J. Eur. Math. Soc. 9, 77–83

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The inverse mean curvature flow and p-harmonic functions

Received October 12, 2005

Abstract. We consider the level set formulation of the inverse mean curvature flow. We establish a connection to the problem of p-harmonic functions and give a new proof for the existence of weak solutions.

1. The problem

For $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be an open set with smooth boundary such that its complement, $\Omega^c = \mathbb{R}^n \setminus \Omega$, is bounded. We study the problem

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u| \qquad \text{in } \Omega, \tag{1}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{2}$$

This can be regarded as a level set formulation of a parabolic evolution problem for hypersurfaces in $\mathbb{R}^n$: Suppose $F : M^{n-1} \times [0, T) \to \mathbb{R}^n$ is a family of embedded hypersurfaces evolving by

$$\frac{\partial F}{\partial t} = -\frac{H}{|H|^2},$$

where H is the mean curvature vector of $M_t = F(M, t)$ (with a sign convention such that round spheres expand under the flow). If a function $u : \Omega \to [0, \infty)$ exists on a certain open set $\Omega \subset \mathbb{R}^n$, such that $u \equiv t$ on M_t, and if this u is sufficiently smooth and satisfies $\nabla u \neq 0$, then it is a solution of (1). If, in addition, $\partial\Omega \subset M_0$, then (2) is satisfied as well.

This evolution problem is called the *inverse mean curvature flow*. It has been studied by Gerhardt [1], Urbas [10], Huisken–Ilmanen [3, 2, 4, 5], Smoczyk [8], and others. The inverse mean curvature flow (on other manifolds than $\mathbb{R}^n$) has been used by Huisken–Ilmanen [3, 4] to prove the Riemannian Penrose inequality from general relativity. Moreover, a theory of weak solutions of (1) was developed in [3, 4], based on a variational principle involving the functionals

$$J_u(w; K) = \int_K (|\nabla w| + w|\nabla u|)\, dx$$

for precompact sets $K \subset \Omega$.

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Definition 1.1. *A function* $u \in C^{0,1}_{\mathrm{loc}}(\Omega)$ *is called a* weak solution *of* (1) *if for every precompact set* $K \subset \Omega$ *and every* $w \in C^{0,1}_{\mathrm{loc}}(\Omega)$ *with* $w = u$ *in* $\Omega \setminus K$, *the inequality*

$$J_u(u; K) \leq J_u(w; K) \tag{3}$$

holds. A weak solution is proper *if*

$$\lim_{|x|\to\infty} u = \infty.$$

One of the main results in [4] is an existence result: For every $\Omega \subset \mathbb{R}^n$ as above, a proper weak solution $u \in C^{0,1}_{\mathrm{loc}}(\overline{\Omega})$ of (1) and (2) exists. Moreover, proper weak solutions of the problem are unique. We give another proof of the existence result in this paper with a completely different method. Our approach is based on an approximation of (1) by the equations

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^p \quad \text{in } \Omega \tag{4}$$

for $p > 1$. We use the following observation: If

$$v = \exp\left(\frac{u}{1-p}\right),$$

then (4) is equivalent to

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0 \quad \text{in } \Omega. \tag{5}$$

This, in contrast to (1), is the Euler–Lagrange equation of a variational problem, even a rather simple one. It is no problem at all to find a function in the homogeneous Sobolev space $\dot{W}^{1,p}(\mathbb{R}^n)$ that solves (5) in Ω and satisfies $v = 1$ in Ω^c. If we can find a limit of such solutions for $p \to 1$, this limit is a natural candidate for a solution of our problem. It turns out that this strategy is successful.

Theorem 1.1. *Suppose* $\Omega \subset \mathbb{R}^n$ *is an open set with smooth boundary, such that* Ω^c *is bounded. For* $p > 1$, *let* $v^{(p)} \in \dot{W}^{1,p}(\mathbb{R}^n)$ *solve*

$$\operatorname{div}(|\nabla v^{(p)}|^{p-2}\nabla v^{(p)}) = 0 \quad \textit{in } \Omega,$$

and $v^{(p)} = 1$ *on* Ω^c. *Then*

$$(1-p)\log v^{(p)} \to u$$

locally uniformly in $\overline{\Omega}$, *where* $u \in C^{0,1}_{\mathrm{loc}}(\overline{\Omega})$ *is a proper weak solution of* (1) *and* (2).

This theorem can be interpreted as a result on the behaviour of special p-harmonic functions (namely the ones giving the p-capacity of Ω^c) as p tends to 1. But of course it also implies in particular that a weak solution of the inverse mean curvature flow exists. The proof turns out to be quite simple and direct. In addition to the stated facts, it also gives a gradient bound and an estimate for the growth of u at infinity. A maximum principle and a comparison principle (for solutions constructed with this method) follow directly from the corresponding facts about p-harmonic functions. All of this, however, has already been proved for proper weak solutions of the inverse mean curvature flow by Huisken–Ilmanen [4], in the case of the gradient estimate even a slightly better result.

The method we use gives a link between two problems of different types: the inverse mean curvature flow on the one hand, which is parabolic and not a variational problem, and p-harmonic functions on the other hand, which are solutions of an archetypal elliptic variational problem. Moreover, we obtain a construction of solutions of (1) and (2) with elliptic rather than parabolic methods, which may be helpful when equation (1) is studied independently of the inverse mean curvature flow, as a problem in its own right.

2. Construction of the solutions

In this section we give the proof of Theorem 1.1. Let thus $\Omega \subset \mathbb{R}^n$ be open with smooth boundary, such that Ω^c is bounded. We denote the open ball in $\mathbb{R}^n$ with centre x_0 and radius r by $B_r(x_0)$. Let $R > 0$ be the supremum of all numbers $r > 0$ such that each $x \in \partial\Omega$ is on a sphere $\partial B_r(x_0)$ with $B_r(x_0) \subset \Omega^c$.

Fix $p > 1$, and suppose that $v \in \dot{W}^{1,p}(\mathbb{R}^n)$ is a minimizer of the functional

$$E_p(w) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla w|^p \, dx$$

among all $w \in \dot{W}^{1,p}(\mathbb{R}^n)$ with $w \geq 1$ in Ω^c. Then v solves equation (5) with boundary data $v = 1$ on $\partial\Omega$. If $B_r(x_0) \subset \Omega^c$, the function

$$w(x) = \left(\frac{|x - x_0|}{r}\right)^{(n-p)/(1-p)}$$

is another solution of (5) with $w \leq 1$ on $\partial\Omega$. Since the equation is subject to a comparison principle (see, e.g., Tolksdorf [9]), we have

$$v(x) \geq \left(\frac{|x - x_0|}{r}\right)^{(n-p)/(1-p)}, \quad x \in \Omega.$$

Similarly, if $B_s(y_0)$ is a ball with $\Omega^c \subset B_s(y_0)$, we conclude

$$v(x) \leq \left(\frac{|x - y_0|}{s}\right)^{(n-p)/(1-p)}, \quad x \in \Omega \setminus \{y_0\}.$$

According to the results of Lewis [6], we have $v \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha > 0$ (depending on n and p). Since $\partial\Omega$ is smooth, we can even show that $v \in C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega})$ by the application of a reflection principle and arguments as in [6].

Now let $B_r(x_0) \subset \Omega$ be a fixed ball. With arguments from J. Moser [7] (which are easily adapted to our situation) or with other standard arguments, we prove the Harnack inequality

$$\sup_{B_{r/2}(x_0)} v \leq C_1 \inf_{B_{r/2}(x_0)} v$$

for a certain constant C_1 that depends only on n and p. If $\eta \in C_0^\infty(\Omega)$ is a cut-off function, we compute

$$\int_\Omega \eta^p |\nabla v|^p \, dx = -p \int_\Omega \eta^{p-1} v |\nabla v|^{p-2} \nabla v \cdot \nabla \eta \, dx$$
$$\leq p \left(\int_\Omega \eta^p |\nabla v|^p \, dx \right)^{(p-1)/p} \left(\int_\Omega v^p |\nabla \eta|^p \, dx \right)^{1/p}.$$

Thus

$$\int_\Omega \eta^p |\nabla v|^p \, dx \leq p^p \int_\Omega v^p |\nabla \eta|^p \, dx.$$

Together with the Harnack inequality this gives

$$r^{p-n} \int_{B_{r/4}(x_0)} |\nabla v|^p \, dx \leq C_2 \inf_{B_{r/2}(x_0)} v^p$$

for a constant C_2 that depends only on n and p. Now we apply the results of Lewis [6] again. They imply the existence of a constant C_3, depending on n and p, such that

$$\sup_{B_{r/8}(x_0)} |\nabla v| \leq \frac{C_3}{r} \inf_{B_{r/2}(x_0)} v.$$

In particular we have

$$\lim_{|x| \to \infty} \frac{|\nabla v|}{v} = 0.$$

Next we define

$$u = (1-p) \log v.$$

If $B_r(x_0) \subset \Omega^c$, we have

$$u(x) \leq (n-p) \log\left(\frac{|x - x_0|}{r} \right), \qquad x \in \Omega, \tag{6}$$

and if $\Omega^c \subset B_s(y_0)$,

$$u(x) \geq (n-p) \log\left(\frac{|x - y_0|}{s} \right), \qquad x \in \Omega \setminus \{y_0\}. \tag{7}$$

We know that $u \in C^{1,\alpha}_{\text{loc}}(\overline{\Omega})$ and

$$\lim_{|x| \to \infty} |\nabla u| = 0.$$

Most importantly, u satisfies equation (4) and $u = 0$ on $\partial\Omega$.

Inequality (6) together with the definition of R implies

$$|\nabla u| \leq \frac{n-p}{R} \quad \text{on } \partial\Omega.$$

Thus for every $\beta > (n-p)/R$, the set

$$\Omega_\beta = \{x \in \Omega : |\nabla u(x)| > \beta\}$$

is a bounded, open set with $\overline{\Omega}_\beta \cap \partial\Omega = \emptyset$. On $\partial\Omega_\beta$, we have $|\nabla u| = \beta$.

Differentiating equation (4), we obtain

$$\operatorname{div}[|\nabla u|^{p-2}\nabla u_{x_i} + (p-2)|\nabla u|^{p-4}(\nabla u \cdot \nabla u_{x_i})\nabla u] = p|\nabla u|^{p-2}\nabla u \cdot \nabla u_{x_i}$$

for $i = 1, \ldots, n$, at least in Ω_β. Thus

$$\begin{aligned} &\operatorname{div}[|\nabla u|^{p-2}u_{x_i}\nabla u_{x_i} + (p-2)|\nabla u|^{p-4}u_{x_i}(\nabla u \cdot \nabla u_{x_i})\nabla u] \\ &\quad = p|\nabla u|^{p-2}u_{x_i}\nabla u \cdot \nabla u_{x_i} + |\nabla u|^{p-2}|\nabla u_{x_i}|^2 + (p-2)|\nabla u|^{p-4}\left(\nabla u \cdot \nabla u_{x_i}\right)^2. \end{aligned}$$

For the function $f = |\nabla u|^2$, this means

$$\operatorname{div}\left[\frac{1}{p}\nabla f^{p/2} + (\nabla u \cdot \nabla f^{p/2-1})\nabla u\right] - \nabla u \cdot \nabla f^{p/2} \geq \frac{p-1}{4} f^{p/2-2}|\nabla f|^2.$$

If we write

$$A = \mathrm{id} + (p-2)\frac{\nabla u \otimes \nabla u}{f},$$

the last inequality is equivalent to

$$\operatorname{div}(f^{p/2-1}A\nabla f) - pf^{p/2-1}\nabla u \cdot \nabla f \geq \frac{p-1}{4} f^{p/2-2}|\nabla f|^2.$$

Since $p > 1$, the matrix A is uniformly positive definite. Regarding $f^{p/2-1}A$ and $-pf^{p/2-1}\nabla u$ as the coefficients of a linear partial differential operator, we find that this operator is elliptic and subject to a maximum principle. Hence

$$\sup_{\Omega_\beta} f \leq \sup_{\partial\Omega_\beta} f \leq \beta^2.$$

We conclude that

$$\sup_{\overline{\Omega}} |\nabla u| \leq \frac{n-p}{R}.$$

We also note that u is a minimizer of the functional

$$J_u^p(w; K) = \int_K \left(\frac{1}{p}|\nabla w|^p + w|\nabla u|^p\right) dx$$

for every precompact set $K \subset \Omega$ in the sense that

$$J_u^p(u; K) \leq J_u^p(w; K) \tag{8}$$

whenever $w \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfies $w = u$ in $\Omega \setminus K$. Indeed, for every such w, we have, by (4),

$$\int_K (u - w)|\nabla u|^p \, dx = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot (\nabla w - \nabla u)\, dx \leq \frac{1}{p} \int_K (|\nabla w|^p - |\nabla u|^p)\, dx.$$

It is now easy to complete the proof of Theorem 1.1. If $v^{(p)} \in \dot{W}^{1,p}(\mathbb{R}^n)$ are solutions of (5) with $v^{(p)} = 1$ in Ω^c, they coincide in Ω with the functions considered above, for the solutions are unique. Thus we have uniform gradient bounds for the functions $u^{(p)} = (1 - p)\log v^{(p)}$, namely

$$|\nabla u^{(p)}| \leq \frac{n-p}{R} \quad \text{in } \overline{\Omega}.$$

There exists a sequence $p_k \to 1$ such that $u^{(p_k)} \to u$ locally uniformly in $\overline{\Omega}$ for a function $u \in C^{0,1}_{\mathrm{loc}}(\overline{\Omega})$. With practically the same arguments that were used in [4] to prove a compactness result for weak solutions of (1), we can show that u is a weak solution of (1). We include a version of these arguments below for completeness. First, however, we note the following: Once it is proved that u is a weak solution of (1), we see that it is proper because of (7). Another result of [4] states that proper weak solutions of the problem are unique. Thus u does not depend on the choice of p_k, and we have in fact $u^{(p)} \to u$ locally uniformly in $\overline{\Omega}$ as $p \to 1$.

It remains to show that (3) holds for every precompact set $K \subset \Omega$ and every $w \in C^{0,1}_{\mathrm{loc}}(\Omega)$ with $w = u$ in $\Omega \setminus K$. Suppose such a set K and such a function w are given. Choose $\eta \in C^\infty_0(\Omega)$ with $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in K, and insert $\eta w + (1 - \eta)u^{(p)}$ as a test function into $J^p_{u^{(p)}}(\,\cdot\,; \operatorname{supp}\eta)$ in inequality (8). It follows that

$$\begin{aligned}
&\int_{\operatorname{supp}\eta} \left(\frac{1}{p}|\nabla u^{(p)}|^p + \eta(u^{(p)} - w)|\nabla u^{(p)}|^p\right) dx \\
&\quad\leq \frac{1}{p} \int_{\operatorname{supp}\eta} |\eta\nabla w + (1 - \eta)\nabla u^{(p)} + (w - u^{(p)})\nabla\eta|^p \, dx \\
&\quad\leq \frac{3^{p-1}}{p} \int_{\operatorname{supp}\eta} (\eta^p|\nabla w|^p + (1 - \eta)^p|\nabla u^{(p)}|^p + |w - u^{(p)}|^p|\nabla\eta|^p)\, dx. \qquad (9)
\end{aligned}$$

Choosing first $w = u$ and letting $p \to 1$, we obtain

$$\int_\Omega \eta|\nabla u|\, dx \geq \limsup_{k\to\infty} \int_\Omega \eta|\nabla u^{(p_k)}|^{p_k}\, dx,$$

and we infer $|\nabla u^{(p_k)}|^{p_k} \to |\nabla u|$ in $L^1_{\mathrm{loc}}(\Omega)$. Considering (9) again, we conclude that (3) holds. This completes the proof of Theorem 1.1.

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