numbers. In this manner Betti established the concept of boundary and generalised Riemann's concept of connectivity. Later, based also on Riemann and Betti, Poincaré made monumental contributions to the development of topology and this is the reason why he is generally acknowledged as the father of this field.

In 1895, in his famous memoir *Analysis situs*, Poincaré established the difference between curves deformable to one another and curves bounding a larger space, respectively leading to the concepts of homotopy, fundamental groups, and homology. Poincaré was the first to discover that topological arguments could be applied to prove the existence of periodic solutions in the three-body problem of celestial mechanics.

Topology constitutes one of the most central fields of mathematics. Notwithstanding its very abstract nature, there is a staggering number of applications of topology to various other fields of Science such as astronomy, physics, biology, computer science and robotics.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

204. Note that in any topological space with an isolated point, any two dense sets must intersect. Show that there is a 0-dimensional, Hausdorff topological space X with no isolated points so that still, there are no disjoint dense sets in X.

(Daniel Soukup, Kurt Gödel Research Center, University of Vienna, Austria)

205. For $X = \{\{x, y\} : x, y \in \mathbb{Q}\}$, find a function $b : X \to \mathbb{N}$ such that

$$\{b(\{x, y\}) : x, y \in B\} = \mathbb{N},$$

whenever $B \subseteq \mathbb{Q}$ is homeomorphic to $\mathbb{Q}$.

(Boriša Kuzeljević, University of Novi sad, Department of Mathematics and Informatics, Serbia)

206. Suppose that $(G, \cdot)$ is a group, with identity element e and (G, τ) is a compact metrisable topological space. Suppose also that $L_g : (G, \tau) \to (G, \tau)$ and $R_g : (G, \tau) \to (G, \tau)$ defined by, $L_g(x) := g \cdot x$ and $R_g(x) := x \cdot g$ for all $x \in G$, are continuous functions. Show that $(G, \cdot, \tau)$ is in fact a topological group.

(Warren B. Moors, Department of Mathematics, The University of Auckland, New Zealand)

207. We will say that a nonempty subset A of a normed linear space $(X, \|\cdot\|)$ is a *uniquely remotal set* if for each $x \in X$,

$$\{y \in A : \|y - x\| = \sup\{\|a - x\| : a \in A\}\}$$

is a singleton. Clearly, nonempty uniquely remotal sets are bounded. Show that if $(X, \|\cdot\|)$ is a finite-dimensional normed linear space and A is a nonempty closed and convex uniquely remotal subset of X, then A is a singleton set.

(Warren B. Moors, Department of Mathematics, The University of Auckland, New Zealand)

208. Let X be any set. A family $\mathcal{F}$ of functions from X to $\{0, 1\}$ is said to separate *countable sets and points* if for every countable set $B \subseteq X$ and every $x \in X \setminus B$, there is a function $f \in \mathcal{F}$ so that $f(x) = 1$ and $f[B] = \{0\}$.
Let κ and λ be infinite cardinals with $\lambda \leq 2^\kappa$. Give $\{0, 1\}$ the discrete topology and $\{0, 1\}^\lambda$ the usual product topology. Show that the following are equivalent:

1. there is a family $\mathcal{F}$ of λ many functions from κ to $\{0, 1\}$ such that $\mathcal{F}$ separates countable sets and points;
2. there is a subspace $X \subseteq \{0, 1\}^\lambda$ of size κ such that every countable subset of X is closed in X.

(Dilip Raghavan, Department of Mathematics, National University of Singapore, Singapore)

209. A subset X of a partial order $(P, \leq)$ is *cofinal* in P if for each $p \in P$ there is an $x \in X$ satisfying $p \leq x$. Let $\beta\omega$ denote the Stone–Čech compactification of the natural numbers, and let ω^* denote the Stone–Čech remainder, $\beta\omega \setminus \omega$. A neighbourhood base $\mathcal{N}_x$ at a point x forms a directed partial order under reverse inclusion. A neighbourhood base $(\mathcal{N}_x, \supseteq)$ is said to be *cofinal* in another neighborhood base $(\mathcal{N}_y, \supseteq)$ if there is a map $f : \mathcal{N}_x \to \mathcal{N}_y$ such that f maps each neighbourhood base at x to a neighborhood base at y. Assume the continuum hypothesis. Show that there are at least two points x, y in ω^* with neighbourhood bases $(\mathcal{N}_x, \supseteq)$ and $(\mathcal{N}_y, \supseteq)$ which are cofinally incomparable; that is, neither is cofinal in the other.

(Natasha Dobrinen, Department of Mathematics, University of Denver, USA)

II An Open Problem, by Simon Donaldson

(Department of Mathematics, Imperial College, London, UK)

210* A problem in 4-manifold topology. This is not a new problem, it has been well known to 4-manifold specialists for the 20 years since the paper [1] of Fintushel and Stern, which is our basic reference. (Other good background references include [2] and [4].) The question involves a simple topological construction, *knot surgery*, introduced by Fintushel and Stern, involving a compact 4-manifold M and a knot K (i.e., an embedded circle in the 3-sphere S^3). We assume that there is an embedded 2-dimensional torus T in M with trivial normal bundle. We fix an identification of a neighbourhood N of T in M with a product $D^2 \times T$, where D^2 is the 2-dimensional disc . Thus the boundary of N is identified with the 3-dimensional torus $T^3 = S^1 \times T = S^1 \times S^1 \times S^1$. Likewise, a tubular neighbourhood ν of the knot K in S^3 can be identified with $D^2 \times K$, with boundary $S^1 \times K = S^1 \times S^1$. Thus the product $Y_K = (S^3 \setminus \nu) \times S^1$ has the same boundary, a 3-torus, as the complement $M \setminus N$ and we define a new compact 4-manifold

$$M_{K,\phi} = (M \setminus N) \cup_\phi Y_K,$$

where the notation means that the two spaces are glued along their common boundary using a diffeomorphism $\phi : \partial N \to \partial Y_K$. This map ϕ is chosen to take the circle ∂D^2 in the boundary of N, which bounds a disc in N, to the "longitude" in the boundary of ν, which is distinguished by the fact that it bounds a surface in the complement $S^3 \setminus \nu$. This condition does not completely fix ϕ but for the case of main interest here it is known that the resulting manifold is independent of the choice of ϕ, so we just write M_K. For the trivial knot K_0 the complement $S^3 \setminus \nu$ is diffeomorphic to $S^1 \times D^2$, so Y_{K_0} is the same as N and M_{K_0} is the same as M – the construction just cuts out N and then puts it back again.

The general problem is this: *For two knots K_1, K_2, when is the 4-manifold M_{K_1} diffeomorphic to M_{K_2}?* But there is no need to be so ambitious so we can ask the following: *Can we find interesting examples of M, K_1, K_2 such that M_{K_1} and M_{K_2} either are, or are not, diffeomorphic?*

The simplest way in which one might detect the effect of this knot surgery is through the fundamental group. For a non-trivial knot K, the fundamental group of the complement $S^3 \setminus \nu$ is a complicated nonabelian group, but it has the property that it is normally generated by the loops in the boundary 2-torus. That is, the only normal subgroup of $\pi_1(S^3 \setminus \nu)$ which contains $\pi_1(\partial\nu)$ is the whole group. It follows that if the complement $M \setminus T$ is simply connected then the same is true of M_K. In particular, this will be true if M is simply connected and there is a 2-sphere Σ in M which meets T transversely in a single point. From now on we restrict attention to the case when the 4-manifold M is the 4-manifold underlying a complex *K3 surface* X and $T \subset K$ is a complex curve. Regarded as complex manifolds there is a huge moduli space of K3 surfaces (only some of which contain complex curves) but it is known that all such pairs (X, T) are equivalent up to diffeomorphism. For one explicit model we could take X to be the quartic surface in $\mathbf{CP}^3$ defined by the equation

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0.$$

If $\kappa \in \mathbf{C}$ is a fourth root of -1 then the line L defined by the equations $z_1 = \kappa z_0, z_3 = \kappa z_2$ lies in X and for a generic plane Π through L the intersection of X with Π is the union of L and a smooth plane curve of degree 3. It is well known that smooth plane cubics are (as differentiable manifolds) 2-dimensional tori, so this gives our torus $T \subset X$, which one can check has trivial normal bundle. Using the manifest symmetries of X we can find another line L' in X which is skew to L and then L' meets T in just one point. A standard general result in complex algebraic geometry (the Lefschetz hyperplane theorem) shows that X is simply connected and since L' is a 2-sphere (as a differentiable manifold) we see that $X \setminus T$ is simply connected. There are many other possible models for (X, T) that one can take, for example using the "Kummer construction" via the quotient of a 4-torus by an involution.

To set our problem in context we recall that, in 1982, Freedman obtained a complete classification of simply connected 4-manifolds up to *homeomorphism*: everything is determined by the homology. At the level of homology all knot complements look the same and it follows that all the manifolds X_K are homeomorphic to the K3 surface X. By contrast the classification up to *diffeomorphism*, which is the setting for our problem, is a complete mystery. The only tools available come from the Seiberg–Witten equations which yield the *Seiberg–Witten invariants*. Ignoring some significant technicalities, these invariants of a smooth 4-manifold M take the form of a finite number of distinguished classes ("basic classes") in the homology $H_2(M)$, with for each basic class β a non-zero integer $SW(\beta)$. So there is a way to show that 4-manifolds are not diffeomorphic, by showing that their Seiberg–Witten invariants are different, but if the Seiberg–Witten invariants are the same one has no technique to decide if the manifolds are in fact diffeomorphic, except for constructing a diffeomorphism by hand, if such exists. The special importance of the K3 surface X appears here in the fact that it has the simplest possible non-trivial Seiberg–Witten invariant: there is just one basic class $0 \in H_2(X)$ and $SW(0) = 1$.

The main result of Fintushel and Stern in [1] is a calculation of the Seiberg–Witten invariants of the knot-surgered manifolds X_K. To explain their result we need to recall the *Alexander polynomial* of a knot K. While the knotting is invisible in the homology of the complement $S^3 \setminus \nu$ we get something interesting by passing to the infinite cyclic cover. The action of the covering transformations makes the 1-dimensional homology of this covering space a module over the group ring of $\mathbf{Z}$, which is the ring $\Lambda = \mathbf{Z}[t, t^{-1}]$ of Laurent series with integer coefficients. One finds that this is a torsion module Λ/I, for a principal ideal $I \subset \Lambda$ and the generator of this ideal I gives the Alexander polynomial $p_K \in \Lambda$. From this point of view p_K is defined up to multiplication by a unit in Λ but there is a way to normalise so that

$$p_K(t) = a_0 + \sum_{i=1}^{g} a_i(t^i + t^{-i}),$$

for integers a_i with $a_0 + 2\sum_{i=1}^{g} a_i = 1$.

Fintushel and Stern show that X_K has basic classes $\pm 2i[T]$, where $[T]$ is the homology class of a "parallel" copy of T in the complement $X \setminus N$ (which is contained in all X_K) and $SW(2i[T]) = a_i$. In other words, the Seiberg–Witten invariants capture exactly the Alexander polynomial of K. It is easy to construct distinct knots with same Alexander polynomial, so our question becomes the following: *if K_1, K_2 are knots with the same Alexander polynomial, are the 4-manifolds X_{K_1}, X_{K_2} diffeomorphic?*

As we have outlined, this question is a prototype – in an explicit and elementary setting – for the fundamental mystery of four-dimensional differential topology. There are also important connections with symplectic topology. A knot is called "fibred" if there is a fibration $\pi : S^3 \setminus \nu \to S^1$, extending the standard fibration on the 2-torus boundary. The fibre S is the complement of a disc in a compact surface of genus g and in this case the Alexander polynomial is just t^{-g} times the characteristic polynomial of the action of the monodromy on $H_1(S)$. In particular the polynomial is "monic", with leading coefficent a_g equal to ± 1. On the other hand there are knots K with monic Alexander polynomial which are not fibred and distinct fibred knots may have the same Alexander polynomial. If K is fibred then one can construct a symplectic structure ω_K on X_K. Conversely if X_K has a symplectic structure then results of Taubes on Seiberg–Witten invariants, combined with the calculation of Fintushel and Stern, show that p_K must be monic. So we have further questions such as

1. *If p_K is monic but K is not fibred, does X_K admit a symplectic structure?*
2. *If K_1, K_2 are fibred knots and (X_{K_1}, ω_{K_1}) is symplectomorphic to (X_{K_2}, ω_{K_2}) are K_1, K_2 equivalent?*

Another question in the same vein as (1) is whether a 4-manifold $S^1 \times Z^3$ admits a symplectic structure if and only if the 3-manifold Z^3 fibres over the circle. This was proved by Friedl and Vidussi [3] and by Kutluhan and Taubes [5] (with an extra technical assumption).

If we take the product $X_K \times S^2$ we move into the realm of high-dimensional geometric topology: the subtleties of 4 dimensions disappear and all the manifolds are diffeomorphic. But in the symplectic theory there are still interesting questions:

For which fibred knots K_1, K_2 are $(X_{K_i} \times S^2, \omega_{K_i} + \omega_{S^2})$ symplectomorphic?

It seems likely that the Alexander polynomials must be the same, using Taubes' result relating the Seiberg–Witten and Gromov–Witten invariants.

References

[1] R. Fintushel and R. Stern, *Knots, links and 4-manifolds*. *Inventiones Math.* **134** (1998), 363–400.

[2] R, Fintushel and R. Stern, *Six lectures on 4-manifolds* in *Low dimensional topology*. IAS/Park City Math. Series, Vol. 15 Amer. Math. Soc. 2009.

[3] S. Friedl and S. Vidussi, Twisted Alexander polynomials detect fibered 3-manifolds. *Annals of Math.* **173** (2011), 1587–1643.

[4] R. Gompf and A. Stipsicz, *Four-manifolds and Kirby calculus*. Grad. Studies in Math. Amer. Math. Soc. (1999)

[5] C. Kutluhan and C. Taubes, *Seiberg–Witten Floer homology and symplectic forms on $S^1 \times M^3$*. *Geometry and Topology* **13** (2009), 493–525.

III Solutions

197. In a game, a player moves a counter on the integers according to the following rules. During each round, a fair die is thrown. If the die shows "5" or "6", the counter is moved up one position and if it shows "1" or "2", it is moved down one position. If the die shows "3" or "4", the counter is moved up one position if the current position is positive, down one position if the current position is negative and stays at the same position if the current position is 0. Let X_n denote the position of the player after n rounds when starting at $X_0 = 1$. Find the probability p that $\lim X_n = +\infty$ and show that $X_n/n \to 1/3$ with probability p and $X_n/n \to -1/3$ with probability $1 - p$.

(Andreas Eberle, Institute for Applied Mathematics, Probability Theory, Bonn, Germany)

Solution by the proposer. For $x \in \mathbb{Z}$, we denote by $p(x)$ the probability that $\lim X_n = +\infty$ if the counter starts at $X_0 = x$. If $x > 0$ then during the first round, the player moves to $x+1$ with probability $2/3$ and to $x - 1$ with probability $1/3$. By conditioning on the first step, we see that for $x > 0$,

$$\begin{aligned} p(x) &= \frac{2}{3}\,\mathbb{P}[\lim X_n = \infty \mid X_1 = x+1] \\ &\qquad\qquad + \frac{1}{3}\,\mathbb{P}[\lim X_n = \infty \mid X_1 = x-1] \\ &= \frac{2}{3}\,p(x+1) + \frac{1}{3}\,p(x-1). \end{aligned}$$

This intuitive argument is made mathematically rigorous by applying the Markov property for the process $(X_n)_{n \in \mathbb{Z}_+}$. Similarly, we can consider the cases $x < 0$ and $x = 0$. We obtain the linear system

$$p(x) = \frac{2}{3}\,p(x+1) + \frac{1}{3}\,p(x-1) \qquad \text{for all } x > 0, \tag{1}$$

$$p(x) = \frac{1}{3}\,p(x+1) + \frac{2}{3}\,p(x-1) \qquad \text{for all } x < 0, \tag{2}$$

$$p(0) = \frac{1}{3}\,p(1) + \frac{1}{3}\,p(0) + \frac{1}{3}\,p(-1). \tag{3}$$

The equation (1) can be rewritten as the difference equation

$$\frac{2}{3}\,(p(x+1) - p(x)) = \frac{1}{3}\,(p(x) - p(x-1)) \qquad \text{for all } x > 0.$$

Thus the general solution of (1) is given by

$$p(x) = a + b \cdot (1 - 2^{-x}) \qquad \text{for } x \geq 0, \tag{4}$$

where a and b are real constants. Similarly, the general solution of (2) is

$$p(x) = c + d \cdot (1 - 2^{x}) \text{ for } x \leq 0.$$

Matching coefficients at $x = 0$ shows that $c = a$, and taking into account (3) implies $d = -b$. Hence

$$p(x) = a - b \cdot (1 - 2^{x}) \qquad \text{for } x \leq 0. \tag{5}$$

Finally, we observe that

$$\lim_{x \to +\infty} p(x) = 1 \qquad \text{and} \qquad \lim_{x \to -\infty} p(x) = 0. \tag{6}$$

To see this let $S_n = \sum_{i=1}^{n} Y_i$ where $Y_i = +1$ if the dice shows "3", "4", "5", or "6" in the ith round, and $Y_i = -1$ otherwise. Then by the law of large numbers, with probability 1, $\lim S_n/n = 1/3$, and thus $\lim S_n = \infty$. Moreover, if $\inf S_n > -X_0$ then $X_0 + S_n$ is always positive, and hence $X_n = X_0 + S_n$ for all n. Thus

$$\liminf_{x \to \infty} p(x) \geq \lim_{x \to \infty} \mathbb{P}[\inf S_n > -x] = \mathbb{P}[\inf S_n > -\infty] = 1.$$

This shows that $\lim_{x\to\infty} p(x) = 1$, and, by a similar argument, $\lim_{x\to-\infty} p(x) = 0$. By (4), (5), and (6), $a + b = 1$ and $a - b = 0$, i.e., $a = b = 1/2$. Hence

$$p(x) = \begin{cases} 1 - 2^{-x-1} & \text{for } x \geq 0, \\ 2^{x-1} & \text{for } x \leq 0. \end{cases}$$

In particular, for $X_0 = 1$ we obtain

$$p = \mathbb{P}[\lim X_n = \infty] = p(1) = 3/4.$$

Moreover, by symmetry,

$$\mathbb{P}[\lim X_n = -\infty] = p(-1) = 1/4 = 1 - p.$$

Hence with probability 1, we have either $\lim X_n = +\infty$ or $\lim X_n = -\infty$. In the first case, $X_n - X_{n-1} = Y_n$ for sufficiently large n, and hence

$$\lim(X_n/n) = \lim(S_n/n) = 1/3.$$

Similarly, in the second case,

$$\lim(X_n/n) = -1/3. \qquad \square$$

Also solved by Mihaly Bencze (Romania), Socratis Varelogiannis (France), and Alexander Vauth (Germany).

198. Let $B := (B_t)_{t\geq 0}$ be Brownian motion in the complex plane. Suppose that $B_0 = 1$.

(a) Let T_1 be the first time that B hits the imaginary axis, T_2 be the first time after T_1 that B hits the real axis, T_3 be the first time after T_2 that B hits the imaginary axis, etc. Prove that, for each $n \geq 1$, the probability that $|B_{T_n}| \leq 1$ is 1/2.

(b) More generally, let ℓ_n be lines through 0 for $n \geq 1$ such that $1 \notin \ell_1$. Let $T_1 := \inf\{t \geq 0\,;\ B_t \in \ell_1\}$ and recursively define $T_{n+1} := \inf\{t > T_n\,;\ B_t \in \ell_{n+1}\}$ for $n \geq 1$. Prove that, for each $n \geq 1$, the probability that $|B_{T_n}| \leq 1$ is 1/2.

(c) In the context of part (b), let α_n be the smaller of the two angles between ℓ_n and ℓ_{n+1}. Show that $\sum_{n=1}^{\infty} \alpha_n = \infty$ iff, for all $\epsilon > 0$, the probability that $\epsilon \leq |B_{T_n}| \leq 1/\epsilon$ tends to 0 as $n \to \infty$.

(d) In the context of part (a), show that

$$\lim_{n\to\infty} \mathbf{P}\Big[\exp(-\delta_n \sqrt{n}) \leq |B_{T_n}| \leq \exp(\delta_n \sqrt{n})\Big] = \int_{-2\delta/\pi}^{2\delta/\pi} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du$$

if $\delta_n \geq 0$ tend to $\delta \in [0, \infty]$.

(Russell Lyons, Department of Mathematics, Indiana University, USA [Partially supported by the National Science Foundation under grant DMS-1612363])

Solution by the proposer. We skip (a) and pass directly to (b). Denote inversion in the unit circle by $\phi(z) := 1/\bar{z}$. It is well known that $W := (\phi(B_t))_{t\geq 0}$ is a time-change of Brownian motion. Since ϕ maps each line ℓ_n to itself, $T_1 = \inf\{t \geq 0\,;\ W_t \in \ell_1\}$ and $T_{n+1} = \inf\{t > T_n\,;\ W_t \in \ell_{n+1}\}$ for $n \geq 1$. Thus, B_{T_n} and W_{T_n} have the same distribution. However, $|B_t| \leq 1$ iff $|W_t| \geq 1$. Since the chance that $|B_{T_n}| = 1$ is 0 for each n, we obtain (b).

In light of (b), the conclusion of (c) is equivalent to $\lim_{n\to\infty} \mathbf{P}[|B_{T_n}| < \epsilon] = 1/2$ for all $\epsilon > 0$.

Let $T := \lim_{n\to\infty} T_n$. If $T < \infty$ a.s., then $\lim_{n\to\infty} B_{T_n} = B_T$, which implies that for some $\epsilon > 0$, $\lim_{n\to\infty} \mathbf{P}[|B_{T_n}| < \epsilon] \neq 1/2$. Now suppose that $T = \infty$ a.s. Neighbourhood recurrence of B shows that for each $\epsilon > 0$, there is some $t < \infty$ such that $|B_t| \leq \epsilon$. Let S_ϵ be the first such time t. The strong Markov property, scaling, rotational symmetry, and part (b) shows that $\mathbf{P}\big[|B_{T_n}| < \epsilon \;\big|\; T_n > S_\epsilon\big] = 1/2$. Because $\lim_{n\to\infty} \mathbf{P}[T_n > S_\epsilon] = 1$, this shows that $\lim_{n\to\infty} \mathbf{P}[|B_{T_n}| < \epsilon] = 1/2$.

It remains to show that if $\sum_{n=1}^{\infty} \alpha_n < \infty$, then $T < \infty$ a.s., whereas if $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $T = \infty$ a.s. If $\limsup_{n\to\infty} \alpha_n > 0$, as in (a), then this is clear from the fact that then a.s. there is no limiting argument of B_{T_n}. In general, we use the skew-product representation of B as $B_t = \exp(X_{H_t} + iY_{H_t})$, where X and Y are independent real Brownian motions started at 0 and $H_t := \int_0^t ds/|B_s|^2$ (the form of H will not matter to us). Note that H_{T_n} are functions of Y and thus independent of X. Also, $H_\infty = \infty$ a.s. It is well known that the expected time for Y to visit either $\alpha > 0$ or $-\beta < 0$ is $\alpha\beta$, whence the expectation of $H_{T_{n+1}} - H_{T_n}$ equals $(\pi - \alpha_n)\alpha_n$. Note that $H_{T_{n+1}} - H_{T_n}$ are independent (and nonnegative). Therefore, if $\sum_{n=1}^{\infty} \alpha_n < \infty$, then $\sum_n (H_{T_{n+1}} - H_{T_n}) < \infty$ a.s., and otherwise (by Kolmogorov's three-series theorem) $\sum_n (H_{T_{n+1}} - H_{T_n}) = \infty$ a.s. This is the same as $H_T < \infty$ a.s. or $H_T = \infty$ a.s. respectively, which in turn is equivalent to $T < \infty$ a.s. or $T = \infty$ a.s.

For part (d), note that $H_{T_n}/n \to \pi^2/4$ in probability by the weak large of large numbers. With $\epsilon_n := \exp(-\delta_n \sqrt{n})$, we have

$$\begin{aligned}\mathbf{P}[\epsilon_n \leq |B_{T_n}| \leq 1/\epsilon_n] &= \mathbf{P}[-\delta_n \sqrt{n} \leq X_{H_{T_n}} \leq \delta_n \sqrt{n}] \\ &= \mathbf{P}\Big[-\frac{2\delta_n}{\pi} \leq X_{\frac{4H_{T_n}}{\pi^2 n}} \leq \frac{2\delta_n}{\pi}\Big]\end{aligned}$$

by Brownian scaling. Now let $n \to \infty$. $\square$

Remark. The proof of (c) could be shortened by using the skew-product representation throughout, but the proof given is more elementary in the context of (a). Part (b) could also be proved with the skew-product representation.

Also solved by Mihaly Bencze (Romania), Sotirios E. Louridas (Greece), and Socratis Varelogiannis (France).

199. Suppose that each carioca (native of Rio de Janeiro) likes at least half of the other 2^{23} cariocas. Prove that there exists a set A of 1000 cariocas with the following property: for each pair of cariocas in A, there exists a *distinct* carioca who likes both of them.

(Rob Morris, IMPA, Rio de Janeiro, Brazil)

Solution by the proposer. Choose 10 random cariocas (possibly with repetition), and consider the set X of cariocas that they all like. Observe that, writing V for the set of all n cariocas, we have

$$\mathbb{E}[|X|] \geq \sum_{v\in V} \left(\frac{d(v)}{n}\right)^{10} \geq 2^{-11} n \geq 2000,$$

where $d(v)$ is the number of cariocas that like carioca v. Indeed, the first inequality follows by linearity of expectation, together with the fact that $v \in X$ if and only if the 10 random cariocas all like v, the second inequality uses the convexity of the function $f(x) = x^{10}$ and the fact that $\sum_{v\in V} d(v) \geq \binom{n}{2}$, and the third holds since $n \geq 2^{23}$.

Now, let Y be the set of pairs of cariocas in X such that fewer than $\binom{1000}{2}$ cariocas like both of them. Observe that

$$\mathbb{E}[|Y|] \leq \binom{n}{2}\left(\frac{\binom{1000}{2}}{n}\right)^{10} \leq \frac{2^{189}}{n^8} \leq 1000,$$

again using the fact that $n \geq 2^{23}$. It follows that $\mathbb{E}[|X| - |Y|] \geq 1000$, and hence there exists a choice of 10 cariocas such that $|X| - |Y| \geq$

1000. Removing one carioca from each pair in X such that fewer than $\binom{1000}{2}$ cariocas like both of them, we obtain a set A of 1000 cariocas, so that no pair in A has this property. But now we can greedily (i.e., one by one) find a distinct carioca w for each $u, v \in A$ such that w likes both u and v, as required.

This proof is due to N. Alon, M. Krivelevich, and B. Sudakov [1, 5], and is based on an earlier idea of W. T. Gowers [3] and (independently) A. V. Kostochka and V. Rödl [4]. It is a simple example of a powerful technique known as *dependent random choice*; see [2]. □

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Also solved by Mihaly Bencze (Romania) and Socratis Varelogiannis (France).

200. Let X, Y, Z be independent and uniformly distributed in $[0, 1]$. What is the probability that three sticks of length X, Y and Z can be assembled together to form a triangle?

(Sebastien Vasey, Department of Mathematics, Harvard University, Cambridge, Massachusetts, USA)

Solution by the proposers. When x, y, z are fixed and $z = \max(x, y, z)$, three sticks of length x, y, and z can be assembled together to form a triangle if and only if $x + y \leq z$. Let A be the event that three sticks of length X, Y, and Z can be assembled together to form a triangle. We want to know the probability of A. Let $M = \max(X, Y, Z)$ We first compute $P(A|Z = M)$. This is

$$P(X + Y \leq Z|Z = M)\,.$$

We condition on Z: for a fixed $z \in [0, 1]$, $P(X + Y \leq z|Z = z = M)$ is

$$P(X + Y \leq z|X \leq z, Y \leq z) = \frac{P(X + Y \leq z)}{P(X \leq z, Y \leq z)}\,.$$

The denominator is z^2. Since the density is uniform, the nominator is the area of the triangle bounded by the x and y axes and the line $y = z - x$. This area is $\frac{z^2}{2}$, and hence we obtain that $P(A|Z = z = M) = \frac{1}{2}$. De-conditioning, we obtain

$$P(A|Z = M) = \int_0^1 \frac{1}{2} \cdot 1dz = \frac{1}{2}\,.$$

Similarly, $P(A|X = M) = P(A|Y = M) = \frac{1}{2}$. By symmetry, X, Y, and Z are equally likely to be maximal, hence $P(X = M) = P(Y = M) = P(Z = M) = \frac{1}{3}$. We conclude that $P(A) = \frac{1}{2}$. □

Also solved by Mihaly Bencze (Romania), Jim Kelesis (Greece), Panagiotis Krasopoulos (Greece), Peter Marioni (USA), Socratis Varelogiannis (France), and Alexander Vauth (Germany).

201. Suppose that each hour, one of the following four events may happen to a certain type of cell: it may die, it may split into two cells, it may split into three cells, or it may remain a single cell. Suppose these four events are equally likely. Start with a population consisting of a single cell. What is the probability that the population eventually goes extinct?

(Sebastien Vasey, Department of Mathematics, Harvard University, Cambridge, Massachusetts, USA)

Solution by the proposer. Let Z_n be the number of cells after n hours (the sequence $Z_0, Z_1, \dots$ is called a branching process). We have that $Z_0 = 1$ and the mass function f_{Z_1} of Z_1 is $f_{Z_1}(k) = \frac{1}{4}$ for $k = 0, 1, 2, 3$. Thus its generating function is

$$G_{Z_1}(s) = \sum_{k=0}^{\infty} f_{Z_1}(k)s^k = \frac{1}{4}\left(1 + s + s^2 + s^3\right).$$

Write $G := G_{Z_1}$, and let $G_n := G_{Z_n}$. We claim that for $n \geq 2$, $G_n = G_{n-1} \circ G$. Indeed, for $1 \leq i \leq Z_{n-1}$, let X_i be the number of cells that the ith cell reproduced into. Then $Z_n = X_1 + X_2 + \cdots + X_{Z_{n-1}}$ and $G_{X_i}(s) = G(s)$. Thus,

$$\begin{aligned} G_n(s) &= \mathbf{E}(s^{Z_n}) = \mathbf{E}\left(\mathbf{E}(s^{Z_n}|Z_{n-1})\right) \\ &= \sum_{m=0}^{\infty} (G(s))^m P(Z_{n-1} = m) = G_{n-1}(G(s)). \end{aligned}$$

We now claim that the probability η of extinction is the least nonnegative solution to the equation $G(s) = s$. Indeed, let $\eta_n := P(Z_n = 0)$. Note that $(\eta_n)_{n\in\mathbb{N}}$ is an increasing sequence with limit η. Moreover, $G_n(0) = \eta_n$, and so in particular $G(0) = \eta_1$, $\eta_2 = G(G(0)) = G(\eta_1)$, and in general $G(\eta_n) = \eta_{n+1}$. G is continuous, so taking the limit on both sides, $G(\eta) = \eta$. Clearly, η is nonnegative, and if η' is another nonnegative solution, then $0 \leq \eta'$, so (using that $G(s)$ is nondecreasing for $s \geq 0$) $\eta_1 = G(0) \leq G(\eta') = \eta'$, and so $\eta_2 = G(\eta_1) \leq G(\eta') = \eta'$, and so on. Thus $\eta_n \leq \eta'$ for all n, and hence $\eta \leq \eta'$.

We have shown that the desired probability of extinction is the least nonnegative solution of $\frac{1}{4}(1 + s + s^2 + s^3) = s$, i.e., of

$$p(s) := \frac{s^3}{4} + \frac{s^2}{4} - \frac{3s}{4} + \frac{1}{4} = 0.$$

To find the roots of p, note that $G(1) = 1$ (probabilities must sum to 1), so after a polynomial division,

$$p(s) = (s - 1)\left(\frac{s^2}{4} + \frac{s}{2} - \frac{1}{4}\right).$$

The second factor has roots $-1 \pm \sqrt{2}$. Discarding the negative solution (and noting that $\sqrt{2} - 1 < 1$), we obtain that the probability of extinction is $\eta = \sqrt{2} - 1$. □

Also solved by Mihaly Bencze (Romania), Jim Kelesis (Greece), Panagiotis Krasopoulos (Greece), and Peter Marioni (USA).

202. We are flipping a fair coin repeatedly and recording the outcomes.

(1) How many coin flips do we need on average to see three tails in a row?

(2) Suppose that we stop when we first see heads, heads, tails (H, H, T) or tails, heads, tails (T, H, T) come up in this order on three consecutive flips. What is the probability that we stop at H, H, T?

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Solution by the proposer. Let $X_k \in \{H, T\}$ denote the outcome of the kth coin flip. By assumption

$$P(X_k = H) = P(X_k = T) = \frac{1}{2}$$

and the random variables $\{X_k, k \geq 1\}$ are independent. For a finite sequence

$$A = (a_1, \dots, a_n),\ a_i \in \{H, T\}$$

we set

$$\tau_A = \inf\{k \geq n : X_{k-n+1} = a_1, X_{k-n+2} = a_2, \dots, X_k = a_n\}$$

to be the first time we see the sequence A appearing as the result of consecutive outcomes in our sequence of coin flips.

The first observation is that for any finite A we have $P(\tau_A < \infty) = 1$ and $E[\tau_A] < \infty$. To prove this, we divide up the infinite coin flip sequence into consecutive blocks of n (the length of A), and set $\widetilde{\tau}_A$ to be the first time we see A appearing in one of these blocks:

$$\widetilde{\tau}_A = \inf\{k \geq 1 : X_{(k-1)n+1} = a_1, X_{(k-1)n+2} = a_2, \dots, X_{kn} = a_n\}.$$

Since the outcomes in non-overlapping blocks are independent, and the probability that we see A in a given block of length n is 2^{-n}, we have

$$P(\widetilde{\tau}_A = j) = 2^{-n}(1-2^{-n})^{j-1} \text{ for } j \geq 1.$$

From this we get

$$P(\widetilde{\tau} < \infty) = \sum_{j=1}^{\infty} P(\widetilde{\tau}_A = j) = 1$$

and

$$E[\widetilde{\tau}_A] = \sum_{j=1}^{\infty} j \cdot P(\widetilde{\tau}_A = j) = 2^n < \infty.$$

By definition $\tau_A \leq n \cdot \widetilde{\tau}_A$, which implies $P(\tau_A < \infty) = 1$ and $E[\tau_A] < \infty$.

Now we turn to the actual questions.

(a) We have to compute $E[\tau_A]$ where $A = (T, T, T)$. Consider the following events:

$$B_1 = \{X_1 = H\}, \qquad B_2 = \{X_1 = T, X_2 = H\},$$
$$B_3 = \{X_1 = T, X_2 = T, X_3 = H\}, \quad B_4 = \{X_1 = X_2 = X_3 = T\}.$$

These are disjoint, and one of them will always happen, i.e., they form a partition of our sample space. Hence we can compute $E[\tau_A]$ by averaging the conditional expectations:

$$E[\tau_A] = \sum_{i=1}^{4} E[\tau_A | B_i] P(B_i).$$

The probabilities $P(B_i)$ can be computed using the independence of the different coin flips: $P(B_1) = 1/2$, $P(B_2) = 1/4$, $P(B_3) = P(B_4) = 1/8$. If $X_1 = X_2 = X_3 = T$ (i.e., B_4 occurs) then $\tau_A = 3$ which means that $E[\tau_A | B_4] = 3$. If the first coin flip is heads (i.e., B_1 occurs) then the first T, T, T sequence will have to start at least at the second coin flip. By the independence of the coin flips this means that τ_A conditioned on B_1 behaves the same way as $\tau_A + 1$, which implies

$$E[\tau_A | B_1] = E[\tau_A + 1] = E[\tau_A] + 1\ .$$

We can show $E[\tau_A | B_2] = E[\tau_A] + 2$ and $E[\tau_A | B_3] = E[\tau_A] + 3$ the same way. This gives

$$E[\tau_A] = \frac{1}{2}(E[\tau_A] + 1) + \frac{1}{4}(E[\tau_A] + 2) + \frac{1}{8}(E[\tau_A] + 3) + \frac{1}{8} \cdot 3,$$

and solving this we get $E[\tau_A] = 14$. (Note that we need $E[\tau_A] < \infty$ for the last step.)

(b) Let $A_1 = (H, H, T)$ and $A_2 = (T, H, T)$. We need to compute $P(\tau_{A_1} < \tau_{A_2})$. Introduce the events

$$C_1 = \{X_1 = X_2 = H\},\ C_2 = \{X_1 = T, X_2 = H\},$$
$$C_3 = \{X_1 = X_2 = T\},\ C_4 = \{X_1 = H, X_2 = T\}.$$

These events form a partition of our sample space, hence we can compute $P(\tau_{A_1} < \tau_{A_2})$ by averaging the corresponding conditional probabilities:

$$P(\tau_{A_1} < \tau_{A_2}) = \sum_{i=1}^{4} P(\tau_{A_1} < \tau_{A_2} | C_i) P(C_i)$$
$$= \frac{1}{4} \sum_{i=1}^{4} P(\tau_{A_1} < \tau_{A_2} | C_i).$$

Given that the first two flips are heads we will have $\tau_{A_1} < \tau_{A_2}$, as the first appearing tails will form a sequence of H, H, T (and T, H, T cannot happen before that). Thus $P(\tau_{A_1} < \tau_{A_2} | C_1) = 1$. Suppose now that $X_1 = T$ and $X_2 = H$. If $X_3 = T$ (which has a conditional probability of 1/2) then $\tau_{A_2} = 3 < \tau_{A_1}$. If $X_3 = H$ then we are in a similar situation as before: H, H, T will come up before T, H, T. Thus $P(\tau_{A_1} < \tau_{A_2} | C_2) = \frac{1}{2}$. A similar argument shows that $P(\tau_{A_1} < \tau_{A_2} | C_3) = P(\tau_{A_1} < \tau_{A_2} | C_4) = \frac{1}{2}$ as well (in both cases one has to consider the first heads appearing after the second flip). This gives

$$P(\tau_{A_1} < \tau_{A_2}) = \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) = \frac{5}{8}. \qquad \square$$

Remarks.

- One can always compute the expectation $E[\tau_A]$ for a given finite sequence of length n by setting up a system of linear equations for the conditional expectations of τ_A with respect to the first $n-1$ possible coin flips. These equations are the consequence of the fact that we only need to 'remember' the last $n-1$ coin flips to check whether we complete the sequence at a given coin flip. This idea can also be used to compute $P(\tau_{A_1} < \tau_{A_2})$ for any two given sequences A_1, A_2.
- It might be surprising to note that there are sequences A_1, A_2 so that $E[\tau_{A_1}] > E[\tau_{A_2}]$ but $P(\tau_{A_1} < \tau_{A_2}) > 1/2$. Moreover, there are sequences A_1, A_2, A_3 so that A_1 is more likely to come up before A_2, A_2 is more likely to come up before A_3, and A_3 is more likely to come up before A_1.
- Using a bit more sophisticated methods (martingales and optional stopping, see, e.g., [1]) one can prove an explicit formula for $E[\tau_A]$. If $A = (a_1, \dots, a_n)$ then

  $$E[\tau_A] = \sum_{k=1}^{n} 2^k \cdot \mathbb{1}(a_{n-k+1} = a_1, a_{n-k+2} = a_2, \dots, a_n = a_k).$$

  Thus the more ways the sequence A can 'overlap' with itself the larger the expected wait time for its first appearance. A similar formula can be derived for $P(\tau_{A_1} < \tau_{A_2})$.

References

[1] David Williams *Probability with Martingales*, Cambridge University Press, 2001.

Also solved by Marcello Galeotti (Italy), Jim Kelesis (Greece), Socratis Varelogiannis (France), and Alexander Vauth (Germany).

We encourage you to submit solutions to the proposed problems and ideas on the open problems. Send your solutions by email to Michael Th. Rassias, Institute of Mathematics, University of Zürich, Switzerland, michail.rassias@math.uzh.ch.

We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *differential equations*.