

Long Time Behavior of Solutions to the Caginalp System with Singular Potential

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Abstract. We consider a nonlinear parabolic system which governs the evolution of the (relative) temperature ϑ and of an order parameter χ . This system describes phase transition phenomena like, e.g., melting-solidification processes. The equation ruling χ is characterized by a singular potential W which forces χ to take values in the interval $[-1, 1]$. We provide reasonable conditions on W which ensure that, from a certain time on, χ stays uniformly away from the pure phases 1 and -1 . Combining this separation property with the Łojasiewicz-Simon inequality, we show that any smooth and bounded trajectory uniformly converges to a stationary state and we give an estimate of the decay rate.

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1. Introduction

Let us consider a material occupying, for any time $t \geq 0$, a bounded and connected domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary Γ . We suppose that the material under consideration has two phases like, for instance, solid and liquid or two different magnetization states, which are subject to temperature variations. Then, we denote by ϑ its (relative) temperature and by χ the order parameter (or phase-field) describing the proportion of either of the phases. A well-known system modelling the evolution of ϑ and χ has been proposed by Caginalp [14] for melting-solidification processes in several classes of materials. Taking all the

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constants equal to 1 for the sake of simplicity, the resulting differential system has the following form:

$$\vartheta_t + \lambda(\chi)_t - \Delta\vartheta = f \quad (1.1)$$

$$\chi_t - \Delta\chi + W'(\chi) = \lambda'(\chi)\vartheta. \quad (1.2)$$

We recall that ϑ is such that $\vartheta = 0$ is the critical value at which the phase transition takes place. Regarding χ , we observe that $\chi = \pm 1$ represent the pure states, while the values inside the open interval $(-1, 1)$ account for local presence of a mixture (mushy region). Moreover, the function $\lambda(\cdot)$ stands for a latent heat density, f is a volumic heat source and W is the potential associated with the phase configuration.

In [14], λ is assumed to be linear and W is supposed to be a double well potential (i.e., $W(y) = (y^2 - 1)^2$). Initial and boundary problems for (1.1)–(1.2) with such a W have been widely analyzed in the existing literature (see, e.g., [13, 39], cf. also [9, 11, 12, 18, 24, 34, 35, 36]). Some papers have also been devoted to the treatment of more general potentials (see, for instance, [17, 45] for well-posedness results). Here we want to give a further contribution along this direction by examining the longtime behavior of solutions when the potential W has a domain contained in $[-1, 1]$. This means that the values outside $[-1, 1]$, which are not physically meaningful, are excluded by setting $W(y) := +\infty$ for $y \notin [-1, 1]$ so justifying the adjective *singular* in the title. Our potential W is the sum of a principal part $\widehat{\beta}$, assumed to be convex and accounting for the presence of the barriers at ± 1 , and of a possibly nonconvex but smooth remainder $\widehat{\gamma}$. In this regard, the derivative W' in (1.2) has in fact to be intended as a subdifferential ∂W in the sense of Convex Analysis (see, e.g., [8, 10]) and relation (1.2) has to be understood as a differential inclusion due to the possibly multivalued character of $\partial W = \partial\widehat{\beta} + \widehat{\gamma}'$. In this general setting, a physically relevant case is, e.g., the double obstacle potential; namely the convex part $\widehat{\beta}$ is the indicator function of $[-1, 1]$, which is defined as identically 0 inside that interval, and $+\infty$ outside. In such a situation, although W is very regular in the open interval $(-1, 1)$, since it coincides with its smooth part $\widehat{\gamma}$, we cannot in general exclude that, at some point (x, t) , the variable $\chi(x, t)$ might take either of the values ± 1 corresponding to the barriers. As we shall see below, it would be interesting to know sufficient conditions on W which ensure that χ stays (uniformly) away from ± 1 from a certain time on. This is exactly the main goal of the present paper. To be more precise, we first endow system (1.1)–(1.2) with the physically reasonable boundary conditions

$$\vartheta = 0 \quad \text{on } \Gamma \times (0, +\infty) \quad (1.3)$$

$$\chi_{\mathbf{n}} = 0 \quad \text{on } \Gamma \times (0, +\infty), \quad (1.4)$$

\mathbf{n} being the outward normal to Γ , and the initial conditions

$$\vartheta(0) = \vartheta_0 \quad \text{in } \Omega \quad (1.5)$$

$$\chi(0) = \chi_0 \quad \text{in } \Omega. \quad (1.6)$$

Our main result will be to find a wide class of data and potentials for which the following separation property holds:

$$\begin{aligned} &\text{there exist } T > 0 \text{ and } \delta \in (0, 1) \text{ such that} \\ &-1 + \delta < \chi(x, t) < 1 - \delta \text{ for all } x \in \Omega, t \geq T, \end{aligned} \quad (\text{S})$$

where (ϑ, χ) is the unique global solution to problem (1.1)–(1.6). It is worth mentioning that the importance of this property has already been analyzed in [40] for the Cahn-Hilliard equation.

In order to understand for which class of potentials we can hope to obtain (S), we introduce the energy functional

$$E(u, v) := \int_{\Omega} \left(\frac{(u - \lambda(v))^2}{2} + \frac{|\nabla v|^2}{2} + W(v) \right) dx \quad (1.7)$$

which can be interpreted as a real valued functional, defined on the space $\mathcal{E} := H^{-1}(\Omega) \times L^2(\Omega)$, provided that the integral is intended to be $+\infty$ when either u is not in $L^2(\Omega)$, or v is not in $H^1(\Omega)$, or $W(u)$ is not summable. We notice anyway that E is bounded from below, and can actually be assumed nonnegative, thanks to the convex character of the principal part of W .

It is now not difficult to verify that, introducing the auxiliary variable $e := \vartheta + \lambda(\chi)$ (i.e., the enthalpy of the system), and setting for the moment $f \equiv 0$, our boundary value problem can be restated in the gradient flow form

$$(e, \chi)_t = -\partial E(e, \chi) \quad \text{for } t \geq 0, \quad (1.8)$$

the symbol ∂ denoting here the subdifferential in the space \mathcal{E} . Owing to the coercivity of E and to the compactness of its sublevels with respect to the \mathcal{E} -norm, it is then apparent that a solution exists at least for initial data (e_0, χ_0) , $e_0 = \vartheta_0 + \lambda(\chi_0)$, of bounded energy, i.e., such that $E(e_0, \chi_0) < +\infty$; moreover, *in absence* of an external source, $E(t) = E(e(t), \chi(t))$ tends to *decrease* along solution paths. Consequently, if at least one of the pure configurations corresponding to ± 1 is a local minimum of the energy, there is no hope that the system may tend to move away from it, i.e., to get the separation property. Conversely, if the leftmost and the rightmost minima of W are well separated from -1 and $+1$, respectively (cf. condition (H7w) below for a rigorous statement of this fact), then the variable χ , as it approaches these minima, moves away from the values ± 1 , a clear indication that (S) might be valid.

In this paper we shall show that, if W is *sufficiently coercive* at the extrema (corresponding, roughly speaking, to the presence of “deep” minima inside $(-1, 1)$), in a way *compatible* with the other data (in particular, with the source f which has to be not too big as t grows), then the solution moves away from ± 1 in the $L^\infty(\Omega)$ -norm. This implies (S) at least for a sufficiently large critical time T . We point out that the *compatibility condition* required to get (S) does not involve, instead, the initial data, which can have arbitrarily large energy and influence the critical time T and δ only.

The proof of this result is achieved by means of a very simple argument. Firstly, we prove uniform parabolic regularization estimates. At this step, we have to take care of the fact that (S) is not yet known, and it might well happen that, for instance, $\chi(t) \equiv 1$ in some region of nonzero spatial measure and for some time $t > 0$. Thus, the estimates have to respect the upper regularity threshold imposed by the presence of the barriers. However, the regularizing effect is sufficiently strong to get, starting from any (arbitrarily small) time $r > 0$, a uniform (in time) bound of the $L^\infty(\Omega)$ -norm of both ϑ and χ . This allows us to compare our χ , for $t \geq r$, with the solutions of the ODEs obtained by suppressing the Laplace operator in (2.13) and replacing the right hand side with (plus and minus) its $L^\infty(\Omega)$ -norm, which is now a uniformly bounded quantity. Now, in order that the solution to each ODE has the right profile, entailing validity of (S), we need $\lambda'(\chi)\vartheta$ be small enough in $L^\infty(\Omega)$, in relation to the coercivity property of W , i.e., the above mentioned compatibility condition.

We have to remark that, when W is “very coercive” at the extrema, i.e., the leftmost and rightmost minima are “infinitely deep” (namely, W is $+\infty$ outside the interval $(-1, 1)$), then no compatibility condition is needed, and the parabolic regularization property is sufficiently strong to guarantee the validity of (S) for any choice of the critical time $T > 0$. Also, no compatibility is needed when the heat source f is L^2 in time. Here the comparison argument works under very general assumptions on W ; however, in this case T is not arbitrary and there might be a transient dynamics where χ is not completely separated from the pure states. If the heat source vanishes after a certain given time, the property (S) allows us to prove a further result of this note, namely, the convergence of each bounded solution of (1.1)–(1.6) to a stationary state. Indeed, it is well known (and actually not difficult to prove) that, as (ϑ_0, χ_0) have finite energy, then there exists the ω -limit of the system, all of whose elements have the form $(0, \chi_\infty)$, χ_∞ being one solution of the homogeneous Neumann problem for the nonlinear elliptic equation (more precisely, inclusion)

$$-\Delta v + \partial W(v) \ni 0.$$

However, if W is not convex, as it happens in our case, it is known that such a problem may have a continuum of solutions (see, e.g., [26, Remark 2.3.13]).

Consequently, it is a nontrivial question to understand if the ω -limit is a singleton χ_∞ , i.e., if all the trajectory $\chi(t)$ converges to this single stationary state as $t \nearrow \infty$. This property has been proved in [2] for the Caginalp system (1.1)–(1.2) with double well potential, subject to homogeneous Dirichlet boundary conditions, provided that W , whose domain is now the whole real line, is analytic on a sufficiently large interval. More recently, in [49] a similar result has been obtained in the more difficult case of homogeneous Neumann boundary conditions. The method used in [2, 49] essentially relies on a sharp growth estimate, originally established by Łojasiewicz [37, 38] for analytic functions of several complex variables. It has been then shown by Simon [46] (see also [32]) that this inequality can be extended to an infinite dimensional setting and used to characterize the ω -limit sets of various types of evolution systems with analytic nonlinearities. However, it is worth reminding that if W is just C^∞ , then the ω -limit set might be a continuum (see [42], cf. also [7, 41]).

Several (nontrivial) variants of the Łojasiewicz-Simon method have been applied in these last years to many other systems of PDEs (see, e.g., [1, 3, 4, 5, 15, 16, 19, 22, 20, 27, 28, 31, 33, 44, 48]), among which phase-field models are just meaningful examples. It is also worth mentioning that a non-smooth version of this inequality has been obtained in [21] and applied to a spatially nonlocal phase-field system. From this point of view, we have to remark that the singleton property for our Caginalp system with singular potential is just a consequence of the separation property (S) combined with some results in [2], provided of course that the restriction of W to the open interval $(-1, 1)$ is analytic and $f(t)$ goes to zero in an appropriate way. Though we do not claim any mathematical originality for this step, we believe nonetheless that the property itself is interesting enough to be stated as a theorem. Moreover, we also give an estimate of the decay rate to equilibrium.

In the next Section 2 we shall state our hypotheses and we present the main results which will be proved in Section 3.

2. Main results

Let us set $H := L^2(\Omega)$ endowed with its standard scalar product (\cdot, \cdot) . The same notation will be also used for $(L^2(\Omega))^3$. Then we introduce

$$V_0 := H_0^1(\Omega), \quad V := H^1(\Omega),$$

and endow them with their standard scalar products, namely

$$(u, v)_{V_0} := (\nabla u, \nabla v), \quad (u, v)_V := (\nabla u, \nabla v) + (u, v).$$

We recall that $V_0 \hookrightarrow V \hookrightarrow H \equiv H' \hookrightarrow V' \hookrightarrow V'_0$, where $V'_0 = H^{-1}(\Omega)$. Let us now consider the linear operators

$$A : V \rightarrow V', \quad \langle Av, z \rangle := (\nabla v, \nabla z) \quad \text{for all } z \in V \quad (2.1)$$

$$B : V_0 \rightarrow V'_0, \quad \langle Bv, z \rangle := (\nabla v, \nabla z) \quad \text{for all } z \in V_0. \quad (2.2)$$

Note that A is weakly elliptic, while B is strongly elliptic thanks to the Poincaré inequality

$$\|v\|_H \leq c_\Omega \|\nabla v\|_H \quad \text{for all } v \in V_0, \quad (2.3)$$

where c_Ω is a suitable positive constant only depending on Ω . We also consider the domains of the H -realizations of the operators A and B

$$D(A) = \{v \in H^2(\Omega) : v_n = 0 \text{ on } \Gamma\}, \quad D(B) = H^2(\Omega) \cap V_0,$$

and we endow them with their graph norms.

Our basic hypotheses on the data are the following:

$$\lambda \in C^2(\mathbb{R}), \quad \gamma \in C^1(\mathbb{R}) \quad (\text{H1})$$

$$\vartheta_0 \in H \quad (\text{H2})$$

$$\beta \subset \mathbb{R} \times \mathbb{R} \text{ is a maximal monotone graph such that } 0 \in \beta(0) \quad (\text{H3})$$

$$\overline{\text{dom}(\beta)} = [-1, 1]. \quad (\text{H4})$$

We let

$$L := \sup_{y \in [-1, 1]} \{\gamma(y), \gamma'(y), \lambda(y), \lambda'(y), \lambda''(y)\}. \quad (2.4)$$

Next, we denote by $\widehat{\gamma}$ a primitive of γ that is nonnegative in $[-1, 1]$, and by $\widehat{\beta} : \mathbb{R} \rightarrow [0, +\infty]$ the convex and lower semicontinuous function such that $\widehat{\beta}(0) = 0$ and $\beta = \partial\widehat{\beta}$. Then, we set $W := \widehat{\beta} + \widehat{\gamma}$, so that $\partial W = \beta + \gamma$. Thanks to (H4), W is identically $+\infty$ outside $[-1, 1]$ and it is never $+\infty$ inside $(-1, 1)$. Thus, our assumptions (H3)–(H4) are just a reformulation of what we required to W in the Introduction. Also, observe that, thanks to (H1) and (H3), there exists a constant $c_\gamma > 0$ such that

$$0 \leq \widehat{\gamma}(y) \leq c_\gamma, \quad 0 \leq W(y) \leq \widehat{\beta}(y) + c_\gamma \leq W(y) + c_\gamma \quad \forall y \in [-1, 1], \quad (2.5)$$

and

$$zy \geq \widehat{\beta}(y) \quad \forall y \in [-1, 1], \quad \forall z \in \beta(y). \quad (2.6)$$

We can now state our assumptions on the initial datum χ_0 , namely,

$$\chi_0 \in V, \quad \widehat{\beta}(\chi_0) \in L^1(\Omega). \quad (\text{H5})$$

Of course, the second condition is equivalent to requiring $W(\chi_0) \in L^1(\Omega)$. It is apparent that the coupling of (H2) and (H5) is a restatement of the finite energy assumption $E(e_0, \chi_0) < +\infty$ mentioned in the Introduction.

Finally, let us detail our assumptions on the heat source. We introduce the Banach space of L^2_{loc} -translation bounded functions with values in a Banach space X :

$$\mathcal{T}^2(X) := \left\{ g \in L^2_{\text{loc}}([0, +\infty); X) : \|g\|_{\mathcal{T}^2(X)}^2 := \sup_{t \geq 0} \int_t^{t+1} \|g(s)\|_X^2 ds < +\infty \right\},$$

and we suppose that

$$f \in \mathcal{T}^2(V). \quad (\text{H6})$$

This assumption is satisfied when f is time-independent (i.e., f is a fixed element of V) and it is essential for getting uniform-in-time estimates, since it does not allow the $L^\infty(\Omega)$ -norm of ϑ to get too large.

Let us now state our first theorem, which is devoted to the well-posedness of the weak formulation of problem (1.1)–(1.6) as well as to some parabolic regularization properties of its solution. The first part of the result is basically known; thus, the proof will be mainly concerned with the second part.

Theorem 2.1. *Let (H1)–(H6) hold. Then there exists a unique triplet (ϑ, χ, ξ) such that*

$$\vartheta \in H_{\text{loc}}^1([0, +\infty); V_0') \cap C^0([0, +\infty); H) \cap L_{\text{loc}}^2([0, +\infty); V_0) \quad (2.7)$$

$$\chi \in H_{\text{loc}}^1([0, +\infty); H) \cap C^0([0, +\infty); V) \cap L_{\text{loc}}^2([0, +\infty); D(A)) \quad (2.8)$$

$$\xi \in L_{\text{loc}}^2([0, +\infty); H) \quad (2.9)$$

$$-1 \leq \chi \leq 1 \quad \text{a.e. in } \Omega \times (0, +\infty), \quad (2.10)$$

$$\widehat{\beta}(\chi) \in L^\infty(0, +\infty; L^1(\Omega)), \quad (2.11)$$

which satisfies, for almost all $t \in (0, +\infty)$,

$$\vartheta_t + \lambda(\chi)_t + B\vartheta = f \quad \text{in } V_0' \quad (2.12)$$

$$\chi_t + A\chi + \xi + \gamma(\chi) = \lambda'(\chi)\vartheta \quad \text{in } V' \quad (2.13)$$

$$\xi \in \beta(\chi) \quad \text{a.e. in } \Omega, \quad (2.14)$$

together with the initial conditions

$$\vartheta(0) = \vartheta_0, \quad \chi(0) = \chi_0, \quad \text{a.e. in } \Omega. \quad (2.15)$$

Moreover, for all $r \in (0, 1]$, there exists a constant K_0 depending on $\vartheta_0, \chi_0, f, \gamma, \lambda$, on r , but neither on β nor on t , such that, for any $t \geq 3r$,

$$\|\widehat{\vartheta}(t)\|_{C(\bar{\Omega})} + \|\vartheta(t)\|_{V_0} + \|\chi(t)\|_{D(A)} + \|\chi_t(t)\|_H + \|\beta^0(\chi(t))\|_H \leq K_0. \quad (2.16)$$

Here by β^0 we denote the minimal section of β (see, e.g., [10]).

Let us examine now the separation property. We have to refine our assumptions on the potential W . We set

$$R^\pm := \begin{cases} \beta^0(\pm 1) + \gamma(\pm 1) & \text{if } \pm 1 \in \text{dom}(\beta) \\ \pm\infty & \text{otherwise.} \end{cases} \quad (2.17)$$

Then, our weak coercivity assumption on W can be stated as

$$R := \min \{R^+, -R^-\} > 0. \quad (\text{H7w})$$

Correspondingly, we say that W is strongly coercive if

$$R := \min \{R^+, -R^-\} = +\infty, \quad (\text{H7s})$$

which is of course equivalent to requiring that $\text{dom}(\beta) = (-1, 1)$.

Remark 2.2. A meaningful example of potential W satisfying (H7s) is the so-called logarithmic potential, namely,

$$W(y) = k_1 \left((1+y) \ln(1+y) + (1-y) \ln(1-y) \right) - \frac{k_2}{2} y^2 + k_3 y + k_4,$$

for $y \in (-1, 1)$, where $k_i \in \mathbb{R}$, $i = 1 \div 4$, with k_1 and k_2 positive. In this case, our model can be viewed as a linearized version (with respect to the absolute temperature around the critical one) of the Penrose-Fife system analyzed in [30] (see also [43]). There, a separation property is established on a finite time interval.

Our main result is

Theorem 2.3. *Assume (H1)–(H6) and let (ϑ, χ) be the solution to (2.7)–(2.15). There exists $R_0 \geq 0$, depending on $\Omega, \lambda, \gamma, f$ such that, if (H7w) holds with $R > R_0$, then the separation property (S) takes place.*

Remark 2.4. As it will appear clear in the proof, the separation time T as well as the distance δ actually depend on R, R_0 and on the initial data.

Next, as mentioned in the Introduction, we have the following couple of consequences related to particular (but significant) cases.

Corollary 2.5. *Let (H1)–(H6) and (H7s) hold. Then, the property (S) holds for any time $T > 0$ and for a suitable $\delta > 0$ depending on T and on all the other data.*

Corollary 2.6. *Let (H1)–(H6) and (H7w) hold. If*

$$f \in L^2([0, +\infty); H), \quad (\text{H8})$$

then Theorem 2.3 is valid with $R_0 = 0$.

Finally, we state the result concerning the long time behavior.

Theorem 2.7. *Assume (H1)–(H6) and (H8). Then, as t tends to $+\infty$, we have*

$$\vartheta(t) \rightarrow 0 \quad \text{strongly in } V \cap C(\bar{\Omega}),$$

and the ω -limit set of the χ -trajectory is contained in the set of solutions to the equation

$$Av + \partial W(v) \ni 0. \quad (2.18)$$

If (H7w) holds, the restriction of W to $(-1, 1)$ is real analytic, and $f \equiv 0$, then the ω -limit set consists of a single point χ_∞ , i.e., as t goes to $+\infty$,

$$\chi(t) \rightarrow \chi_\infty \quad \text{strongly in } V \cap C(\bar{\Omega}). \quad (2.19)$$

Moreover, there exist $\rho \in (0, \frac{1}{2})$, $t^* > 0$, and a constant c depending on ρ, λ, γ, L and Ω such that, for any $t > t^*$,

$$\|\vartheta(t)\|_H + \|\chi(t) - \chi_\infty\|_H \leq c(1+t)^{-\frac{\rho}{1-2\rho}}. \quad (2.20)$$

Remark 2.8. Theorem 2.7 still holds when f is asymptotically autonomous in the following sense (see [16, 31]):

$$\sup_{t \geq 0} t^{1+\varpi} \int_t^\infty \|f(s)\|^2 ds < \infty,$$

for some $\varpi > 0$. One can argue, for instance, as in [25]. Of course, the decay rate will also depend on ϖ .

Remark 2.9. Let $f \equiv 0$ for the sake of simplicity. It is worth observing that the estimates obtained in the subsequent proof of Theorem 2.1 (see Section 3) along with the arguments used in [43, Section 4] allow us to construct a strongly continuous semigroup $S(t)$ on the phase space

$$\mathcal{X} := \left\{ (u, v) \in H \times V : \hat{\beta}(v) \in L^1(\Omega) \right\},$$

endowed with the complete metric

$$d_{\mathcal{X}}((u_1, v_1), (u_2, v_2)) := \|u_1 - u_2\|_H + \|v_1 - v_2\|_V + \int_{\Omega} |\hat{\beta}(v_1) - \hat{\beta}(v_2)|,$$

such that $(\vartheta(t), \chi(t)) = S(t)(\vartheta_0, \chi_0)$ solves (2.12)–(2.15). Moreover, still arguing as in [43], we can prove that $S(t)$ is dissipative (i.e., it admits a bounded absorbing set in \mathcal{X}) and it possesses a (compact and connected) global attractor \mathcal{A} which is bounded in $\mathcal{H} = V_0 \times D(A)$. Moreover, since the system has a global Lyapunov functional (see below (3.1)), then it is well known that

$$\mathcal{A} = \left\{ \begin{array}{l} \text{the trajectory } \mathbf{z}(t) \text{ originated from } (\vartheta_0, \chi_0) \text{ is} \\ (\vartheta_0, \chi_0) \in \mathcal{H} : \text{complete and bounded, and } \lim_{t \rightarrow -\infty} \mathbf{z}(t) = (0, \chi_\infty) \\ \text{in } \mathcal{X}, \text{ for some } \chi_\infty \text{ satisfying (2.18).} \end{array} \right\}$$

Thus, if W satisfies the assumptions of Theorem 2.7, we can conclude that \mathcal{A} consists of all the pairs $(\vartheta_0, \chi_0) \in \mathcal{H}$ which originate either constant or heteroclinic orbits.

Remark 2.10. As it will be clear from the proof, the *singleton property* (2.19) for the ω -limit does not directly depend on Theorem 2.3, since the properties of the solutions to (2.18) (cf. (3.42) below) and the precompactness in $C(\bar{\Omega})$ of χ -trajectories (which follows from (2.8) and (2.16)) are enough to get (S) at least from a sufficiently large T . Nevertheless, Theorem 2.3 permits to get a quantitative estimate of the time T after which the trajectories start being separated and, consequently, the Łojasiewicz-Simon inequality can be applied. We also remark that, in the statement of Theorem 2.7, analyticity of W might be required only in an open interval containing all the zeros of W' . We supposed it holds in $(-1, 1)$ just to avoid further technicalities in the proof.

3. Proofs

3.1. Proof of Theorem 2.1. If λ is linear, existence and uniqueness follows from [45, Theorem 5.1]. On the other hand, the arguments used to get the quoted result can be extended to the case of a nonlinear λ satisfying (H1) with no essential difficulties. Therefore, in the sequel we focus on the proof of property (2.16). For the sake of simplicity, we shall perform only formal a priori estimates. However, these can be justified rigorously by using, for instance, a suitable approximation scheme based on the Faedo-Galerkin method combined with a Yosida regularization of β .

First estimate. Let us test (2.12) by ϑ , (2.13) by χ_t , and take the sum. Noting that a couple of terms cancels out and using Young and Poincaré inequalities, we then get

$$\frac{d}{dt} \left(\frac{\|\vartheta\|_H^2}{2} + \frac{\|\nabla\chi\|_H^2}{2} + \int_{\Omega} W(\chi) \, dx \right) + \frac{\|\vartheta\|_{V_0}^2}{2} + \|\chi_t\|_H^2 \leq \frac{c_{\Omega}^2}{2} \|f\|_H^2. \quad (3.1)$$

Let us now test (2.13) by χ . Using (H1), (2.5), (2.6), the uniform bound $-1 \leq \chi \leq 1$, and, again, Young and Poincaré inequalities, we obtain

$$\frac{d}{dt} \frac{\|\chi\|_H^2}{2} + \|\nabla\chi\|_H^2 + \int_{\Omega} W(\chi) \, dx \leq \frac{\|\vartheta\|_{V_0}^2}{4} + c. \quad (3.2)$$

We point out that, here and below, the symbol c will be used to denote a generic positive constant, which may vary from line to line (even within the same formula), which depends on $\lambda, \gamma, L, \Omega$ at most. The generic constant additionally depending on $\|f\|_{\mathcal{T}^2(V)}$ will be indicated by c_f .

Adding together (3.1) and (3.2), using once more the Poincaré inequality, and adding to both sides the H -norm of χ (which is uniformly bounded), we

get

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\|\vartheta\|_H^2}{2} + \frac{\|\nabla\chi\|_H^2}{2} + \int_{\Omega} W(\chi) \, dx \right) + \frac{\|\vartheta\|_{V_0}^2}{8} \\ & + \frac{1}{8c_{\Omega}^2} \|\vartheta\|_H^2 + \|\chi_t\|_H^2 + \|\chi\|_V^2 + \int_{\Omega} W(\chi) \, dx \leq c(1 + \|f\|_H^2). \end{aligned} \quad (3.3)$$

Denoting now (cf. (1.7))

$$E(t) := \frac{\|\vartheta\|_H^2}{2} + \frac{\|\nabla\chi\|_H^2}{2} + \int_{\Omega} W(\chi) \, dx, \quad (3.4)$$

we can rewrite (3.3) as

$$\frac{d}{dt} E(t) + \varepsilon E(t) + \frac{\|\vartheta(t)\|_{V_0}^2}{8} + \|\chi_t(t)\|_H^2 \leq c(1 + \|f(t)\|_H^2), \quad (3.5)$$

where we have set $\varepsilon := \min\{1, 1/4c_{\Omega}^2\}$. Hence, using the Gronwall Lemma in differential form (cf., e.g., [23, Lemma 2.5]), we deduce, for any $t > 0$,

$$\begin{aligned} E(t) & \leq E(0)e^{-\varepsilon t} + c \left(\varepsilon^{-1} + \int_0^t \|f(s)\|_V^2 e^{-\varepsilon(t-s)} \, ds \right) \\ & \leq E(0)e^{-\varepsilon t} + c_f. \end{aligned} \quad (3.6)$$

Furthermore, integrating (3.5) between t and $t+r$, where $t \geq 0$ and $r \in (0, 1]$, and taking (3.6) into account, it is not difficult to infer

$$\begin{aligned} E(t+r) + \int_t^{t+r} \left(\frac{1}{8} \|\vartheta(s)\|_{V_0}^2 + \|\chi_t(s)\|_H^2 \right) \, ds \\ \leq E(t) + c(r + \|f\|_{L^2(t, t+r; H)}^2) \\ \leq E(0)e^{-\varepsilon t} + c_f. \end{aligned} \quad (3.7)$$

Second estimate. Let us now test (2.12) by ϑ_t . We easily obtain

$$\frac{\|\vartheta_t\|_H^2}{2} + \frac{d}{dt} \frac{\|\vartheta\|_{V_0}^2}{2} \leq \frac{1}{2} \int_{\Omega} (f - \chi'(\chi)\chi_t)^2 \, dx, \quad (3.8)$$

so that, recalling (3.7), the *uniform* Gronwall Lemma (cf., e.g., [47, Lemma III.1.1]) yields

$$\|\vartheta(t+r)\|_{V_0}^2 \leq c(E(0)e^{-\varepsilon t} + c_f)(1+r^{-1}) \quad \text{for all } t \geq 0. \quad (3.9)$$

Next, integrating (3.8) over $(t, t+r)$, we also get

$$\int_t^{t+r} \|\vartheta_t(s)\|_H^2 \, ds \leq c(E(0)e^{-\varepsilon t} + c_f)(1+r^{-1}) \quad \text{for all } t \geq r. \quad (3.10)$$

Third estimate. A formal differentiation with respect to time of (2.13) gives

$$\chi_{tt} + A\chi_t + W''(\chi)\chi_t = \lambda''(\chi)\chi_t\vartheta + \lambda'(\chi)\vartheta_t, \quad (3.11)$$

where, of course, the second derivative W'' does not make sense in the present setting, but still the procedure can be made rigorous by regularizing β . Now, let us test (3.11) by χ_t . Thanks to the monotonicity of β , we have

$$\frac{d}{dt} \frac{\|\chi_t\|_H^2}{2} + \|\nabla\chi_t\|_H^2 \leq \int_{\Omega} (-\gamma'(\chi)\chi_t^2 + \lambda''(\chi)\chi_t^2\vartheta + \lambda'(\chi)\vartheta_t\chi_t) dx. \quad (3.12)$$

Using (H1), estimating the second term as follows,

$$\int_{\Omega} \lambda''(\chi)\chi_t^2\vartheta dx \leq \frac{\|\chi_t\|_V^2}{2} + c\|\vartheta\|_{V_0}^2\|\chi_t\|_H^2, \quad (3.13)$$

and adding $\frac{1}{2}\|\chi_t\|_H^2$ to both sides of (3.12), we easily deduce

$$\frac{d}{dt} \|\chi_t\|_H^2 + \|\chi_t\|_V^2 \leq c(1 + \|\vartheta\|_{V_0}^2)\|\chi_t\|_H^2. \quad (3.14)$$

Thus, on account of (3.7) and (3.9), a further use of the uniform Gronwall Lemma gives

$$\begin{aligned} \|\chi_t(t+r)\|_H^2 &\leq c(E(0)e^{-\varepsilon t} + c_f) [r^{-1} + c(E(0)e^{-\varepsilon t} + c_f)(1+r^{-1})] \\ &\leq c(E(0)e^{-\varepsilon t} + r^{-1} + c_f)^2 \quad \text{for all } t \geq r, \end{aligned} \quad (3.15)$$

and integrating (3.14) over $(t, t+r)$ for $t \geq 2r$, thanks to (3.15) we have

$$\int_t^{t+r} \|\chi_t(s)\|_V^2 ds \leq c(E(0)e^{-\varepsilon t} + r^{-1} + c_f)^2 \quad \text{for all } t \geq 2r. \quad (3.16)$$

Fourth estimate. A comparison argument in relation (2.12), together with (3.7), (3.10), (3.16), and well-known elliptic regularity results, immediately yields

$$\int_t^{t+r} \|\vartheta(s)\|_{D(B)}^2 ds \leq c(E(0)e^{-\varepsilon t} + r^{-1} + c_f)^2, \quad \text{for all } t \geq 2r. \quad (3.17)$$

Fifth estimate. Testing (2.13) by $A\chi$, using the monotonicity of β , and relying on estimates (3.6) and (3.15), we easily get

$$\|\chi(t)\|_{D(A)}^2 \leq c(E(0)e^{-\varepsilon t} + r^{-1} + c_f)^2 \quad \text{for all } t \geq 2r. \quad (3.18)$$

Next, a comparison in (2.13) immediately permits us to obtain

$$\|\beta^0(\chi(t))\|_H^2 \leq c(E(0)e^{-\varepsilon t} + r^{-1} + c_f)^2 \quad \text{for all } t \geq 2r. \quad (3.19)$$

Let us point out that this comparison argument, which is rigorous as far as β is a smooth function (so that $\beta \equiv \beta^0$), is just formal when β is multivalued. This procedure is fully justified, e.g., in [43, Proof of Theorem 2.16], where it is also explained why the minimal section β^0 appears.

Sixth estimate. Let us observe that ϑ solves

$$\vartheta_t + B\vartheta = g \quad \text{a.e. in } \Omega \times (0, +\infty),$$

where $g := f - \lambda'(\chi)\chi_t$. Recalling (H6) and using (3.7), (3.16), (3.18), and (2.4) it is not difficult to get

$$\begin{aligned} \int_t^{t+r} \|g(s)\|_V^2 \, ds &\leq c_f + c \int_t^{t+r} \|\chi_t(s)\|_V^2 (1 + \|\chi(s)\|_V \|\chi(s)\|_{D(A)}) \, ds \\ &\leq (E(0)e^{-\varepsilon t} + r^{-1} + c_f)^{\frac{7}{2}} \quad \text{for all } t \geq 2r. \end{aligned} \quad (3.20)$$

Thus, $g(\cdot)$ is in V . However, in general it does not belong to $V_0 = D(B^{1/2})$. Nevertheless, if $\sigma \in (0, \frac{1}{4})$, $V \subset D(B^\sigma)$ with continuous embedding. Thus, testing (2.12) by $B^{2\sigma}\vartheta$ and using (H6), (3.17), and the uniform Gronwall lemma, we infer

$$\|\vartheta(t)\|_{D(B^{(1+2\sigma)/2})}^2 \leq c(E(0)e^{-\varepsilon t} + r^{-1} + c_f)^{\frac{7}{2}} \quad \text{for all } t \geq 3r. \quad (3.21)$$

Let us point out that, as $\sigma \in (0, \frac{1}{4})$, it is

$$D(B^{(1+2\sigma)/2}) \subset V \quad \text{with compact embedding.} \quad (3.22)$$

However, in three dimensions of space $D(B^{(1+2\sigma)/2})$ is not included into $C(\bar{\Omega})$. Thus, a further bound is required.

Seventh estimate. Here we estimate $\|\vartheta(t)\|_{C(\bar{\Omega})}$ for sufficiently large t . Let us recall (3.20) and notice that, by (3.7) and (2.4) it is also

$$\int_t^{t+r} \|g(s)\|_H^2 \, ds \leq E(0)e^{-\varepsilon t} + c_f \quad \text{for all } t \geq 0. \quad (3.23)$$

Then, let us take, just for the sake of simplicity, $r = 1$ and consider $t \geq 3$. Let $N = N(t) \in \mathbb{N}$ such that $t \in (N, N + 1]$ and decompose $\vartheta = \vartheta^1 + \vartheta^2$, where

$$\vartheta_t^1 + B\vartheta^1 = g^1, \quad \vartheta^1(0) = \vartheta_0 \quad (3.24)$$

$$\vartheta_t^2 + B\vartheta^2 = g^2, \quad \vartheta^2(0) = 0. \quad (3.25)$$

Here

$$g^1 := g\zeta([0, N-1]), \quad g^2 := g\zeta((N-1, N+1)),$$

$\zeta(J)$ denoting the characteristic function of the interval J . For the solution of the equation (3.24) we have the representation formula

$$\vartheta^1(t) = e^{Bt}\vartheta_0 + \int_0^{N-1} e^{B(t-s)}g^1(s) \, ds.$$

Let $X^\alpha = D(B^\alpha)$ endowed with the graph norm $\|\cdot\|_\alpha$. Then $X^\alpha \subset C(\bar{\Omega})$ for $\alpha > \frac{3}{4}$ and the following estimate holds:

$$\|B^\alpha e^{Bt}u\|_H \leq \kappa t^{-\alpha} e^{-\mu t} \|u\|_H, \quad (3.26)$$

for some $\kappa > 0$ and some $\mu > 0$. It is now sufficient to estimate $\|B^\alpha \vartheta(t)\|_H$, the H -norm being estimated by (2.3) and (3.9). We have

$$B^\alpha \vartheta^1(t) = B^\alpha e^{Bt}\vartheta_0 + \int_0^{N-1} B^\alpha e^{B(t-s)}g^1(s) \, ds.$$

The first term decays exponentially by (3.26). Concerning the second, on account of (3.23) and (3.26), we infer

$$\begin{aligned} \int_0^{N-1} \|B^\alpha e^{B(t-s)}g^1(s)\|_H \, ds &\leq c \sum_{n=0}^{N-2} \int_n^{n+1} e^{-\mu(t-s)} \|g^1(s)\|_H \, ds \\ &\leq c \sum_{n=0}^{N-2} e^{-\mu(N-n-1)} \int_n^{n+1} \|g^1(s)\|_H \, ds \\ &\leq c \sum_{n=0}^{N-2} e^{-\mu(N-n-1)} (E(0)e^{-\varepsilon n} + c_f)^{\frac{1}{2}} \\ &\leq cE(0)e^{-\mu(N-1)} \frac{e^{(\mu-\frac{\varepsilon}{2})(N-1)} - 1}{e^{\mu-\varepsilon} - 1} + c_f \\ &\leq cE(0)e^{-mt} + c_f, \end{aligned} \quad (3.27)$$

where we have set $m := \min\{\mu, \frac{\varepsilon}{2}\}$ and the last constants c, c_f depend, of course, also on ε, μ . Therefore, for any $t \geq 3$ we deduce

$$\|\vartheta^1(t)\|_{X^\alpha}^2 \leq cE(0)e^{-mt} + c_f. \quad (3.28)$$

Consider now equation (3.25) and observe that (3.20) implies

$$\|g^2\|_{L^2((N-1, N+1); L^6(\Omega))} \leq c(E(0)e^{-\varepsilon t} + c_f)^{\frac{7}{4}}.$$

Thus we can apply the maximal regularity result [29] to obtain

$$\begin{aligned} \|\vartheta^2\|_{L^2((0,N+1);W^{2,6}(\Omega))} + \|\vartheta_t^2\|_{L^2((0,N+1);L^6(\Omega))} &\leq c\|g^2\|_{L^2((N-1,N+1);L^6(\Omega))} \\ &\leq c(E(0)e^{-\varepsilon t} + c_f)^{\frac{7}{4}}. \end{aligned}$$

Then, the embedding theorem [6, Chapter III, Theorem 4.10.2] entails

$$\|\vartheta^2\|_{BUC([0,N+1];(L^6(\Omega),W^{2,6}(\Omega))_{1/2,2})} \leq c(E(0)e^{-\varepsilon t} + c_f)^{\frac{7}{4}}. \quad (3.29)$$

We recall that the interpolation space $(L^6(\Omega), W^{2,6}(\Omega))_{1/2,2} = \mathcal{B}_{6,2}^1(\Omega)$ is continuously embedded into $C(\bar{\Omega})$. This gives a bound for $\|\vartheta^2(t)\|_{C(\bar{\Omega})}$ which is still independent of N . Taking $\alpha > \frac{3}{4}$ and summing up (3.28) and (3.29), we finally deduce

$$\|\vartheta(t)\|_{C(\bar{\Omega})} \leq M_1 E(0)^{\frac{1}{2}} e^{-\frac{mt}{2}} + M_2 E(0)^{\frac{7}{4}} e^{-\frac{7\varepsilon t}{4}} + M_f, \quad (3.30)$$

for some constant $M_1, M_2, M_f > 0$, among which only M_f depends on f , and for any $t \geq 3$. This completes the proof of Theorem 2.1 in the case of $r = 1$ (cf. (2.16)). The modifications required to deal with a general $r > 0$ will be outlined below. \square

3.2. Proof of the separation property (Theorem 2.3). Recalling (2.4), we set

$$R_0 = LM_f. \quad (3.31)$$

Then, if $R > R_0$, i.e., the compatibility condition holds, on account of (3.30) we can choose $T_0 \geq 3$ such that

$$LM_1 E(0)^{\frac{1}{2}} e^{-\frac{mT_0}{2}} + LM_2 E(0)^{\frac{7}{4}} e^{-\frac{7\varepsilon T_0}{4}} < R - R_0. \quad (3.32)$$

Therefore the function

$$u \mapsto \phi(u) := L \max\{|\vartheta(x, t)|, x \in \bar{\Omega}, t \geq T_0\} - \partial W(u) \quad (3.33)$$

is strictly negative in a left neighborhood of $u = 1$. We remark once more that ϕ is actually a multifunction. Nevertheless, we shall still reason as if ϕ were smooth, since considering the general (multivalued) case gives notation problems only, but no further technical complications.

Consequently, if $1 \in D(\beta)$, then we can compare χ with the solution \bar{u} to the Cauchy problem

$$\begin{cases} \bar{u}' = \phi(\bar{u}) \\ \bar{u}(T_0) = 1, \end{cases} \quad (3.34)$$

and the comparison principle immediately entails, for any $t \geq T_0$,

$$\chi(x, t) \leq \bar{u}(t) \quad \forall x \in \bar{\Omega}. \quad (3.35)$$

If (H7s) holds, then (3.34) can be modified by taking instead

$$\begin{cases} \bar{u}' = \phi(\bar{u}) \\ \bar{u}(T_0) = \max\{|\chi(x, t)|, x \in \bar{\Omega}, t \geq T_0\}, \end{cases} \quad (3.36)$$

where the right hand side of the last equation is a uniformly bounded quantity in view of estimate (3.18). In both cases, thanks to the properties of ϕ , it is clear that the upper bound in (S) holds by choosing any $T > T_0$ and

$$\delta = \delta^+ := 1 - (\max\{u : \phi(u) = 0\} \vee \bar{u}(T)). \quad (3.37)$$

Next, the procedure can be repeated (with obvious adjustments) in order to get the lower bound in (S) with respect to a possibly different number $\delta^- > 0$. Coupling the two bounds leads exactly to (S) with $\delta := \min\{\delta^+, \delta^-\}$. This ends the proof. \square

Proof of Corollary 2.5. We proceed along the same lines as before. However, we have to refine the Seventh Estimate in the proof of Theorem 2.1 by making explicit the dependence on $r > 0$ (and not just take $r = 1$). We do not give the technical details. The main difference is that now one has to take a generic $t \geq 3r$ and choose $N \in \mathbb{N}$ such that $t \in (Nr, (N+1)r]$. Then, one replaces the domains of integration accordingly and repeats all the subsequent procedure, taking care of the dependence on r of the various constants. The evident conclusion is that the constant R_0 defined in (3.31) is itself depending on r as well (and, of course, it blows up as $r \searrow 0$). Nevertheless, since we now have the strong coercivity (H7s), the function ϕ defined in (3.33) will be strictly negative in a left neighborhood of $u = 1$ whatever is the value taken for $T_0 > 0$ (namely, we do not need to make the choice (3.32)). At this point, we conclude as before by a comparison argument with respect to the system (3.36). \square

Proof of Corollary 2.6. The assertion follows by repeating once more the argument from (3.33) and noting that $\vartheta(t) \rightarrow 0$ as $t \rightarrow +\infty$ in the space $C(\bar{\Omega})$, which is proved in the next Lemma 3.1. \square

3.3. Proof of Theorem 2.7. Recalling (3.1) and using (H8) together with (3.6), we deduce

$$\|\vartheta\|_{L^2(0,+\infty;V_0)} + \|\chi_t\|_{L^2(0,+\infty;H)} \leq c_f. \quad (3.38)$$

Then, thanks to (2.16) and (3.38), from equation (2.12) we infer

$$\vartheta_t, B\vartheta \in L^2((1, +\infty); H).$$

Thus, using the uniform bound (3.21) and the compact immersion (3.22), it is not difficult to deduce that

$$\vartheta(t) \rightarrow 0 \quad \text{in } V \quad \text{as } t \rightarrow +\infty. \quad (3.39)$$

Furthermore, on account of (3.28) and (3.29) it is

$$\vartheta \in C([3, +\infty); X^\alpha + \mathcal{B}_{6,2}^1(\Omega)).$$

Since both the spaces X^α (for $\alpha > \frac{3}{4}$) and $\mathcal{B}_{6,2}^1(\Omega)$ are compactly embedded into $C(\bar{\Omega})$ we readily obtain

$$\vartheta(t) \rightarrow 0 \quad \text{in } C(\bar{\Omega}) \quad \text{as } t \rightarrow +\infty.$$

On account of the above considerations and keeping (3.18) in mind, one easily realizes that the following lemma holds.

Lemma 3.1. *Let the hypotheses (H1)–(H6) and (H8) be satisfied. Then*

$$\vartheta(t) \rightarrow 0 \quad \text{in } V_0 \cap C(\bar{\Omega}) \quad \text{as } t \rightarrow +\infty. \quad (3.40)$$

Moreover, any sequence of times $t_n \rightarrow +\infty$ contains a subsequence (not relabeled) such that

$$\chi(t_n) \rightarrow \chi_\infty \quad \text{strongly in } V \cap C(\bar{\Omega}), \quad (3.41)$$

where χ_∞ is a solution to (2.18).

Hence, to conclude the proof we have to show, thanks to the analyticity of W , the whole convergence of $\chi(t)$ to χ_∞ (not just along subsequences). Recall that we assume $f \equiv 0$ just for the sake of simplicity (cf. Remark 2.8, however). We proceed in the same way as in [2], so we just outline the argument.

We first observe that the separation property (S) holds with $R_0 = 0$, due to Corollary 2.6, for some $T > 0$ and some $\delta > 0$. On the other hand, a comparison argument entails

$$\max_{x \in \Omega} |\chi_\infty(x)| < 1.$$

More precisely, since the set of all stationary solutions is compact in $C(\bar{\Omega})$, then there exists $\delta_0 > 0$ such that

$$\max_{x \in \Omega} |\chi_\infty(x)| \leq 1 - \delta_0, \quad (3.42)$$

for any stationary solution χ_∞ , and $|W'(y)| > 0$ for $|y| > 1 - \delta_0$. We can thus modify W' outside of $(-1 + \frac{\delta_1}{2}, 1 - \frac{\delta_1}{2})$, with $\delta_1 := \min\{\delta, \delta_0\}$, in such a way that

$$\begin{aligned} \mathcal{F}(y) &= W'(y) \quad \forall y \in [-1 + \delta_1, 1 - \delta_1], \\ |\mathcal{F}(y)| &\leq c, \quad |\mathcal{F}'(y)| \leq c \quad \forall y \in \mathbb{R}. \end{aligned}$$

To conclude the proof it suffices to show that there exists $T_1 > T$ such that $\chi_t \in L^1(T_1, +\infty; H)$. In fact, consider first the energy inequality (3.1). It is clear that E defined by (3.4) is non-increasing in $(0, +\infty)$. Therefore, on account

of (3.40), we have that $E(t)$ converges, as t goes to $+\infty$, to $I_\infty := I(\chi_\infty)$, where I is the functional defined by

$$I(v) := \frac{1}{2} \|\nabla v\|_H^2 + \int_{\Omega} W(v) \, dx.$$

Integrate now the energy inequality (3.1) with respect to time from $t > 0$ to $+\infty$. This gives

$$\int_t^{+\infty} \left(\frac{\|\vartheta(s)\|_{V_0}^2}{2} + \|\chi_t(s)\|_H^2 \right) ds \leq \frac{\|\vartheta(t)\|_H^2}{2} + I(\chi(t)) - I_\infty. \quad (3.43)$$

Since χ_∞ is smooth enough, we can estimate $I(\chi(t)) - I_\infty$ by a modified version of the Łojasiewicz-Simon inequality. Indeed, following [2, Proposition 4.2], we have that, for any $P > 0$, there exist $\rho \in (0, \frac{1}{2})$ and $\delta_2 = \delta_2(P) \in (0, \delta_1)$, $c_P > 0$ such that

$$\|A(\bar{\chi}) + \partial W(\bar{\chi})\|_{V'} \geq c_P |I(\bar{\chi}) - I(\chi_\infty)|^{1-\rho}, \quad (3.44)$$

for all $\bar{\chi} \in V$ such that $\|\bar{\chi} - \chi_\infty\|_H < \delta_2$ and $|I(\bar{\chi}) - I(\chi_\infty)| < P$.

Thus, from (3.43) we infer (cf. also (2.3))

$$\int_t^{+\infty} (\|\chi_t(s)\|_H^2 + \|\vartheta(s)\|_{V_0}^2) \, ds \leq c (\|\chi_t(t)\|_H^2 + \|\vartheta(t)\|_{V_0}^2)^{\frac{1}{2-2\rho}},$$

for any $t \in \mathcal{M} := \{t > 0 : \|\chi(t) - \chi_\infty\|_H < \delta_2\}$. This implies (see [19, Lemma 7.1], cf. also [2, Lemma 5.1]) $\|\chi_t(\cdot)\|_H \in L^1(\mathcal{M})$, which combined with (3.41) and a simple contradiction argument yields $(T_1, +\infty) \subset \mathcal{M}$ for some $T_1 > T$ large enough. Hence, we obtain $\|\chi_t(\cdot)\|_H \in L^1(T_1, +\infty)$ which entails

$$\chi(t) \rightarrow \chi_\infty \quad \text{strongly in } H, \quad \text{as } t \rightarrow +\infty.$$

Thus, on account of (3.41), the second assertion of Theorem 2.7 follows.

To prove the decay rate estimate, we adapt an argument used, for instance, in [28]. On account of (3.1), we have

$$\frac{d}{dt} E(t) \leq -c (\|\vartheta(t)\|_H^2 + \|\chi_t(t)\|_H^2) \quad \forall t \geq 0. \quad (3.45)$$

Using again (3.44), recalling (2.4), and taking advantage of equation (2.13) and (3.39), we have, for $t > 0$ large enough,

$$\begin{aligned} E(t) - I_\infty &\leq c \left(\|\vartheta(t)\|_H^2 + \|A\chi(t) + \partial W(\chi(t))\|_H^{\frac{1}{1-\rho}} \right) \\ &\leq c (\|\vartheta(t)\|_H + \|\lambda'(\chi(t))\vartheta(t) - \chi_t(t)\|_H)^{\frac{1}{1-\rho}} \\ &\leq c (\|\vartheta(t)\|_H + \|\chi_t(t)\|_H)^{\frac{1}{1-\rho}}. \end{aligned}$$

Thus, we deduce

$$(E(t) - I_\infty)^{2(1-\rho)} \leq c(\|\vartheta(t)\|_H^2 + \|\chi_t(t)\|_H^2). \quad (3.46)$$

Inequalities (3.45) and (3.46) then yield $\frac{d}{dt}(E(t) - I_\infty) + c(E(t) - I_\infty)^{2(1-\rho)} \leq 0$, so that

$$E(t) - I_\infty \leq c(1+t)^{-\frac{1}{1-2\rho}}. \quad (3.47)$$

On the other hand, we have $-\frac{d}{dt}(E(t) - I_\infty)^\rho = -\rho\frac{d}{dt}E(t)(E(t) - I_\infty)^{\rho-1}$. Thus, thanks to (3.45) and (3.46), we infer

$$-\frac{d}{dt}(E(t) - I_\infty)^\rho \geq c(\|\vartheta(t)\|_H + \|\chi_t(t)\|_H).$$

This entails $\frac{1}{c}(E(t) - I_\infty)^\rho \geq \int_t^{+\infty} (\|\vartheta(s)\|_H + \|\chi_t(s)\|_H) ds$, and (2.20) follows from (3.47). \square

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