Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 25 (2006), 237–248

Existence of Periodic Solutions of a Class of Planar Systems

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Abstract. In this paper, we consider the existence of periodic solutions for the following planar system:

$$Ju' = \nabla H(u) + G(u) + h(t)\,,$$

where the function $H(u) \in C^3(\mathbb{R}^2\backslash\{0\}, \mathbb{R})$ is positive for $u \neq 0$ and positively (q, p)-quasi-homogeneous of quasi-degree pq, $G : \mathbb{R}^2 \to \mathbb{R}^2$ is local Lipschitz and bounded, $h \in L^\infty(0, 2\pi)$ is 2π-periodic and J is the standard symplectic matrix.

Keywords. Periodic solutions, resonance, planar systems
Mathematics Subject Classification (2000). Primary 34C, secondary 34C15, 34C25

1. Introduction

We consider in this paper the existence of periodic solutions for the following planar system

$$Ju' = \nabla H(u) + G(u) + h(t), \qquad ('= d/dt) \tag{1}$$

where $H(u) \in C^3(\mathbb{R}^2\backslash\{0\}, \mathbb{R})$ is positive for $u \neq 0$ and positively (q, p)-quasi-homogeneous of quasi-degree pq, that is, for any $u = (x, y)^T \in \mathbb{R}^2$, $\lambda > 0$,

$$H(\lambda^q x, \lambda^p y) = \lambda^{pq} H(x, y),$$

here $p > 1$ and q is the conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$, the function $G : \mathbb{R}^2 \to \mathbb{R}^2$ is local Lipschitz and bounded, $h = (h_1, h_2) \in L^\infty(0, 2\pi)$ is 2π-periodic and $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is the standard symplectic matrix.

If $G \equiv 0$, system (1) reduces to a Hamiltonian system

$$x' = \frac{\partial \bar{H}}{\partial y}, \quad y' = -\frac{\partial \bar{H}}{\partial x}$$

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with Hamiltonian function $\bar{H} = H(u) + \langle h, u\rangle$.

Under above conditions, it is easy to see that the origin is an isochronous center for the autonomous system

$$Ju' = \nabla H(u)\,, \tag{2}$$

that is, all solutions of (2) are periodic with the same minimal period, which we denote by τ.

For example, let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$H(x,\, y) = \frac{\alpha(x^+)^p + \beta(x^-)^p}{p} + \frac{|y|^q}{q}, \quad G \equiv 0, \quad h(t) = (-f(t),\, 0)$$

with $\alpha > 0$, $\beta > 0$ and satisfy

$$D_p\left(\frac{1}{\alpha^{\frac{1}{p}}} + \frac{1}{\beta^{\frac{1}{p}}}\right) = \frac{2}{n}\,,$$

where the positive constant $D_p > 0$ will be given at the end of this paper (see Example 1) . Then system (1) reduces to

$$x' = \phi_q(y), \quad y' = -\alpha\phi_p(x^+) + \beta\phi_p(x^-) + f(t)$$

which is equivalent to the second order p-Laplacian

$$(\phi_p(x'))' + \alpha\phi_p(x^+) - \beta\phi_p(x^-) = f(t),$$

where $\phi_p(x) = |x|^{p-2}x,\ p > 1$ is a constant and $x^+ = \max\{x,\, 0\}$ is the positive part of x, $x^- = \max\{-x,\, 0\}$ is the negative part of x.

The existence of periodic solutions for second order differential equation

$$x'' + f(x)x' + g(x) = f(t)$$

has aroused the interests of many mathematicians (see, for example, the references [1 – 8] and references therein). Recently, Capietto and Wang [3] studied the following asymmetric nonlinear equation:

$$x'' + f(x)x' + ax^+ - bx^- + g(x) = p(t). \tag{3}$$

Assume $F(x) = \int_0^x f(s)ds$ and $g(x)$ are bounded and $p(t)$ is 2π-periodic and continuous, $a,\, b$ are positive constants satisfying the resonance condition $\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{2}{n}$. Let $\phi(t)$ be the solution of the initial value problem

$$x'' + ax^+ - bx^- = 0, \quad x(0) = 0,\ x'(0) = 1.$$

Assume in addition that the limits $\lim_{x\to\pm\infty} F(x) = F(\pm\infty)$ and $\lim_{x\to\pm\infty} g(x) = g(\pm\infty)$ exist. They showed that (3) has at least one 2π-periodic solution provided that either the function

$$\Sigma_1(\theta) = \frac{n}{\pi}\left[\frac{g(+\infty)}{a} - \frac{g(-\infty)}{b}\right] - \frac{1}{2\pi}\int_0^{2\pi} p(t)\phi(\theta+t)\,dt$$

or the function

$$\Sigma_2(\theta) = \frac{n}{\pi}[F(+\infty) - F(-\infty)] - \frac{1}{2\pi}\int_0^{2\pi} p(t)\phi'(\theta+t)\,dt$$

is of constant sign.

More recently, Fonda [6] considered system (1) with $p = q = 2$, $G \equiv 0$. He gave a general description of the dynamics of the solutions, for example, the existence and multiplicity of 2π-periodic solutions, boundedness and unboundedness of solutions. In this paper, inspired by the works of Fonda, Capietto and Wang, we shall consider the existence of a 2π-periodic solution of system (1). The results of this paper generalize and refine some results of [3] and [6].

Let $S(t) = (S_1(t),\, S_2(t))$ be the solution of (2) satisfying $H(S(t)) \equiv 1$ for all $t \in \mathbb{R}$ and has minimal positive period τ. In this paper, we denote by $\langle a,\, b\rangle$ the scalar product of vectors of a, b.

If we define $\frac{2\pi}{n}$-periodic functions $\lambda_1(\theta)$ and $\mu_0(\theta)$ as

$$\lambda_1(\theta) = \begin{cases} \frac{1}{q}\left(\int_0^{2\pi} h_2(t)S_2(\theta+t)\,dt + n\Phi_1(\theta)\right), & \text{if } p > 2 \\ \frac{1}{2}\left(\int_0^{2\pi} \langle h(t),\, S(\theta+t)\rangle\,dt + n\Phi_2(\theta)\right), & \text{if } p = 2 \\ \frac{1}{p}\left(\int_0^{2\pi} h_1(t)S_1(\theta+t)\,dt + n\Phi_3(\theta)\right), & \text{if } 1 < p < 2 \end{cases} \tag{4}$$

and

$$\mu_0(\theta) = \begin{cases} -\frac{1}{p}\left(\int_0^{2\pi} h_2(t)S_2'(\theta+t)\,dt + n\Psi_1(\theta)\right), & \text{if } p > 2 \\ -\frac{1}{2}\left(\int_0^{2\pi} \langle h(t),\, S'(\theta+t)\rangle\,dt + n\Psi_2(\theta)\right), & \text{if } p = 2 \\ -\frac{1}{q}\left(\int_0^{2\pi} h_1(t+\theta)S_1'(\theta+t)\,dt + n\Psi_3(\theta)\right), & \text{if } 1 < p < 2, \end{cases} \tag{5}$$

where

$$\begin{aligned}\Phi_1(\theta) = g_2(+\infty,\, +\infty)\int_{I_{++}} S_2(\theta+t)\,dt + g_2(+\infty,\, -\infty)\int_{I_{+-}} S_2(\theta+t)\,dt \\ + g_2(-\infty,\, +\infty)\int_{I_{-+}} S_2(\theta+t)\,dt + g_2(-\infty,\, -\infty)\int_{I_{--}} S_2(\theta+t)\,dt\end{aligned}$$

$$\begin{aligned}\Phi_2(\theta) = g_1(+\infty,\, +\infty)\int_{I_{++}} S_1(\theta+t)\,dt + g_1(+\infty,\, -\infty)\int_{I_{+-}} S_1(\theta+t)\,dt \\ + g_1(-\infty,\, +\infty)\int_{I_{-+}} S_1(\theta+t)\,dt + g_1(-\infty,\, -\infty)\int_{I_{--}} S_1(\theta+t)\,dt \\ + g_2(+\infty,\, +\infty)\int_{I_{++}} S_2(\theta+t)\,dt + g_2(+\infty,\, -\infty)\int_{I_{+-}} S_2(\theta+t)\,dt \\ + g_2(-\infty,\, +\infty)\int_{I_{-+}} S_2(\theta+t)\,dt + g_2(-\infty,\, -\infty)\int_{I_{--}} S_2(\theta+t)\,dt\end{aligned}$$

$$\begin{aligned}\Phi_3(\theta) = g_1(+\infty,\, +\infty)\int_{I_{++}} S_1(\theta+t)\,dt + g_1(+\infty,\, -\infty)\int_{I_{+-}} S_1(\theta+t)\,dt \\ + g_1(-\infty,\, +\infty)\int_{I_{-+}} S_1(\theta+t)\,dt + g_1(-\infty,\, -\infty)\int_{I_{--}} S_1(\theta+t)\,dt\end{aligned}$$

and

$$\begin{aligned}\Psi_1(\theta) = -g_2(+\infty,\, +\infty)\int_{I_{++}} \frac{\partial H}{\partial S_1}(\theta+t)\,dt - g_2(+\infty,\, -\infty)\int_{I_{+-}} \frac{\partial H}{\partial S_1}(\theta+t)\,dt \\ - g_2(-\infty,\, +\infty)\int_{I_{-+}} \frac{\partial H}{\partial S_1}(\theta+t)\,dt - g_2(-\infty,\, -\infty)\int_{I_{--}} \frac{\partial H}{\partial S_1}(\theta+t)\,dt\end{aligned}$$

$$\begin{aligned}\Psi_2(\theta) = g_1(+\infty,\, +\infty)\int_{I_{++}} \frac{\partial H}{\partial S_2}(\theta+t)\,dt + g_1(+\infty,\, -\infty)\int_{I_{+-}} \frac{\partial H}{\partial S_2}(\theta+t)\,dt \\ + g_1(-\infty,\, +\infty)\int_{I_{-+}} \frac{\partial H}{\partial S_2}(\theta+t)\,dt + g_1(-\infty,\, -\infty)\int_{I_{--}} \frac{\partial H}{\partial S_2}(\theta+t)\,dt \\ - g_2(+\infty,\, +\infty)\int_{I_{++}} \frac{\partial H}{\partial S_1}(\theta+t)\,dt - g_2(+\infty,\, -\infty)\int_{I_{+-}} \frac{\partial H}{\partial S_1}(\theta+t)\,dt \\ - g_2(-\infty,\, +\infty)\int_{I_{-+}} \frac{\partial H}{\partial S_1}(\theta+t)\,dt - g_2(-\infty,\, -\infty)\int_{I_{--}} \frac{\partial H}{\partial S_1}(\theta+t)\,dt\end{aligned}$$

$$\begin{aligned}\Psi_3(\theta) = g_1(+\infty,\, +\infty)\int_{I_{++}} \frac{\partial H}{\partial S_2}(\theta+t)\,dt + g_1(+\infty,\, -\infty)\int_{I_{+-}} \frac{\partial H}{\partial S_2}(\theta+t)\,dt \\ + g_1(-\infty,\, +\infty)\int_{I_{-+}} \frac{\partial H}{\partial S_2}(\theta+t)\,dt + g_1(-\infty,\, -\infty)\int_{I_{--}} \frac{\partial H}{\partial S_2}(\theta+t)\,dt\end{aligned}$$

with

$$\begin{aligned}I_{++} &= \left\{t \in [0,\, \tfrac{2\pi}{n}] : S_1(\theta+t) > 0,\, S_2(\theta+t) > 0\right\} \\ I_{+-} &= \left\{t \in [0,\, \tfrac{2\pi}{n}] : S_1(\theta+t) > 0,\, S_2(\theta+t) < 0\right\} \\ I_{-+} &= \left\{t \in [0,\, \tfrac{2\pi}{n}] : S_1(\theta+t) < 0,\, S_2(\theta+t) > 0\right\} \\ I_{--} &= \left\{t \in [0,\, \tfrac{2\pi}{n}] : S_1(\theta+t) < 0,\, S_2(\theta+t) < 0\right\},\end{aligned}$$

then we have the following result:

Theorem 1. *Assume* $\frac{2\pi}{\tau} = n \in \mathbb{N}$, $H \in C^3(\mathbb{R}^2, \mathbb{R})$, $h = (h_1, h_2) \in L^\infty(0, 2\pi)$, $G(u) = (g_1(x, y), g_2(x, y)) \in C(\mathbb{R}^2; \mathbb{R}^2)$ *are local Lipschitz and bounded. Moreover, let the limits*

$$\lim_{x,y\to\pm\infty} g_1(x,y) = g_1(\pm\infty,\pm\infty), \qquad \lim_{x,y\to\pm\infty} g_2(x,y) = g_2(\pm\infty,\pm\infty)$$

exist and assume that there exists a constant $\sigma_0 > 0$ *such that the following limits hold:*

$$\lim_{x,y\to\pm\infty} [g_i(x,y) - g_i(\pm\infty,\pm\infty)](x^2+y^2)^{\sigma_0} = 0, \quad i = 1,2.$$

Then system (1) *has at least one* 2π*-periodic solution provided that the function* $\lambda_1(\theta)$ *or the function* $\mu_0(\theta)$ *is of constant sign.*

If we define another two 2π-periodic functions $\lambda_{1+\sigma}(\theta)$ and $\mu_1(\theta)$ as follows: if $p > 2$, $\sigma = p - 2$,

$$\lambda_{1+\sigma}(\theta) = \begin{cases} \frac{1}{p}\int_0^{2\pi} h_1(t)S_1(\theta+t)\,dt, & 0<\sigma<1 \\ \frac{1}{p}\int_0^{2\pi} h_1(t)S_1(\theta+t)\,dt \\ \quad +c_p\int_0^{2\pi} h_2(t)S_2'(\theta+t)\int_0^t h_2(\tau)S_2(\theta+\tau)\,d\tau\,dt, & \sigma=1 \\ c_p\int_0^{2\pi} h_2(t)S_2'(\theta+t)\int_0^t h_2(\tau)S_2(\theta+\tau)\,d\tau\,dt, & \sigma>1 \end{cases}$$

if $p = 2$,

$$\lambda_{1+\sigma}(\theta) = \lambda_2(\theta) = \lambda_1(\theta)\lambda_1'(\theta) \equiv 0 \quad \forall\,\theta \in \mathbb{R},$$

$$\mu_1(\theta) = -\frac{1}{4}\int_0^{2\pi} \langle S''(\theta+t),\, h(t)\rangle \int_0^t \langle S(\theta+\tau),\, h(\tau)\rangle\,d\tau\,dt$$

if $1 < p < 2$, $\sigma = \frac{2-p}{p-1}$,

$$\lambda_{1+\sigma}(\theta) = \begin{cases} \frac{1}{q}\int_0^{2\pi} h_2(t)S_2(\theta+t)\,dt, & 0<\sigma<1 \\ \frac{1}{q}\int_0^{2\pi} h_2(t)S_2(\theta+t)\,dt \\ \quad +c_q\int_0^{2\pi} h_1(t)S_1'(\theta+t)\int_0^t h_1(\tau)S_1(\theta+\tau)\,d\tau\,dt, & \sigma=1 \\ c_q\int_0^{2\pi} h_1(t)S_1'(\theta+t)\int_0^t h_1(\tau)S_1(\theta+\tau)\,d\tau\,dt, & \sigma>1, \end{cases}$$

where $c_p = \frac{(p-2)(p-1)}{p^2} > 0$ for $p > 2$ and $c_q = \frac{2-p}{p^2} > 0$ for $1 < p < 2$, then we have

Theorem 2. *Let the conditions on* H, h *in Theorem* 1 *hold. Assume* $G(u) = \lambda_1(\theta) \equiv 0$, *where* $\lambda_1(\theta)$ *is given by* (4) *with* $\Phi_i(\theta) \equiv 0$, $i = 1, 2, 3$. *Then system* (1) *has at least one* 2π*-periodic solution provided that the function* $\lambda_{1+\sigma}(\theta)$ *(for* $p > 1$*) or the function* $\mu_1(\theta)$ *(for* $p = 2$*) is of constant sign.*

2. Generalized polar coordinates transformation

Since H is positively $(q,\, p)$-quasi-homogeneous of quasi-degree pq, we have for any $\lambda > 0$ and $u = (x,\, y)^T \in \mathbb{R}^2$,

$$H(\lambda^q x,\, \lambda^p y) = \lambda^{pq} H(x,\, y). \tag{6}$$

Taking the derivative of both sides of (6) with respect to λ and then letting $\lambda = 1$, we obtain the generalized Euler's identity

$$\frac{1}{p}\, x\, \frac{\partial H(x,\, y)}{\partial x} + \frac{1}{q}\, y\, \frac{\partial H(x,\, y)}{\partial y} = H(x,\, y). \tag{7}$$

For $r > 0$, $\theta(mod\, 2\pi) \in \mathbb{R}$, we define the generalized polar coordinates transformation $P : (r,\, \theta) \to u$ as

$$P: \;\; u = (x,\, y)^T = \left(r^{\frac{1}{p}} S_1(\theta),\, r^{\frac{1}{q}} S_2(\theta) \right)^T. \tag{8}$$

Then the map P is a diffeomorphism from the half plane $\{r > 0\}$ to $\mathbb{R}^2 \backslash \{(0, 0)\}$ and is area-preserving: $dx \wedge dy = -dr \wedge d\theta$, the functions $r,\, \theta$ are of C^2 as far as $u(t)$ does not cross the origin. By assumption, for all $r > 0$,

$$H\left(r^{\frac{1}{p}} S_1,\, r^{\frac{1}{q}} S_2 \right) = r H(S_1,\, S_2),$$

we get

$$\frac{\partial H}{\partial x} \frac{\partial x}{\partial S_1} = r \frac{\partial H}{\partial S_1}, \quad \frac{\partial H}{\partial y} \frac{\partial y}{\partial S_2} = r \frac{\partial H}{\partial S_2},$$

which implies

$$\frac{\partial H}{\partial x} = r^{1-\frac{1}{p}} \frac{\partial H}{\partial S_1}, \quad \frac{\partial H}{\partial y} = r^{1-\frac{1}{q}} \frac{\partial H}{\partial S_2}.$$

This is equivalent to

$$\nabla H(u) = \left(r^{1-\frac{1}{p}} \frac{\partial H}{\partial S_1},\, r^{1-\frac{1}{q}} \frac{\partial H}{\partial S_2} \right). \tag{9}$$

Substituting (8) into (1) and using (9), we obtain

$$r' J \frac{\partial u}{\partial r} + \theta' J \frac{\partial u}{\partial \theta} = \left(r^{1-\frac{1}{p}} \frac{\partial H}{\partial S_1},\, r^{1-\frac{1}{q}} \frac{\partial H}{\partial S_2} \right) + (G + h). \tag{10}$$

By the generalized Euler's identity (7) and by using $\langle Ju,\, u \rangle = 0$ for any $u \in \mathbb{R}^2$, a scalar product in (10) with $\frac{\partial u}{\partial r}$ yields

$$\begin{aligned} \theta' \left\langle J \frac{\partial u}{\partial \theta},\, \frac{\partial u}{\partial r} \right\rangle &= \left(\frac{1}{p} S_1 \frac{\partial H}{\partial S_1} + \frac{1}{q} S_2 \frac{\partial H}{\partial S_2} \right) + \left\langle (G + h)\,,\, \frac{\partial u}{\partial r} \right\rangle \\ &= 1 + \left\langle (G + h)\,,\, \frac{\partial u}{\partial r} \right\rangle. \end{aligned}$$

But it is not difficult to verify $\langle J\frac{\partial u}{\partial \theta}, \frac{\partial u}{\partial r}\rangle = r^{\frac{1}{p}+\frac{1}{q}-1} = 1$, we get therefore $\theta' = 1 + \langle (G+h), \frac{\partial u}{\partial r}\rangle$. Similarly, a scalar product of (10) with $\frac{\partial u}{\partial \theta}$ yields $r' = -\langle (G+h), \frac{\partial u}{\partial \theta}\rangle$. We get therefore

$$\theta' = 1 + \left\langle (G+h), \frac{\partial u}{\partial r}\right\rangle, \quad r' = -\left\langle (G+h), \frac{\partial u}{\partial \theta}\right\rangle, \tag{11}$$

where $h = h(t)$ and u is given by (8).

Now we discuss (11) according to $p > 2$, $p = 2$ and $1 < p < 2$, separately. Let $\rho = r^{\frac{1}{p}}$ for $p > 2$, then (11) is changed into the form

$$\theta' = 1 + \left\langle G+h, \frac{1}{p}\rho^{-(p-1)}\frac{\partial u}{\partial \rho}\right\rangle, \quad \rho' = -\left\langle G+h, \frac{1}{p}\rho^{-(p-1)}\frac{\partial u}{\partial \theta}\right\rangle, \tag{12}$$

where $u = (\rho S_1(\theta), \rho^{p-1}S_2(\theta))$. Similarly, we let $\rho = r^{\frac{1}{2}}$ for $p = 2$ and $\rho = r^{\frac{1}{q}}$ for $1 < p < 2$, we can obtain similar forms of above approximation. For $\rho_0 \gg 1$, by the boundedness of S, S', G and h, for any $t \in [0, 2\pi]$, we obtain

$$\rho(t) = \rho_0 + O(1), \quad \rho^{-1}(t) = \rho_0^{-1} + O(\rho_0^{-2}), \quad \theta(t) = \theta_0 + t + O(\rho_0^{-1}), \tag{13}$$

where by $O(1)$ we mean a function $a(\rho, t)$ which is bounded uniformly in (ρ, t) for $\rho > 0$, $t \in [0, 2\pi]$ and by $O(\rho^{-k})$ we mean a function $b(\rho, t)$ such that $\|\rho^k b(\rho, t)\|$ is bounded for $\rho > 0$, $t \in [0, 2\pi]$ and $k > 0$.

Substituting (13) in to (12) and integrating over $[0, 2\pi]$ with respect to t, we get, by analyzing the cases $p > 2$, $p = 2$ and $1 < p < 2$ separately, the following asymptotic expression:

$$\theta_1 = \theta_0 + 2\pi + \lambda_1(\theta_0)\rho_0^{-1} + o(\rho_0^{-1}), \quad \rho_1 = \rho_0 + \mu_0(\theta_0) + o(1), \tag{14}$$

where by $o(1)$ we mean a function $A(\rho, \theta)$ which is 2π-periodic in θ and satisfies $\lim_{\rho\to+\infty} A(\rho, \theta) = 0$ uniformly for $\theta \in \mathbb{R}$, and by $o(\rho^{-k})$ we mean a function $B(\rho, \theta)$ which is 2π-periodic in θ and satisfies $\lim_{\rho\to+\infty} \rho^k B(\rho, \theta) = 0$ uniformly in $\theta \in \mathbb{R}$ for $k > 0$. $\lambda_1(\theta)$ and $\mu_0(\theta)$ are given in (4) and (5) respectively.

In case $G \equiv 0$, substituting (13) in to (12) and integrating over $[0, t] \subset [0, 2\pi]$ with respect to t, we get

$$\begin{aligned} \theta(t) &= \theta_0 + t + \lambda_1(\theta_0, t)\rho_0^{-1} + o(\rho_0^{-1}) \\ \rho(t) &= \rho_0 + \mu_0(\theta_0, t) + o(1) \\ \rho^{-1}(t) &= \rho_0^{-1} - \mu_0(\theta_0, t)\rho_0^{-2} + o(\rho_0^{-2}), \end{aligned} \tag{15}$$

where

$$\lambda_1(\theta,\,t)=\begin{cases}\frac{1}{q}\int_0^t h_2(\tau)S_2(\theta+\tau)\,d\tau, & \text{if } p>2\\ \frac{1}{2}\int_0^t\langle h(\tau),\,S(\theta+\tau)\rangle\,d\tau, & \text{if } p=2\\ \frac{1}{p}\int_0^t h_1(\tau)S_1(\theta+\tau)\,d\tau, & \text{if } 1<p<2\end{cases}\tag{16}$$

$$\mu_0(\theta,\,t)=\begin{cases}-\frac{1}{p}\int_0^t h_2(\tau)S_2'(\theta+\tau)\,d\tau, & \text{if } p>2\\ -\frac{1}{2}\int_0^t\langle h(\tau),\,S'(\theta+\tau)\rangle\,d\tau, & \text{if } p=2\\ -\frac{1}{q}\int_0^t h_1(\tau)S_1'(\theta+\tau)\,d\tau, & \text{if } 1<p<2\,.\end{cases}\tag{17}$$

Substituting (15)–(17) in to (12) and integrating over $[0,\,2\pi]$ with respect to t, under the assumption $\lambda_1(\theta)\equiv 0$, after some elementary calculations we get $\mu_0(\theta)=\lambda_1'(\theta)\equiv 0$ and

$$\begin{aligned}\theta_1&=\theta_0+2\pi+\lambda_{1+\sigma}(\theta_0)\rho_0^{-(1+\sigma)}+o\big(\rho_0^{-(1+\sigma)}\big)\\ \rho_1&=\rho_0+\mu_\sigma(\theta_0)\rho_0^{-\sigma}+o(\rho_0^{-\sigma}),\end{aligned}\tag{18}$$

where $\lambda_{1+\sigma}(\theta)$ is given in Theorem 2. Moreover, we have the following relations:

$$\lambda_2(\theta)=\lambda_1(\theta)\lambda_1'(\theta)\equiv 0\quad\text{if } p=2$$

$$\mu_\sigma(\theta)=\begin{cases}-\lambda_{1+\sigma}'(\theta), & 2<p\le 3,\ \sigma=p-2>0\\ -\frac{1}{p-2}\lambda_2'(t), & p>3,\end{cases}\tag{19}$$

$\mu_1(\theta)$ is given in Theorem 2 if $p=2$, and

$$\mu_\sigma(\theta)=\begin{cases}-\lambda_{1+\sigma}'(\theta), & \frac{3}{2}\le p<2,\ \sigma=\frac{2-p}{p-1}>0\\ -\frac{p-1}{2-p}\lambda_2'(\theta), & 1<p<\frac{3}{2}\,.\end{cases}\tag{20}$$

Combining the above discussions, we obtain the following lemmata.

Lemma 1. *Let* $\frac{2\pi}{\tau}=n\in\mathbb{N}$ *and the conditions of Theorem* 1 *hold. Then for* $\rho_0\gg 1$, *the Poincaré map*

$$P:(\theta_0,\,\rho_0)\to(\theta_1,\,\rho_1)=\big(\theta(2\pi;\theta_0,\rho_0),\,\rho(2\pi;\theta_0,\rho_0)\big)$$

of the solution of (11) *with initial value* $(\theta_0,\,\rho_0)$ *has the asymptotic expression of* (14) *where* $\rho=r^p,\,r^{\frac{1}{2}}$ *or* r^q *according to* $p>2,\,p=2$ *and* $1<p<2$, *respectively,* $\lambda_1(\theta)$, $\mu_0(\theta)$ *are given in* (4) *and* (5) *respectively.*

Lemma 2. *Let* $\frac{2\pi}{\tau}=n\in\mathbb{N}$ *and the conditions of Theorem* 2 *hold. Then under the assumption* $\lambda_1(\theta)\equiv 0$ *and for* $\rho_0\gg 1$, *the Poincaré map*

$$P:(\theta_0,\,\rho_0)\to(\theta_1,\,\rho_1)=\big(\theta(2\pi;\theta_0,\rho_0),\,\rho(2\pi;\theta_0,\rho_0)\big)$$

of the solution of (11) *with initial value* $(\theta_0,\,\rho_0)$ *has the asymptotic expression of* (18) *where* $\rho=r^p,\,r^{\frac{1}{2}}$ *or* r^q *according to* $p>2,\,p=2$ *and* $1<p<2$, *respectively,* $\lambda_{1+\sigma}(\theta)$, $\mu_1(\theta)$ *are given in Theorem* 2 *and* $\mu_\sigma(\theta)$ *satisfies* (19) *or* (20). *Moreover, for* $p=2$, $\lambda_2(\theta)=\lambda_1(\theta)\lambda_1'(\theta)\equiv 0$.

3. Proof of Theorems

Proof of Theorem 1. The proof of Theorem 1 is similar to the proof of Theorem A in [3], so we only sketch it.

From Lemma 1, the Poincaré map of the solutions of (11) has the form of (14). If λ_1 is of constant sign, then there exists a constant $c_0 > 0$ such that $\lambda_1(\theta) \geq c_0$ or $\lambda_1(\theta) \leq -c_0$. Therefore, the image (θ_0, ρ_1) of (θ_0, ρ_0) under the map P does not lie on the ray $\theta = \theta_0$ if ρ_0 is large enough. By the Poincaré–Bohl Theorem (see [9]), the map P possesses at least one fixed point, which implies that system (11) and hence system (1) has at least one 2π-periodic solution.

If $\mu_0(\theta)$ is of constant sign, then there exists a constant $c_1 > 0$ such that either (i) $\mu_0(\theta) \leq -c_1 < 0$ or (ii) $\mu_0(\theta) \geq c_1 > 0$ for all $\theta \in \mathbb{R}$. In case (i), we have $\rho_1 < \rho_0$ for ρ_0 large enough. Therefore, the Brouwer fixed theorem ensures the existence of a fixed point of the map of P. Hence system (11) and therefore system (1) has a 2π-periodic solution. In case (ii), we see the map P^{-1} has the corresponding property of P, therefore P^{-1} has a fixed point, which implies that system (11) and therefore system (1) as at least one 2π-periodic solution. □

Proof of Theorem 2. It follows from Lemma 2 that the Poincaré map of the solutions of (11) has the form of (18). The rest of the proof of Theorem 2 is similar to that of Theorem 1, so we omit it. □

Remark 1. If $\lambda_1(\theta) \equiv 0$, the results of [3] and [6] can not be applied here since by (19) and (20), for $p \neq 2$, the function $\mu_\sigma(\theta)$ is either identically zero or changes signs at least two times in $[0, 2\pi)$, by its 2π-periodicity.

Example 1. Let us consider the following planar Hamilton system

$$\begin{aligned} x' &= \ a^+\phi_q(y^+) - a^-\phi_q(y^-) - F(x) + h_2(t) \\ y' &= -b^+\phi_p(x^+) + b^-\phi_p(x^-) - g(x) - h_1(t), \end{aligned} \tag{21}$$

where $a^\pm$, $b^\pm$ are positive constants satisfying

$$D_p\left(\frac{1}{(a^+)^{\frac{1}{q}}(b^+)^{\frac{1}{p}}} + \frac{1}{(a^+)^{\frac{1}{q}}(b^-)^{\frac{1}{p}}} + \frac{1}{(a^-)^{\frac{1}{q}}(b^+)^{\frac{1}{p}}} + \frac{1}{(a^-)^{\frac{1}{q}}(b^-)^{\frac{1}{p}}}\right) = \frac{4}{n} \tag{22}$$

with

$$D_p = \frac{1}{p^{\frac{1}{q}}q^{\frac{1}{p}}}\, B\left(\tfrac{1}{p}, \tfrac{1}{q}\right),$$

where $B(\lambda, \mu) = \int_0^1 t^{\lambda-1}(1-t)^{\mu-1}\,dt$ is the β function for $\lambda, \mu > 0$ and $x^\pm = \max\{\pm x, 0\}$, $y^\pm = \max\{\pm y, 0\}$, $n \in \mathbb{N}, F(x)$, $g(x) \in C$ are bounded and the limits $\lim_{x\to\pm\infty} F(x) = F(\pm\infty)$ and $\lim_{x\to\pm\infty} g(x) = g(\pm\infty)$ exist,

$h_1(t)$, $h_2(t) \in L^\infty(0, 2\pi)$ are 2π-periodic, $\phi_p(u) = |u|^{p-2}u$ for $p>1$. Especially, let $a^+ = a^- = 1$, $b^+ = \alpha$, $b^- = \beta, F = h_1 \equiv 0$, $h_2(t) = e(t)$, then (21) reduces to

$$(\phi_p(x'))' + \alpha\phi_p(x^+) - \beta\phi_p(x^-) + g(x) = e(t)$$

and (22) reduces to

$$D_p\left(\frac{1}{\alpha^{\frac{1}{p}}} + \frac{1}{\beta^{\frac{1}{p}}}\right) = \frac{2}{n}.$$

Let $(S(t), C(t))$ be the solution of the initial value problem

$$\begin{aligned} x' &= a^+\phi_q(y^+) - a^-\phi_q(y^-), \qquad & x(0) &= 0 \\ y' &= -b^+\phi_p(x^+) + b^-\phi_p(x^-), \qquad & y(0) &= q^{\frac{1}{q}}(a^+)^{-\frac{1}{q}}. \end{aligned}$$

Then it is easy to verify the equation

$$H(S(t), C(t)) \equiv 1 \quad \forall\, t \in \mathbb{R},$$

where

$$H(x, y) = \frac{[b^+(x^+)^p + b^-(x^-)^p]}{p} + \frac{[a^+(y^+)^q + a^-(y^-)^q]}{q}.$$

Let $\tau = \frac{2\pi}{n}$ and $\lambda_1(\theta)$, $\mu_0(\theta)$ be defined in Theorem 1, it is not difficult to obtain

$$\lambda_1(\theta) = \begin{cases} \frac{1}{q}\left(\int_0^{2\pi} h_2(t)C(\theta+t)\,dt + n\Phi_1(\theta)\right), & p > 2 \\ \frac{1}{2}\left(\int_0^{2\pi} [h_1(t)S(\theta+t) + h_2(t)C(\theta+t)]\,dt + n\Phi_2(\theta)\right), & p = 2 \\ \frac{1}{p}\left(\int_0^{2\pi} h_1(t)S(\theta+t)\,dt + n\Phi_3(\theta)\right), & 1 < p < 2, \end{cases}$$

where

$$\begin{aligned} \Phi_1(\theta) &= -F(+\infty)\int_{S(\theta+t)>0} C(\theta+t)\,dt - F(-\infty)\int_{S(\theta+t)<0} C(\theta+t)\,dt \\ \Phi_2(\theta) &= -F(+\infty)\int_{S(\theta+t)>0} C(\theta+t)\,dt - F(-\infty)\int_{S(\theta+t)<0} C(\theta+t)\,dt \\ &\quad + g(+\infty)\int_{S(\theta+t)>0} S(\theta+t)\,dt + g(-\infty)\int_{S(\theta+t)<0} S(\theta+t)\,dt \\ \Phi_3(\theta) &= g(+\infty)\int_{S(\theta+t)>0} S(\theta+t)\,dt + g(-\infty)\int_{S(\theta+t)<0} S(\theta+t)\,dt, \end{aligned}$$

and

$$\mu_0(\theta) = \begin{cases} -\frac{1}{p}\left(\int_0^{2\pi} h_2(t)C'(\theta+t)\,dt + n\Psi_1(\theta)\right), & p > 2 \\ -\frac{1}{2}\left(\int_0^{2\pi} [h_1(t)S'(\theta+t) + h_2(t)C'(\theta+t)]\,dt + n\Psi_2(\theta)\right), & p = 2 \\ -\frac{1}{q}\left(\int_0^{2\pi} h_1(t)S'(\theta+t)\,dt + n\Psi_3(\theta)\right), & 1 < p < 2, \end{cases}$$

where

$$\Psi_1(\theta) = -F(+\infty)\int_{S(\theta+t)>0} C'(\theta+t)\,dt - F(-\infty)\int_{S(\theta+t)<0} C'(\theta+t)\,dt$$

$$\Psi_2(\theta) = -F(+\infty)\int_{S(\theta+t)>0} C'(\theta+t)\,dt - F(-\infty)\int_{S(\theta+t)<0} C'(\theta+t)\,dt$$
$$+ g(+\infty)\int_{S(\theta+t)>0} S'(\theta+t)\,dt + g(-\infty)\int_{S(\theta+t)<0} S'(\theta+t)\,dt$$

$$\Psi_3(\theta) = g(+\infty)\int_{S(\theta+t)>0} S'(\theta+t)\,dt + g(-\infty)\int_{S(\theta+t)<0} S'(\theta+t)\,dt$$

with $C' = C'(\theta+t) = -b^+\phi_p(S^+(\theta+t)) + b^-\phi_p(S^-(\theta+t))$.

Let $p=2$, $a^+ = a^- = 1$, $b^+ = a$, $b^- = b$, $h_1(t) = p(t)$, $h_2(t) \equiv 0$, then (21) reduces to (3) with $F(x) = \int_0^t f(s)ds$.

Example 2. Let $p = 2$, $\alpha = \beta = n = 1$, $H(x,y) = \frac{1}{2}(x^2+y^2)$, $h(t) = (h_1(t), h_2(t))^T = (1,\,1)^T$ and

$$g_1(x,y) = g_2(x,y) = \Big(\frac{\pi}{2} + \arctan x\Big)\Big(\frac{\pi}{2} + \arctan y\Big).$$

Then by Theorem 1, it is not difficult to show that $S_1(t) = \sin t$, $S_2(t) = \cos t$ and

$$\lambda_1(\theta) = \frac{\pi^2}{2}\left[\int_{I_{++}} \sin(t+\theta)\,dt + \int_{I_{++}} \cos(t+\theta)\,dt\right] > 0$$

for all $\theta \in \mathbb{R}$. Hence Theorem 1 implies that system (1) has at least one 2π-periodic solution.

Example 3. Let $p > 2$, $\alpha = \beta = n = 1$, $H(x,y) = \frac{1}{p}(|x|^p + |y|^p)$, $h(t) = (h_1(t), h_2(t))^T = (1,\,1)^T$ and $g_1(x,y) = g_2(x,y) \equiv 0$. Then by Theorem 2, it is not difficult to show that $\lambda_1(\theta) \equiv 0$ and, for $\sigma = p-2 \geq 1$, we have

$$\lambda_{1+\sigma} = c_p\int_0^{2\pi} S_2'(\theta+t)\int_0^2 S_2(\tau+\theta)\,d\tau\,dt = c_p\int_0^{2\pi} S_1^2(\theta+t)\,dt > 0$$

for all $\theta \in \mathbb{R}$, where $c_p > 0$ is a constant. Theorem 2 implies that system (1) has at least one 2π-periodic solution.

Remark 2. Similar to Theorem B in [3], we can prove the following result: If the functions λ_1 and μ_0 have zeros and all the zeros are simple and the zeros of λ_1 and μ_0 are different, moreover, if the signs of μ_0 at the zeros of λ_1 in $[0,\,\frac{2\pi}{n})$ do not change or change more than two times, then system (1) has a 2π-periodic solution.

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Received May 25, 2004; revised December 20, 2004