

Solved and Unsolved Problems

Michael Th. Rassias (University of Zürich, Switzerland)

Newton has shown us that a law is only a necessary relation between the present state of the world and its immediately subsequent state. All the other laws since discovered are nothing else; they are in sum, differential equations..

Henri Poincaré (1854–1912)

The present column is devoted to Partial Differential Equations (PDEs). The study of PDEs has proved to have a tremendously wide spectrum of applications to various domains, from the study of black holes to mathematical finance. Such equations can be used to describe and quantitatively investigate various and diverse phenomena such as heat, sound, elasticity, fluid dynamics, quantum mechanics, etc.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

211. Recall that a smooth function $u : \mathbf{R}^2 \rightarrow \mathbf{R}$ is called harmonic if

$$\Delta u(x, y) := \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0, \text{ for any } (x, y) \in \mathbf{R}^2.$$

Determine all harmonic polynomials in two real variables.

(Giovanni Bellettini, Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Siena, Italia, and ICTP International Centre for Theoretical Physics, Mathematics Section, Trieste, Italy)

212 Reaction-diffusion systems of the form

$$u_t = Du_{xx} + g(u) + \mu Mu, \quad (x, t) \in \mathbf{R} \times (0, \infty),$$

where

$$u(x, t) \in \mathbf{R}^n, \quad g_i(u) = r_i u_i \left(1 - \sum_{j=1}^n \alpha_j u_j \right), \quad r_i, \alpha_i > 0, \\ i = 1, \dots, n, \quad \mu > 0,$$

and D and M are constant $n \times n$ matrices such that D is positive-definite diagonal and M has strictly positive off-diagonal elements and zero column sums, arise in the modelling of the population densities of n phenotypes of a species that diffuse, compete both within a phenotype and with other phenotypes, and may mutate from one phenotype to another. Denoting the Perron-Frobenius

eigenvalue of a matrix Q by $\eta_{PF}[Q]$ and assuming that the n phenotypes spread together into an unoccupied spatial region at the μ -dependent speed

$$c(\mu) := \inf_{\beta > 0} \eta_{PF} \left[\beta D + \beta^{-1} (\text{diag}(r_1, \dots, r_n) + \mu M) \right],$$

which is determined by the linearisation of the reaction-diffusion system about the extinction steady state $u = (0, \dots, 0) \in \mathbf{R}^n$, prove that spreading speed $c(\mu)$ is a non-increasing function of μ .

(Elaine Crooks, Department of Mathematics, College of Science, Swansea University, Swansea, UK)

213. Consider the second-order PDE with non-constant coefficients,

$$u_{xx} - x^2 u_{yy} = 0.$$

Find at least one family of solutions.

(Jonathan Fraser, School of Mathematics and Statistics, The University of St Andrews, Scotland)

214. Let u solve

$$(\Delta + 200^2 xy^2)u = 1$$

on the triangle $T = \{(x, y) : 0 < x < 1, 0 < y < 1 - x\}$ with zero Dirichlet conditions:

$$u(x, 0) = u(0, y) = u(x, 1 - x) = 0.$$

What are the first 10 significant digits of $u(0.1, 0.2)$?

(Sheehan Olver, Department of Mathematics, Imperial College, London, UK)

215. Let u be an entire harmonic function in \mathbf{R}^n , satisfying $u(x) \geq -c(1 + |x|^m)$ for some constants $c > 0$ and $m \in \mathbf{N}$. Show that u is a polynomial of degree less or equal to m .

(Gantumur Tsogtgerel, McGill University, Department of Mathematics and Statistics, Montreal, Canada)

216. Let $f : [0, \infty) \rightarrow (0, \infty)$ be a continuous function satisfying $f(x) \rightarrow 0$ as $x \rightarrow \infty$, and let

$$\Omega = \{(x, y) \in \mathbf{R}^2 : x > 0, 0 < y < f(x)\}.$$

Exhibit an unbounded function u in Ω , such that $u \in H^k(\Omega)$ for all $k \geq 0$. Here $H^k(\Omega)$ is the standard Sobolev space of functions whose partial derivatives of all orders up to k are square integrable.

(Gantumur Tsogtgerel, McGill University, Department of Mathematics and Statistics, Montreal, Canada)

II Open Problem. New rigorous developments regarding the Fokas method and an open problem
 by A. S. Fokas (DAMTP, University of Cambridge, UK) and T. Özsan¹ (Department of Mathematics, Izmir Institute of Technology, Turkey)

Initial-boundary value problems for nonlinear Schrödinger type equations

Consider the following nonlinear Schrödinger equation (NLS) on a domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$ with $p > 0$, $\kappa \in \mathbb{R} - \{0\}$, $\sigma \in \{2, 4\}$, disregarding for the moment initial and boundary conditions (b.c.):

$$i\partial_t u + (-\Delta)^{\frac{\sigma}{2}} u + \kappa |u|^p u = 0, x \in \Omega, t \in (0, T). \quad (1)$$

This equation is the *classical* NLS when $\sigma = 2$ and the *biharmonic* NLS when $\sigma = 4$.

It is easy to see that if $\Omega = \mathbb{R}^n$, then

$$u_\epsilon(x, t) \doteq \epsilon^{-\frac{\sigma}{p}} u(\epsilon^{-1}x, \epsilon^{-\sigma}t)$$

defines an invariant scaling of the above equation. Namely, u solves (1) on $(0, T)$ iff u_ϵ solves (1) on $(0, \epsilon^\sigma T)$. Moreover,

$$\|u_\epsilon(0)\|_{H_x^s} = \epsilon^{\frac{n}{2} - \frac{\sigma}{p} - s} \|u(0)\|_{H_x^s}.$$

Therefore, if $s < s^* \doteq \frac{n}{2} - \frac{\sigma}{p}$, then both $\|u_\epsilon(0)\|_{H_x^s}$ and the life span of u_ϵ vanish as $\epsilon \rightarrow 0^+$. This suggests that the problem is locally illposed for $s < s^*$ and locally wellposed otherwise, where local wellposedness is in the Hadamard’s sense (existence, uniqueness, and uniform continuity with respect to data for some $T > 0$). It is generally easier to establish such results for $0 \leq s < s^*$ whenever $s^* > 0$ (L^2 -*supercritical*) or for $s < s^* = 0$ (L^2 -*critical*). For instance, at least for the focusing problems for NLS ($\sigma = 2, \kappa < 0$), one can simply reduce the problem to one of blow-up in arbitrarily small time in the case $s^* \geq 0$ by constructing a blow-up solution and rescaling it. However, if $p < \frac{2\sigma}{n}$, then $s^* < 0$, in which case an explicit blow-up solution cannot be constructed. Therefore, one wonders what is the range of s for which local wellposedness fails when $s^* < 0$ (L^2 -*subcritical*). The answer to this question for the Cauchy problem in \mathbb{R}^n is that wellposedness fails in H_x^s indeed for any $s < \max(0, s^*)$ [2]. This is proven by showing that the solution operator is no longer uniformly continuous. These observations (see [2] for further details) motivate us to consider the local wellposedness problem for (1) with respect to the above ranges also in the case of domains with a boundary.

If $\partial\Omega \neq \emptyset$, then (1) also requires appropriate boundary conditions (and compatibility conditions if s is sufficiently large that traces exist) for wellposedness to hold. Recent papers treating the half-space case $\Omega = \mathbb{R}_+^n$ ($n = 1, 2$) for the problem (1) obtained wellposedness for nonnegative s . For instance, in the one dimensional case with $\Omega = \mathbb{R}_+$, the natural space for the data of NLS subject to Dirichlet b.c. $u|_{x=0} = g$ turns out to be $(u(0), g) \in H_x^s(\mathbb{R}_+) \times H_t^{\frac{2s+1}{4}}(0, T)$, see for instance [3], [6], and [8]. On the other hand, this space for the biharmonic NLS subject to Dirichlet–Neumann b.c. $u|_{x=0} = g, u_x|_{x=0} = h$ becomes $(u(0), g, h) \in H_x^s(\mathbb{R}_+) \times H_t^{\frac{2s+3}{8}}(0, T) \times H_t^{\frac{2s+1}{8}}(0, T)$ [7]. In the two dimensional case, the spaces for boundary data turn out to be of Bourgain type [1], [5].

One of the effective methods for the treatment of the above half-space problems is the so-called *Uniform Transform Method* (a.k.a. Fokas method) [4]. It has been shown by many researchers that the

Fokas method is a powerful tool for solving initial – (inhomogeneous) boundary-value problems. Although this method was initially introduced for obtaining formal representation formulas for solutions, it has been shown recently that it can also be used to obtain rigorous wellposedness results in the fractional Sobolev and Bourgain spaces. Initially, nonlinear dispersive partial differential equations (PDEs) with power type nonlinearities such as NLS were treated at the high regularity level with this method by obtaining estimates in the $L_t^\infty H_x^s$ norm with $s > 1/2$, see, e.g., [3] and [5]. In this setting, H_x^s becomes a Banach algebra (i.e., $|uv|_{H_x^s} \lesssim |u|_{H_x^s} |v|_{H_x^s}$) and therefore handling the nonlinearities via contraction is relatively easier. Unfortunately, in the low regularity setting $s \leq \frac{1}{2}$, H_x^s loses its algebra structure and estimates in the $L_t^\infty H_x^s$ norm are not good enough for performing the associated nonlinear analysis. The classical method in the theory of nonlinear dispersive PDEs for dealing with this difficulty is to prove Strichartz type estimates which measure the size and decay of solutions in mixed norm function spaces $L_t^q W_x^{s,r}$, where (q, r) satisfies a special *admissibility* condition intrinsic to the underlying evolution operator. However, proving these inequalities for inhomogeneous initial boundary value problems is generally more difficult than proving them for the corresponding Cauchy problems on the whole space \mathbb{R}^n . It is well known that Strichartz estimates holding on \mathbb{R}^n may fail on a general domain $\Omega \subset \mathbb{R}^n$ or on a manifold M with or without boundary and some loss in regularity is indispensable even in nice and smooth geometries. Researchers have used quite technical tools in order to prove these estimates for inhomogeneous initial boundary value problems even in low dimensional settings. The second author has recently shown, in connection with the biharmonic NLS [7], that the kernel of the integral formula obtained by the Fokas method representing the solution has a nice space-time structure for applying the elementary tools of harmonic analysis such as *Van der Corput lemma* to prove decay properties in the time variable, which eventually yields necessary Strichartz estimates. The time decay of the kernel in Fokas’s integral formula for the solution of the boundary value problem can also be used to prove Strichartz estimates for a wide range of dispersive PDEs, at least in the half-space case.

The literature mentioned above on the local wellposedness for the inhomogeneous boundary value problems for the classical NLS in fractional spaces covers the half-space case in dimensions $n = 1, 2$ and the finite interval case $\Omega = (0, L)$ in dimension $n = 1$. In the latter case, it was found that the boundary data must be taken from $H_t^{\frac{s+1}{2}}(0, T)$ in order to establish the local wellposedness at the level of $H_x^s(0, L)$ [8]. One observes that boundary input was associated with a smoother space compared to the half-line problem in order to get well-posedness in $H_x^s(0, L)$. To the best of our knowledge, there is no work which establishes the local wellposedness for NLS on bounded rectangular domains in $2 + 1$ and higher dimensional settings. Therefore, we would like to end this short note with the following open problem which might be of interest to researchers in analysis of PDEs.

Note

1. T. Özsan’s research is supported by TÜBİTAK 1001 Grant #117F449.

217*. **Open Problem.** Let $\Omega = (a, b) \times (c, d)$ be a rectangle in \mathbb{R}^2 , and consider the NLS in (1) ($\sigma = 2$) with Dirichlet b.c. on all sides of $\partial\Omega$ and initial datum $u_0 \in H^s(\Omega)$. Determine the maximal range of s and the (optimal) function spaces for boundary data for which the local wellposedness for (1) holds true in $H_x^s(\Omega)$.

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III Solutions

204. Note that in any topological space with an isolated point, any two dense sets must intersect. Show that there is a 0-dimensional, Hausdorff topological space X with no isolated points so that still, there are no disjoint dense sets in X .

(Daniel Soukup, Kurt Gödel Research Center, University of Vienna, Austria)

Solution by the proposer.

First proof. Take the set of rational numbers \mathbb{Q} and consider the set \mathcal{T} of all possible 0-dimensional topologies τ on \mathbb{Q} that have no isolated point. For example, the usual Euclidean topology is in \mathcal{T} . Now, note that any chain in \mathcal{T} has an upper bound; indeed, the union of an increasing chain of such topologies forms a basis for an element in \mathcal{T} . Hence, by Zorn’s lemma, there must be a maximal element τ in \mathcal{T} .

We claim that any two τ -dense subsets D, E of \mathbb{Q} must meet. First, note that neither D nor E can have isolated points; indeed, if U is open and $U \cap D$ is a singleton x then $U \setminus \{x\}$ is a non-empty open set that avoids D . But now, if D and E are disjoint, then the topology generated by $\tau \cup \{D, E\}$ is still in \mathcal{T} and a proper extension of τ . \square

Such spaces, with no disjoint dense sets, are called *irresolvable* and the above result was first proved by Hewitt in 1943.² Studying the degrees of resolvability, i.e., the maximal number of pairwise disjoint dense sets in spaces, is still an active area of research.³ In fact, any dense-in-itself compact or metrizable space is *maximally resolvable*, i.e., contains as many pairwise disjoint dense sets as the minimal size of a non-empty open set.⁴

Let us present another, more constructive argument for the existence of irresolvable spaces.

Second proof. We construct a countable, dense subset

$$X = \{x_n : n \in \mathbb{N}\}$$

of the product $2^{2^{\aleph_0}}$ so that X is also irresolvable (in the subspace topology). We proceed by an induction of length 2^{\aleph_0} and at step α , we will specify the coordinates $x_n(\alpha)$. Moreover, we will make sure that

$$X \upharpoonright \alpha = \{x_n \upharpoonright \alpha : n < \omega\}$$

is always dense in 2^α .

Define

$$\{x_n \upharpoonright \omega : n < \omega\}$$

to be an arbitrary dense subset of 2^ω . Now, list all infinite, co-infinite subsets of ω as

$$\{I_\alpha : \omega \leq \alpha < 2^{\aleph_0}\}.$$

These correspond to partitions of X and we will make sure at step α that

$$X_{I_\alpha} = \{x_n : n \in I_\alpha\} \text{ and } X \setminus X_{I_\alpha}$$

cannot both be dense in the final space X . Suppose we defined

$$X \upharpoonright \alpha = \{x_n \upharpoonright \alpha : n < \omega\}$$

already. Now, consider the set $X_{I_\alpha} \upharpoonright \alpha$ and its complement in $X \upharpoonright \alpha$. If both these sets are dense in $X \upharpoonright \alpha$, or equivalently in 2^α , then we simply put $x_n(\alpha) = 0$ if and only if $n \in I_\alpha$. Note that our set $X \upharpoonright \alpha + 1$ remained dense in $2^{\alpha+1}$ and $X_{I_\alpha} \upharpoonright \alpha + 1$ is now clopen in $X \upharpoonright \alpha + 1$. In limit steps of the induction, we simply take unions of the functions $x_n \upharpoonright \alpha$ that we constructed already. This finishes the construction.

It should be clear that X is irresolvable. Indeed, if $A \subset X$ is dense and co-dense then $A \upharpoonright \alpha$ is dense and co-dense in $X \upharpoonright \alpha$ for any $\alpha < 2^{\aleph_0}$. Hence, if $X_{I_\alpha} = A$ then at step α , we must have made $A \upharpoonright \alpha + 1$ clopen. In turn, A is clopen as well, a contradiction. \square

Notes

2. E. Hewitt, A problem of set-theoretic topology. *Duke Math. J.* **10** (1943), 309–333.
 3. Juhász, I., Soukup, L., & Szentmiklóssy, Z. (2006). D-forced spaces: A new approach to resolvability. *Topology and its Applications* **153**(11), 1800–1824.
 4. Ceder, J. (1964). On maximally resolvable spaces. *Fundamenta Mathematicae* **55**(1), 87–93.

Also solved by John N. Daras (Greece), Socratis Varelogiannis (France), Alexander Vauth (Germany)

205. For $X = \{\{x, y\} : x, y \in \mathbb{Q}\}$, find a function $b : X \rightarrow \mathbb{N}$ such that

$$\{b(\{x, y\}) : x, y \in B\} = \mathbb{N},$$

whenever $B \subseteq \mathbb{Q}$ is homeomorphic to \mathbb{Q} .

(Boriša Kuzeljević, University of Novi Sad, Department of Mathematics and Informatics, Serbia)

Solution by the proposer. This solution is by James Baumgartner. First fix an enumeration of $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, fix a set

$$N(q_n, q_0), N(q_n, q_1), \dots, N(q_n, q_n)$$

of pairwise disjoint neighbourhoods of q_0, \dots, q_n , respectively. Now define a function $f : X \rightarrow \mathbb{N}$. For $\{q_m, q_n\}$ in X : if $m < n$ and there is $i < m$ so that $q_n \in N(q_m, q_i)$, then let

$$f(\{q_m, q_n\}) = \{q_i, q_m\}.$$

Otherwise let $f((q_m, q_n))$ undefined. Denote

$$f^l(\{x, y\}) = f(\{x, y\}), \text{ and } f^{n+1}(\{x, y\}) = f(f^n(\{x, y\}))$$

for each $n \geq 1$ and $\{x, y\} \in X$. By definition of f , for a fixed $x, y \in \mathbb{Q}$, there is $n \geq 1$ for which $f^n(\{x, y\})$ is undefined. Define $b(\{x, y\})$ to be the least n such that $f^{n+1}(\{x, y\})$ is undefined. Suppose that $B \subseteq \mathbb{Q}$ is homeomorphic to \mathbb{Q} . We prove by induction on $l \in \mathbb{N}$ that

$$\{0, \dots, 2l - 1\} \subseteq \{b(\{x, y\}) : x, y \in B\}.$$

This will finish the proof.

Let $l = 1$. There is $q_n \in B \cap N(q_0, q_0)$ for $n > 0$. Note that $f(\{q_0, q_n\})$ is undefined. Also, $N(q_n, q_0) \cap B$ is infinite since q_0 is a limit point of B . So if $x_k \in N(q_n, q_0) \cap B$ and $k > n$, then

$$f(\{q_n, q_k\}) = \{q_0, q_n\}.$$

Hence $b(\{q_0, q_n\}) = 0$, while $b(\{q_n, q_k\}) = 1$.

Now suppose that $l \geq 1$ and that

$$\{0, \dots, 2l - 1\} \subseteq \{b(\{x, y\}) : x, y \in B\}.$$

By inductive hypothesis, there are q_m and q_n in B such that $b(\{q_m, q_n\}) = 2l - 1$. Suppose that $m < n$. Since q_m, q_n are limit points of B , there are

$$q_i \in N(q_n, q_m) \cap B \text{ and } q_j \in N(q_i, q_n) \cap B,$$

where $j > i > n$. Now

$$f(\{q_i, q_j\}) = \{q_n, q_i\} \text{ and } f(\{q_n, q_i\}) = \{q_m, q_n\},$$

so

$$b(\{q_n, q_i\}) = 2l \text{ and } b(\{q_i, q_j\}) = 2l + 1.$$

□

Also solved by Mihaly Bencze (Romania), Socratis Varelogiannis (France).

206. Suppose that (G, \cdot) is a group, with identity element e and (G, τ) is a compact metrisable topological space. Suppose also that $L_g : (G, \tau) \rightarrow (G, \tau)$ and $R_g : (G, \tau) \rightarrow (G, \tau)$ defined by, $L_g(x) := g \cdot x$ and $R_g(x) := x \cdot g$ for all $x \in G$, are continuous functions. Show that (G, \cdot, τ) is in fact a topological group.

(Warren B. Moors, Department of Mathematics, The University of Auckland, New Zealand)

Solution by the proposer. Let $\pi : G \times G \rightarrow G$ be defined by $\pi(h, g) := h \cdot g$ for all $(h, g) \in G \times G$. We will first show that there exists an element $h_0 \in G$ such that π is continuous at (h_0, e) . Let $(V_n : n \in \mathbb{N})$ be a countable base for the topology on (G, τ) . For each $(m, n) \in \mathbb{N} \times \mathbb{N}$, let

$$F_{(m,n)} := \{g \in G : L_g(V_m) \subseteq \overline{V_n}\}.$$

Then, since each R_g is continuous, each set $F_{(m,n)}$ is closed. For each $(m, n) \in \mathbb{N} \times \mathbb{N}$, let $D_{(m,n)} := \text{Bd}(F_{(m,n)}) = F_{(m,n)} \setminus \text{int}(F_{(m,n)})$. Then each $D_{(m,n)}$ is closed and has no interior.

We claim that π is continuous at each point of

$$\left(G \setminus \bigcup_{(m,n) \in \mathbb{N} \times \mathbb{N}} D_{(m,n)}\right) \times G;$$

which is nonempty, by the Baire category theorem. Let

$$(h_0, g) \in \left(G \setminus \bigcup_{(m,n) \in \mathbb{N} \times \mathbb{N}} D_{(m,n)}\right) \times G$$

and let W be an open neighbourhood of $\pi(h_0, g)$. By appealing to the regularity of (G, τ) there exists an $n \in \mathbb{N}$ such that

$$\pi(h_0, g) \in V_n \subseteq \overline{V_n} \subseteq W.$$

Since L_{h_0} is continuous at g there exists an $m \in \mathbb{N}$ such that $g \in V_m$ and $L_{h_0}(V_m) \subseteq V_n$. Hence, $h_0 \in F_{(m,n)}$ and so

$$h_0 \in F_{(m,n)} \setminus \bigcup_{(m',n') \in \mathbb{N} \times \mathbb{N}} D_{(m',n')} \subseteq F_{(m,n)} \setminus D_{(m,n)} \subseteq \text{int}(F_{(m,n)}).$$

Let $U := \text{int}(F_{(m,n)})$. Then $h_0 \in U$ and

$$\pi(U \times V_m) \subseteq \overline{V_n} \subseteq W.$$

This shows that π is continuous at each point of $\{h_0\} \times G$. In particular, at (h_0, e) . We now show that π is continuous at any point of $G \times G$. Let (x, y) be any point of $G \times G$ and let $(x_n : n \in \mathbb{N})$ be a sequence in G converging to x and let $(y_n : n \in \mathbb{N})$ be a sequence in G converging to y . Then, $(h_0 \cdot x^{-1} \cdot x_n : n \in \mathbb{N})$ converges to h_0 and $(y_n \cdot y^{-1} : n \in \mathbb{N})$ converges to e . Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} (h_0 \cdot x^{-1} \cdot x_n) \cdot (y_n \cdot y^{-1}) &= \lim_{n \rightarrow \infty} \pi((h_0 \cdot x^{-1} \cdot x_n), y_n \cdot y^{-1}) \\ &= \pi(h_0, e) \\ &= h_0. \end{aligned}$$

and so

$$\begin{aligned} x \cdot y &= (x \cdot h_0^{-1}) \cdot (h_0) \cdot y \\ &= (x \cdot h_0^{-1}) \cdot \left(\lim_{n \rightarrow \infty} (h_0 \cdot x^{-1} \cdot x_n) \cdot (y_n \cdot y^{-1})\right) \cdot y \text{ by above} \\ &= \lim_{n \rightarrow \infty} \left((x \cdot h_0^{-1}) \cdot (h_0 \cdot x^{-1} \cdot x_n) \cdot (y_n \cdot y^{-1}) \cdot y\right) \quad (**) \\ &= \lim_{n \rightarrow \infty} x_n \cdot y_n. \end{aligned}$$

Note that $(**)$ follows from the continuity of the function, $g \mapsto (x \cdot h_0^{-1}) \cdot g \cdot y$. Hence, we have that $\lim_{n \rightarrow \infty} x_n \cdot y_n = x \cdot y$. It now only remains to show that inversion $I : (G, \tau) \rightarrow (G, \tau)$ defined by, $I(x) := x^{-1}$ for all $x \in G$, is continuous on G . In fact, since (G, τ) is compact it is sufficient to show that the graph of I is closed. However,

$$\begin{aligned} \text{Graph}(I) &= \{(x, y) \in G \times G : y = x^{-1}\} \\ &= \{(x, y) \in G \times G : x \cdot y = e\} \\ &= \pi^{-1}(\{e\}); \end{aligned}$$

which is closed, since $\{e\}$ is closed and π is continuous. □

Also solved by Sotirios Louridas (Greece), Alexander Vauth (Germany)

207. We will say that a nonempty subset A of a normed linear space $(X, \|\cdot\|)$ is a *uniquely remotal set* if for each $x \in X$,

$$\{y \in A : \|y - x\| = \sup\{\|a - x\| : a \in A\}\}$$

is a singleton. Clearly, nonempty uniquely remotal sets are bounded. Show that if $(X, \|\cdot\|)$ is a finite-dimensional normed linear space and A is a nonempty closed and convex uniquely remotal subset of X , then A is a singleton set.

(Warren B. Moors, Department of Mathematics, The University of Auckland, New Zealand)

Solution by the proposer. Let A be a nonempty uniquely remotal subset of a finite dimensional normed linear space $(X, \|\cdot\|)$. For each $a \in A$, let $r_a : X \rightarrow [0, \infty)$ be defined by, $r_a(x) := \|x - a\|$ for all $x \in X$. Let $r : X \rightarrow [0, \infty)$ be defined by, $r(x) := \sup_{a \in A} r_a(x)$ for all $x \in X$. Then r is 1-Lipschitz and convex, as it is the pointwise supremum of a family of 1-Lipschitz convex functions. Since A is a nonempty uniquely remotal we can define a function $f_A : X \rightarrow A$ (called the *farthest point mapping*) by,

$$\{f_A(x)\} := \{y \in A : \|y - x\| = r(x)\} \quad \text{for all } x \in X.$$

Since A is closed and bounded, A is compact (in the norm topology). Thus, to show that f_A is continuous, it is sufficient to show that f_A has a closed graph. To this end, suppose that $x = \lim_{n \rightarrow \infty} x_n$ and $y := \lim_{n \rightarrow \infty} f_A(x_n)$. Then, $y \in A$, since A is closed and

$$\begin{aligned} r(x) &= \lim_{n \rightarrow \infty} r(x_n) \\ &= \lim_{n \rightarrow \infty} \|f_A(x_n) - x_n\| \\ &= \left\| \lim_{n \rightarrow \infty} (f_A(x_n) - x_n) \right\| \\ &= \|y - x\|. \end{aligned}$$

Therefore, $y = f_A(x)$. We now apply Brouwer’s fixed-point theorem to the continuous function $(f_A)|_A : A \rightarrow A$ to obtain a fixed-point $x_0 \in A$. That is, $f_A(x_0) = x_0$. Since x_0 is the “farthest point in A ” from x_0 , we must have that $A = \{x_0\}$. \square

Also solved by Mihaly Bencze (Romania), Socratis Varelogiannis (France).

208. Let X be any set. A family \mathcal{F} of functions from X to $\{0, 1\}$ is said to separate *countable sets and points* if for every countable set $B \subseteq X$ and every $x \in X \setminus B$, there is a function $f \in \mathcal{F}$ so that $f(x) = 1$ and $f[B] = \{0\}$.

Let κ and λ be infinite cardinals with $\lambda \leq 2^\kappa$. Give $\{0, 1\}$ the discrete topology and $\{0, 1\}^\lambda$ the usual product topology. Show that the following are equivalent:

1. there is a family \mathcal{F} of λ many functions from κ to $\{0, 1\}$ such that \mathcal{F} separates countable sets and points;
2. there is a subspace $X \subseteq \{0, 1\}^\lambda$ of size κ such that every countable subset of X is closed in X .

(Dilip Raghavan, Department of Mathematics, National University of Singapore, Singapore)

Solution by the proposer. The proof of (1) \implies (2) just requires reinterpreting the functions, but the proof of (2) \implies (1) uses the coding that is used in the proof that large independent families exist.

(1) \implies (2): Let $\{f_\xi : \xi < \lambda\}$ be a 1-1 enumeration of the family \mathcal{F} . Thus for each $\xi < \lambda$, $f_\xi : \kappa \rightarrow \{0, 1\}$. Now for each $\alpha < \kappa$, we define a function $g_\alpha : \lambda \rightarrow \{0, 1\}$ by stipulating that $g_\alpha(\xi) = f_\xi(\alpha)$, for each $\xi < \lambda$. Suppose $B \subseteq \kappa$ is countable and $\beta \in \kappa \setminus B$. By hypothesis there exists $\xi < \lambda$ such that $f_\xi(\beta) = 1$ and $f_\xi(\alpha) = 0$, for all $\alpha \in B$. Thus $g_\beta(\xi) = 1$ and $g_\alpha(\xi) = 0$, for all $\alpha \in B$. So

$$U = \{g \in \{0, 1\}^\lambda : g(\xi) = 1\}$$

is an open neighbourhood of g_β which has empty intersection with $\{g_\alpha : \alpha \in B\}$. This shows that $\{g_\alpha : \alpha < \kappa\}$ is a collection of κ many distinct points of $\{0, 1\}^\lambda$ with the property that every countable subset of it is relatively closed. This proves (2).

(2) \implies (1): Let $\{g_\alpha : \alpha < \kappa\}$ be a 1-1 enumeration of X . Thus for each $\alpha < \kappa$, $g_\alpha : \lambda \rightarrow \{0, 1\}$. Let

$$L = \{\langle s, H \rangle : s \subseteq \lambda \text{ is a finite set and } H \subseteq \{0, 1\}^s\}.$$

The cardinality of L is λ . We will now produce a family

$$\{f_{\langle s, H \rangle} : \langle s, H \rangle \in L\}$$

of functions from κ to $\{0, 1\}$ which separates countable sets from points. For a fixed $\langle s, H \rangle \in L$, define $f_{\langle s, H \rangle} : \kappa \rightarrow \{0, 1\}$ by stipulating that for each $\alpha < \kappa$, $f_{\langle s, H \rangle}(\alpha) = 1$ if and only if $g_\alpha \upharpoonright s \in H$. Suppose $B \subseteq \kappa$ is countable and $\beta \in \kappa \setminus B$. By hypothesis $\{g_\alpha : \alpha \in B\}$ is closed in X , and so g_β is not in the closure of $\{g_\alpha : \alpha \in B\}$. Therefore we can find a finite set $s \subseteq \lambda$ such that the open neighbourhood

$$U = \{g \in \{0, 1\}^\lambda : g \upharpoonright s = g_\beta \upharpoonright s\}$$

of g_β misses $\{g_\alpha : \alpha \in B\}$. Let

$$H = \{g_\beta \upharpoonright s\} \subseteq \{0, 1\}^s.$$

So $\langle s, H \rangle \in L$. Now since $g_\beta \upharpoonright s \in H$, we have $f_{\langle s, H \rangle}(\beta) = 1$. On the other hand, for each $\alpha \in B$, $g_\alpha \notin U$, and so $g_\alpha \upharpoonright s \neq g_\beta \upharpoonright s$. Hence for all $\alpha \in B$, $g_\alpha \upharpoonright s \notin H$, whence $f_{\langle s, H \rangle}(\alpha) = 0$. So the function $f_{\langle s, H \rangle}$ separates B from β . Now

$$\{f_{\langle s, H \rangle} : \langle s, H \rangle \in L\}$$

is a family of *at most* λ many functions from κ to $\{0, 1\}$ which separates countable sets from points. Since the hypothesis is that $\lambda \leq 2^\kappa$, we may enlarge this family, if necessary, by adding λ many distinct functions from κ to $\{0, 1\}$ to produce a family of exactly λ many functions from κ to $\{0, 1\}$ which separates countable sets from points. \square

Also solved by Mihaly Bencze (Romania), John N. Daras (Greece), Sotirios Louridas (Greece).

209. A subset X of a partial order (P, \leq) is *cofinal* in P if for each $p \in P$ there is an $x \in X$ satisfying $p \leq x$. Let $\beta\omega$ denote the Stone–Čech compactification of the natural numbers, and let ω^* denote the Stone–Čech remainder, $\beta\omega \setminus \omega$. A neighbourhood base \mathcal{N}_x at a point x forms a directed partial order under reverse inclusion. A neighbourhood base $(\mathcal{N}_x, \supseteq)$ is said to be *cofinal* in another neighborhood base $(\mathcal{N}_y, \supseteq)$ if there is a map $f : \mathcal{N}_x \rightarrow \mathcal{N}_y$ such that f maps each neighbourhood base at x to a neighborhood base at y . Assume the Continuum Hypothesis. Show that there are at least two points x, y in ω^* with neighbourhood bases $(\mathcal{N}_x, \supseteq)$ and $(\mathcal{N}_y, \supseteq)$ which are cofinally incomparable; that is, neither is cofinal in the other.

(Natasha Dobrinen, Department of Mathematics, University of Denver, USA)

Solution by the proposer. Recall that the points in ω^* are nonprincipal ultrafilters. Let Fin denote the set of all finite nonempty subsets of \mathbb{N} . An ultrafilter \mathcal{U} is *selective* if for any collection $\{U_s : s \in \text{Fin}\}$ of members of \mathcal{U} , there is a selector $X \in \mathcal{U}$ such that for each $s \in \text{Fin}$, $X/s := X \setminus (\max(s) + 1) \subseteq U_s$. Assuming the Continuum Hypothesis, we can build selective ultrafilters by transfinite recursion. As any ultrafilter partially ordered by reverse inclusion is Dedekind complete, one need only consider cofinal maps which are monotone: $Y \supseteq X$ implies $f(Y) \supseteq f(X)$. Identify the collection of all subsets of the natural numbers with the Cantor space C via their indicator functions. Continuity of cofinal maps is with respect to the topology on C .

Claim 1 If \mathcal{U} is selective, then for any monotone cofinal map $f : \mathcal{U} \rightarrow \mathcal{V}$, there is an $X \in \mathcal{U}$ such that f is continuous when restricted to $\{U \in \mathcal{U} : U \subseteq X\}$.

Proof. Given \mathcal{U}, \mathcal{V} and f , for each finite set $s \subseteq \mathbb{N}$, take a set $X_s \in \mathcal{U}$ satisfying the following: $X_s = X_s/s$, and for all $k \leq \max(s)$, $k \in f(s \cup X_s)$ if and only if $k \in f(Y)$ for each $Y \in \mathcal{U}$ with s an initial segment of Y . By monotonicity of f , such an X_s in \mathcal{U} exists. Since \mathcal{U} is selective, there is a member $U \in \mathcal{U}$ such that for each finite set s , $U/s \subseteq X_s$. Then f is continuous when restricted to $\{U \in \mathcal{U} : U \subseteq X\}$: Given $U \subseteq X$ in \mathcal{U} , for any k , let s be any initial segment of U for which $k \leq \max(s)$. Then $k \in f(U) \iff k \in f(s \cup X) \iff k \in f(s \cup X_s)$. \square

Claim 2 There are two selective ultrafilters which are cofinally incomparable.

Proof. Fix an enumeration $\langle f_\alpha : \alpha < \omega_1 \rangle$ of all monotone continuous maps from the Cantor space into itself. An equivalent form of selective ultrafilter is that for each partition of \mathbb{N} into infinitely many pieces, either one piece is in the ultrafilter, or else there is a member of the ultrafilter which intersects each piece exactly once. Fix an enumeration $\langle P_\alpha : \alpha < \omega_1 \rangle$ of all partitions $\{P_\alpha^n : n < \omega\}$ of ω into infinitely many pieces. We construct a sequence of countable filter bases (closed under finite intersection) via transfinite recursion on ω_1 .

Let $\mathcal{U}_0 = \mathcal{V}_0 = \mathcal{F}r$, the Fréchet filter of cofinite sets of natural numbers. For $\alpha < \omega_1$, given countable filter bases \mathcal{U}_α and \mathcal{V}_α , extend them to filter bases $\mathcal{U}_{\alpha+1}$ and $\mathcal{V}_{\alpha+1}$ as follows: If there is an n such that $P_\alpha^n \in \mathcal{U}_\alpha$, let $\mathcal{U}'_\alpha = \mathcal{U}_\alpha$. Otherwise, there is an infinite set X such that for each n , $|X \cap P_\alpha^n| = 1$ and the set $\mathcal{U}_\alpha \cup \{X\}$ generates a proper filter; let \mathcal{U}'_α be the filter base consisting of all intersections of finitely many members of $\mathcal{U}_\alpha \cup \{X\}$. In a similar manner, construct \mathcal{V}'_α .

Since \mathcal{U}'_α is countable, it has a pseudointersection; that is, an infinite set U such that $U \setminus Y$ is finite for each $Y \in \mathcal{U}'_\alpha$. Like-

wise, there is a pseudointersection V for \mathcal{V}'_α . If $V \setminus f_\alpha(U)$ is infinite, let $\mathcal{U}_{\alpha+1} = \mathcal{U}'_\alpha$, and let $\mathcal{V}_{\alpha+1}$ be the filter base generated by $\mathcal{V}'_\alpha \cup \{V \setminus f_\alpha(U)\}$. Otherwise, $V \setminus f_\alpha(U)$ is finite. If there is an infinite subset $V' \subseteq V$ such that $V' \setminus f_\alpha(X)$ is finite for each infinite $X \subseteq U$, then f_α cannot be a cofinal map into any ultrafilter containing V' . In this case, let $\mathcal{U}_{\alpha+1} = \mathcal{U}'_\alpha$ and let $\mathcal{V}_{\alpha+1}$ be the filter base generated by $\mathcal{V}'_\alpha \cup \{V'\}$. The final case is that for each infinite $V' \subseteq V$, there is an infinite $U' \subseteq U$ such that $V' \setminus f_\alpha(U')$ is infinite. In particular, there is an infinite $U' \subseteq U$ such that $V \setminus f_\alpha(U)$ is infinite. In this case, let $\mathcal{U}_{\alpha+1}$ be the filter base generated by $\mathcal{U}'_\alpha \cup \{U'\}$ and $\mathcal{V}_{\alpha+1}$ to be the filter base generated by $\mathcal{V}'_\alpha \cup \{V \setminus f_\alpha(U)\}$.

If $\alpha < \omega_1$ is a limit ordinal, take \mathcal{U}_α to be the union of the \mathcal{U}_β , for $\beta < \alpha$; likewise for \mathcal{V}_α . Once the sequences of filter bases $\langle \mathcal{U}_\alpha : \alpha < \omega_1 \rangle$ and $\langle \mathcal{V}_\alpha : \alpha < \omega_1 \rangle$ are constructed, let \mathcal{U} be an ultrafilter extending $\bigcup_{\alpha < \omega_1} \mathcal{U}_\alpha$ and let \mathcal{V} be an ultrafilter extending $\bigcup_{\alpha < \omega_1} \mathcal{V}_\alpha$. By the construction, \mathcal{U} and \mathcal{V} are selective ultrafilters, and there is no monotone continuous function mapping one cofinally into the other. \square

Also solved by Alexander Vauth (Germany).

Note. A much lengthier construction of 2^{\aleph_1} many cofinally incomparable selective ultrafilters appeared in a paper of Dobrinen and Todorcevic, in 2011. However, the short and straightforward construction of two cofinally incomparable selective ultrafilters provided here did not previously appear in the literature.

We encourage you to submit solutions to the proposed problems and ideas on the open problems. Send your solutions by email to Michael Th. Rassias, Institute of Mathematics, University of Zürich, Switzerland, michail.rassias@math.uzh.ch.

We also solicit your new problems with their solutions for the next “Solved and Unsolved Problems” column, which will be devoted to *Analytic Number Theory*.