



Solid mechanics. — *A theory of plastic splashing*, by PIERO VILLAGGIO.

ABSTRACT. — The fall on the ground of a highly deformable body is followed by a remarkable spread of the material on the plane of fall. Here we propose a model for determining the extent of the splashed mass and the time of arrest after the first contact with the ground.

KEY WORDS: Plastic impact; uniform energy method.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 74C15.

1. INTRODUCTION

When a perfectly plastic body falls on a rigid ground it tends to spread horizontally along the surface of impact and, at the same time, to flatten in the vertical direction. Since the material is plastic, the body will maintain its final configuration after the impact ceases. If the squeeze is relevant, we call the phenomenon “squashing” and observe it when, for instance, a rotten fruit falls on the floor, an avalanche reaches the foot of a mountain, or a block of ductile metal is dynamically forged as described by Thomsen *et al.* [8, Ch. 14].

A rigorous theory of splashing is prohibitively difficult since the rapid change of shape of the splashing body occurs by propagation of plastic waves inside the material. But to account for plastic waves is hard even for one-dimensional bodies. In order to simplify the problem a semi-statical theory of impact is commonly accepted. It is based on an energy balance between the initial kinetic energy and the dissipation energy spent at the end of splashing. But this approach too is not easy since it requires the construction of a stress-strain field satisfying equilibrium, plasticity conditions, and incompressibility at the same time.

However, it is possible to avoid these technical difficulties by treating the splashing of bodies of relatively simple geometry and applying the so called method of “uniform energy” in metal processing (Thomsen *et al.* [8, 10.3]). We consider the vertical fall of a circular disk of given radius a_0 and given height h_0 (Fig. 1(a)) onto a smooth rigid horizontal plane. Since the material is perfectly plastic the kinetic energy of the falling body, endowed with a velocity u_0 , will be converted into plastic dissipation energy and the disk will assume a new squeezed configuration with a smaller height h_1 and a larger radius a_1 . These quantities are unknown, but the condition of incompressibility requires $\pi a_0^2 h_0 = \pi a_1^2 h_1$. Also the time taken by the disk to reach the splashed shape (b) is unknown.

The treatment of impact problems by equating the energies of an elastic body before and after the first instant of collision was first introduced by Cox [2] in his approximate theory of longitudinally struck rods. The extension of Cox’s method to plastic materials is

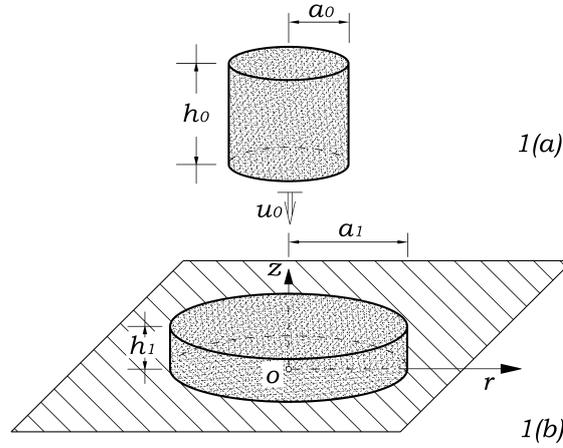


FIG. 1. (a) The disk just before the impact. (b) The completely splashed disk.

immediate for rods, but it is not easy for three-dimensional bodies. The method of uniform energy offers an ingenious device for overcoming this difficulty.

It may appear that a so simplistic theory applied to a particular geometry catches only the order of magnitude of the expected effects and not their exact values, but a comparison with experiments done on the impact of projectiles on rigid targets show that the theoretical previsions are also accurate. A less precise answer can be instead expected for the fall of avalanches, which are described by a different mathematical model. But even in this case the outcome of the uniform energy method is in surprising accord with observations.

2. BASIC ASSUMPTIONS AND ENERGY BALANCE

As soon as the lower face of the disk touches the rigid plane, stresses and strain arise in the interior of the disk, but their exact evaluation is impossible. Hence some assumptions are expedient.

Let us consider the disk in its final splashed state shown in Fig. 1(b). With reference to a system of cylindrical coordinates having the origin O placed at the centre of the bottom face, the z -axis coinciding with the axis of the disk, and the r, θ -axes placed on the plane, the cylindrical disk occupies the region $0 \leq r \leq a$, $0 \leq z \leq h_1$, $0 \leq \theta \leq 2\pi$. We assume the strain state to be defined by three normal strain components $\varepsilon_r, \varepsilon_z, \varepsilon_t$ in the r, z, θ directions respectively. Since these strains are large it is convenient to define them in the logarithmic form (cf. Thomsen *et al.* [8, p. 60])

$$(1) \quad \varepsilon_z = \ln \frac{h_1}{h_0}, \quad \varepsilon_r = \varepsilon_t = \ln \frac{a_0}{a_1}.$$

Note that the hypothesis of conservation of volume ($\pi a_0^2 h_0 = \pi a_1^2 h_1$) yields

$$(2) \quad \ln \frac{h_1}{h_0} + 2 \ln \frac{a_0}{a_1} = 0,$$

or

$$(3) \quad \varepsilon_t = \varepsilon_r = -\frac{\varepsilon_z}{2}.$$

After the first contact a state of stress arises in the disk. Its exact evaluation is again involved, since, beside the (dynamic) field equations, also the boundary conditions on the faces must be satisfied. However, in order to avoid these difficulties we consider the simple distribution of stresses

$$(4) \quad \sigma_z = C = \text{const}, \quad \sigma_r = \sigma_t = \tau = 0,$$

which satisfies the (statical) field equation but not all the boundary conditions. In particular, $\sigma_z = C = \text{const}$ violates the expected requirement that σ_z vanishes on the upper face $z = h_1$, that is free. The constant C in (4) must be determined by the condition of plasticity. If the material obeys the Huber–Hencky criterion of plasticity (see Nadai [5, p. 210]), the condition is

$$(5) \quad \sigma_z = C = -\sigma_{V_0},$$

where σ_{V_0} is a material constant. The negative sign indicates that σ_z is a compression.

In order to evaluate the dissipation energy it is first necessary to write the stress deviator associated with the stress state (4):

$$(6) \quad \sigma'_r = \sigma'_t = \frac{1}{3}\sigma_{V_0}, \quad \sigma'_z = -\frac{2}{3}\sigma_{V_0}, \quad \tau' = 0.$$

Hence the dissipation energy per unit volume is (cf. Szabó [7, §16])

$$dW_p = \sigma'_z \varepsilon_z + \sigma'_r \varepsilon_r + \sigma'_t \varepsilon_t,$$

and the total dissipation energy is

$$(7) \quad W_p = \int \int \int_V (\sigma'_z \varepsilon_z + \sigma'_r \varepsilon_r + \sigma'_t \varepsilon_t) dV,$$

where $V (= V_0)$ denotes the volume. Substitution of (1) and (6) into (7) yields

$$(8) \quad W_p = \sigma_{V_0} \left[-\frac{2}{3} \ln \frac{h_1}{h_0} + \frac{2}{3} \ln \frac{a_0}{a_1} \right] V_0 = -\sigma_{V_0} \pi a_0^2 h_0 \ln \frac{h_1}{h_0}.$$

On the other hand, the kinetic energy of the disk before the impact is $T = \frac{1}{2} \rho V_0 u_0^2$, where ρ is the density and u_0 the velocity. Since conservation of energy requires the equation

$$T_0 = \frac{1}{2} \rho V_0 u_0^2 = W_p = -\sigma_{V_0} V_0 \ln \frac{h_1}{h_0},$$

we obtain

$$(9) \quad h_1 = h_0 \exp\left(-\frac{\rho u_0^2}{2 \sigma_{V_0}}\right).$$

As a numerical illustration let us consider the fall of a cylindrical mass of initial height $h_0 = 1$ m, radius $a_0 = 1$ m, velocity $u_0 = 10$ m/sec, density $\rho \sim 10^2$ Kg sec²/m⁴ (the density of water), and yields stress $\sigma_{V_0} = 10^4$ Kg/m² (that of a soft clay; Schleicher [6, p. 42]). Then formula (9) gives $h_1 \simeq e^{-1/2} = 0.606$ m, and $a_1 = a_0 \sqrt{\frac{h_0}{h_1}} \simeq 1.285$ m.

It is interesting to compare this result with the outcomes of the experiments performed by Taylor (cf. Goldsmith [3, p. 187]) on lead bullets fired against an armor plate. For lead we take $\rho \sim 10^3$ Kg sec²/m⁴, $\sigma_{V_0} = 5 \cdot 10^6$ Kg/m². For a lead cylinder, fired with velocity $v_0 = 10^2$ m/sec against a rigid target, the ratio h_1/h_0 is about 0.5, whereas formula (9) yields the value 0.330.

3. THE TIME OF SPLASHING

The energy method gives the values of the dimensions of the disk (h_1, a_1) at the end of the process of splashing, but does not estimate the velocity with which splashing propagates. In order to answer this question it is necessary to make recourse to the momentum balance equation with respect to the vertical motion (parallel to the z -axis) of the disk. Let us concentrate the mass of the body in its instantaneous center of mass $r = 0, z = h(t)/2$ (Fig. 1(b)) where $h(t)$ is now a function of time. The reacting force, according to (5), is $P = -\sigma_{V_0} \pi a^2$. Hence the equation of motion is

$$(10) \quad M \frac{\ddot{h}}{2} + P = \rho \pi a^2 h \frac{\ddot{h}}{2} + \sigma_{V_0} \pi a^2 = 0,$$

which can be reduced to

$$(11) \quad \ddot{h} + \frac{2\sigma_{V_0}}{\rho} \frac{1}{h} = 0,$$

with the initial conditions $h(0) = h_0, \dot{h}(0) = u_0$. The implicit solution of (11), taking account of the initial conditions, is

$$(12) \quad \int_h^{h_0} \frac{dh}{\sqrt{u_0^2 - \frac{4\sigma_{V_0}}{\rho} \ln \frac{h}{h_0}}} = t,$$

and the time T at which h reaches its final value h_1 is given by

$$(13) \quad \int_{h_1}^{h_0} \frac{dh}{\sqrt{u_0^2 - \frac{4\sigma_{V_0}}{\rho} \ln \frac{h}{h_0}}} = T.$$

The integral on the left hand side of (13) is not elementarily calculable, but may be roughly estimated from above and below (cf. Kauderer [4, §43]). Since the logarithmic term under the square root is positive and satisfies the inequalities

$$0 \leq -\ln \frac{h}{h_0} \leq -\ln \frac{h_1}{h_0} = \frac{\rho}{2} \frac{u_0^2}{\sigma_{V_0}},$$

we obtain from (13) the upper bound

$$T^+ = \int_{h_1}^{h_0} \frac{dh}{u_0} = \frac{h_0 - h_1}{u_0} = \frac{h_0}{u_0} \left(1 - \exp\left(-\frac{\rho}{2} \frac{u_0^2}{\sigma_{V_0}}\right) \right),$$

and the lower bound

$$T^- = \int_{h_1}^{h_0} \frac{dh}{\sqrt{u_0^2 - \frac{4\sigma_{V_0}}{\rho} \ln \frac{h_1}{h_0}}} = \frac{h_0 - h_1}{\sqrt{3}u_0} = \frac{h_0}{\sqrt{3}u_0} \left(1 - \exp\left(-\frac{\rho}{2} \frac{u_0^2}{\sigma_{V_0}}\right) \right).$$

It may be useful to exemplify these formulae with the numerical data chosen at the end of Section 2. The bounds are

$$T^- = 0.0227 \text{ sec}, \quad T^+ = 0.0394 \text{ sec}.$$

The surprising conclusion deriving from these rough estimates is that complete splashing occurs in a short time even for relatively soft bodies like that considered in the example. The time of complete arrest has the order of magnitude of three hundredths of a second. With the same numerical data we find that the mean horizontal propagation velocity of the disk is about 10 m/sec, which explains, *inter alia*, the rapid spread of avalanches. The measured value of the propagation velocity of the edge of an avalanche in the deposition zone ranges from 8 to 20 m/sec (cf. Bozhinskiy and Losev [1, p. 175]).

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