ERME Column

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ERME Thematic Working Groups

The European Society for Research in Mathematics Education (ERME), holds a bi-annual conference (CERME), in which research is presented and discussed in Thematic Working Groups (TWG). We will continue the initiative of introducing the working groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for research mathematicians. Our aim is to extend the ERME community with new participants, who may benefit from hearing about research methods and findings and who may contribute to future CERMEs.

Introducing CERME's Thematic Working Group 3 – Algebraic thinking

Thematic Working Group 3 (TWG3) focuses on a prominent area of research in mathematics education: algebraic thinking. This working group featured at all (bi-annual) CERME conferences except CERME 2. Typically, 20-30 papers are presented at the conferences with authors from all over Europe, and to a minor extent from the Americas and Africa. These numbers reflect the timeless interest of researchers into the teaching and learning of algebra and their consensus on the important role of algebraic thinking for improving students' mathematical knowledge and performance from primary through to university level.

As a basis for algebraic thinking, most of the studies reported can be seen to draw on a conceptualisation of school algebra as consisting of three interrelated principal activities: generational activity (i.e. the creation of algebraic expressions and equations); transformational activity (i.e. syntactically guided manipulations of formalisms); and global/meta-level activity (i.e. activities for which algebra is used as a tool) [1]. This view is in contrast to the "traditional" image of algebra as a set of procedures that are taught and learned in the middle school and are disconnected from other mathematical domains, students' earlier experiences with mathematics and college/university-level content. Like the discipline as a whole, the papers presented in TWG3 are characterised by diversity regarding their specific algebraic topics and their theoretical and methodological approaches. One of the central issues addressed in TWG3 has been the nature and characteristics of algebraic thinking, indicating the groups' interest in understanding students' ways of using algebraic tools, rather than in algebra as an epistemic body of knowledge. Participants have examined the notion of algebraic thinking from a multitude of theoretical, historical, and epistemological perspectives. Specifically, a number of papers used conceptual frameworks (developed with algebraic thinking as their momentum) as models for describing the structure of algebra and algebraic thinking, as well as clarifying particular algebra learning goals and instructional activities. Other papers have used general theories of teaching and learning as lenses to analyse algebra (e.g. semiotic theory, genetic epistemology, theory of sense and reference, theory of mediating tools, cognitive theory of instrument use). These theories assist in further explaining how individuals acquire and deploy algebraic abilities. Further, some papers have used holistic theories that have guided the instructional design of the studies reported (e.g. the theory of didactical situation in mathematics, the anthropological theory of the didactic, variation theory). This multitude of theories and research perspectives reveal that many questions remain open, such as what kind of research problems in the teaching and learning of algebra are related to which theoretical frameworks, and to what extent are these various perspectives complementary or contradictory.

A recurrent theme in TWG3 has been the process of generalisation, which is considered as central to algebraic thinking and at the very heart of mathematical activity. Virtually all papers have touched upon it to some extent, while some papers have addressed generalisation more explicitly. A considerable number of papers have examined individuals' abilities to detect figural and numerical patterns, as well as generalising those patterns and representing them symbolically.

An important breakthrough in the last 1–2 decades has been the success of early algebra. In the past, researchers tended to assume that a certain level of cognitive development was needed before symbolic algebra could be understood, yet recent research has shown that primary school students are able to use symbolic algebraic thinking tools in a sensible manner. The importance of generalisation already explained above has been supported by these works. Furthermore, many papers on early algebra have viewed the establishment of generalisations as a key characteristic of early algebraic thinking and as a foundation for developing understanding of the more formal algebra in later years and at university level. Moreover, these papers have presented curricular and instructional approaches that were found to facilitate students' moving from intuitive ways of thinking about mathematical relationships and structure to more formalised ways.

A considerable number of papers in TWG3 focus on students' understanding and different thinking levels about fundamental algebraic concepts, such as functions, equations, equivalence and the equals sign. Especially, students' difficulties, errors and misconceptions have been a central theme. For example, students tend to interpret the equal sign as an operational sign that says: calculate what's on the left. As a consequence, young students may be baffled by problems with unknowns such as $3+\_=8$ (many think the answer is 11), and even students in higher grades, where $3+5=8$ is recognised as a valid equation, may consider $8=3+5$ to be invalid. Discussions within the TWG have highlighted the possible impact of such difficulties on students' transition to university.

Interestingly, even some basic epistemic issues remain unclear. For example, can two equations be considered equivalent if one is over a field (i.e. the variable represents a number) and the other is over a polynomial ring, and thus have different semantics?

Another interest of researchers in TWG3 has been the nature and objective of activities that promote the development of algebraic thinking. A number of papers have presented teaching experiments and design research studies that suggest ways for approaching important algebraic concepts. These studies have provided evidence of ways in which students' algebraic thinking evolves within appropriate classroom environments. Some of these studies discuss task design and the use of technological tools, while others focus on teachers' roles and actions for prompting students to develop and apply algebraic processes and reasoning.

A limited number of studies draw on methodological and theoretical approaches from complementary fields such as cognitive science. For example, the notion of embodied cognition has been examined through the electroencephalographic brain activity of university students while performing algebraic, geometric and numerical reasoning tasks. Preliminary results suggest that bodily movements have a positive effect on the cognitive processing of demanding mathematical tasks. We suggest that this line of studies might further be extended and developed, as recent results in brain science show that people with amazing accomplishments in mathematics have more communication between different areas of the brain [2–3]. What encourages brain connections is when we see mathematics in different ways: e.g. as numbers, visuals, words, algebraic expressions, algorithms, gestures [2]. The three principal activities involved in algebraic thinking mentioned above (generational, transformational, global/meta-level activity) presuppose creating, manipulating and transforming between different representations of mathematical objects. In this way, these activities encourage brain connections, a matter we believe should be of interest to teachers in school as well as to university mathematicians.

Overall, the corpus of work on algebraic thinking at CERME discloses the rich and varied perspectives according to which the participants have been examining its nature, its learning and its teaching. The chapter on algebraic thinking in the book *Developing Research in Mathematics Education* gives more details on these issues [4]. The spirit of communication and collaboration, as well as the structure of TWG discussions at CERME, have proven to be effective for spotting critical issues and inspiring further research. TWG3 remains an active group that contributes to our understanding of how the development of algebraic thinking can be supported so that more students around the world can gain access to algebra as a valuable tool in their everyday and academic lives.

References

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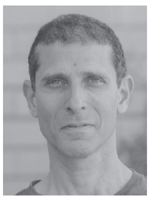

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