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Sylvia Serfaty

Vortex collisions and energy-dissipation rates in the Ginzburg–Landau heat flow

Part I: Study of the perturbed Ginzburg–Landau equation

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Abstract. We study vortices for solutions of the perturbed Ginzburg–Landau equations $\Delta u + (u/\varepsilon^2)(1 - |u|^2) = f_\varepsilon$ where f_ε is estimated in L^2. We prove upper bounds for the Ginzburg–Landau energy in terms of $\|f_\varepsilon\|_{L^2}$, and obtain lower bounds for $\|f_\varepsilon\|_{L^2}$ in terms of the vortices when these form "unbalanced clusters" where $\sum_i d_i^2 \neq (\sum_i d_i)^2$.

These results will serve in Part II of this paper to provide estimates on the energy-dissipation rates for solutions of the Ginzburg–Landau heat flow, which allow one to study various phenomena occurring in this flow, including vortex collisions; they allow in particular extending the dynamical law of vortices beyond collision times.

Keywords. Ginzburg–Landau equation, Ginzburg–Landau vortices, vortex dynamics, vortex collisions

1. Introduction and statement of the main results

1.1. Presentation of the problem

In this paper, we study the forced Ginzburg–Landau equation

$$\begin{cases} \Delta u + \dfrac{u}{\varepsilon^2}(1 - |u|^2) = f_\varepsilon & \text{in } \Omega, \\ u = g \text{ (resp. } \dfrac{\partial u}{\partial \nu} = 0) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f_ε is a forcing right-hand side which is *given in* $L^2(\Omega)$. Here Ω is a two-dimensional domain, assumed to be smooth, bounded and simply connected, and u is a *complex-valued* function, assumed to satisfy either one of the boundary conditions

$$u = g \quad \text{on } \partial\Omega \tag{1.2}$$

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S. Serfaty: Courant Institute of Mathematical Sciences, 251 Mercer St., New York, NY 10012, USA; e-mail: serfaty@cims.nyu.edu

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with g a fixed regular map from Ω to $\mathbb{S}^1$, in which case we also assume that Ω is strictly starshaped with respect to a point; or

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

in which case no further assumption is made. This equation with $f_\varepsilon = 0$ is the standard Ginzburg–Landau equation, which has been intensively studied, in the asymptotic limit $\varepsilon \to 0$, in particular since the work of Bethuel–Brezis–Hélein [BBH].

Our motivation for studying the L^2 perturbed equation, which we will develop in Part II of this paper [S1], is to study the two-dimensional parabolic Ginzburg–Landau equation:

$$\begin{cases} \dfrac{\partial_t u}{|\log \varepsilon|} = \Delta u + \dfrac{1}{\varepsilon^2} u(1-|u|^2) & \text{in } \Omega \times \mathbb{R}_+, \\ u(\cdot, 0) = u_\varepsilon^0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

with the same boundary conditions as above. However, the results we present here have an interest of their own and can be read independently of Part II.

The Ginzburg–Landau heat flow is an L^2 gradient flow (or steepest descent) for the Ginzburg–Landau functional

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \right). \tag{1.5}$$

This energy functional is a simplified version (without magnetic field) of the Ginzburg–Landau model of superconductivity. Such functionals also appear in other models from physics: for superfluidity, nonlinear optics, Bose–Einstein condensates; and the complex-valued function u, called “order parameter”, plays the role of a condensed wave function.

In this model, the interesting objects are the *vortices*, or the zero-set of the complex-valued function u carrying a topological degree: since u is complex-valued, it can have a nonzero integer degree around each of its zeroes. Vortices can also be seen as having a “core”, where $|u|$ is small, of characteristic length scale ε; and a “tail” where $|u|$ is close to 1, but the phase of u still carries a lot of energy; they can be clearly extracted in the asymptotic limit $\varepsilon \to 0$.

Vortices in the Ginzburg–Landau model have been the object of intensive studies, generally in the asymptotic limit $\varepsilon \to 0$ where they become point singularities, in particular since the work of [BBH] on (1.5), under the assumption $E_\varepsilon(u) \le C|\log \varepsilon|$ (bounding the possible number of vortices); refer also to [SS2] for the analysis of the full model with magnetic field. In both cases, some Γ-convergence type results were obtained.

A very precise description of the vortices and of the energy of (nonminimizing) solutions of the Ginzburg–Landau equation, i.e. (1.1) with $f_\varepsilon \equiv 0$, was given by Comte and Mironescu in [CM1, CM2]. We are interested here in generalizing these results, and in studying how much the situation can differ from the $f_\varepsilon \equiv 0$ case. Since we are interested in studying vortex collisions for solutions of (1.4), we focus on understanding static situations where vortices are very close to each other. We will characterize “pathological vortex situations” for (1.1) as those for which we have a group of vortices which are far from the others, and have degrees d_i with $(\sum_i d_i)^2 \neq \sum_i d_i^2$ in the group, which we call an *unbalanced cluster of vortices*.

We study the equation (1.1) with an L^2 perturbation term because (1.4) is precisely an L^2 gradient flow for (1.5), and thus for u_ε solving (1.4), we have

$$-\frac{d}{dt}E_\varepsilon(u_\varepsilon(x,t)) = |\log\varepsilon| \int_\Omega \left|\Delta u_\varepsilon + \frac{1}{\varepsilon^2}u_\varepsilon(1-|u_\varepsilon|^2)\right|^2.$$

Thus, if we write that (1.4) holds with $f_\varepsilon = \partial_t u_\varepsilon/|\log\varepsilon|$, we precisely see that $|\log\varepsilon|\,\|f_\varepsilon\|^2_{L^2(\Omega)}$ is the energy-dissipation rate for solutions of (1.4). This will be crucially used in Part II [S1] and this motivates our need for estimates, in particular lower bounds, on $\|f_\varepsilon\|_{L^2}$. If $\|f_\varepsilon\|_{L^2}$ is large, then the energy dissipates fast in the flow (1.4), thus decreasing to a point which allows one to rule out certain configurations (for example if E_ε decreases so much that $E_\varepsilon \leq C$ then there can be no more vortices). On the other hand, if f_ε is small, then (1.1) can be seen as a small perturbation of the Ginzburg–Landau equation

$$\begin{cases} -\Delta u = \dfrac{1}{\varepsilon^2}u(1-|u|^2) & \text{in } \Omega, \\ u = g \quad \text{or} \quad \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

for which a number of qualitative facts about vortices are known. The idea is thus to use this alternative in a *quantitative* way, in order to deduce from the static study information on vortex collisions or other pathological situations in the dynamics.

More precisely, it is known that if u is a solution of Ginzburg–Landau in the plane, with vortices (a_i, d_i), then we must have

$$\Big(\sum_i d_i\Big)^2 = \sum_i d_i^2, \tag{1.7}$$

which is equivalent to the fact that $\sum_{i\neq j} d_i d_j = 0$, or to the fact that the forces exerted by the vortices balance each other. This follows from suitable applications of the Pohozaev identity, as in [BMR]. Similarly, as seen in [BBH, CM1], if u_ε, a solution of (1.6) in a bounded domain, has some vortices a_i of degree d_i accumulating (as $\varepsilon \to 0$) around a single point p, then the same rule $(\sum_i d_i)^2 = \sum_i d_i^2$ holds. Now, if u_ε is a configuration with say, two vortices, one of degree 1, one of degree -1, at a distance $o(1)$ as $\varepsilon \to 0$ (which is what happens during a vortex collision of a $+1$ with a -1) then this rule is obviously violated (and it is the same for any situation with $(\sum_i d_i)^2 \neq \sum_i d_i^2$), so we can trace how much it is violated in the Pohozaev identity for (1.1), and get a lower bound for $\|f_\varepsilon\|_{L^2}$. The technique thus relies on some adaptations of the Pohozaev identities with error term f_ε. Observe that Pohozaev identities have already been widely used in the context of Ginzburg–Landau statics and dynamics ([BMR, BBH, BCPS, RuS, SS2]). Some similar results and the "balanced cluster" condition (1.7) also appear in the recent preprint of Bethuel–Orlandi–Smets [BOS] (see Theorem 5) on the parabolic Ginzburg–Landau equation.

1.2. Main results on (1.1)

Before stating the results, let us make a few assumptions. Since we are going to consider nice initial data u_ε^0 for (1.4) with a fixed number of vortices as $\varepsilon \to 0$, and since the energy decreases during the flow, it is natural to restrict ourselves to

$$E_\varepsilon(u_\varepsilon) \le M|\log\varepsilon| \tag{1.8}$$

and

$$|u_\varepsilon| \le 1, \qquad |\nabla u_\varepsilon| \le M/\varepsilon. \tag{1.9}$$

It is well known that (1.4) is well-posed and that if these estimates are true for u_ε^0, they remain satisfied at all times for solutions of (1.4).

We sometimes assume in addition that

$$\|f_\varepsilon\|^2_{L^2(\Omega)} \le 1/\varepsilon^\beta \quad \text{for some } \beta < 2. \tag{1.10}$$

If this assumption is not true, then clearly we have a large lower bound on $\|f_\varepsilon\|_{L^2}$. If (1.10) holds, then after blow-up at the scale ε, solutions of (1.1) converge to solutions of Ginzburg–Landau in the plane

$$-\Delta U = U(1-|U|^2)$$

which enables us to define a "good collection of vortices" a_i with degrees d_i (depending on ε) for u_ε. Without going into full details of what it means and how they are found, these are points such that the balls $B_i := B(a_i, R_\varepsilon\varepsilon)$ with some $1 \ll R_\varepsilon \le |\log\varepsilon|$ are disjoint and cover all the zeroes of u_ε, and $d_i = \deg(u_\varepsilon, \partial B(a_i, R_\varepsilon\varepsilon)) \ne 0$. We can then give a more precise definition (although we will mostly use a slightly weaker condition, see Theorem 2).

Definition 1. *The a_i's and d_i's being as above, we say that u_ε has a* cluster of vortices at the scale l at x_0 *if*

$$B(x_0, l) \cap \{a_i\} \ne \emptyset, \tag{1.11}$$

$$\operatorname{dist}(\{a_i : a_i \notin B(x_0, l)\}, B(x_0, l)) \gg l \quad \text{as } \varepsilon \to 0. \tag{1.12}$$

We say u_ε has an unbalanced cluster of vortices at the scale l at x_0 *if the previous conditions hold and if*

$$\sum_{i\,:\,a_i\in B(x_0,l)} d_i^2 \ne \Big(\sum_{i\,:\,a_i\in B(x_0,l)} d_i\Big)^2.$$

Once these vortices are found, this allows us to define a canonical harmonic phase θ in $\Omega_\varepsilon := \Omega \setminus \bigcup_{i=1}^n B(a_i, R_\varepsilon\varepsilon)$ as the harmonic conjugate of Φ solving $-\Delta\Phi = 2\pi\sum_i d_i\delta_{a_i}$ with suitable boundary conditions. Once this is done, denoting by φ the phase of u_ε, i.e. $u = \rho e^{i\varphi}$ in Ω_ε, we may consider the phase excess $\psi = \varphi - \theta$. The first main result consists in evaluating the energy excess (due to both the phase excess and the modulus of u), in terms of only one natural quantity: the L^2 norm of f_ε, the natural norm to consider for the study of the parabolic flow.

The method is inspired by that of Comte–Mironescu in [CM1, CM2]; however, their result was for the case $f_\varepsilon \equiv 0$, and used some precise L^∞ and decay estimates for solutions of Ginzburg–Landau away from the vortices. Here we retrieve the result with the only control on $\|f_\varepsilon\|_{L^2}$ and no a priori bounds other than (1.8) and (1.9). We obtain in addition a scaled version of the estimate, localized in any (small) ball. The result is

Theorem 1. *Let u_ε satisfy* (1.1) *and* (1.8)–(1.10)*. The a_i, d_i, B_i being as above, we have*

$$\int_{\Omega_\varepsilon} |\nabla\psi_\varepsilon|^2 \le o(1) + C\|f_\varepsilon\|^2_{L^2(\Omega)}, \tag{1.13}$$

$$\int_{\Omega_\varepsilon} \left(|\nabla|u_\varepsilon||^2 + \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2}\right) \le o(1 + \|f_\varepsilon\|^2_{L^2(\Omega)}). \tag{1.14}$$

For any $x \in \overline{\Omega}$, and any $l \gg \varepsilon\sqrt{|\log\varepsilon|}$, we have

$$\int_{\Omega\cap B(x,l)\setminus\bigcup_i B_i} |\nabla\psi_\varepsilon|^2 \le \min(C + Cl^2\log^2 l\|f_\varepsilon\|^2_{L^2(\Omega)}, o(1 + l^2\log^4 l\|f_\varepsilon\|^2_{L^2(\Omega)})), \tag{1.15}$$

and

$$\int_{\Omega\cap B(x,l)\setminus\bigcup_i B_i} |\nabla|u_\varepsilon||^2 + \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2} \le o(1) + o(l^2\log^2 l\|f_\varepsilon\|^2_{L^2(\Omega)}). \tag{1.16}$$

Moreover,

$$\forall\alpha < 1, \quad \alpha\pi\sum_{i=1}^n d_i^2 \le \frac{E_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} + C|\log\varepsilon|^{7/2}\varepsilon^{1-\alpha}\|f_\varepsilon\|_{L^2(\Omega)} + o(1), \tag{1.17}$$

and

$$\begin{aligned}\pi\sum_{i=1}^n d_i^2\log\frac{1}{\varepsilon} + W_{\mathbf{d}}(a_1,\ldots,a_n) + \sum_{i=1}^n \gamma(V_i) + o(1) \le E_\varepsilon(u_\varepsilon)\\ \le \pi\sum_{i=1}^n d_i^2\log\frac{1}{\varepsilon} + W_{\mathbf{d}}(a_1,\ldots,a_n) + \sum_{i=1}^n \gamma(V_i) + C\|f_\varepsilon\|^2_{L^2(\Omega)} + o(1),\end{aligned} \tag{1.18}$$

where the V_i's are the (limiting) blown-up profiles of u_ε around a_i at scale ε, $W_{\mathbf{d}}$ is the renormalized energy function introduced in [BBH]*, relative to the collection of degrees* **d** *(see definition in* (3.3)*) and the $\gamma(V_i)$ are constants equal when $d_i = \pm 1$ to a universal constant γ introduced in* [BBH].

Moreover, all the constants C and $o(1)$ above depend only on β, M, Ω and g (if applicable).

This result allows us to bound the phase excess (with scaled versions of it, cf. (1.15)–(1.16)), and in turn to bound the energy excess in terms of $\|f_\varepsilon\|_{L^2}$ and of the vortices of u_ε only, in (1.18). This way, it provides a lower bound for $\|f_\varepsilon\|_{L^2}$ and it allows us, for solutions of (1.4), to bound from below the energy-dissipation rate, and to bound from above the number of vortices through (1.17). Let us mention that the "energy-quantization" result for solutions of (1.4) shown in [BOS] (Theorem 6 and appendix) is equivalent at leading order to (1.18).

From this first theorem, we may implement the Pohozaev strategy described above and obtain the following.

Theorem 2. *There exist constants $l_0 > 0$ and $K_0 > 0$ such that, assuming that u_ε is as in Theorem 1, and that there exists a nonempty subcollection $\{B_i\}_{i=1}^k$ of the balls $\{B_i\}$ which are included in $B(x_0, l/2)$, $\varepsilon\sqrt{|\log\varepsilon|} \ll l < l_0$ as $\varepsilon \to 0$, and such that for some $K > K_0$, either*

(i) *$B(x_0, Kl) \subset \Omega$ and $B(x_0, Kl)$ intersects no other ball in the collection $\{B_i\}$, and*

$$\sum_{i=1}^k d_i^2 \neq \Big(\sum_{i=1}^k d_i\Big)^2, \tag{1.19}$$

or

(ii) *$x_0 \in \partial\Omega$ and $B(x_0, Kl)$ intersects no other ball in the collection $\{B_i\}$.*

Then

$$\|f_\varepsilon\|^2_{L^2(\Omega)} \geq \min\left(\frac{C}{l^2|\log\varepsilon|}, \frac{C}{l^2\log^2 l}\right). \tag{1.20}$$

All the constants above depend only on β, M, Ω and g.

This is exactly the desired lower bound on $\|f_\varepsilon\|^2_{L^2}$: it shows it blows up like $1/(l^2|\log\varepsilon|)$ in most cases, as the scale of the unbalanced cluster of vortices l gets small.

As a byproduct, in the case $f_\varepsilon = 0$ we retrieve

Corollary 1.1. *Let u_ε be solutions of the Ginzburg–Landau equation* (1.6) *such that $E_\varepsilon(u_\varepsilon) \leq C|\log\varepsilon|$. Then there exists a constant $l_0 > 0$ such that for ε small enough, u_ε has no unbalanced cluster of vortices at any scale $l < l_0$; and has no vortex at distance $< l_0$ from the boundary.*

Some sharper (but of the same order) lower bounds for $\|f_\varepsilon\|^2_{L^2}$ will be given in Proposition 5.1 in [S1], by blowing up at the scale l in the case where l is not too small ($\log^4 l \leq C|\log\varepsilon|$).

Observe that all these results (in particular (1.20)) can be viewed as obtaining lower bounds for the higher-order energy functional $F_\varepsilon(u) = \int_\Omega |\Delta u + (u/\varepsilon^2)(1 - |u|^2)^2|^2$ under the assumption $E_\varepsilon(u_\varepsilon) \leq C|\log\varepsilon|$. It was proved in [Li, SS1] that (denoting here and in the rest of the paper by $(\cdot,\cdot)$ the scalar product in $\mathbb{C}$ identified with $\mathbb{R}^2$) if $\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon) \rightharpoonup 2\pi\sum_{i=1}^n D_i\delta_{p_i}$ as $\varepsilon \to 0$ (i.e. the limiting vortices of u_ε as $\varepsilon \to 0$ are the p_i's with degrees D_i), then

$$\liminf_{\varepsilon\to 0}(|\log\varepsilon| F_\varepsilon(u_\varepsilon)) \geq \frac{1}{\pi}\sum_{i=1}^n |\nabla_i W_{\mathbf{D}}(p_1, \dots, p_n)|^2,$$

This is the lower bound part of a Γ-convergence result (the upper bound should not be hard to prove). The lower bounds we obtain here (and in Proposition 5.1 of Part II) are in agreement with this, but in general sharper since they involve the locations and degrees of the vortices at the ε level, and blow up when these get very close.

Let us point out that such a study of forced equations, with its "dual" Γ-convergence point of view, was performed for the Allen–Cahn equation (the same equation as (1.1) but with real-valued functions—an important model for phase transitions) with a lower bound by the Wilmore functional (see [To, RS] and the references therein). We are not aware of any other singularly perturbed equation for which this has been done.

In this first paper, we start by performing a "Pohozaev ball construction" which is an adaptation of that done in [SS2] but with nonzero error term f_ε. This allows us to bound the number of vortices and define a good collection of vortices. Then we prove Theorems 1 and 2.

In the second part [S1], we will present applications of both of these theorems to the dynamics and collisions of vortices under (1.4).

2. A "Pohozaev ball construction" for (1.1) and applications

This construction, which is a combination of the Pohozaev identity with the ball-growth method of Jerrard/Sandier, consists in an adjustment of the one presented in [SS2], taking into account the nonzero right-hand side in (1.1). The main result is

Proposition 2.1. *Let u_ε satisfy* (1.1) *and* (1.8)–(1.10). *Then*

$$\int_{\{x\in\Omega\,:\,|u(x)|\le 1-1/|\log\varepsilon|^2\}} \frac{(1-|u|^2)^2}{\varepsilon^2} \le C, \tag{2.1}$$

where C depends only on β in (1.10), *M, Ω and g.*

2.1. Pohozaev identities for (1.1)

The Pohozaev identity consists in multiplying (1.1) by $x\cdot\nabla u$ and integrating by parts. However, because of the boundary conditions, we will need a more general version of it, as in [SS2, Chapter 5].

Introducing the associated stress-energy tensor

$$T_{ij} = \frac{1}{2}\left(|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right)\delta_{ij} - (\partial_i u, \partial_j u), \tag{2.2}$$

an easy computation yields

$$\operatorname{div} T_{ij} = -\left(\partial_j u, \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2)\right) = -(\partial_j u, f_\varepsilon), \tag{2.3}$$

where $\operatorname{div} T_{ij}$ denotes $\sum_{i=1}^2 \partial_i T_{ij}$. Multiplying the relation (2.3) by a vector field X, we find

Lemma 2.1. *Let u satisfy* (1.1). *For any open subset U of Ω and any smooth vector field X, we have*

$$\int_{\partial U} \sum_{i,j} X_j \nu_i T_{ij} = \int_U \sum_{i,j} (\partial_i X_j) T_{ij} - \int_U (f_\varepsilon, X \cdot \nabla u) \tag{2.4}$$

where ν denotes the outer unit normal to ∂U and the indices i, j run over $\{1, 2\}$.

The most standard Pohozaev identity follows by applying this in $U = \Omega \cap B(x_0, s)$ to $X = x - x_0$; it yields

$$\frac{1}{2} \int_{\partial(B(x_0,s)\cap\Omega)} \left((x - x_0) \cdot \nu \left(\left| \frac{\partial u}{\partial \nu} \right|^2 - \left| \frac{\partial u}{\partial \tau} \right|^2 - \frac{(1 - |u|^2)^2}{2\varepsilon^2} \right) + (x - x_0) \cdot \tau \left(\frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial \nu} \right) \right) + \int_{B(x_0,s)\cap\Omega} \frac{(1 - |u|^2)^2}{2\varepsilon^2} = \int_{B(x_0,s)\cap\Omega} (f_\varepsilon, (x - x_0) \cdot \nabla u). \tag{2.5}$$

In particular, if $B(x_0, s)$ does not intersect $\partial\Omega$, one obtains

$$\int_{\partial B(x_0,s)} \left| \frac{\partial u}{\partial r} \right|^2 + \frac{1}{s} \int_{B(x_0,s)} \frac{(1 - |u|^2)^2}{\varepsilon^2} = \int_{\partial B(x_0,s)} \left(\left| \frac{\partial u}{\partial \tau} \right|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2} \right) + \frac{1}{s} \int_{B(x_0,s)} (f_\varepsilon, (x - x_0) \cdot \nabla u). \tag{2.6}$$

We deduce the following lemma.

Lemma 2.2. *Let u satisfy* (1.1). *Then, if R is such that $B(x_0, R) \subset \Omega$ and $0 < r < R$, we have*

$$\int_r^R \frac{1}{s} \int_{B(x_0,s)} \frac{(1 - |u|^2)^2}{\varepsilon^2} \, ds + \int_{B(x_0,R)\setminus B(x_0,r)} \left| \frac{\partial u}{\partial r} \right|^2 = \int_{B(x_0,R)\setminus B(x_0,r)} \left(\left| \frac{\partial u}{\partial \tau} \right|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2} \right) + \int_r^R \frac{1}{s} \int_{B(x_0,s)} ((x - x_0) \cdot \nabla u, f_\varepsilon) \, ds, \tag{2.7}$$

with

$$\left| \int_r^R \frac{1}{s} \int_{B(x_0,s)} (f_\varepsilon, (x - x_0) \cdot \nabla u) \, ds \right| \le \int_{B(x_0,R)\setminus B(x_0,r)} \left| \frac{\partial u}{\partial r} \right|^2 + \frac{R^2}{4} \int_{B(x_0,R)\setminus B(x_0,r)} |f_\varepsilon|^2 + r \log \frac{R}{r} \| f_\varepsilon \|_{L^2(B(x_0,r))} \| \nabla u \|_{L^2(B(x_0,r))} . \tag{2.8}$$

Proof. (2.7) follows from integrating (2.6) for $s \in [r, R]$. For (2.8), we write

$$\left|\int_r^R \frac{1}{s}\int_{B(x_0,s)} (f_\varepsilon, (x-x_0)\cdot\nabla u)\right| \\ \leq \int_r^R \frac{1}{s}\left(\int_{B(x_0,r)} |x-x_0|\left|\frac{\partial u}{\partial \nu}\right||f_\varepsilon| + \int_{B(x_0,s)\setminus B(x_0,r)} |x-x_0|\left|\frac{\partial u}{\partial \nu}\right||f_\varepsilon|\right) ds \\ \leq r\log\frac{R}{r}\int_{B(x_0,r)}\left|\frac{\partial u}{\partial \nu}\right||f_\varepsilon| + R\int_{B(x_0,R)\setminus B(x_0,r)}\left|\frac{\partial u}{\partial \nu}\right||f_\varepsilon|. \tag{2.9}$$

Inserting the fact that for every $\lambda > 0$,

$$\int_{B(x_0,R)\setminus B(x_0,r)}\left|\frac{\partial u}{\partial \nu}\right||f_\varepsilon| \leq \frac{1}{2\lambda}\int_{B(x_0,R)\setminus B(x_0,r)}\left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{\lambda}{2}\int_{B(x_0,R)\setminus B(x_0,r)} |f_\varepsilon|^2 \tag{2.10}$$

applied to $\lambda = R/2$, we are led to (2.8). ◻

Another standard relation consists in writing in the Dirichlet case, as in [BBH], a global Pohozaev identity using (2.4) on the whole Ω. Using the fact that Ω is strictly starshaped, one obtains

Lemma 2.3. *Let Ω be strictly starshaped and let u satisfy* (1.1) *with $u = g$ on $\partial\Omega$. Then*

$$\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 + \int_\Omega \frac{(1-|u|^2)^2}{\varepsilon^2} \leq C(1 + \|\nabla u\|_{L^2(\Omega)}\|f_\varepsilon\|_{L^2(\Omega)}) \tag{2.11}$$

where the constant C depends only on Ω and g.

Proof. Assume Ω is strictly starshaped with respect to the point x_0 (hence $(x-x_0)\cdot\nu \geq \beta > 0$ on $\partial\Omega$), and apply (2.4) to $U = \Omega$ and $X = x - x_0$. This yields

$$\frac{1}{2}\int_{\partial\Omega}\beta\left|\frac{\partial u}{\partial \nu}\right|^2 + \int_\Omega \frac{(1-|u|^2)^2}{2\varepsilon^2} \\ \leq C\int_{\partial\Omega}\left(\left|\frac{\partial g}{\partial \tau}\right|^2 + \left|\left(\frac{\partial u}{\partial \nu}, \frac{\partial g}{\partial \tau}\right)\right|\right) + \int_\Omega |x-x_0|\,|f_\varepsilon|\,|\nabla u|, \tag{2.12}$$

from which the result follows easily. ◻

2.2. Proof of Proposition 2.1—interior case

For simplicity, we start with the proof of Proposition 2.1 assuming no balls intersect $\partial\Omega$.

Proof of Proposition 2.1. The proof is a ball construction that is very similar to that presented in [SS2, Chapter 4]. Following [Sa1], since (1.8) holds, by the coarea formula, one may cover the set $\{x : |u(x)| \leq 1 - 1/|\log\varepsilon|^2\}$ by a finite union of disjoint closed balls $B_i(0)$ of radii r_i such that $\sum_i r_i \leq C\varepsilon|\log\varepsilon|^3$. We increase all these balls in parallel according to the method of Jerrard and Sandier, presented for example in [SS2], which yields:

Lemma 2.4. *For every $t \geq 0$ there exists a finite collection $\mathcal{B}(t)$ of disjoint closed balls such that:*

(i) $\mathcal{B}(0) = \{B_i(0)\}_i$.
(ii) $r(\mathcal{B}(t)) = e^t r(\mathcal{B}(0))$ *for every* $t \geq 0$, *where* $r(\mathcal{B}(t))$ *denotes the sum of the radii of the balls in the collection.*
(iii) *For every* $t \geq s$,

$$\bigcup_{B \in \mathcal{B}(s)} B \subset \bigcup_{B \in \mathcal{B}(t)} B.$$

There exists a finite set $T \subset \mathbb{R}_+$ *such that if* $[t_1, t_2] \subset \mathbb{R}_+ \setminus T$, *then* $\mathcal{B}(t_2) = e^{t_2 - t_1}\mathcal{B}(t_1)$, *where* $\lambda\mathcal{B}$ *denotes the collection of balls obtained from* $\mathcal{B}$ *by keeping the same centers and multiplying all the radii by* λ.

The times $t \in T$ correspond to "merging times" when some of the balls have intersecting closures. Assuming first the balls remain disjoint through the growing, we may apply (2.7) to $r = r_i$ and $R = e^t r_i$ to find

$$\int_{r_i}^{e^t r_i} \frac{1}{s} \int_{B_i(\log(s/r_i))} \frac{(1-|u|^2)^2}{\varepsilon^2}\, ds \leq \int_{B_i(t)} \left(|\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \right) + e^t r_i \|f_\varepsilon\|_{L^2(B_i(t))} \|\nabla u\|_{L^2(B_i(t))}, \tag{2.13}$$

where

$$\int_r^R \frac{1}{s} \int_{B(x_0,s)} ((x - x_0)\cdot \nabla u, f_\varepsilon)\, ds \leq R \int_{B(x_0,R)} |f_\varepsilon|\,|\nabla u| \leq R \|f_\varepsilon\|_{L^2(B_i(t))} \|\nabla u\|_{L^2(B_i(t))}.$$

We easily deduce

$$t \int_{B_i(0)} \frac{(1-|u|^2)^2}{\varepsilon^2} \leq 2E_\varepsilon(u, B_i(t)) + r(B_i(t)) \|f_\varepsilon\|_{L^2(B_i(t))} \|\nabla u\|_{L^2(B_i(t))}, \tag{2.14}$$

where we write, for any set U,

$$E_\varepsilon(u, U) = \frac{1}{2} \int_U \left(|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \right).$$

Now these relations add up nicely over all balls in the collection $\mathcal{B}(t)$, including through possible merging of balls, and we have for every t, and every $B_k(t) \in \mathcal{B}(t)$,

$$t \int_{\bigcup_{i\,:\,B_i(0)\subset B_k(t)} B_i(0)} \frac{(1-|u|^2)^2}{\varepsilon^2} \leq 2E_\varepsilon(u, B_k(t)) + r(B_k(t)) \|f_\varepsilon\|_{L^2(B_k(t))} \|\nabla u\|_{L^2(B_k(t))}. \tag{2.15}$$

Summing this over k, using (1.8), and applying this relation to $t = \alpha \log(1/\varepsilon)$ for some $0 < \alpha < 1$, we find

$$\alpha|\log\varepsilon|\int_{\bigcup_i B_i(0)}\frac{(1-|u|^2)^2}{\varepsilon^2}\le 2E_\varepsilon\Big(u,\bigcup_k B_k(t)\Big)+C\varepsilon^{1-\alpha}\|f_\varepsilon\|_{L^2(\bigcup_k B_k(t))}|\log\varepsilon|^{7/2}, \tag{2.16}$$

where we observe that $r(\mathcal{B}(t))\le C\varepsilon^{1-\alpha}|\log\varepsilon|^3$. Since (1.10) is satisfied, we may choose $\alpha>0$ such that $2-2\alpha-\beta>0$, and find

$$\int_{\bigcup_i B_i(0)}\frac{(1-|u|^2)^2}{\varepsilon^2}\le\frac{2E_\varepsilon(u)}{\alpha|\log\varepsilon|}+o(1)\le C. \tag{2.17}$$

We conclude that $\int_{\bigcup_i B_i(0)}(1-|u|^2)^2/\varepsilon^2\le C$ and since the $B_i(0)$ were constructed to cover the set $\{x\in\Omega : |u(x)|\le 1-1/|\log\varepsilon|^2\}$ we deduce (2.1). This proof is valid if none of the balls $B_k(t)$ intersects $\partial\Omega$. ◻

2.3. Proof of Proposition 2.1—boundary issues

The method follows that of [SS2, Chapter 4], with the only modifications due to the f_ε term. We sketch the main steps.

2.3.1. Dirichlet case. In the Dirichlet case, instead of using (2.6) and (2.7), we use (2.5). Decomposing $\partial(B(x_0,s)\cap\Omega)$ into $\partial B(x_0,s)\cap\Omega$ and $\partial\Omega\cap B(x_0,s)$, we find

$$\begin{aligned}\int_{\partial B(x_0,s)\cap\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2+\frac1s\int_{B(x_0,s)\cap\Omega}\frac{(1-|u|^2)^2}{\varepsilon^2}&\le\int_{\partial B(x_0,s)\cap\Omega}\left(\left|\frac{\partial u}{\partial\tau}\right|^2+\frac{(1-|u|^2)^2}{2\varepsilon^2}\right)\\&\quad+\frac1s\int_{\partial\Omega\cap B(x_0,s)}(x-x_0)\cdot\nu\left(\left|\frac{\partial g}{\partial\tau}\right|^2-\left|\frac{\partial u}{\partial\nu}\right|^2-(x-x_0)\cdot\tau\left(\frac{\partial g}{\partial\tau},\frac{\partial u}{\partial\nu}\right)\right)\\&\quad+\frac1s\int_{B(x_0,s)\cap\Omega}2((x-x_0)\cdot\nabla u,f_\varepsilon).\end{aligned}\tag{2.18}$$

Using Lemma 2.3, we deduce

$$\begin{aligned}\int_{\partial B(x_0,s)\cap\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2+\frac1s\int_{B(x_0,s)\cap\Omega}\frac{(1-|u|^2)^2}{\varepsilon^2}&\le\int_{\partial B(x_0,s)\cap\Omega}\left(\left|\frac{\partial u}{\partial\tau}\right|^2+\frac{(1-|u|^2)^2}{2\varepsilon^2}\right)\\&\quad+C(1+\|\nabla u\|_{L^2(\Omega)}\|f_\varepsilon\|_{L^2(\Omega)}).\end{aligned}\tag{2.19}$$

Integrating gives

$$\begin{aligned}&\int_r^R\frac1s\int_{B(x_0,s)\cap\Omega)}\frac{(1-|u|^2)^2}{\varepsilon^2}\,ds+\int_{(B(x_0,R)\setminus B(x_0,r))\cap\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2\\&\quad\le\int_{(B(x_0,R)\setminus B(x_0,r))\cap\Omega}\left(\left|\frac{\partial u}{\partial\tau}\right|^2+\frac{(1-|u|^2)^2}{2\varepsilon^2}\right)+CR(1+\|\nabla u\|_{L^2(\Omega)}\|f_\varepsilon\|_{L^2(\Omega)})\end{aligned}\tag{2.20}$$

and we may reproduce the proof above with this relation instead of (2.7).

2.3.2. Neumann case. In the Neumann case, we extend u by performing a reflection with respect to $\partial\Omega$. Let $\tilde{\Omega}$ denote a large enough tubular neighborhood of Ω, i.e. $\Omega \subset \tilde{\Omega}$. Let ψ be a smooth mapping of Ω onto the unit disc. It can be extended to a mapping from $\tilde{\Omega}$ to a domain strictly containing the unit disc. Let then $\mathcal{R}$ denote the reflection with respect to the unit circle defined in complex coordinates by $\mathcal{R}(z) = z/|z|^2$. The mapping $\varphi = \psi^{-1} \circ \mathcal{R} \circ \psi$ then maps $\tilde{\Omega} \setminus \Omega$ to Ω. One can check that it is the identity on $\partial\Omega$, that it is C^2 in $\tilde{\Omega} \setminus \Omega$, and that $D\varphi(x)$ converges to the orthogonal symmetry relative to the tangent to $\partial\Omega$ at x_0 as $x \to x_0 \in \partial\Omega$, at a rate bounded by $C\,\mathrm{dist}(x, \partial\Omega)$.

We can then extend u with $\partial u/\partial\nu = 0$ on $\partial\Omega$, by setting $\overline{u} = u$ in Ω and

$$\overline{u}(x) = u(\varphi(x)) \quad \text{if } x \in \tilde{\Omega} \setminus \Omega.$$

Since $D\varphi$ converges to a reflection with respect to the boundary as $x \to \partial\Omega$ and $\partial u/\partial\nu = 0$ on $\partial\Omega$, we find that $\overline{u}$ is C^1 in $\tilde{\Omega}$. We also define $\bar{f}_\varepsilon = f_\varepsilon(\varphi(x))$ in $\tilde{\Omega} \setminus \Omega$ and $\bar{f}_\varepsilon = f_\varepsilon$ in Ω. We will use the same proof as above through ball growth in $\tilde{\Omega}$ for $\overline{u}$. The relation (2.7) still applies inside Ω. For the balls that intersect $\partial\Omega$, we need to replace it with a variant for $\overline{u}$.

Let $B(x_0, s)$ be a ball intersecting $\partial\Omega$ and let $D_1 = B(x_0, s) \cap \Omega$ and $D_2 = B(x_0, s) \setminus \Omega$. From (2.5), we have

$$\int_{D_1} \frac{(1-|u|^2)^2}{\varepsilon^2} = \int_{\partial\Omega \cap B(x_0,s)} (x - x_0)\cdot\nu \left(\left|\frac{\partial u}{\partial\tau}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \right)$$
$$+ \int_{\partial B(x_0,s)\cap\Omega} s \left(\left|\frac{\partial u}{\partial\tau}\right|^2 - \left|\frac{\partial u}{\partial\nu}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \right) + \int_{D_1} ((x - x_0)\cdot\nabla u, f_\varepsilon). \tag{2.21}$$

In order to get the analogue in D_2, we apply (2.4) in $D_2' = \varphi(D_2)$ with $X(\varphi(x)) = D\varphi(x)(x - x_0)$. Arguing as in [SS2], this leads to

$$\int_{D_2'} \frac{(1-|u|^2)^2}{\varepsilon^2}(1 + O(s)) = \int_{B(x_0,s)\cap\partial\Omega} (x - x_0)\cdot\nu \left(\left|\frac{\partial u}{\partial\tau}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \right)$$
$$+ s\int_{\partial D_2'\cap\Omega} \left(\left|\frac{\partial u}{\partial\tau}\right|^2 - \left|\frac{\partial u}{\partial\nu}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \right) + O\left(s^2 \int_{\partial D_2'} E_\varepsilon(u, \partial D_2') \right) + \int_{D_2'} (f_\varepsilon, X\cdot\nabla u). \tag{2.22}$$

If we add this to the relation (2.21), the contributions on $\partial\Omega$ cancel and we find

$$\int_{D_1\cap D_2'} \frac{(1-|u|^2)^2}{\varepsilon^2}(1 + O(s)) = s\int_{\partial D_1 \cup \partial D_2'} \left(\left|\frac{\partial u}{\partial\tau}\right|^2 - \left|\frac{\partial u}{\partial\nu}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \right)$$
$$+ O(s^2 E_\varepsilon(u, \partial D_2')) + O\left(s\int_{D_1\cup D_2'} |\nabla u|\,|f_\varepsilon| \right).$$

After a change of variables, and since φ approaches a reflection, we find (as in [SS2])

$$\int_{B(x_0,s)} \frac{(1-|\overline{u}|^2)^2}{\varepsilon^2} = s\int_{\partial B(x_0,s)} \left(\left|\frac{\partial \overline{u}}{\partial \tau}\right|^2 - \left|\frac{\partial \overline{u}}{\partial \nu}\right|^2 + \frac{(1-|\overline{u}|^2)^2}{2\varepsilon^2}\right)$$
$$+ O\left(s\int_{B(x_0,s)} \frac{(1-|\overline{u}|^2)^2}{\varepsilon^2}\right) + O(s^2 E_\varepsilon(u, \partial B(x_0,s))) + O\left(s\int_{B(x_0,s)} |\nabla \overline{u}|\,|\bar{f}_\varepsilon|\right).$$

Dividing by s and integrating, we find

$$\int_r^R \frac{1}{s}\int_{B(x_0,s)} \frac{(1-|\overline{u}|^2)^2}{\varepsilon^2}\,ds \le \int_{B(x_0,R)\setminus B(x_0,r)} \left(|\nabla \overline{u}|^2 + \frac{(1-|\overline{u}|^2)^2}{2\varepsilon^2}\right)$$
$$+CRE_\varepsilon(u, B(x_0,R)\setminus B(x_0,r)) + R\int_{B(x_0,R)} \frac{(1-|\overline{u}|^2)^2}{\varepsilon^2} + CR\|\nabla \overline{u}\|_{L^2(\tilde{\Omega})}\|\bar{f}_\varepsilon\|_{L^2(\tilde{\Omega})}. \tag{2.23}$$

Replacing (2.7) by this relation, and increasing the balls the same way, we are led to the same result for $\overline{u}$, and Proposition 2.1 is proved.

2.4. Application: construction of the vortex collection

We now show how to define a good collection of vortex-balls for solutions of (1.1).

If $|u(x_0)| < 1/2$, the assumption (1.9) implies standardly that $|u| \le 3/4$ in some ball $B(x_0, \lambda\varepsilon)$ and thus that there exists a constant $\mu > 0$ such that

$$\int_{B(x_0,\lambda\varepsilon)} \frac{(1-|u|^2)^2}{\varepsilon^2} \ge \mu. \tag{2.24}$$

If we use this, the result of Proposition 2.1 suffices to conclude as in [BBH] that the set $\{|u(x)| \le 3/4\}$ can be covered by a bounded (independently of ε) number of disjoint balls of centers a_i and radii $R\varepsilon$ (where R is fixed), and changing R if necessary, we may always assume that $|a_i - a_j| \gg \varepsilon$ for $i \ne j$. We may also assume that each ball contains a point x_0 where $|u(x_0)| < 1/2$ (otherwise the ball can simply be removed from the collection).

The next step is to perform a blow-up analysis. If (1.10) is satisfied, then the perturbation term f_ε in (1.1) disappears after blow-up at the scale ε, and we can use the known results on (1.6).

Lemma 2.5. *Let u_ε satisfy* (1.1) *and* (1.8)–(1.10). *If a_ε is a sequence of points such that* $\text{dist}(a_\varepsilon, \partial\Omega) \gg \varepsilon$ *and* $\deg(u, \partial B(a_\varepsilon, R\varepsilon)) = d$, *then up to a subsequence, $v_\varepsilon(x) = u_\varepsilon(a_\varepsilon + \varepsilon x)$ converges uniformly over compact subsets of $\mathbb{R}^2$ as $\varepsilon \to 0$ to a solution U of*

$$-\Delta U = U(1-|U|^2) \quad \text{in } \mathbb{R}^2 \tag{2.25}$$

with

$$\int_{\mathbb{R}^2} (1-|U|^2)^2 = 2\pi d^2. \tag{2.26}$$

If a_ε is such that $\text{dist}(a_\varepsilon, \partial\Omega) \le C\varepsilon$ *then up to a subsequence, $v_\varepsilon(x) = u_\varepsilon(a_\varepsilon + \varepsilon x)$ converges locally uniformly to a constant of modulus* 1.

Proof. Setting $v_\varepsilon(x) = u_\varepsilon(a_\varepsilon + \varepsilon x)$ we have

$$\Delta v_\varepsilon + v_\varepsilon(1 - |v_\varepsilon|^2) = \varepsilon^2 f_\varepsilon(a_\varepsilon + \varepsilon x), \tag{2.27}$$

and we also know that $|\nabla v_\varepsilon(x)| \le C$ and $|v_\varepsilon| \le 1$. Thus, v_ε is compact in L^∞ by Ascoli's theorem. But we have $\|\varepsilon^2 f_\varepsilon(a_\varepsilon + \varepsilon x)\|_{L^2(B_R)} \le \varepsilon \|f_\varepsilon\|_{L^2(\Omega)} \le o(1)$ by (1.10); thus Δu_ε is strongly compact in $L^2(B_R)$ for every R. In the first case, up to a subsequence, we thus find that v_ε converges locally uniformly and in H^2_{loc} to a solution U of (2.25). It was proved in [BMR] that under our assumptions, (2.26) holds. In the case $\mathrm{dist}(a_\varepsilon, \partial\Omega) \le C\varepsilon$, up to translation and taking a subsequence, we find that v_ε converges to a solution of $-\Delta U = U(1 - |U|^2)$ on the half-plane $\mathbb{R}^2_+$, with boundary condition either $|U| = 1$ or $\partial U/\partial \nu = 0$. In the Dirichlet case, a result of Sandier [Sa2] allows us to conclude that U is a constant; in the Neumann case, a simple reflection yields a solution to (2.25) of degree zero, hence a constant of modulus 1 (from [BMR]). □

For a solution U of (2.25), following [BMR], we have

$$\frac{1}{2}\int_{B(0,R)} \left(|\nabla U|^2 + \frac{(1 - |U|^2)^2}{2} \right) = \pi d^2 \log R + \gamma(U) \quad \text{as } R \to \infty \tag{2.28}$$

where d is the degree of U and $\gamma(U)$ is a constant depending on the solution. When $d = \pm 1$, it has been proved by Mironescu [M] that there exists a unique solution to (2.25) (up to multiplication by a constant of modulus 1), which is the radial solution of (2.25) and then $\gamma(U) = \gamma$, a universal constant first defined in [BBH].

Proposition 2.2. *Let u_ε satisfy* (1.1) *and* (1.8)–(1.10)*. Then, after extraction of a sequence $\varepsilon \to 0$, we can find $R_\varepsilon \to +\infty$ with $R_\varepsilon \le C|\log \varepsilon|$ and a family of balls $B_i = B(a_i, R_\varepsilon \varepsilon)$, $i = 1, \dots, n$, with a_i depending on ε and n bounded independently of ε, such that*

(i) $\|1 - |u_\varepsilon|\,\|_{L^\infty(\Omega \setminus \bigcup_i B(a_i, R_\varepsilon \varepsilon))} \to 0$ *as* $\varepsilon \to 0$.

(ii) $|a_i - a_j| \gg R_\varepsilon \varepsilon$ *for* $i \ne j$ *and* $\mathrm{dist}(a_i, \partial\Omega) \gg R_\varepsilon \varepsilon$ *for every* i.

(iii) *The* $d_i = \deg(u, \partial B(a_i, R_\varepsilon \varepsilon))$ *are all nonzero.*

(iv)
$$\lim_{\varepsilon \to 0} \left\| u_\varepsilon - V_i\left(\frac{\cdot - a_i}{\varepsilon}\right) \right\|_{L^\infty(B(a_i, R_\varepsilon \varepsilon))} = 0 \tag{2.29}$$

where V_i is some solution of degree d_i of (2.25).

(v)
$$\lim_{\varepsilon \to 0} \int_{\partial B(a_i, R_\varepsilon \varepsilon)} \left| \frac{\partial |u_\varepsilon|}{\partial \nu} \right| = 0, \qquad \int_{\partial B(a_i, R_\varepsilon \varepsilon)} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right| \le C. \tag{2.30}$$

(vi)
$$\lim_{\varepsilon \to 0} \int_{B(a_i, R_\varepsilon \varepsilon)} \frac{(1 - |u_\varepsilon|^2)^2}{\varepsilon^2} = 2\pi d_i^2, \tag{2.31}$$

$$\lim_{\varepsilon \to 0} \left(E_\varepsilon(u_\varepsilon, B(a_i, R_\varepsilon \varepsilon)) - \frac{1}{2}\int_{B(0, R_\varepsilon)} \left(|\nabla V_i|^2 + \frac{(1 - |V_i|^2)^2}{2} \right) \right) = 0. \tag{2.32}$$

Moreover, for every $\alpha < 1$, and every subset I of $[1, n]$, we have

$$\alpha\pi \sum_{i\in I} d_i^2 \le \frac{E_\varepsilon(u_\varepsilon, \bigcup_{i\in I} B(a_i, R_\varepsilon \varepsilon^{1-\alpha}))}{|\log\varepsilon|} + C|\log\varepsilon|^{7/2}\varepsilon^{1-\alpha}\|f_\varepsilon\|_{L^2(\Omega)} + o(1). \tag{2.33}$$

Proof. We have already found the points a_i. In view of Lemma 2.5, we may blow up around them to find that $v_\varepsilon(x) = u_\varepsilon(a_i + \varepsilon x)$ converges in $H^2(B(0, R))$ for every $R > 0$ to some solution V_i of (2.25). Now, following [CM2], we may find by an abstract argument $R_\varepsilon \to \infty$, with $R_\varepsilon \le C|\log\varepsilon|$ and (ii), such that $\{|u_\varepsilon| \le 3/4\} \subset \bigcup_{i=1}^k B(a_i, R_\varepsilon\varepsilon)$ and

$$\lim_{\varepsilon\to 0} \|v_\varepsilon - V_i\|_{H^2(B(0,R_\varepsilon))} = 0.$$

We deduce (2.29)–(2.32): H^2 convergence implies $H^{1/2}$ convergence of the derivatives on the boundary (by trace). (iii) comes from the fact that if $d_i = 0$ then (2.31) contradicts (2.24). But the choice of $3/4$ was arbitrary, the same can be done to cover $\{|u_\varepsilon| \ge m\}$ for any $m < 1$. By a diagonal argument, one can then obtain (i).

It remains to prove (2.33). In the previous subsection, we may apply the method of Proposition 2.1 with initial balls $B_i(0)$ equal to the $B(a_i, R_\varepsilon\varepsilon)$, and deduce exactly as in (2.16) that for every $\alpha < 1$,

$$\alpha|\log\varepsilon| \int_{\bigcup_{i\,:\,B_i(0)\subset B_k} B_i(0)} \frac{(1-|u|^2)^2}{\varepsilon^2} \le 2E_\varepsilon(u, B_k) + C\varepsilon^{1-\alpha}\|f_\varepsilon\|_{L^2(B_k)}|\log\varepsilon|^{7/2},$$

where the B_k are disjoint balls with sum of radii $\le e^{\alpha|\log\varepsilon|}R_\varepsilon\varepsilon = R_\varepsilon\varepsilon^{1-\alpha}$. Combining this with (2.31) leads to (2.33) and thus to (1.17). □

3. Canonical phase and energy lower bounds

We introduce the Green kernel $G(x, y)$ which solves

$$\begin{cases} -\Delta_x G(x, y) = \delta_y & \text{in } \Omega, \\ \dfrac{\partial G}{\partial \nu} = \left(ig, \dfrac{\partial g}{\partial \tau}\right) \quad \text{(resp. } G = 0 \text{ for Neumann boundary condition)} & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

and $S(x, y)$ defined by

$$S(x, y) = 2\pi G(x, y) + \log|x - y|. \tag{3.2}$$

It is standard that G is symmetric, and S is a C^1 function in $\Omega \times \Omega$. Also the renormalized energy W, as introduced in [BBH], can be written with these notations as

$$\begin{aligned} W_{\mathbf{d}}(a_1, \dots, a_n) = {} & -\pi \sum_{i\ne j} d_i d_j \log|a_i - a_j| + \pi \sum_{i,j} d_i d_j S(a_i, a_j) \\ & + \frac{1}{2}\int_{\partial\Omega} \Big(-\sum_i d_i \log|x - a_i| + \sum_i d_i S(x, a_i)\Big)(ig, \partial_\tau g) \end{aligned} \tag{3.3}$$

where the a_i's are distinct points in Ω, the d_j's are integers, and the last integral is taken to be 0 in the Neumann case. If there are no vortices then we consider instead

$$W_0 = \int_\Omega |\nabla\Phi|^2 \tag{3.4}$$

where $\Phi = 0$ in the Neumann case, and Φ is a harmonic function with $\partial\Phi/\partial\nu = (ig, \partial g/\partial\tau)$ on $\partial\Omega$ in the Dirichlet case.

3.1. Estimates for the canonical phase

The balls $B_i = B(a_i, R_\varepsilon\varepsilon)$ being given by Proposition 2.2, we define $\Omega_\varepsilon = \Omega \setminus \bigcup_i B(a_i, R_\varepsilon\varepsilon)$. We consider

$$\begin{cases} -\Delta\Phi = 2\pi \sum_i d_i\delta_{a_i} & \text{in } \Omega, \\ \dfrac{\partial\Phi}{\partial\nu} = \left(ig, \dfrac{\partial g}{\partial\tau}\right) \quad (\text{resp. } \Phi = 0 \text{ for Neumann}) & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

with $\int_\Omega \Phi = 0$ in the Dirichlet case.

Then we consider θ, the "canonical phase associated to the (a_i, d_i)", the harmonic conjugate of Φ in Ω_ε. It is not single-valued, but $e^{i\theta}$ is well defined. Observe that θ depends implicitly on ε since the points a_i do. We will use the estimate

$$|\nabla\theta(x)| \le C/r \quad \text{where} \quad r = \text{dist}(x, \{a_i\} \cup \partial\Omega), \tag{3.6}$$

and the following result:

Lemma 3.1. *Let $B(b_j, \rho_j)$ be any finite collection of disjoint balls (bounded in number as $\varepsilon \to 0$), with $\rho_j \ge R_\varepsilon\varepsilon$ depending on ε, such that*

(i) $\bigcup_i B(a_i, R_\varepsilon\varepsilon) \subset \bigcup_j B(b_j, \rho_j)$,
(ii) $\rho_j \ll |b_i - b_j|$ *for every* $i \ne j$ *and* $\rho_j \ll \text{dist}(\bigcup_i\{b_i\}, \partial\Omega)$,
(iii) $\forall a_i \in B(b_j, \rho_j)$, $|a_i - b_j| \ll \rho_j$

(these hypotheses allow in particular taking $b_i = a_i$ and $\rho_i = R_\varepsilon\varepsilon$). Then

$$\frac{1}{2}\int_{\Omega\setminus\bigcup_j B(b_j,\rho_j)} |\nabla\theta|^2 = \pi\sum_i D_i^2 \log\frac{1}{\rho_j} + W_{\mathbf{D}}(b_1, \dots, b_n) + o(1) \tag{3.7}$$

where $D_j = \deg(e^{i\theta}, \partial B(b_j, \rho_j)) = \sum_{i\,:\,a_i\in B(b_j,\rho_j)} d_i$.

Proof. The proof is quite standard, similar to results in [BBH] (except that here the a_i depend on ε) or to [SS2, Chap. 10]. From (3.2), we have

$$\Phi(x) = -\sum_{i=1}^n d_i \log|x - a_i| + \sum_{i=1}^n d_i S(x, a_i). \tag{3.8}$$

Using the fact that $|a_i - b_j| \ll \rho_j \ll |b_j - b_k|$ for $j \neq k$, and $a_i \in B(b_j, \rho_j)$, and computing explicitly, we find

$$\frac{\partial \Phi(x)}{\partial \nu} = \frac{-D_j}{\rho_j}(1 + o(1)) \quad \text{on } \partial B(b_j, \rho_j), \tag{3.9}$$

$$|\nabla \Phi(x)| \leq \frac{C}{\min_j |x - b_j|} \quad \text{in } \Omega \setminus \bigcup_j B(b_j, \rho_j). \tag{3.10}$$

For assertion (3.7), integrating by parts using (3.5), we first have

$$\begin{aligned} \frac{1}{2}\int_{\Omega\setminus\bigcup_j B(b_j,\rho_j)} |\nabla\theta|^2 &= \frac{1}{2}\int_{\Omega\setminus\bigcup_j B(b_j,\rho_j)} |\nabla\Phi|^2 \\ &= -\frac{1}{2}\sum_j \int_{\partial B(b_j,\rho_j)} \Phi\frac{\partial\Phi}{\partial\nu} + \frac{1}{2}\int_{\partial\Omega} \Phi\frac{\partial\Phi}{\partial\nu}, \end{aligned} \tag{3.11}$$

where ν denotes the outer unit normal to $\partial B(b_j, \rho_j)$. Inserting (3.8), we find

$$\begin{aligned} -\int_{\partial B(b_j,\rho_j)} \Phi\frac{\partial\Phi}{\partial\nu} &= -\int_{\partial B(b_j,\rho_j)} \Bigg(-\sum_{i\,:\,a_i\in B(b_j,\rho_j)} d_i \log|x - a_i| \\ &\qquad - \sum_{i\,:\,a_i\notin B(b_j,\rho_j)} d_i \log|x - a_i| + \sum_i d_i S(x, a_i)\Bigg)\frac{\partial\Phi}{\partial\nu} \\ &= -\int_{\partial B(b_j,\rho_j)} \Bigg(D_j \log\frac{1}{\rho_j} - \sum_{k\neq j} D_k \log|b_j - b_k| + \sum_k D_k S(x, b_k) + o(1)\Bigg)\frac{\partial\Phi}{\partial\nu}, \end{aligned} \tag{3.12}$$

where we have used the continuity of S, and the facts that on $\partial B(b_j, \rho_j)$, if $a_i \in B(b_j, \rho_j)$,

$$\log|x - a_i| = \log|x - b_j| + \log\left|1 + \frac{b_j - a_i}{x - b_j}\right| = \log|x - b_j| + o(1) = \log\rho_j + o(1),$$

because $|x - b_j| = \rho_j \gg |a_i - b_j|$; and similarly if $a_i \in B(b_k, \rho_k)$, then $\log|x - a_i| = \log|b_j - b_k| + o(1)$ on $\partial B(b_j, \rho_j)$. On the other hand,

$$-\int_{\partial B(b_j,\rho_j)} \frac{\partial\Phi}{\partial\nu} = \int_{B(b_j,\rho_j)} -\Delta\Phi = 2\pi D_j.$$

Inserting this into (3.12) and using the regularity of S, we get

$$\begin{aligned} -\int_{\partial B(b_j,\rho_j)} \Phi\frac{\partial\Phi}{\partial\nu} &= 2\pi D_j^2 \log\frac{1}{\rho_j} - 2\pi\sum_{k\neq j} D_j D_k \log|b_j - b_k| \\ &\quad + 2\pi\sum_k D_j D_k S(b_j, b_k) + o(1). \end{aligned} \tag{3.13}$$

Combining (3.11) and (3.13), we conclude that

$$\frac{1}{2}\int_{\Omega\setminus\bigcup_j B(b_j,\rho_j)} |\nabla\Phi|^2 = 2\pi\sum_j D_j^2 \log\frac{1}{\rho_j} - 2\pi\sum_{k\neq j} D_j D_k \log|b_j - b_k|$$
$$+ 2\pi\sum_{j,k} D_j D_k S(b_j,b_k) + \frac{1}{2}\int_{\partial\Omega}\Big(-\sum_j \log|x-b_j| + \sum_j D_j S(x,b_j)\Big)(ig,\partial_\tau g) + o(1) \tag{3.14}$$

and thus (3.7) holds. □

With the same kind of techniques, we can get the following result, which will be useful later on.

Lemma 3.2. *Let B_i be a family of balls as in Proposition* 2.2*, and let θ be the canonical phase associated to the (a_i, d_i)'s. If B_R and B_l are two concentric balls such that $B_{2R}\setminus B_{l/2}$ is included in Ω and does not intersect any of the balls B_i, then*

$$2\pi\Big(\sum_{i\,:\,B_i\subset B_l} d_i\Big)^2 \log\frac{R}{l} \le \int_{B_R\setminus B_l} |\nabla\theta|^2 \le 2\pi\Big(\sum_{i\,:\,B_i\subset B_l} d_i\Big)^2\log\frac{R}{l} + O(1), \tag{3.15}$$

$$\int_{B_R\setminus B_l}\left|\frac{\partial\theta}{\partial r}\right|^2 = O(1). \tag{3.16}$$

If B_R and B_l are two concentric balls centered on $\partial\Omega$ such that $B_{2R}\setminus B_{l/2}$ does not intersect any of the B_i's, then

$$\int_{B_R\setminus B_l} |\nabla\theta|^2 = O(1). \tag{3.17}$$

Here the $O(1)$ depends only on Ω, on the number of points a_i and on some upper bound on $\sum_i |d_i|$.

Proof. Let us first deal with the interior case, and the left-hand side inequality in (3.15). Since $B_{2R}\setminus B_{l/2}$ does not intersect any ball, θ is well defined in $B_R\setminus B_l$ and the degree is constant equal to $\sum_{i\,:\,B_i\subset B_l} d_i$; that is, for every $l\le r\le R$, we have

$$\int_{\partial B_r}\frac{\partial\theta}{\partial\tau} = 2\pi\sum_{i\,:\,B_i\subset B_l} d_i.$$

Then, using the Cauchy–Schwarz inequality, we have

$$\int_{B_R\setminus B_l}|\nabla\theta|^2 \ge \int_l^R\int_{\partial B_r}\left|\frac{\partial\theta}{\partial\tau}\right|^2 dr$$
$$\ge \int_l^R \frac{1}{2\pi r}\Big(\int_{\partial B_r}\frac{\partial\theta}{\partial\tau}\Big)^2 dr = 2\pi\Big(\sum_{i\,:\,B_i\subset B_l} d_i\Big)^2\log\frac{R}{l}. \tag{3.18}$$

This proves the left-hand inequality in (3.15).

For the other inequality, for both the interior and boundary case, let us evaluate

$$\int_{B_R\setminus B_l} |\nabla\theta|^2 = \int_{B_R\setminus B_l} |\nabla\Phi|^2.$$

Clearly, since $B_R \setminus B_l$ does not contain any a_i, integrating by parts yields

$$\int_{B_R\setminus B_l} |\nabla\Phi|^2 = \int_{\partial(B_R\setminus B_l)} \Phi\frac{\partial\Phi}{\partial\nu}, \tag{3.19}$$

where ν is the outer unit normal to each disc. Let us now study Φ closer. For each point a_i, let us denote by a_i^* its symmetric image with respect to $\partial\Omega$ (there might be several choices, but it does not matter). Let then

$$\Psi(x) = -\sum_i d_i \log|x-a_i| + \sum_i d_i \log|x-a_i^*|. \tag{3.20}$$

One can check that Ψ and $\partial\Psi/\partial\nu$ remain bounded on $\partial\Omega$ by some constant independent of the a_i's. On the other hand, $\Delta(\Phi-\Psi)=0$ in Ω, so in view of the boundary conditions for Φ (see (3.5)) we find, by the maximum principle, that $\Phi-\Psi$ is bounded in Ω (in both Dirichlet and Neumann cases). Thus, we may write

$$\Phi(x) = -\sum_i d_i \log|x-a_i| + \sum_i d_i \log|x-a_i^*| + O(1). \tag{3.21}$$

Let us now first focus on the interior case. Denote by x_0 the center of the balls B_l and B_R, and let $x \in \partial B_l$ and $a_i \in B_{l/2}$. We have

$$\log|x-a_i| = \log|x-x_0| + \log\left|1-\frac{a_i-x_0}{x-x_0}\right|.$$

But, since $a_i \in B_{l/2}$, we have $\left|\frac{a_i-x_0}{x-x_0}\right| \le \frac{1}{2}$, thus $\log\left|1-\frac{a_i-x_0}{x-x_0}\right|$ remains uniformly bounded and we can write $\log|x-a_i| = \log l + O(1)$. Assume now that $x \in \partial B_l$ and $a_i \notin B_{l/2}$; that means that a_i is outside of B_{2R}. Then

$$\log|x-a_i| = \log|a_i-x_0| + \log\left|1-\frac{x-x_0}{a_i-x_0}\right|$$

and since $|x-x_0| \le R$ and $|a_i-x_0| \ge 2R$ we have $\left|\frac{x-x_0}{a_i-x_0}\right| \le \frac{1}{2}$ and thus $\log\left|1-\frac{x-x_0}{a_i-x_0}\right|$ remains uniformly bounded. The same holds for a_i^* which is always in $\Omega \setminus B_{2R}$. We can thus write

$$\begin{aligned}\Phi(x) = &-\sum_{i\,:\,B_i\subset B_{l/2}} d_i \log l - \sum_{i\,:\,B_i\subset\Omega\setminus B_{2R}} d_i (\log|a_i-x_0| - \log|a_i^*-x_0|)\\ &+ O(1) \quad \text{for } x \in \partial B_l.\end{aligned} \tag{3.22}$$

Similarly, for $x \in \partial B_R$, we have

$$\Phi(x) = -\sum_{i\,:\,B_i\subset B_{l/2}} d_i \log R - \sum_{i\,:\,B_i\subset\Omega\setminus B_{2R}} d_i (\log|a_i-x_0| - \log|a_i^*-x_0|) + O(1).$$

Thus,

$$\begin{aligned}\int_{\partial B_l}\Phi\frac{\partial\Phi}{\partial\nu}&\\&=\Big(-\sum_{i\,:\,B_i\subset B_{l/2}}d_i\log l-\sum_{i\,:\,B_i\subset\mathbb{R}^2\setminus B_{2R}}d_i\log\frac{|a_i-x_0|}{|a_i^*-x_0|}\Big)\int_{\partial B_l}\frac{\partial\Phi}{\partial\nu}+O\Big(\int_{\partial B_l}\Big|\frac{\partial\Phi}{\partial\nu}\Big|\Big)\\&=\Big(-\sum_{i\,:\,B_i\subset B_{l/2}}d_i\log l-\sum_{i\,:\,B_i\subset\mathbb{R}^2\setminus B_{2R}}d_i\log\frac{|a_i-x_0|}{|a_i^*-x_0|}\Big)\Big(-2\pi\sum_{j/B_j\subset B_{l/2}}d_j\Big)+O(1),\end{aligned}\tag{3.23}$$

where we have used (3.5) and the estimate (3.6) or in other words $|\nabla\Phi|\le C/l$ on ∂B_l. Similarly,

$$\int_{\partial B_R}\Phi\frac{\partial\Phi}{\partial\nu}=\Big(-\sum_{i\,:\,B_i\subset B_{l/2}}d_i\log R-\sum_{i\,:\,B_i\subset\mathbb{R}^2\setminus B_{2R}}d_i\log\frac{|a_i-x_0|}{|a_i^*-x_0|}\Big)\int_{\partial B_l}\frac{\partial\Phi}{\partial\nu}+O(1).$$

Subtracting those two relations and returning to (3.19), we find

$$\int_{B_R\setminus B_l}|\nabla\Phi|^2=2\pi\Big(\sum_{i\,:\,B_i\subset B_{l/2}}d_i\Big)^2\log\frac{R}{l}+O(1).$$

This finishes the proof of (3.15). Comparing it to (3.18), we conclude that (3.16) must hold.

For the boundary case, one may check with similar ideas that (3.21) implies that Φ is bounded (independently of the location of the points and R and l) in $B_R\setminus B_l$. Therefore

$$\int_{\partial(B_R\setminus B_l)}\Phi\frac{\partial\Phi}{\partial\nu}=\int_{\partial\Omega\cap(B_R\setminus B_l)}\Phi\frac{\partial\Phi}{\partial\nu}+O\Big(\int_{\partial B_R\cap\Omega}\Big|\frac{\partial\Phi}{\partial\nu}\Big|\Big)+O\Big(\int_{\partial B_l\cap\Omega}\Big|\frac{\partial\Phi}{\partial\nu}\Big|\Big).$$

The contribution on $\partial\Omega$ is zero in the Neumann case and is bounded in the Dirichlet case (in view of the bound on Φ and the boundary condition on $\partial\Phi/\partial\nu$). The contributions on ∂B_R and ∂B_l are bounded by the same argument as above (using $|\nabla\Phi|\le C/R$ or C/l). We conclude that (3.17) holds. ◻

3.2. *Lower bounds on the energy*

Returning to u_ε, we introduce $\rho_\varepsilon=|u_\varepsilon|$, and φ_ε such that

$$u_\varepsilon=\rho_\varepsilon e^{i\varphi_\varepsilon}\quad\text{in }\Omega_\varepsilon.\tag{3.24}$$

We also introduce the phase-excess $\psi_\varepsilon=\varphi_\varepsilon-\theta$ in Ω_ε, and observe it is a single-valued function. Afterwards, we most often drop the subscripts ε. We claim that from (2.29), for each i, there exists a constant c_i such that

$$\psi=\varphi-\theta=c_i+o(1)\quad\text{on }\partial B(a_i,R_\varepsilon\varepsilon).\tag{3.25}$$

Also, $\psi=\text{const}$ on $\partial\Omega$ in the case of the Dirichlet boundary condition, and $\partial\psi/\partial\nu=0$ on $\partial\Omega$ in the case of the Neumann boundary condition.

In the next section, we will work alternatively in Ω_ε or in $B(x,l)\backslash\bigcup_i B_i$ where $B(x,l)$ is some ball of radius l (possibly depending on ε) included in Ω such that $\partial B(x,l)\subset\Omega_\varepsilon$. In what follows, D_ε denotes either Ω_ε or any subset of Ω_ε of the form $\Omega\cap B(x,l)\backslash\bigcup_i B_i$, and D denotes Ω or $B(x,l)\cap\Omega$ respectively.

Lemma 3.3. *Assume u_ε satisfies* (1.1) *and* (1.8)–(1.10)*, and hence the results of Proposition* 2.2*. Then*

$$E_\varepsilon(u_\varepsilon)\geq\frac{1}{2}\int_{\Omega_\varepsilon}|\nabla\theta|^2+\pi\sum_i d_i^2\log R_\varepsilon+\sum_i\gamma(V_i)+o(1)$$
$$=\pi\sum_i d_i^2\log\frac{1}{\varepsilon}+W_{\mathbf{d}}(a_1,\ldots,a_n)+\sum_i\gamma(V_i)+o(1)\qquad(3.26)$$

where V_i is given by (2.29)*.*

Proof. This follows arguments of [BMR, CM1]. Let $D_\varepsilon=\Omega_\varepsilon$ or $D_\varepsilon=B_l\setminus\bigcup_i B_i$. We claim that

$$E_\varepsilon(u_\varepsilon,D_\varepsilon)\geq E_\varepsilon(e^{i\theta},D_\varepsilon)+\frac{1}{2}\int_{D_\varepsilon}\rho^2|\nabla\psi|^2+\frac{1}{5}\int_{D_\varepsilon}\frac{(1-\rho^2)^2}{\varepsilon^2}+\int_{D_\varepsilon}\rho^2\nabla\theta\cdot\nabla\psi+o(1).\qquad(3.27)$$

Indeed,

$$\int_{D_\varepsilon}\rho^2|\nabla\varphi|^2=\int_{D_\varepsilon}\rho^2|\nabla(\theta+\psi)|^2=\int_{D_\varepsilon}(\rho^2|\nabla\theta|^2+\rho^2|\nabla\psi|^2+2\rho^2\nabla\theta\cdot\nabla\psi).\qquad(3.28)$$

But, by Cauchy–Schwarz,

$$\left|\int_{D_\varepsilon}(\rho^2-1)|\nabla\theta|^2\right|\leq\varepsilon\left(\int_{D_\varepsilon}\frac{(1-\rho^2)^2}{\varepsilon^2}\right)^{1/2}\left(\int_{D_\varepsilon}|\nabla\theta|^4\right)^{1/2}.\qquad(3.29)$$

By definition of θ, we have $\int_{\Omega_\varepsilon}|\nabla\theta|^4\leq C/(R_\varepsilon\varepsilon)^2$, thus, since $R_\varepsilon\to+\infty$,

$$\int_{D_\varepsilon}(\rho^2-1)|\nabla\theta|^2=o\left(\left(\int_{D_\varepsilon}\frac{(1-\rho^2)^2}{\varepsilon^2}\right)^{1/2}\right).\qquad(3.30)$$

Hence,

$$\int_{D_\varepsilon}\rho^2|\nabla\theta|^2\geq\int_{D_\varepsilon}|\nabla\theta|^2+o\left(\int_{D_\varepsilon}\frac{(1-\rho^2)^2}{\varepsilon^2}\right)+o(1).\qquad(3.31)$$

This yields (3.27). Arguing similarly, we also have

$$\left|\int_{D_\varepsilon}(\rho^2-1)\nabla\theta\cdot\nabla\psi\right|\leq\varepsilon\|\nabla\theta\|_{L^\infty(D_\varepsilon)}\left(\int_{D_\varepsilon}\frac{(1-\rho^2)^2}{\varepsilon^2}\right)^{1/2}\left(\int_{D_\varepsilon}|\nabla\psi|^2\right)^{1/2}$$
$$\leq\frac{C}{R_\varepsilon}\int_{D_\varepsilon}\left(\frac{(1-\rho^2)^2}{\varepsilon^2}+|\nabla\psi|^2\right)$$
$$\leq o\left(\int_{D_\varepsilon}\left(\frac{(1-\rho^2)^2}{\varepsilon^2}+|\nabla\psi|^2\right)\right)\qquad(3.32)$$

where we have used (3.6). Combining this with (3.27), we find

$$E_\varepsilon(u_\varepsilon, D_\varepsilon) \geq E_\varepsilon(e^{i\theta}, D_\varepsilon) + \int_{D_\varepsilon} \nabla\theta \cdot \nabla\psi + o(1). \tag{3.33}$$

Specializing to $D_\varepsilon = \Omega_\varepsilon$, using the fact that θ is harmonic, we also have

$$\int_{\Omega_\varepsilon} \nabla\theta \cdot \nabla\psi = \sum_i \int_{\partial B_i} \psi \frac{\partial\theta}{\partial\nu}.$$

Inserting (3.25), and using $\int_{\partial B_i} \frac{\partial\theta}{\partial\nu} = \int_{\partial B_i} \frac{\partial\Phi}{\partial\tau} = 0$, we find that

$$\int_{\Omega_\varepsilon} \nabla\theta \cdot \nabla\psi = o(1) \sum_i \int_{\partial B_i} \left|\frac{\partial\theta}{\partial\nu}\right|.$$

Using (3.6) again, we conclude that $\int_{\Omega_\varepsilon} \nabla\theta \cdot \nabla\psi = o(1)$ and hence, from (3.33), we get

$$E_\varepsilon(u_\varepsilon, \Omega_\varepsilon) \geq E_\varepsilon(e^{i\theta}, \Omega_\varepsilon) + o(1). \tag{3.34}$$

On the other hand, from (2.32) and (2.28), we have

$$E_\varepsilon(u_\varepsilon, B_i) = \pi d_i^2 \log R_\varepsilon + \gamma(V_i) + o(1). \tag{3.35}$$

Adding to (3.34) and combining with Lemma 3.1, we have the result. □

As a corollary, we get the lower bound:

Lemma 3.4. *Assume that u_ε satisfies the results of Proposition 2.2 and $B(b_k, \rho_k)$ is a family of balls satisfying hypotheses* (i)–(iii) *of Lemma* 3.1. *Let the p_j's be the points of accumulation of the a_i's with nonzero total degree. Then, with the same notations as in Lemma* 3.1,

$$E_\varepsilon(u_\varepsilon) \geq \pi \sum_k |D_k|\,|\log\varepsilon| + W_{\mathcal{D}}(p_j) - \pi \sum_j \sum_{k\neq k' : b_k\to p_j,\, b_{k'}\to p_j} D_k D_{k'} \log|b_k - b_{k'}| + \Big(\sum_k |D_k|\Big)\gamma + o(1), \tag{3.36}$$

where $\mathcal{D}_j = \sum_{b_k\to p_j} D_k$. Moreover, if there is equality in (3.36) *then each $D_k = \pm 1$ and each $B(b_k, \rho_k)$ contains only one a_i.*

Proof. From the results of Proposition 2.2, we have a family of small balls $B(a_i, d_i)$. Applying Lemma 3.3, we have

$$E_\varepsilon(u_\varepsilon) \geq \pi \sum_i d_i^2 \log\frac{1}{\varepsilon} + W_{\mathbf{d}}(a_1, \dots, a_n) + \sum_i \gamma(V_i) + o(1). \tag{3.37}$$

On the other hand, one can check that

$$W_{\mathbf{d}}(a_1,\dots,a_n) = W_{\mathcal{D}}(p_j) - \pi \sum_j \sum_{i\neq i' : a_i, a_i' \to p_j} d_i d_{i'} \log|a_i - a_{i'}| + o(1). \tag{3.38}$$

For each given j, let us now study

$$\pi \sum_{i : a_i \to p_j} d_i^2 |\log\varepsilon| - \pi \sum_{i\neq i' : a_i, a_i' \to p_j} d_i d_{i'} \log|a_i - a_{i'}|.$$

The points a_i converging to the same p_j belong to several of the $B(b_k,\rho_k)$. It is again easy to check from the properties of the $B(b_k,\rho_k)$ that

$$\begin{aligned} -\pi \sum_{i\neq i' : a_i, a_i' \to p_j} d_i d_{i'} \log|a_i - a_{i'}| &= -\pi \sum_{k\neq k' : b_k, b_{k'} \to p_j} D_k D_{k'} \log|b_k - b_{k'}| \\ &\quad - \pi \sum_{k : b_k \to p_j} \Big(\sum_{i\neq i' : a_i, a_{i'} \in B(b_k,\rho_k)} d_i d_{i'} \log|a_i - a_{i'}| \Big) + o(1). \end{aligned} \tag{3.39}$$

So we are led to studying, for each k,

$$\pi \sum_{i : a_i \in B(b_k,\rho_k)} d_i^2 |\log\varepsilon| - \pi \sum_{i\neq i' : a_i, a_{i'} \in B(b_k,\rho_k)} d_i d_{i'} \log|a_i - a_{i'}|. \tag{3.40}$$

We examine the a_i's belonging to one $B(b_k,\rho_k)$. Let l_1 be the smallest distance between two of the a_i's. Let us group together all the a_i's that are at distance $O(l_1)$ from each other. This makes several clusters of points. Over each cluster $\mathcal{C}_m$, since the total number of vortices is bounded, we have

$$\begin{aligned} -\pi \sum_{i\neq i' \in \mathcal{C}_m} d_i d_{i'} \log|a_i - a_{i'}| &= -\pi \Big(\sum_{i\neq i' \in \mathcal{C}_m} d_i d_{i'} \Big) \log l_1 + O(1) \\ &= \pi \Big(\sum_{i\in\mathcal{C}_m} d_i^2 - \Big(\sum_{i\in\mathcal{C}_m} d_i \Big)^2 \Big) \log l_1 + O(1). \end{aligned}$$

Therefore,

$$\begin{aligned} \pi \sum_{i\in\mathcal{C}_m} d_i^2 \log\frac{1}{\varepsilon} - \pi \sum_{i\neq i' \in \mathcal{C}_m} d_i d_{i'} \log|a_i - a_{i'}| \\ = \pi \sum_{i\in\mathcal{C}_m} d_i^2 \log\frac{l_1}{\varepsilon} - \pi \Big(\sum_{i\in\mathcal{C}_m} d_i \Big)^2 \log l_1 + O(1). \end{aligned} \tag{3.41}$$

We now need to sum this over all m's and add the interactions between the clusters themselves, which have total degree $\delta_m^1 = \sum_{i\in\mathcal{C}_m} d_i$. Since they are all at a distance $\gg l_1$ from each other, we may consider $l_2 \gg l_1$ the minimum of their distance. Let us again group the clusters into clusters of size $O(l_2)$, at a distance $l_3 \gg l_2$ from the others.

The interaction within each cluster of size l_2 can be counted as $-\pi\sum\delta^1_m\delta^1_{m'}\log l_2 = \pi(\sum(\delta^1_m)^2 - (\sum\delta^1_m)^2)\log l_2$. Adding up over all clusters of size l_2, we find an energy

$$\pi\sum_i d_i^2\log\frac{l_1}{\varepsilon} + \pi\sum_m(\delta^1_m)^2\log\frac{l_2}{l_1} - \pi\Big(\sum_m\delta^1_m\Big)^2\log l_2 + O(1).$$

Again it remains to add this over all clusters of clusters, and add the interaction between them, which is at the scale $l_3 \gg l_2$, etc. Iterating this process (which stops after a finite number of steps since the total number of balls is bounded) we are left with an energy bounded from below by

$$\begin{aligned}\pi\sum_{i\,:\,a_i\in B(b_k,\rho_k)} d_i^2|\log\varepsilon| - \pi\sum_{i\neq i'\,:\,a_i,a_{i'}\in B(b_k,\rho_k)} d_i d_{i'}\log|a_i-a_{i'}| \\ \geq \pi\sum_{i\,:\,a_i\in B(b_k,\rho_k)} d_i^2\log\frac{l_1}{\varepsilon} + \pi\sum_m(\delta^1_m)^2\log\frac{l_2}{l_1} \\ + \pi\sum_{m'}(\delta^2_{m'})^2\log\frac{l_3}{l_2} + \cdots + \pi D_k^2\log\frac{1}{l_q} + O(1),\end{aligned} \tag{3.42}$$

where D_k is the total degree on $\partial B(b_k,\rho_k)$ and each δ^h, the total degree of a cluster at scale l_h, is the sum of the degrees over all the clusters at scale l_{h-1} that it contains. In other words, we have $\sum_i d_i^2 \geq \sum_i |d_i| \geq |D_k|$ and similarly $\sum_m(\delta^h_m)^2 \geq \sum_m|\delta^h_m| \geq |D_k|$. This means we can bound (3.42) from below by

$$\begin{aligned}\pi\sum_{i\,:\,a_i\in B(b_k,\rho_k)} d_i^2|\log\varepsilon| - \pi\sum_{i\neq i'\,:\,a_i,a_{i'}\in B(b_k,\rho_k)} d_i d_{i'}\log|a_i-a_{i'}| \\ \geq \pi|D_k|\left(\log\frac{l_1}{\varepsilon} + \log\frac{l_2}{l_1} + \cdots + \log\frac{1}{l_q}\right) + O(1) = \pi|D_k|\log\frac{1}{\varepsilon} + O(1).\end{aligned} \tag{3.43}$$

Moreover, this inequality is sharp if and only if $\sum_i d_i^2 = \sum_i|d_i| = |D_k|$ and $\sum_m(\delta^h_m)^2 = \sum_m|\delta^h_m|$ for every h. The first relation implies that each d_i is equal to ±1, the sign being equal to that of D_k. The second relation implies that each $\delta^h_m = \pm1$, which means that there is only one cluster at each scale, so in fact there can be at most one vortex a_i of degree ±1 in $B(b_k,\rho_k)$, and $D_k = \pm1$ (or 0). In that case, the lower bound above can be replaced simply by $\pi|D_k|\,|\log\varepsilon|$. If this is not the case, then we have dropped some term in (3.42) of size $\pi\log(l_h/l_{h-1})$ which tends to $+\infty$ by construction of the l_j's. Thus, in all cases, we may replace (3.43) by

$$\pi\sum_{i\,:\,a_i\in B(b_k,\rho_k)} d_i^2|\log\varepsilon| - \pi\sum_{i\neq i'\,:\,a_i,a_{i'}\in B(b_k,\rho_k)} d_i d_{i'}\log|a_i-a_{i'}| \geq \pi|D_k|\log\frac{1}{\varepsilon} + R_\varepsilon, \tag{3.44}$$

where $R_\varepsilon \to +\infty$, unless $D_k = \pm 1$ or 0, with at most one vortex of degree ± 1 in each $B(b_k, \rho_k)$, in which case $R_\varepsilon = 0$. Combining this with (3.37), (3.38) and (3.40), we find

$$E_\varepsilon(u_\varepsilon) \geq \pi \sum_k |D_k|\,|\log \varepsilon| + W_{\mathcal{D}}(p_j) - \pi \sum_j \sum_{k \neq k' : b_k \to p_j, b_{k'} \to p_j} D_k D_{k'} \log |b_k - b_{k'}| + R_\varepsilon + \sum_i \gamma(V_i) + o(1). \tag{3.45}$$

If $R_\varepsilon \to +\infty$ this implies the desired relation (3.36). If not then all the small vortices are of degree ± 1, so $\gamma(V_i) = \gamma$ for each i, which again implies (3.36). □

4. The substitution lemma and Theorem 1

This section is inspired by the analysis of [CM1, CM2]. It needs to be readjusted to the case where only (1.1) is known, and also to be localized in small balls. The main result we obtain by this method is the following (we recall we work in D_ε which is alternatively $\Omega_\varepsilon = \Omega \setminus \bigcup_i B_i$ or $B(x, l) \setminus \bigcup_i B_i$).

Proposition 4.1. *Assume u_ε satisfies* (1.1) *and* (1.8)–(1.10) *and the results of Proposition* 2.2*. Then, with the same notations as above, as $\varepsilon \to 0$,*

$$\int_{\Omega_\varepsilon} \left(|\nabla \psi|^2 + |\nabla \rho|^2 + \frac{1}{2\varepsilon^2}(1 - \rho^2)^2 \right) \leq C \|f_\varepsilon\|^2_{L^2(\Omega)} + o(1), \tag{4.1}$$

and

$$E_\varepsilon(u_\varepsilon, \Omega_\varepsilon) \leq E_\varepsilon(e^{i\theta}, \Omega_\varepsilon) + C\|f_\varepsilon\|^2_{L^2(\Omega)} + o(1). \tag{4.2}$$

Let x be a given point in $\overline{\Omega}$, and

$$F(l) = \int_{B(x,l)\cap\Omega\setminus\bigcup_i B_i} \left(|\nabla \psi|^2 + \frac{1}{2}|\nabla \rho|^2 + \frac{2}{5}\frac{(1 - \rho^2)^2}{\varepsilon^2} \right). \tag{4.3}$$

If either

(i) $x \in \Omega$, $l \leq \operatorname{dist}(x, \partial\Omega)$ *and* $\operatorname{dist}(\bigcup_i \{a_i\}, \partial B(x, l)) \gg \varepsilon\sqrt{|\log \varepsilon|}$, *or*
(ii) $x \in \partial\Omega$ *and* $\operatorname{dist}(\bigcup_i \{a_i\}, \partial B(x, l)) \gg \varepsilon\sqrt{|\log \varepsilon|}$,

then the function F satisfies a relation of the form

$$F(l) \leq \frac{l + Kl^2}{2} F'(l) + K(l\|f_\varepsilon\|_{L^2(B(x,l))} + 1)\sqrt{F(l)} + o(1), \tag{4.4}$$

where K (and the constant in $o(1)$ above) is a constant depending only on β, M, Ω and g.

The proof requires many steps which we separate into lemmas.

Lemma 4.1 (Substitution lemma). *Under the hypotheses of Proposition* 4.1,

$$E_\varepsilon(e^{i\theta}, D_\varepsilon) = E_\varepsilon(u_\varepsilon, D_\varepsilon) + \frac{1}{2}\int_{D_\varepsilon} (f_\varepsilon, e^{i\varphi})\frac{1}{\rho}(1-\rho^2) + \frac{1}{2}\int_{D_\varepsilon} \rho^2\left|\nabla\frac{1}{\rho}\right|^2 + \frac{1}{2}\int_{D_\varepsilon} |\nabla\psi|^2$$
$$- \int_{D_\varepsilon} \nabla\varphi\cdot\nabla\psi + \frac{1}{4\varepsilon^2}\int_{D_\varepsilon}(1-\rho^2)^2 + \frac{1}{2}\int_{\partial D}\left(\frac{1}{\rho}-\rho\right)\frac{\partial\rho}{\partial\nu} + o(1), \qquad (4.5)$$

where we recall $u = \rho e^{i\varphi}$ *in* Ω_ε.

Proof. For any real-valued functions ζ and $1/2 \le \eta \le 4/3$ in D_ε, we may consider $v = \eta e^{i\zeta} u_\varepsilon = \eta\rho e^{i\varphi+\zeta}$, and we have

$$E_\varepsilon(v, D_\varepsilon) = \frac{1}{2}\int_{D_\varepsilon}\left(|\nabla(\rho\eta)|^2 + \rho^2\eta^2|\nabla\varphi+\nabla\zeta|^2 + \frac{1}{2\varepsilon^2}(1-\eta^2\rho^2)^2\right). \qquad (4.6)$$

Expanding all the terms, we find

$$E_\varepsilon(v, D_\varepsilon) = E_\varepsilon(u_\varepsilon, D_\varepsilon) + \frac{1}{2}\int_{D_\varepsilon}\Big((\eta^2-1)|\nabla\rho|^2 + \rho^2|\nabla\eta|^2 + 2\eta\rho\nabla\rho\cdot\nabla\eta$$
$$+ \rho^2(\eta^2-1)|\nabla\varphi|^2 + \rho^2\eta^2|\nabla\zeta|^2 + 2\rho^2\eta^2\nabla\varphi\cdot\nabla\zeta$$
$$+ \frac{1}{2\varepsilon^2}(-2\rho^2(1-\rho^2)(\eta^2-1) + \rho^4(1-\eta^2)^2)\Big). \qquad (4.7)$$

But, taking the scalar product of (1.1) with $e^{i\varphi}$ yields

$$-\Delta\rho + \rho|\nabla\varphi|^2 = \frac{\rho}{\varepsilon^2}(1-\rho^2) + (f_\varepsilon, e^{i\varphi}) \quad \text{in } D_\varepsilon. \qquad (4.8)$$

Multiplying (4.8) by $(\eta^2-1)\rho$ and integrating, we find

$$-\int_{\partial D_\varepsilon}(\eta^2-1)\rho\frac{\partial\rho}{\partial\nu} + \int_{D_\varepsilon}\Big((\eta^2-1)|\nabla\rho|^2 + 2\eta\rho\nabla\eta\cdot\nabla\rho$$
$$+ \rho^2(\eta^2-1)|\nabla\varphi|^2 + \frac{\rho^2}{\varepsilon^2}(1-\rho^2)(1-\eta^2)\Big) = \int_{D_\varepsilon}(f_\varepsilon, e^{i\varphi})\rho(\eta^2-1). \qquad (4.9)$$

Inserting this into (4.7), and using (2.30), we find

$$E_\varepsilon(v, D_\varepsilon) = E_\varepsilon(u, D_\varepsilon) + \frac{1}{2}\int_{D_\varepsilon}(f_\varepsilon, e^{i\varphi})\rho(\eta^2-1)$$
$$+ \frac{1}{2}\int_{D_\varepsilon}\left(\rho^2|\nabla\eta|^2 + \rho^2\eta^2|\nabla\zeta|^2 + \frac{\rho^4}{2\varepsilon^2}(1-\eta^2)^2 + 2\rho^2\eta^2\nabla\varphi\cdot\nabla\zeta\right)$$
$$+ \frac{1}{2}\int_{\partial D}(\eta^2-1)\rho\frac{\partial\rho}{\partial\nu} + o(1). \qquad (4.10)$$

Choosing specifically $\zeta = -\psi$ and $\eta = 1/\rho$, we find (4.5). □

Lemma 4.2. *Under the same hypotheses,*

$$\int_{D_\varepsilon}(\rho^2-1)\left(\frac{1}{2}|\nabla\psi|^2+\nabla\theta\cdot\nabla\psi\right)+\frac{2}{5}\int_{D_\varepsilon}\frac{(1-\rho^2)^2}{\varepsilon^2}+\frac{1}{2}\int_{D_\varepsilon}|\nabla\rho|^2 \leq C\int_{\partial D}|1-\rho^2|\left|\frac{\partial\rho}{\partial\nu}\right|+o(1) \tag{4.11}$$

and

$$E_\varepsilon(u_\varepsilon,D_\varepsilon)\leq E_\varepsilon(e^{i\theta},D_\varepsilon)+C\int_{\partial D}|1-\rho^2|\left|\frac{\partial\rho}{\partial\nu}\right|+\int_{D_\varepsilon}\nabla\varphi\cdot\nabla\psi+o(1). \tag{4.12}$$

Proof. Adding up the relations (3.27) and (4.5), we find

$$\begin{aligned}0\geq{}&\frac{1}{2}\int_{D_\varepsilon}(\rho^2+1)|\nabla\psi|^2+\left(\frac{1}{5}+\frac{1}{4}\right)\int_{D_\varepsilon}\frac{(1-\rho^2)^2}{\varepsilon^2}\\&+\frac{1}{2}\int_{D_\varepsilon}\rho^2\left|\nabla\frac{1}{\rho}\right|^2+\int_{D_\varepsilon}(\rho^2\nabla\theta-\nabla\varphi)\cdot\nabla\psi\\&-\frac{1}{2}\int_{\partial D}\left|\frac{1}{\rho}-\rho\right|\left|\frac{\partial\rho}{\partial\nu}\right|-C\int_{D_\varepsilon}|f_\varepsilon|\,|\rho^2-1|+o(1).\end{aligned} \tag{4.13}$$

Hence, splitting φ as $\theta+\psi$, we get

$$\int_{D_\varepsilon}(\rho^2-1)\left(\nabla\theta\cdot\nabla\psi+\frac{1}{2}|\nabla\psi|^2\right)+\frac{2}{5}\int_{D_\varepsilon}\frac{(1-\rho^2)^2}{\varepsilon^2}+\frac{1}{2}|\nabla\rho|^2 \leq C\int_{\partial}|1-\rho^2|\left|\frac{\partial\rho}{\partial\nu}\right|+C\int_{D_\varepsilon}|f_\varepsilon|\,|\rho^2-1|+o(1), \tag{4.14}$$

where $\int_{D_\varepsilon}|f_\varepsilon|\,|\rho^2-1|\leq\|f_\varepsilon\|_{L^2(D_\varepsilon)}\|\rho^2-1\|_{L^2(\Omega)}\leq C\varepsilon|\log\varepsilon|\,\|f_\varepsilon\|_{L^2(D_\varepsilon)}\leq o(1)$ by (1.10), hence the result.

Similarly (4.5) implies (4.12). □

Lemma 4.3. *Under the same hypotheses,*

$$\left|\int_{\Omega_\varepsilon}\rho^2\nabla\varphi\cdot\nabla\psi\right|\leq C\|f_\varepsilon\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega_\varepsilon)}+o(1), \tag{4.15}$$

$$\left|\int_{\Omega_\varepsilon}(\nabla\theta-\rho^2\nabla\varphi)\cdot\nabla\psi\right|\leq C\|f_\varepsilon\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega_\varepsilon)}+o(1), \tag{4.16}$$

and

$$\begin{aligned}&\left|\int_{B(x,l)\cap\Omega\setminus\bigcup_i B_i}(\nabla\theta-\rho^2\nabla\varphi)\cdot\nabla\psi\right|\\&\quad\leq\frac{l+Cl^2}{2}\int_{\partial B(x,l)\cap\Omega}\left(|\nabla\psi|^2+\frac{2(1-\rho^2)^2}{5\varepsilon^2}\right)\\&\qquad+Cl\|f_\varepsilon\|_{L^2(B(x,l))}\|\nabla\psi\|_{L^2(B(x,l)\cap\Omega\setminus\bigcup_i B_i)}+o(1).\end{aligned} \tag{4.17}$$

Proof. Taking the inner product of (1.1) and iu, we find

$$\operatorname{div}(\rho^2\nabla\varphi) = (f_\varepsilon, iu). \tag{4.18}$$

Thus,

$$\operatorname{div}(\rho^2\nabla\varphi - \nabla\theta) = (f_\varepsilon, iu) \quad \text{in } D_\varepsilon.$$

We now let $\overline{\psi}$ denote, if D is a ball not intersecting $\partial\Omega$, the average value of ψ on ∂D; if D is a ball intersecting $\partial\Omega$, in the Dirichlet case, the constant value of ψ on $\partial\Omega$, and in the Neumann case, the mean value of ψ on $\partial D \cap \Omega$; finally, if $D = \Omega$, the value of ψ on $\partial\Omega$ in the Dirichlet case, and the average of ψ on $\partial\Omega$ in the Neumann case.

Let us then multiply (4.18) by $\psi - \overline{\psi}$, and integrate by parts. We find

$$\int_{D_\varepsilon} (\nabla\theta - \rho^2\nabla\varphi)\cdot\nabla\psi = \int_{\partial D_\varepsilon} (\psi - \overline{\psi})\left(\frac{\partial\theta}{\partial\nu} - \rho^2\frac{\partial\varphi}{\partial\nu}\right) + O\left(\int_{D_\varepsilon} |f_\varepsilon|\,|\psi - \overline{\psi}|\right). \tag{4.19}$$

Moreover,

$$\int_{\partial B_i} (\psi - \overline{\psi})\frac{\partial\theta}{\partial\nu} = \int_{\partial B_i} (\psi - c_i)\frac{\partial\theta}{\partial\nu} + \int_{\partial B_i} (c_i - \overline{\psi})\frac{\partial\Phi}{\partial\tau} = o(1)$$

by (3.6) and (3.25). Also,

$$\begin{aligned}\int_{\partial B_i} (\psi - \overline{\psi})\rho^2\frac{\partial\varphi}{\partial\nu} &= \int_{\partial B_i} (\psi - c_i)\rho^2\frac{\partial\varphi}{\partial\nu} + \int_{\partial B_i} (c_i - \overline{\psi})\rho^2\frac{\partial\varphi}{\partial\nu}\\ &= o(1) - (c_i - \overline{\psi})\int_{B_i} (f_\varepsilon, iu),\end{aligned}$$

where we have used (2.30) and (4.18). We may always extend ψ inside $D \cap \bigcup_i B_i$ to a function $\tilde{\psi}$ in such a way that $\int_{B(a_i,R_\varepsilon\varepsilon)} |\nabla\tilde{\psi}|^2 \le C\int_{B(a_i,2R_\varepsilon\varepsilon)\setminus B(a_i,R_\varepsilon\varepsilon)} |\nabla\psi|^2$ (see for example [BMR]), so that we have $\int_D |\nabla\tilde{\psi}|^2 \le C\int_{D_\varepsilon} |\nabla\psi|^2$. Moreover, we can do it in such a way that

$$\|\tilde{\psi} - c_i\|_{L^\infty(B_i)} \le \|\psi - c_i\|_{L^\infty(\partial B_i)} = o(1).$$

Using this, we find

$$\begin{aligned}\left|(c_i - \overline{\psi})\int_{B_i} (f_\varepsilon, iu)\right| &\le \int_{B_i} |\tilde{\psi} - \overline{\psi}|\,|f_\varepsilon| + \int_{B_i} |\tilde{\psi} - c_i|\,|f_\varepsilon|\\ &\le \int_{B_i} |\tilde{\psi} - \overline{\psi}|\,|f_\varepsilon| + o(R_\varepsilon\varepsilon\|f_\varepsilon\|_{L^2(\Omega)}).\end{aligned}$$

We may thus conclude from (4.19) and (1.10) (combined with $R_\varepsilon \le |\log\varepsilon|$) that

$$\int_{D_\varepsilon} (\nabla\theta - \rho^2\nabla\varphi)\cdot\nabla\psi = O\left(\int_D |f_\varepsilon|\,|\tilde{\psi} - \overline{\psi}|\right) + \int_{\partial D} (\psi - \overline{\psi})\left(\frac{\partial\theta}{\partial\nu} - \rho^2\frac{\partial\varphi}{\partial\nu}\right) + o(1). \tag{4.20}$$

In the case $D = \Omega$, in view of the boundary conditions ($\psi = \overline{\psi}$ or $\partial\varphi/\partial\nu = 0$), the second term on the right-hand side vanishes identically, so

$$\int_{\Omega_\varepsilon} (\nabla\theta - \rho^2\nabla\varphi)\cdot\nabla\psi = O\left(\int_\Omega |f_\varepsilon|\,|\tilde{\psi} - \overline{\psi}|\right) + o(1). \tag{4.21}$$

Following the exact same steps, we can deduce that

$$\int_{\Omega_\varepsilon} \rho^2\nabla\varphi\cdot\nabla\psi = O\left(\int_\Omega |f_\varepsilon|\,|\tilde{\psi} - \overline{\psi}|\right) + o(1). \tag{4.22}$$

But, by a Poincaré type inequality, we always have

$$\int_D |f_\varepsilon|\,|\tilde{\psi} - \overline{\psi}| \le C|D|\,\|f_\varepsilon\|_{L^2(D)}\|\nabla\psi\|_{L^2(D_\varepsilon)} \tag{4.23}$$

where $|D|$ denotes the half-diameter of D (a constant if $D = \Omega$ and l if $D = B(x,l)$). (Recall that if D intersects $\partial\Omega$, then it is a ball centered at a point of the boundary, essentially a half-disc if l is small, by smoothness of $\partial\Omega$). From (4.21) and (4.22), we already deduce that (4.16) and (4.15) hold.

It remains to bound the other term of the right-hand side of (4.20). In the case $D = B(x,l)\cap\Omega$ (the only one left to consider) we observe that since $\varphi = \theta + \psi$, in view of the boundary conditions and the choice of $\overline{\psi}$, we have

$$\int_{\partial D}(\psi - \overline{\psi})\left(\frac{\partial\theta}{\partial\nu} - \rho^2\frac{\partial\varphi}{\partial\nu}\right) = \int_{\partial B(x,l)\cap\Omega}(\psi-\overline{\psi})(1-\rho^2)\frac{\partial\theta}{\partial\nu} - \int_{\partial B(x,l)\cap\Omega}(\psi-\overline{\psi})\rho^2\frac{\partial\psi}{\partial\nu}. \tag{4.24}$$

Let us now distinguish between the cases where $B(x,l)$ intersects $\partial\Omega$ and not. If $D = B(x,l)\subset\Omega$, we may use, as in [BMR], a sharp scaled Poincaré inequality on $\partial B(x,l)$: observe that

$$\int_{\partial B(x,l)} |\psi - \overline{\psi}|^2 \le l^2\int_{\partial B(x,l)}\left|\frac{\partial\psi}{\partial\tau}\right|^2$$

and

$$\left(\int_{\partial B(x,l)}\left|\frac{\partial\psi}{\partial\tau}\right|^2\right)^{1/2}\left(\int_{\partial B(x,l)}\left|\frac{\partial\psi}{\partial\nu}\right|^2\right)^{1/2} \le \frac{1}{2}\int_{\partial B(x,l)}|\nabla\psi|^2.$$

Inserting this this into the above, and using $\rho\le 1$, we are led to

$$\begin{aligned}\left|\int_{\partial B(x,l)}(\psi - \overline{\psi})\rho^2\frac{\partial\psi}{\partial\nu}\right| &\le l\left(\int_{\partial B(x,l)}\left|\frac{\partial\psi}{\partial\tau}\right|^2\right)^{1/2}\left(\int_{\partial B(x,l)}\left|\frac{\partial\psi}{\partial\nu}\right|^2\right)^{1/2}\\ &\le \frac{l}{2}\int_{\partial B(x,l)}|\nabla\psi|^2.\end{aligned} \tag{4.25}$$

If $D = B(x,l)\cap\Omega$ and $x\in\partial\Omega$, then we may calculate explicitly

$$\min_{h\in H_0^1([0,L])}\frac{\int_0^L (h')^2}{\int_0^L h^2} = \frac{\pi^2}{L^2} \tag{4.26}$$

and

$$\min_{\int_0^L h=0} \frac{\int_0^L (h')^2}{\int_0^L h^2} = \frac{\pi^2}{L^2}. \tag{4.27}$$

Applying this to the curve $\partial B(x,l) \cap \Omega$ parametrized by arclength, we find, using (4.26) in the Dirichlet case and (4.27) in the Neumann case, in view of the choice of $\overline{\psi}$,

$$\int_{\partial B(x,l)\cap\Omega} |\psi - \overline{\psi}|^2 \leq \frac{|\partial B(x,l)\cap\Omega|^2}{\pi^2} \int_{\partial B(x,l)\cap\Omega} \left|\frac{\partial \psi}{\partial \tau}\right|^2, \tag{4.28}$$

where $|\partial B(x,l) \cap \Omega|$ denotes the length of $\partial B(x,l) \cap \Omega$. Using the fact that $\partial\Omega$ is smooth, we can write

$$|\partial B(x,l) \cap \Omega| \leq \pi l + Cl^2$$

(that is, $\partial B(x,l) \cap \Omega$ tends to a half-circle as $l \to 0$). Inserting this into (4.28), we find, in place of (4.25),

$$\begin{aligned} \left| \int_{\partial B(x,l)} (\psi - \overline{\psi})\rho^2 \frac{\partial \psi}{\partial \nu} \right| &\leq \left((l^2 + Cl^3) \int_{\partial B(x,l)\cap\Omega} \left|\frac{\partial \psi}{\partial \tau}\right|^2 \right)^{1/2} \left(\int_{\partial B(x,l)\cap\Omega} \left|\frac{\partial \psi}{\partial \nu}\right|^2 \right)^{1/2} \\ &\leq \frac{1}{2}(l + Cl^2) \int_{\partial B(x,l)\cap\Omega} |\nabla\psi|^2. \end{aligned} \tag{4.29}$$

On the other hand, for both cases (boundary and interior), using (3.6), we have $|\nabla\theta| \ll C/\sqrt{|\log\varepsilon|}\varepsilon$ on $\partial B(x,l)$, hence

$$\begin{aligned} &\left| \int_{\partial B(x,l)\cap\Omega} (\psi - \overline{\psi})(1-\rho^2) \frac{\partial \theta}{\partial \nu} \right| \\ &\qquad \leq \frac{o(1)}{\sqrt{|\log\varepsilon|}} \left(\int_{\partial B(x,l)\cap\Omega} |\psi - \overline{\psi}|^2 \int_{\partial B(x,l)\cap\Omega} \frac{(1-\rho^2)^2}{\varepsilon^2} \right)^{1/2} \\ &\qquad \leq \frac{o(1)}{\sqrt{|\log\varepsilon|}} \left(l \int_{B(x,l)} |\nabla\psi|^2 \int_{\partial B(x,l)\cap\Omega} \frac{(1-\rho^2)^2}{\varepsilon^2} \right)^{1/2} \end{aligned}$$

where we have used a trace inequality. Using the fact that $\int_{\Omega_\varepsilon} |\nabla\psi|^2 \leq \int_\Omega |\nabla u|^2 + \int_{\Omega_\varepsilon} |\nabla\theta|^2 \leq C|\log\varepsilon|$, we deduce

$$\left| \int_{\partial B(x,l)} (\psi - \overline{\psi})(1-\rho^2) \frac{\partial \theta}{\partial \nu} \right| \leq o(1)\left(1 + \frac{l}{2} \int_{\partial B(x,l)} \frac{(1-\rho^2)^2}{\varepsilon^2} \right).$$

Combining this with (4.20) and (4.23), we conclude that (4.17) holds. ◻

Lemma 4.4. *Under the same hypotheses, we have the estimates*

$$\left|\int_{D_\varepsilon}(\rho^2-1)\left(2\nabla\theta+\frac{3}{2}\nabla\psi\right)\cdot\nabla\psi\right| \le C\|\nabla\psi\|_{L^2(D_\varepsilon)}+o(1), \tag{4.30}$$

$$\left|\int_{\Omega_\varepsilon}(\rho^2-1)\left(2\nabla\theta+\frac{3}{2}\nabla\psi\right)\cdot\nabla\psi\right| \le o\left(\int_{\Omega_\varepsilon}\frac{1}{2\varepsilon^2}(1-\rho^2)^2+\int_{\Omega_\varepsilon}|\nabla\psi|^2\right)+o(1). \tag{4.31}$$

Proof. For the first relation, let us write

$$\begin{aligned}
&\int_{D_\varepsilon}(1-\rho^2)|\nabla\psi|^2\\
&\le \int_{D_\varepsilon\cap\{|u|\ge 1-1/|\log\varepsilon|^2\}}(1-\rho^2)|\nabla\psi|^2+\int_{D_\varepsilon\cap\{|u|\le 1-1/|\log\varepsilon|^2\}}(1-\rho^2)|\nabla\psi|^2\\
&\le \frac{C}{|\log\varepsilon|^2}\int_{\Omega_\varepsilon}|\nabla\psi|^2+\left(\int_{D_\varepsilon\cap\{|u|\le 1-1/|\log\varepsilon|^2\}}\frac{(1-\rho^2)^2}{\varepsilon^2}\right)^{1/2}\left(\int_{D_\varepsilon}|\nabla\psi|^2\right)^{1/2}
\end{aligned} \tag{4.32}$$

where we have used the fact that $|\nabla\psi|\le C/\varepsilon$. Now, applying the estimate $\int_{\Omega_\varepsilon}|\nabla\psi|^2 \le C|\log\varepsilon|$ and combining the above with the result of Proposition 2.1, we conclude that $\int_{D_\varepsilon}(1-\rho^2)|\nabla\psi|^2\le o(1)+C\|\nabla\psi\|_{L^2(D_\varepsilon)}$. A similar reasoning (using $|\nabla\theta|\le C/\varepsilon$ in Ω_ε) works for $\int_{D_\varepsilon}(1-\rho^2)\nabla\theta\cdot\nabla\psi$, and we deduce (4.30).

The other relation is a direct consequence of (3.32) and $\rho\ge 1-o(1)$ in Ω_ε. □

Lemma 4.5. *Under the same hypotheses,*

$$\int_{B(x,l)\cap\Omega\setminus\bigcup_i B_i}\left(|\nabla\rho|^2+\frac{(1-\rho^2)^2}{\varepsilon^2}\right) \le C\int_{\partial B(x,l)\cap\Omega}|1-\rho^2|\,|\nabla\rho|+o\left(1+\int_{B(x,l)\cap\Omega\setminus\bigcup_i B_i}|\nabla\psi|^2\right), \tag{4.33}$$

and

$$\int_{\Omega_\varepsilon}\left(|\nabla\rho|^2+\frac{(1-\rho^2)^2}{\varepsilon^2}\right)\le o(1)\left(1+\int_{\Omega_\varepsilon}|\nabla\psi|^2\right). \tag{4.34}$$

Proof. Indeed, returning to (4.11), we find

$$\int_{D_\varepsilon}\left(|\nabla\rho|^2+\frac{(1-\rho^2)^2}{\varepsilon^2}\right)\le C\int_{\partial D_\varepsilon}|1-\rho^2|\left|\frac{\partial\rho}{\partial\nu}\right|+o(1) + C\left|\int_{D_\varepsilon}(\rho^2-1)\nabla\theta\cdot\nabla\psi\right|+C\int_{D_\varepsilon}|1-\rho^2|\,|\nabla\psi|^2.$$

Using (3.32) and the fact that $|1-\rho^2|=o(1)$ in D_ε (from Proposition 2.2(i)) and (2.30) to get rid of the terms on ∂B_i, we easily find that (4.33) and (4.34) hold. □

These relations will be used later. We are now in a position to give the full

Proof of Proposition 4.1. The control of the excess energy comes from (4.11). We just observe that

$$(\rho^2-1)\left(\nabla\theta\cdot\nabla\psi+\frac{1}{2}|\nabla\psi|^2\right) = |\nabla\psi|^2+(\nabla\theta-\rho^2\nabla\varphi)\cdot\nabla\psi+(\rho^2-1)\left(2\nabla\theta\cdot\nabla\psi+\frac{3}{2}|\nabla\psi|^2\right), \tag{4.35}$$

and combine (4.11) with (4.16) and (4.31), to find

$$(1-o(1))\int_{\Omega_\varepsilon}|\nabla\psi|^2+(1-o(1))\int_{\Omega_\varepsilon}\frac{2}{5\varepsilon^2}(1-\rho^2)^2+\int_{\Omega_\varepsilon}\frac{1}{4}|\nabla\rho|^2 \leq C\|\nabla\psi\|_{L^2(\Omega)}\|f_\varepsilon\|_{L^2(\Omega)}+o(1). \tag{4.36}$$

The relation (4.1) follows directly. Similarly, using (4.30), we are led to

$$\int_{B(x,l)\cap\Omega}\left(|\nabla\psi|^2+\frac{1}{2}|\nabla\rho|^2+\frac{2}{5\varepsilon^2}(1-\rho^2)^2\right) \leq \frac{l+Cl^2}{2}\int_{\partial B(x,l)}|\nabla\psi|^2+\frac{1}{2}|\nabla\rho|^2+\frac{2}{5\varepsilon^2}(1-\rho^2)^2 + C\|\nabla\psi\|_{L^2(B(x,l)\setminus\bigcup_i B_i)}(l\|f_\varepsilon\|_{L^2(B(x,l))}+1)+o(1). \tag{4.37}$$

Setting $F(l)=\int_{B(x,l)\cap\Omega\setminus\bigcup_i B_i}\big(|\nabla\psi|^2+\frac{1}{2}|\nabla\rho|^2+\frac{2}{5}\frac{(1-\rho^2)^2}{\varepsilon^2}\big)$, we deduce that F satisfies (4.4).

Also, returning to (4.12), and using (4.15) and the same other arguments, and combining this with (4.1), we find (4.2). □

4.1. ODE approach

Here, we give estimates for F given that it satisfies the differential inequality (4.4).

Lemma 4.6. *Let f be a nondecreasing function on an interval $[r,R]$ with $R\leq L$, satisfying*

$$\forall x\in[r,R],\quad f(x)\leq\frac{x+cx^2}{2}f'(x)+g(x)\sqrt{f(x)}+b \tag{4.38}$$

for some continuous function g. Then

$$f(r)\leq C\frac{r^2}{R^2}f(R)+2r^2(G(R)-G(r))^2+2b\left(1-\frac{r}{R}\right), \tag{4.39}$$

where G is an antiderivative of $g(x)/x^2$ and $C=2(1+Lc)^2$.

Proof. Dividing (4.38) by $x^2\sqrt{f(x)}$, we find

$$\frac{\sqrt{f(x)}}{x^2} \leq \frac{f'(x)}{2x\sqrt{f(x)}} + \frac{cf'(x)}{2\sqrt{f(x)}} + \frac{g(x)}{x^2} + \frac{b}{x^2\sqrt{f(x)}}.$$

Setting $h(x) = \frac{\sqrt{f(x)}}{x}$, we observe that $h'(x) = \frac{f'(x)}{2x\sqrt{f(x)}} - \frac{\sqrt{f(x)}}{x^2}$, and thus

$$0 \leq h'(x) + c(\sqrt{f})'(x) + \frac{g(x)}{x^2} + \frac{b}{x^2\sqrt{f(x)}}. \tag{4.40}$$

Integrating between r and R, and using the monotonicity of f for the last term, we find

$$\frac{\sqrt{f(r)}}{r} \leq \frac{\sqrt{f(R)}}{R} + c\sqrt{f(R)} - c\sqrt{f(r)} + G(R) - G(r) + \frac{b}{\sqrt{f(r)}}\left(\frac{1}{r} - \frac{1}{R}\right).$$

Thus

$$\frac{\sqrt{f(r)}}{r} \leq \sqrt{f(R)}\left(\frac{1}{R} + \frac{Lc}{R}\right) + G(R) - G(r) + \frac{b}{\sqrt{f(r)}}\left(\frac{1}{r} - \frac{1}{R}\right).$$

We observe that this is of the form $\lambda^2 \leq a_1\lambda + a_2$ where $\lambda = \sqrt{f(r)}$. Using the fact that for such an equation we have $\lambda^2 \leq a_1^2 + 2a_2$, we deduce

$$f(r) \leq \left(\frac{r(1+Lc)}{R}\sqrt{f(R)} + r(G(R) - G(r))\right)^2 + 2b\left(1 - \frac{r}{R}\right),$$

and the relation (4.39) follows directly. □

4.2. *Proof of Theorem 1*

Proof of (1.13), (1.14), (1.18) *and* (1.17). (1.13) follows directly from (4.1), and (1.14) from (1.13) combined with (4.34). For (1.18), we start from (4.2), which, combined with Lemma 3.1 (applied to the $B(a_i, R_\varepsilon\varepsilon)$), yields

$$E_\varepsilon(u_\varepsilon, \Omega_\varepsilon) \leq \pi \sum_i d_i^2 \log\frac{1}{R_\varepsilon\varepsilon} + W_{\mathbf{d}}(a_1, \ldots, a_n) + C\|f_\varepsilon\|_{L^2}^2 + o(1).$$

But, from (2.32) and (2.28), we find

$$E_\varepsilon(u_\varepsilon, B(a_i, R_\varepsilon\varepsilon)) = \pi d_i^2 \log R_\varepsilon + \gamma(V_i).$$

By Lemma 3.3, the result (1.18) follows.

Finally, (1.17) was proved in Proposition 2.2.

Localized estimates. We recall that $l \gg \varepsilon\sqrt{|\log \varepsilon|}$, so we can find a quantity $\sqrt{|\log \varepsilon|}\varepsilon \ll Q_\varepsilon \ll l$. Let us first consider the boundary case, i.e. $x \in \partial\Omega$, and let $F(l)$ be defined as in (4.3). Since the number of points a_i remains bounded by some n_0, the set $S = \{l \in \mathbb{R} : \partial B(x,l) \cap \bigcup_i B(a_i, Q_\varepsilon) \neq \emptyset\}$ is a finite union of fewer than n_0 intervals, with total length $\leq CQ_\varepsilon$. Let us write $S \cap [0, R] = [t_1, t_1'] \cup [t_2, t_2'] \cup \cdots \cup [t_k, t_k']$, where $t_1 < t_1' < t_2 < t_2' < \cdots < t_{k+1} = R \leq 1$. Assume now l is given, $l \in [t_i', t_{i+1}]$, and (4.4) holds in that interval. We may use Lemma 4.6 with $f = F$, $g(l) = Kl\|f_\varepsilon\|_{L^2(B(x,R))} + 1$ and b the $o(1)$ found in (4.4). Then $G(l) = K \log l\|f_\varepsilon\|_{L^2(B(x,R))} - 1/l$, thus we find

$$F(l) \leq C\frac{l^2}{t_{i+1}^2}F(t_{i+1}) + 2\left(l\log\frac{t_{i+1}}{l}\|f_\varepsilon\|_{L^2(B(x,R))} + 1\right)^2 + 2b\left(1 - \frac{t_{i+1}}{l}\right).$$

But $F(t_{i+1}) \leq F(t_{i+1}')$ and $t_{i+1} \leq 1$, so

$$F(l) \leq C\frac{l^2}{t_{i+1}^2}F(t_{i+1}') + 4\left(1 + l^2\log^2\frac{1}{l}\|f_\varepsilon\|^2_{L^2(B(x,R))}\right) + 2b. \tag{4.41}$$

Similarly, using (4.4) on $[t_{i+1}', t_{i+2}]$, we have

$$F(t_{i+1}') \leq C\frac{(t_{i+1}')^2}{t_{i+2}^2}F(t_{i+2}') + 4\left(1 + (t_{i+1}')^2\log^2\frac{1}{t_{i+1}'}\|f_\varepsilon\|^2_{L^2(B(x,R))}\right) + 2b. \tag{4.42}$$

The same relation holds for any $i \leq j \leq k+1$. Now observe that since $t_{j+1} \geq t_{j+1}' - Q_\varepsilon$, we have

$$\frac{l^2}{t_{i+1}^2}\frac{(t_{i+1}')^2}{t_{i+2}^2}\cdots\frac{(t_k')^2}{R} \leq \frac{l^2}{R^2}\left(1 + \frac{CQ_\varepsilon}{l}\right)^{2n_0} \leq \frac{Cl^2}{R^2}$$

in view of the assumption $l \gg Q_\varepsilon$. Using this and combining all the relations of the type (4.42), we are led, after some calculations, to

$$F(l) \leq C\frac{l^2}{R^2}F(R) + Cl^2\log^2\frac{1}{l}\|f_\varepsilon\|^2_{L^2(B(x,R))} + C \tag{4.43}$$

where C is a constant (depending on n_0). On the other hand, from (4.1), we have $F(R) \leq C\|f_\varepsilon\|^2_{L^2(\Omega)} + o(1)$, thus, taking $R = 1$,

$$F(l) \leq Cl^2\|f_\varepsilon\|^2_{L^2(\Omega)}\log^2 l + C. \tag{4.44}$$

If l belongs to some interval $[t_i, t_i']$, then we may get (4.44) for t_i', and using $F(l) \leq F(t_i')$ and $t_i' \leq l + Q_\varepsilon \leq 2l$, we deduce that a relation like (4.44) still holds.

For the interior case, let $x \in \Omega$ and let $R = \mathrm{dist}(x, \partial\Omega)$. Denote F by F_x to keep track of the center point. If $l \geq R$, then there exists $x_0 \in \partial\Omega$ such that $B(x,l) \subset B(x_0, 2l)$ and thus $F_x(l) \leq F_{x_0}(2l)$, and the result follows from the boundary case. If $l \leq R$, then, arguing exactly as above, since $B(x, R) \subset \Omega$ and (4.4) holds in the interior case, we can get (4.43) similarly, that is,

$$F_x(l) \le C\frac{l^2}{R^2}F_x(R) + Cl^2\log^2\frac{1}{l}\|f_\varepsilon\|^2_{L^2(B(x,R))} + C. \tag{4.45}$$

If $R \ge 1$, using (4.1), we are done. If not, we can find $x_0 \in \partial\Omega$ such that $B(x, R) \subset B(x_0, 2R)$, thus, using the result (4.44) for the boundary case, we have

$$F_x(R) \le F_{x_0}(2R) \le CR^2\|f_\varepsilon\|^2_{L^2}\log^2\frac{1}{R} + C.$$

Combining this with $R \ge l$ and (4.45), we find

$$F_x(l) \le Cl^2\|f_\varepsilon\|^2_{L^2}\log^2\frac{1}{l} + C,$$

that is, (4.44) is proved in the interior case as well, and we always have

$$\int_{B(x,l)\cap\Omega\setminus\bigcup_i B_i}\left(|\nabla\psi_\varepsilon|^2 + |\nabla\rho|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}\right) \le C + Cl^2\log^2 l\|f_\varepsilon\|^2_{L^2(\Omega)}. \tag{4.46}$$

In order to prove (1.16), let us use (4.33) on $B(x, s)$:

$$\int_{B(x,s)\cap\Omega\setminus\bigcup_i B_i}\left(|\nabla\rho|^2 + \frac{(1-\rho^2)^2}{\varepsilon^2}\right) \le C\int_{\partial B(x,s)\cap\Omega}|1-\rho^2|\,|\nabla\rho| \\ + o(1) + o\left(\int_{B(x,s)\cap\Omega\setminus\bigcup_i B_i}|\nabla\psi|^2\right). \tag{4.47}$$

Let us recall that $l \gg \varepsilon\sqrt{|\log\varepsilon|} \gg \varepsilon$. Thus, from (4.47),

$$\int_{B(x,s)\cap\Omega\setminus\bigcup_i B_i}\left(|\nabla\rho|^2 + \frac{(1-\rho^2)^2}{\varepsilon^2}\right) \le C\varepsilon\int_{\partial B(x,s)\cap\Omega}\left(|\nabla\rho|^2 + \frac{(1-\rho^2)^2}{\varepsilon^2}\right) \\ + o(1) + o\left(\int_{B(x,s)\cap\Omega\setminus\bigcup_i B_i}|\nabla\psi|^2\right).$$

Integrating this relation for $s \in [l, 2l]$, we easily deduce that

$$l\int_{B(x,l)\cap\Omega\setminus\bigcup_i B_i}\left(|\nabla\rho|^2 + \frac{(1-\rho^2)^2}{\varepsilon^2}\right) \le C\varepsilon\int_{B(x,2l)\cap\Omega}\left(|\nabla\rho|^2 + \frac{(1-\rho^2)^2}{\varepsilon^2}\right) \\ + o(1) + o\left(l\int_{B(x,2l)\cap\Omega\setminus\bigcup_i B_i}|\nabla\psi|^2\right).$$

Inserting (4.46) and the fact that $\int_{B_i}\left(|\nabla\rho|^2 + \frac{(1-\rho^2)^2}{2\varepsilon^2}\right) = O(1)$ (from [BMR] for example), we are led to

$$\int_{B(x,l)\setminus\bigcup_i B_i}\left(|\nabla\rho|^2 + \frac{(1-\rho^2)^2}{\varepsilon^2}\right) \le \left(C\frac{\varepsilon}{l} + o(1)\right)\left(l^2\log^2\frac{1}{l}\|f_\varepsilon\|^2_{L^2(\Omega)} + C\right) + o(1)$$

and since we assumed $\varepsilon/l \to 0$, we conclude that (1.16) holds.

We now prove that the second upper bound in (1.15) holds. For that, let us return to the proof of Proposition 4.1. Inserting (1.16) into (4.32), we have

$$\int_{D_\varepsilon} |1-\rho^2|\,|\nabla\psi|^2 \le o(1)\left(1 + l\log\frac{1}{l}\|f_\varepsilon\|_{L^2(B(x,2l))}\right)\|\nabla\psi\|_{L^2(D_\varepsilon)} + o(1). \tag{4.48}$$

In place of (4.37) we can now write

$$\begin{aligned}\int_{B(x,l)\cap\Omega\setminus\bigcup_i B_i} &\left(|\nabla\psi|^2 + \frac{1}{2}|\nabla\rho|^2 + \frac{2}{5\varepsilon^2}(1-\rho^2)^2\right)\\ &\le \frac{l+Cl^2}{2}\int_{\partial B(x,l)\cap\Omega}\left(|\nabla\psi|^2 + \frac{1}{2}|\nabla\rho|^2 + \frac{2}{5\varepsilon^2}(1-\rho^2)^2\right)\\ &\quad + C\|\nabla\psi\|_{L^2((B(x,l)\setminus\bigcup_i B_i))}\, o\left(l\log\frac{1}{l}\|f_\varepsilon\|_{L^2(\Omega)} + 1\right) + o(1).\end{aligned} \tag{4.49}$$

Then, we apply the same reasoning as before, i.e. use (4.39) this time with $g(l) = c(l\log\frac{1}{l}\|f_\varepsilon\|_{L^2(\Omega)} + 1)$, where $c = o(1)$, and the same method. Since $G(l)$, the antiderivative for g/l^2, is equal to $c(-(\log^2 l)/2 - 1/l)$, we find in the end, in place of (4.44),

$$F(l) \le o(1)(l^2\log^4 l\|f_\varepsilon\|^2_{L^2(\Omega)} + 1),$$

and we may conclude as before that (1.15) holds.

Remark 4.1. When $f_\varepsilon = 0$, Theorem 1 reproves the result of [CM2] without the need of L^∞ estimates on $1-|u|^2$ in Ω_ε.

5. Proof of Theorem 2

As we mentioned, the proof relies on the Pohozaev identity as in (2.7) or as in [BMR], combined with Lemma 3.2.

5.1. Interior case

Case $\sum_{i=1}^k d_i^2 > (\sum_i d_i)^2$. We denote by B_R the ball centered at x_0 of radius R. Let us apply Lemma 2.2 with $r = l$ and $R \le Kl/2$ so that $B_{2R}\setminus B_{l/2}$ intersects no B_i. Setting $f(s) = \int_{B_s\cap\Omega}(1-|u|^2)^2/\varepsilon^2$, and combining (2.7) and (2.8), we find

$$\begin{aligned}\int_{B_R\setminus B_l}\left|\frac{\partial u}{\partial r}\right|^2 + \int_l^R \frac{f(s)}{s}\,ds &\le \int_{B_R\setminus B_l}\left(\left|\frac{\partial u}{\partial\tau}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}\right) + \int_{B_R\setminus B_l}\left|\frac{\partial u}{\partial r}\right|^2\\ &\quad + \frac{R^2}{4}\int_{B_R\setminus B_l}|f_\varepsilon|^2 + l\log\frac{R}{l}\|f_\varepsilon\|_{L^2(B_l)}\|\nabla u\|_{L^2(B_l)},\end{aligned}$$

hence

$$\int_l^R \frac{f(s)}{s}\,ds \le \int_{B_R\setminus B_l}\left(\left|\frac{\partial u}{\partial \tau}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}\right) + \frac{R^2}{4}\|f_\varepsilon\|^2_{L^2(\Omega)} + Cl\log\frac{R}{l}\sqrt{|\log\varepsilon|}\|f_\varepsilon\|_{L^2(B_l)}. \tag{5.1}$$

But, in $B_R \setminus B_l$, we have

$$\left|\frac{\partial u}{\partial \tau}\right|^2 \le |\nabla u|^2 = |\nabla \rho|^2 + \rho^2|\nabla\theta + \nabla\psi|^2.$$

We claim that

$$\int_{B_R\setminus B_l}|\nabla\theta|^2 - CR^2\log^2\frac{1}{R}\|f_\varepsilon\|^2_{L^2} - C \le \int_{B_R\setminus B_l}\rho^2|\nabla\theta+\nabla\psi|^2 \le \int_{B_R\setminus B_l}|\nabla\theta|^2 + CR^2\log^2\frac{1}{R}\|f_\varepsilon\|^2_{L^2} + C. \tag{5.2}$$

Assuming this holds, let us insert this relation into (5.1), and use (1.16). We are led to

$$\int_l^R \frac{f(s)}{s}\,ds \le \int_{B_R\setminus B_l}|\nabla\theta|^2 + Cl\log\frac{R}{l}\sqrt{|\log\varepsilon|}\|f_\varepsilon\|_{L^2(B_l)} + CR^2\log^2\frac{1}{R}\|f_\varepsilon\|^2_{L^2(\Omega)} + C. \tag{5.3}$$

Now observe that for all $s \ge l$,

$$f(s) \ge \int_{B_l}\frac{(1-|u|^2)^2}{\varepsilon^2} \ge 2\pi\sum_{i=1}^k d_i^2 - o(1)$$

in view of (2.31). Thus, using the relation $x \le x^2+1$ and (3.15), and inserting this into (5.3), we obtain

$$\Big(2\pi\sum_{i=1}^k d_i^2 - o(1)\Big)\log\frac{R}{l} \le 2\pi\Big(\sum_{i=1}^k d_i\Big)^2\log\frac{R}{l} + C + CR^2\log^2\frac{1}{R}\|f_\varepsilon\|^2_{L^2(\Omega)} + Cl\log\frac{R}{l}\sqrt{|\log\varepsilon|}\|f_\varepsilon\|_{L^2(\Omega)}.$$

But we assumed $(\sum_{i=1}^k d_i)^2 < \sum_{i=1}^k d_i^2$, and because these involve integers, the difference is at least 1. We deduce

$$\log\frac{R}{l} - C \le CR^2\log^2\frac{1}{R}\|f_\varepsilon\|^2_{L^2(\Omega)} + Cl\log\frac{R}{l}\sqrt{|\log\varepsilon|}\|f_\varepsilon\|_{L^2(\Omega)}, \tag{5.4}$$

where again the constants depend only on β, M, Ω, and g.

We then distinguish two cases. Either the first term on the right-hand side is less than the second, in which case we deduce

$$\log \frac{R}{l} - C \le Cl \log \frac{R}{l} \sqrt{|\log \varepsilon|} \| f_\varepsilon \|_{L^2(\Omega)}$$

and taking $R = K_0 l/2$ with K_0 large enough (K_0 thus depends only on β, M, Ω and g), we find

$$\| f_\varepsilon \|^2_{L^2(\Omega)} \ge \frac{C}{l^2 |\log \varepsilon|}. \tag{5.5}$$

In the other case, $CR^2 \log^2 \frac{1}{R} \| f_\varepsilon \|^2_{L^2} \ge l \log \frac{R}{l} \sqrt{|\log \varepsilon|} \| f_\varepsilon \|_{L^2(\Omega)}$. Taking again $R = K_0 l/2$, we find

$$\| f_\varepsilon \|^2_{L^2(\Omega)} \ge \frac{C}{R^2 \log^2 \frac{1}{R}} = \frac{C}{K_0^2 l^2 \log^2 \frac{1}{K_0 l}}. \tag{5.6}$$

The theorem is thus proved in this case.

Proof of (5.2). As in the proof of Theorem 1, we can extend ψ inside B_l in such a way that

$$\int_{B_R} |\nabla \psi|^2 \le C \int_{B_R \setminus B_l} |\nabla \psi|^2 \le CR^2 \log^2 \frac{1}{R} \| f_\varepsilon \|^2_{L^2} + C$$

(from Theorem 1). Then, using the fact that $\int_{\partial B_R} \frac{\partial \theta}{\partial \nu} = \int_{\partial B_l} \frac{\partial \theta}{\partial \nu} = 0$, we get

$$\begin{aligned}
\int_{B_R \setminus B_l} \rho^2 |\nabla \theta + \nabla \psi|^2 &= \int_{B_R \setminus B_l} \rho^2 |\nabla \theta|^2 + \rho^2 |\nabla \psi|^2 + 2 \int_{B_R \setminus B_l} \nabla \theta \cdot \nabla \psi \\
&\quad + 2 \int_{B_R \setminus B_l} (\rho^2 - 1) \nabla \theta \cdot \nabla \psi \\
&= \int_{B_R \setminus B_l} \rho^2 |\nabla \theta|^2 + \rho^2 |\nabla \psi|^2 + 2 \int_{B_R \setminus B_l} (\rho^2 - 1) \nabla \theta \cdot \nabla \psi \\
&\quad + 2 \int_{\partial B_R} \frac{\partial \theta}{\partial \nu} (\psi - \psi_R) - 2 \int_{\partial B_l} \frac{\partial \theta}{\partial \nu} (\psi - \psi_l),
\end{aligned}$$

where ψ_R and ψ_l are the averages of ψ on ∂B_R and ∂B_l respectively. On the other hand, by the trace theorem and Theorem 1,

$$\int_{\partial B_l} |\psi - \psi_l| \le Cl \| \nabla \psi \|_{L^2(B_l)} \le Cl^2 \log^2 \frac{1}{l} \| f_\varepsilon \|_{L^2} + o(1),$$

while $|\nabla \theta| \le C/l$ on ∂B_l, thus

$$\left| \int_{B_l} \frac{\partial \theta}{\partial \nu} (\psi - \psi_l) \right| \le Cl \log \frac{1}{l} \| f_\varepsilon \|_{L^2} + C$$

and the same holds on ∂B_R. Arguing as in Lemma 4.4, we also have

$$\int_{B_R \setminus B_l} (\rho^2 - 1)(|\nabla \theta|^2 + 2 \nabla \theta \cdot \nabla \psi) \le CR \log \frac{1}{R} \| f_\varepsilon \|_{L^2} + C \le CR^2 \log^2 \frac{1}{R} \| f_\varepsilon \|^2_{L^2} + C.$$

Using (1.15) again, we deduce that (5.2) holds.

Case $(\sum_{i=1}^k d_i)^2 \sum_{i=1}^k d_i^2$. We start again from (2.7) and (2.8) and are led to

$$\int_{B_R\setminus B_l}\left(\left|\frac{\partial u}{\partial \tau}\right|^2+\frac{(1-|u|^2)^2}{2\varepsilon^2}\right)\le \int_l^R \frac{f(s)}{s}\,ds+\int_{B_R\setminus B_l}\left|\frac{\partial u}{\partial r}\right|^2+\frac{R^2}{4}\int_{B_R\setminus B_l}|f_\varepsilon|^2 + Cl\log\frac{R}{l}\sqrt{|\log\varepsilon|}\,\|f_\varepsilon\|_{L^2(B_l)}. \tag{5.7}$$

First, using (3.16) and Theorem 1, we have

$$\int_{B_R\setminus B_r}\left(\left|\frac{\partial \rho}{\partial r}\right|^2+\left|\frac{\partial}{\partial r}(\theta+\psi)\right|^2\right)\le C+CR^2\log^2\frac{1}{R}\|f_\varepsilon\|^2_{L^2(\Omega)}. \tag{5.8}$$

On the other hand, from (5.2), we have

$$\int_{B_R\setminus B_l}|\nabla u|^2\ge\int_{B_R\setminus B_l}|\nabla\theta|^2-CR^2\log^2\frac{1}{R}\|f_\varepsilon\|^2_{L^2}-C.$$

But if we combine this with (5.8), we must have

$$\int_{B_R\setminus B_l}\left|\frac{\partial u}{\partial \tau}\right|^2\ge\int_{B_R\setminus B_l}|\nabla\theta|^2-CR^2\log^2\frac{1}{R}\|f_\varepsilon\|^2_{L^2}-C.$$

Combining this with (3.15) and inserting it and (5.8) into (5.7), we are led to

$$2\pi\Big(\sum_{i=1}^k d_i\Big)^2\log\frac{R}{l}\le\int_l^R\frac{f(s)}{s}\,ds+C+CR^2\log^2\frac{1}{R}\|f_\varepsilon\|^2_{L^2(\Omega)} + Cl\log\frac{R}{l}\sqrt{|\log\varepsilon|}\,\|f_\varepsilon\|_{L^2(\Omega)}. \tag{5.9}$$

Meanwhile for all $s\le R$,

$$\begin{aligned} f(s)&=\int_{\bigcup_{i=1}^k B_i}\frac{(1-|u|^2)^2}{\varepsilon^2}+\int_{B_s\setminus\bigcup_{i=1}^k B_i}\frac{(1-|u|^2)^2}{\varepsilon^2}\\ &\le 2\pi\sum_{i=1}^k d_i^2+o\Big(R^2\log^2\frac{1}{R}\|f_\varepsilon\|^2_{L^2}\Big)+o(1) \end{aligned}\tag{5.10}$$

where we have used (2.31) and (1.16). After integrating, this yields

$$2\pi\Big(\sum_{i=1}^k d_i\Big)^2\log\frac{R}{l}\le\Big(2\pi\sum_{i=1}^k d_i^2+o(1)R^2\log^2\frac{1}{R}\|f_\varepsilon\|^2_{L^2}+o(1)\Big)\log\frac{R}{l} + C+CR^2\log^2\frac{1}{R}\|f_\varepsilon\|^2_{L^2(\Omega)}+Cl\log\frac{R}{l}\sqrt{|\log\varepsilon|}\,\|f_\varepsilon\|_{L^2(\Omega)},$$

and hence

$$2\pi\Big(\sum_{i=1}^k d_i\Big)^2-2\pi\sum_{i=1}^k d_i^2\le CR^2\log^2\frac{1}{R}\|f_\varepsilon\|^2_{L^2}+Cl\sqrt{|\log\varepsilon|}\,\|f_\varepsilon\|_{L^2(\Omega)}+\frac{C}{\log\frac{R}{l}}+o(1).$$

Since the left-hand side is at least equal to 2π, we find, if $R = K_0 l/2$ with K_0 large enough, that

$$C \leq CR^2 \log^2 \frac{1}{R} \|f_\varepsilon\|_{L^2}^2 + Cl\sqrt{|\log \varepsilon|}\|f_\varepsilon\|_{L^2}.$$

Distinguishing two cases as previously, we may conclude that

$$\|f_\varepsilon\|_{L^2}^2 \geq \min\left(\frac{C}{l^2|\log \varepsilon|}, \frac{C}{l^2 \log^2 \frac{1}{l}}\right).$$

5.2. Boundary case

The proof is roughly the same. Assuming $R < 1/2$, we may use (2.20) or (2.23) to get in any case

$$\int_l^R \frac{f(s)}{s} \leq C \int_{(B_R \setminus B_l)\cap\Omega} \left(|\nabla u|^2 + \frac{(1-|u|^2)^2}{\varepsilon^2}\right) + CR(1 + \sqrt{|\log \varepsilon|}\|f_\varepsilon\|_{L^2(\Omega)}). \tag{5.11}$$

Arguing as in the interior case and using (3.17), we get

$$\int_{(B_R \setminus B_l)\cap\Omega} \left(|\nabla u|^2 + \frac{(1-|u|^2)^2}{\varepsilon^2}\right) \leq C + CR^2 \log^2 \frac{1}{R}\|f_\varepsilon\|_{L^2}^2$$

and also

$$\int_l^R \frac{f(s)}{s} \geq \left(2\pi \sum_i d_i^2 - o(1)\right) \log \frac{R}{l} \geq \pi \log \frac{R}{l}.$$

Inserting this into (5.11), we find

$$\pi \log \frac{R}{l} \leq C + CR^2 \log^2 \frac{1}{R}\|f_\varepsilon\|_{L^2}^2 + R\sqrt{|\log \varepsilon|}\|f_\varepsilon\|_{L^2}$$

and arguing as above, we deduce, taking $R = K_0 l/2$ with K_0 large enough, that

$$\|f_\varepsilon\|_{L^2}^2 \geq \min\left(\frac{C}{R^2|\log \varepsilon|}, \frac{C}{R^2 \log^2 \frac{1}{R}}\right),$$

from which the result follows.

Applying Theorem 2 in the case $f_\varepsilon = 0$, i.e. for a solution of Ginzburg–Landau, we obtain Corollary 1.1.

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