Solved and Unsolved Problems

Michael Th. Rassias (University of Zürich, Switzerland)

The present column is devoted to Algebra.

As for everything else, so for a mathematical theory:
beauty can be perceived but not explained.
Arthur Cayley (1821–1895)

I Six (plus one) new problems – solutions solicited

226. Let $\mathbb{C}^n$ stand for the space of complex *column* n-vectors, and let $\mathbb{M}_n$ stand for the space of complex $n \times n$ matrices.
The inner product $\langle x|y\rangle$ of $x, y \in \mathbb{C}^n$ is defined as

$$\langle x|y\rangle = x^* y \quad \text{(matrix product)}.$$

Therefore $\langle x|y\rangle$ is linear in y and conjugate linear in x.
Let A, B be $n \times m$ complex matrices. Write them as

$$A = [a_1, \ldots, a_m] \quad \text{and} \quad B = [b_1, \ldots, b_m]$$

with $a_j, b_j \in \mathbb{C}^n$ $(j = 1, 2, \ldots, m)$.
Then it is immediate from the definition of matrix multiplication that

$$A^* B = \left[\langle a_j|b_k\rangle\right]_{j,k=1}^m \in \mathbb{M}_m.$$

Show the following relation:

$$AB^* = \sum_{j=1}^m a_j b_j^* \in \mathbb{M}_n$$

where each product $a_j b_j^*$ $(j = 1, \ldots, n)$ is a rank-one matrix in $\mathbb{M}_n$.

(T. Ando, Hokkaido University, Sapporo, Japan)

227. Let p and q be two distinct primes with $q > p$ and G a group of exponent q for which the map $f_p : G \to G$ defined by $f_p(x) = x^p$, for all $x \in G$, is an endomorphism. Show that G is an abelian group.

(Dorin Andrica and George Cătălin Ţurcaş, Babeş-Bolyai University, Cluj-Napoca, Romania)

228. Let $(G, \cdot)$ be a group with the property that there is an integer $n \geq 1$ such that the map $f_n : G \to G$, $f_n(x) = x^n$ is injective and the map $f_{n+1} : G \to G$, $f_{n+1}(x) = x^{n+1}$ is a surjective endomorphism. Prove that G is an abelian group.

(Dorin Andrica and George Cătălin Ţurcaş, Babeş-Bolyai University, Cluj-Napoca, Romania)

229. Let A and $B \in \mathrm{Mat}_k(K)$ be two matrices over a field K. We say that A and B are *similar* if there exists an invertible matrix $C \in \mathrm{GL}_k(K)$ such that $B = C^{-1}AC$.
Let A and $B \in \mathrm{GL}_k(\mathbb{Q})$ be two similar invertible matrices over the field of rational numbers $\mathbb{Q}$. Assume that for some integer l, $A^{l+1}B = BA^l$. Then A and B are the identity matrices.

(Andrei Jaikin-Zapirain, Departamento de Matemáticas, Universidad Autónoma de Madrid & Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, Spain, and Dmitri Piontkovski, Faculty of Economic Sciences, Moscow Higher School of Economics, Russia)

230. We are trying to hang a picture on a wall. The picture has a piece of string attached to it forming a loop, and there are 3 nails in the wall that we can wrap the string around. We want to hang the picture so that it does not fall down, but it will upon removal of any of the 3 nails.

(Dawid Kielak, Mathematical Institute, University of Oxford, UK)

231. Given a natural number n and a field k, let $M_n(k)$ be the full $n \times n$ matrix algebra over k. A matrix $(a_{ij}) \in M_n(k)$ is said to be centrosymmetric if

$$a_{ij} = a_{n+1-i,n+1-j}$$

for $1 \leq i, j \leq n$. Let $C_n(k)$ denote the set of all centrosymmetric matrices in $M_n(k)$. Then $C_n(k)$ is a subalgebra of $M_n(k)$, called centrosymmetric matrix algebra over k of degree n. Centrosymmentric matrices have a long history (see [1, 5]) and applications in many areas, such as in Markov processes, engineering problems and quantum physics (see [2, 3, 4, 6]). In the representation theory of algebras, a fundamental problem for a finite-dimensional algebra is to know if it has finitely many nonisomorphic indecomposable modules (or in other terminology, representations). In our case, the concrete problem on $C_n(k)$ reads as follows.
Does $C_n(k)$ have finitely many nonisomorphic indecomposable modules? If yes, what is the number?

(Changchang Xi, School of Mathematical Sciences, Capital Normal University, 100048 Beijing, China, and College of Mathematics and Information Science, Henan Normal University, 453007 Xinxiang, Henan, China)

References

[1] A. C. Aitken, *Determinants and Matrices*. Oliver and Boyd, Edinburgh, 1939.
[2] A. R. Collar, On centrosymmetric and centroskew matrices. *Quart. J. Mech. Appl. Math.* **15** (1962), 265–281.
[3] L. Datta and S. D. Morgera, On the reducibility of centrosymmetric matrices – Applications in engineering problems. *Circuits Systems Sig. Proc.* **8**(1) (1989), 71–96.
[4] I. J. Good, The inverse of a centrosymmetric matrix. *Technometrics* **12** (1970), 925–928.
[5] T. Muir, *A Treatise on the Theory of Determinants*. Revised and enlarged by William H. Metzler, Dover Publications, Inc., New York, 1933/1960.
[6] J. R. Weaver, Centrosymmetric (cross-symmetric) matrices, their basic properties, eigenvalues, and eigenvectors. *Amer. Math. Monthly* **92**(10) (1985), 711–717.
[7] C. C. Xi and S. J. Yin, Cellularity of centrosymmetric matrix algebras and Frobenius extensions. *Linear Algebra Appl.* **590** (2020), 317–329.

An additional interesting problem (not intimately connected to algebra). Intervals of monotonic changes in the polynomial are located between the roots of its derivative. A derivative of a polynomial is also a polynomial, although of a lesser degree. Using these considerations, construct an algorithm for calculating the real roots of the quadratic equation. Improve it to calculate the real roots of the polynomial of the third, fourth and generally arbitrary degree.

(Igor Kostin, Moscow, Russian Federation)

II (A) **Open problems,** by Maxim Kontsevich (Institut des Hautes Études Scientifiques, Bures-sur-Yvette, France)

Consider the group ring A of a free finitely generated group (i.e., noncommutative Laurent polynomials) with coefficients in a field $\mathbf{k}$ of characteristic zero. Denote by $\tau : A \to \mathbf{k}$ the "trace map" given by the coefficient of monomial 1 ().

Theorem For any $a \in A$ the following series is *algebraic*:

$$G = G(a) := \exp\Big(-\sum_{n\geq 1} \tau(a^n)\frac{t^n}{n}\Big) = 1 + \cdots \in \mathbf{k}[[t]]$$

The rationale for the minus sign is that if we replace algebra A by $Mat(N \times N, \mathbf{k})$ and τ by the usual trace, the resulting series $G = \det(1 - at)$ is polynomial.

Example For $a = x_1 + x_1^{-1} + x_2 + x_2^{-1} + \cdots + x_m + x_m^{-1}$ we have

$$G = ((f+1)/2)^m/((mf + m - 1)/(2m-1))^{m-1},$$
$$f := \sqrt{1 - 4(2m-1)t^2}.$$

In 2007 I found a ridiculous proof in three steps (basically no progress afterwards):

(1) It is known that $F := G'/G = -\sum_{n\geq 1} \tau(a^n)t^{n-1}$ is algebraic (see, e.g., Corollary 6.7.2 in [10], I learned about it from a paper [6] where it was rediscovered). This is a corollary of the theory of algebraic formal languages developed by N. Chomsky and M. Schützenberger in the 60s [4].
(2) Assume that all coefficients of a are integers (it is not a severe restriction), then it is easy to see that coefficients of G are integers (it follows solely from the fact that the free group is torsion-free).
(3) Then we have (a) $G \in \mathbb{Z}[[t]]$, (b) $dG/dt = FG$, i.e., G is a flat section of a line bundle with connection on an algebraic curve over $\mathbb{Q}$. Property (a) implies that the p-curvature of this connection is 0 for almost all primes p, thus fitting to the realm of general Grothendieck–Katz conjecture. By a result of Yves André [1], [2] based on ideas of D. V. and G. V. Chudnovsky [5], or by later results of J.-B. Bost [3], series G is also algebraic.

If we replace the free group by a finite group Γ, the resulting series G is again algebraic, a fractional power of a polynomial. This follows from the fact that after an extension of scalars $\mathbf{k}' \supset \mathbf{k}$, the group ring of the finite group is a direct sum of matrix algebras, and that the canonical trace on the group algebra is proportional to the matrix trace on each matrix algebra summand $Mat(N_i \times N_i, \mathbf{k}')$ rescaled by $N_i/|\Gamma| \in \mathbb{Q}_{>0}$.

For a finitely-generated free abelian group, the series F can be calculated by residue formula, and is holonomic, i.e., it satisfies a non-trivial algebraic linear differential equation. Neveretheless, the series G in this case is typically not holonomic, in particular not algebraic. For nonhyperbolic groups the situation is much more tricky, see [7] where it was shown that the analog of generating series F can already be nonholonomic for the arithmetic group $SL(4, \mathbb{Z})$.

232*. Find a purely combinatorial proof of algebraicity of G, not based on results from number theory. In particular, what is an adequate upper bound on the degree of the equation satisfied by G? Is it true that the function G considered as a multi-valued function is either $1 - ct$ for some constant $c \in \mathbf{k}$, or it does not attain zero value for $t \neq 0$? Also, is it true that the values of all branches of G at $t = 0$ belong to the set $\{0, 1\} \subset \mathbb{P}^1$?

References

[1] Y. André, *G-Functions and Geometry*. Aspects of Mathematics, 13, Friedr. Vieweg & Sohn, Braunschweig, 1989.
[2] Y. André, Sur la conjecture des p-courbures de Grothendieck et Katz. *Geometric Aspects of Dwork Theory*, Vol. I, II, 55–112, Walter de Gruyter GmbH & Co. KG, Berlin, 2004.
[3] J.-B. Bost, Algebraic leaves of algebraic foliations over number fields. *Publ. Math. Inst. Hautes Études Sci.* **93** (2001), 161–221.
[4] N. Chomsky and M.P. Schützenberger, The algebraic theory of context-free languages. *Computer Programming and Formal Systems*, pp. 118–161, North-Holland, Amsterdam, 1963.
[5] D.V. Chudnovsky and G.V. Chudnovsky, Applications of Padé approximations to the Grothendieck conjecture on linear differential equations. *Number Theory* (New York, 1983–84), 52–100, Lecture Notes in Math. 1135, Springer, Berlin, 1985.
[6] S. Garoufalidis and J. Bellissard, Algebraic G-functions associated to matrices over a group-ring. arXiv:0708.4234v4.
[7] S. Garrabrant and I. Pak, Words in linear groups, random walks, automata and P-recursiveness. arXiv:1505.06508.
[8] C. Kassel and C. Reutenauer, Algebraicity of the zeta function associated to a matrix over a free group algebra. *Algebra & Number Theory* **8**(2)(2014), 497–511.
[9] M. Kontsevich, Noncommutative identities. Talk at Mathematische Arbeitstagung 2011, Bonn. arXiv:1109.2469v1.
[10] R.P. Stanley, *Enumerative combinatorics*, vol. 2, with a foreword by Gian-Carlo Rota and Appendix 1 by Sergey Fomin, Cambridge Studies in Advanced Mathematics 62, Cambridge University Press, Cambridge, 1999.

II (B) **Open problems on Iwahori–Hecke algebras,** by George Lusztig (Department of Mathematics, M.I.T., Cambridge, USA)

Below we state four open problems (see (233*)–(236*)) on Iwahori–Hecke algebras.

1. Let I be a finite set and let $(m_{ij})_{(i,j)\in I\times I}$ be a symmetric matrix whose diagonal entries are 1 and whose nondiagonal entries are integers ≥ 2 or ∞. Let W be the group with generators $\{s_i; i \in I\}$ and relations $(s_i s_j)^{m_{ij}} = 1$ for any i, j such that $m_{ij} < \infty$; this is a Coxeter group. (Examples of Coxeter groups are the Weyl groups of simple Lie algebras; these are finite groups. Other examples are the affine Weyl groups which are almost finite.) For $w \in W$ let $|w|$ be the smallest integer $n \geq 0$ such that w is a product of n generators $s_i, i \in I$. We assume that we are given a weight function $L : W \to \mathbf{N}$ that is a function such that $L(w) > 0$ for all $w \in W - \{1\}$ and

$$L(ww') = L(w) + L(w')$$

for any w, w' in W such that

$$|ww'| = |w| + |w'|.$$

(For example, $w \mapsto |w|$ is a weight function.) Let $A = \mathbf{Z}[v, v^{-1}]$ where v is an indeterminate. Let H be the free A-module with basis $\{T_w; w \in W\}$. There is a unique structure of associative A-algebra on H for which

$$(T_{s_i} + v^{-L(s_i)})(T_{s_i} - v^{L(s_i)}) = 0$$

for $i \in I$ and

$$T_w T_{w'} = T_{ww'}$$

for any w, w' in W such that $|ww'| = |w| + |w'|$; this is the Iwahori–Hecke algebra associated to W, L.

For $c \in \mathbf{C} - \{0\}$ let $H_c = \mathbf{C} \otimes AH$ where $\mathbf{C}$ is viewed as an A-algebra via the ring homomorphism $A \to \mathbf{C}, v \mapsto c$. Now H_c is also referred to as an Iwahori–Hecke algebra.

233*. Show that the algebras associated in [10] to a supercuspidal representation of a parabolic subgroup of a p-adic reductive group are (up to extension by a group algebra of a small finite group) of the form H_q where q is a power of p, with H associated to an affine Weyl group W and with L in the collection Σ_W of weight functions on W described in [4, §17], [5], [6].

For example, if W is of affine type F_4, then Σ_W consists of all L whose values on $\{s_i; i \in I\}$ are $(1, 1, 1, 1, 1)$ or $(1, 1, 1, 2, 2)$ or $(2, 2, 2, 1, 1)$ or $(1, 1, 1, 4, 4)$; if W is of affine type G_2, then Σ_W consists of all L whose values on $\{s_i; i \in I\}$ are $(1, 1, 1)$ or $(1, 1, 3)$ or $(3, 3, 1)$ or $(1, 1, 9)$.

The statement analogous to (233*) for groups with connected centre over a finite field F_q instead of p-adic groups is known to hold, without the words in parenthesis; in that case, W is a Weyl group and Σ_W consists of the weight functions on W described in [3, p. 35].

2. There is a unique group homomorphism$^-$: $H \to H$ such that

$$\overline{v^n T_w} T_{w^{-1}} = v^{-n}$$

for $n \in \mathbf{Z}, w \in W$; it is a ring isomorphism. Let

$$H_{\leq 0} = \sum_{w \in W} \mathbf{Z}[v^{-1}] T_w \subset H.$$

For any $w \in W$ there is a unique element $c_w \in H_{\leq 0}$ such that

$$\bar{c}_w = c_w$$

and

$$c_w - T_w \in v^{-1} H_{\leq 0}$$

(see [2], [7]). Then $\{c_w; w \in W\}$ is an A-basis of H. For x, y, z in W we define $f_{x,y,z} \in A$, $h_{x,y,z} \in A$ by

$$T_x T_y = \sum_{z \in W} f_{x,y,z} T_z,$$

$$c_x c_y = \sum_{z \in W} h_{x,y,z} c_z.$$

234*. Show that there exists an integer $N \geq 0$ such that for any x, y, z in W we have $v^{-N} f_{x,y,z} \in \mathbf{Z}[v^{-1}]$.

(See [7, 13.4]) If W is finite this is obvious. If W is an affine Weyl group, this is known.

We will now assume that (234*) holds. With N as in (234*), we see that

$$v^{-N} h_{x,y,z} \in \mathbf{Z}[v^{-1}]$$

for any x, y, z in W. It follows that for any $z \in W$ there is a unique integer $a(z) \geq 0$ such that

$$h_{x,y,z} \in v^{a(z)} \mathbf{Z}[v^{-1}]$$

for all x, y in W and

$$h_{x,y,z} \notin v^{a(z)-1} \mathbf{Z}[v^{-1}]$$

for some x, y in W. Hence for x, y, z in W there is a well-defined integer $\gamma_{x,y,z^{-1}}$ such that

$$h_{x,y,z} = \gamma_{x,y,z^{-1}} v^{a(z)} \mod v^{a(z)-1} \mathbf{Z}[v^{-1}].$$

Let J be the free abelian group with basis $\{t_w; w \in W\}$. For x, y in W we set

$$t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z.$$

(This is a finite sum.)

235*. Show that this defines an (associative) ring structure on J (without 1 in general).

Assume now that W is a Weyl group or an affine Weyl group and $L = \|$. In this case, (235*) is known to be true and the ring J does have a unit element.

More generally, assume that W is an affine Weyl group and $L \in \Sigma_W$ (see (233*)); in this case there is a (conjectural) geometric description [8, 3.11], of the elements c_w. From this one should be able to deduce (235*) as well as the well-definedness of the $\mathbf{C}$-algebra homomorphism $H_q \to \underline{J}$ in [7, 18.9], where H_q is as in (233*) and $\underline{J} = \mathbf{C} \otimes J$ is independent of q. One should expect that the irreducible (finite dimensional) $\underline{J}$-modules, when viewed as H_q-modules, form a basis of the Grothendieck group of H_q-modules. (This is indeed so if $L = \|$.) This should provide a construction of the "standard modules" of H_q which, unlike the construction in [5],[6], does not involve the geometry of the dual group.

3. Assume that W is finite and that $L = \|$. Let C be a conjugacy class in W; let C_{min} be the set of all $w \in C$ such that $|w|$ is minimal. For $w \in C$ let $N^w \in A$ be the trace of the A-linear map $H \to H$,

$$h \mapsto v^{2|w|} T_w h T_{w^{-1}}.$$

We have $N^w \in \mathbf{Z}[v^2]$. (Note that $N^w|_{v=1}$ is the order of the centralizer of w in W.) From [1] one can deduce that for $w \in C_{min}$, N^w depends only on C, not on w. We say that C is *positive* if $N^w \in \mathbf{N}[v^2]$. For example, if C is an elliptic regular conjugacy class (in the sense of [11]) then C is positive (see [9]). If W is of type A_n, the positive conjugacy classes are $\{1\}$ and the class of the Coxeter element. In the case where W is a Weyl group of exceptional type, a complete list of positive conjugacy classes in W is given in [9].

236*. Make a list of all positive conjugacy classes in W, assuming that W is a Weyl group of type B_n or D_n.

References

[1] M. Geck and G. Pfeiffer, *Characters of Finite Coxeter Groups and Iwahori–Hecke Algebras*. Clarendon Press, Oxford, 2000.

[2] D.Kazhdan and G.Lusztig, Representations of Coxeter groups and Hecke algebras, *Inv. Math.* 53 (1979), 165–184.

[3] G. Lusztig, Representations of finite Chevalley groups. *Regional Conf. Ser. in Math.* 39. Amer. Math. Soc., 1978.

[4] G. Lusztig, Intersection cohomology methods in representation theory. *Proc. Int. Congr. Math. Kyoto 1990*. Springer Verlag, 1991.

[5] G. Lusztig, Classification of unipotent representations of simple p-adic groups. *Int. Math. Res. Notices* (1995), 517–589.

[6] G. Lusztig, Classification of unipotent representations of simple p-adic groups, II. *Represent. Th.* 6 (2002), 243–289.

[7] G. Lusztig, *Hecke algebras with unequal parameters*, CRM Monograph Ser. 18, Amer. Math. Soc., 2003. Additional material in version 2 (2014). arXiv:math/0208154.

[8] G. Lusztig, Nonsplit Hecke algebras and perverse sheaves. *Selecta Math.* 22 (2016), 1953–1986.

[9] G. Lusztig, Positive conjugacy classes in Weyl groups. arXiv:1805.03772.

[10] M. Solleveld, Endomorphism algebras and Hecke algebras for reductive p-adic groups. arXiv:2005.07899.

[11] T. A. Springer, Regular elements of finite reflection groups. *Invent. Math.* 25 (1974), 159–193.

III Solutions

218.
Determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{2^n - 1},$$

where φ is the Euler's totient function.

(Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania)

Solution by the proposer. Let $(a_n)n \geq 1$ be a sequence of real numbers. From the equality

$$\frac{x^n}{1-x^n} = x^n + x^{2n} + \cdots + x^{kn} + \cdots, |x| < 1$$

we derive

$$\sum_{n=1}^{\infty} \frac{a_n x^n}{1-x^n} = \sum_{n=1}^{\infty} A_n x^n, \tag{1}$$

where

$$A_n = \sum_{d|n} a_d$$

and assuming that the power series in the right-hand side of (1) is convergent for $|x| < 1$. This is the main idea of the so-called Lambert series with the related identities.

Now, using the well-known Gauss' identity $\sum_{d|n} \varphi(d) = n$, the formula (1) yields

$$\sum_{n=1}^{\infty} \frac{\varphi(n) x^n}{1-x^n} = \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}, |x| < 1. \tag{2}$$

Setting $x = \frac{1}{2}$ implies $\sum_{n=1}^{\infty} \frac{\varphi(n)}{2^n-1} = 2$.

Also solved by George Miliakos (Sparta, Greece), Moubinool Omarjee (Paris, France), Richard G.E. Pinch (Gloucestershire, UK), Rudolf Rupp (Nuremberg, Bavaria, Germany), Muhammad Thoriq (Yogyakarta, Indonesia), Socratis Varelogiannis (France) and James J. Ward (Galway, Ireland).

219[a]. Let $\omega(n)$ denote the number of distinct prime factors of a non-zero natural number n.

(i) Prove that $\sum_{n \leq x} \omega(n) = x \log\log x + O(x)$.
(ii) Prove that $\sum_{n \leq x} \omega(n)^2 = x(\log\log x)^2 + O(x \log\log x)$.
(iii) Using (i) and (ii), prove that $\sum_{n \leq x} (\omega(n) - \log\log x)^2 = O(x \log\log x)$.
(iv) Using (iii), prove that $\sum_{n \leq x} (\omega(n) - \log\log n)^2 = O(x \log\log x)$.
(v) Using (iv), prove that $\omega(n)$ has normal order $\log\log n$, i.e., for every $\varepsilon > 0$,

$$\#\{n \leq x : (1-\varepsilon)\log\log n < \omega(n) < (1+\varepsilon)\log\log n\} \sim x \quad (\text{as } x \to \infty).$$

a. Parts (i)–(v) of Problem 219 are extracted from a proof by Paul Turán (1910–1976), published in 1934, of a theorem of G. H. Hardy (1877–1947) and S. Ramanujan (1887–1920), published in 1917; see references below:

[1] G. H. Hardy and S. Ramanujan, The normal number of prime factors of a number *n*. *Quart. J. Math.* **48** (1917), 76–92.
[2] P. Turán, On a Theorem of Hardy and Ramanujan. *J. London Math. Soc.* **9**(4) (1934), 274–276.

(Alina Carmen Cojocaru, Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, USA, and Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, Romania)

Solution by the proposer. In what follows, the letters p, p_1, p_2 denote primes.

(i) Using the definition of $\omega(n)$ and Mertens' theorem, we obtain

$$\sum_{n \leq x} \omega(n) = \sum_{p \leq x} \left[\frac{x}{p}\right] = x \sum_{p \leq x} \frac{1}{p} + O(x) = x \log\log x + O(x).$$

(ii) As above, using the definition of $\omega(n)$ and Mertens' theorem, we obtain

$$\begin{aligned}\sum_{n \leq x} \omega(n)^2 &= \sum_{n \leq x} \sum_{p_1|n} \sum_{p_2|n} 1 = \sum_{p_1 \leq x} \sum_{p_2 \leq x} \sum_{\substack{n \leq x \\ p_1|n,\, p_2|n}} 1 \\ &= \sum_{\substack{p_1, p_2 \leq x \\ p_1 \neq p_2}} \left[\frac{x}{p_1 p_2}\right] + \sum_{p \leq x} \left[\frac{x}{p}\right] \\ &= \sum_{\substack{p_1, p_2 \\ p_1 p_2 \leq x}} \left[\frac{x}{p_1 p_2}\right] + O(x \log\log x) \\ &= x \sum_{\substack{p_1, p_2 \\ p_1 p_2 \leq x}} \frac{1}{p_1 p_2} + O(x \log\log x).\end{aligned}$$

Now observe that

$$\left(\sum_{p \leq \sqrt{x}} \frac{1}{p}\right)^2 \leq \sum_{\substack{p_1, p_2 \\ p_1 p_2 \leq x}} \frac{1}{p_1 p_2} \leq \left(\sum_{p \leq x} \frac{1}{p}\right)^2$$

and, again by Mertens' theorem,

$$\sum_{p \leq \sqrt{x}} \frac{1}{p} = \log\log \sqrt{x} + O(1) = \log\log x + O(1).$$

Thus

$$\sum_{n \leq x} \omega(n)^2 = x(\log\log x)^2 + O(x \log\log x).$$

(iii) Expanding the square and using parts (i), (ii), we deduce that

$$\begin{aligned}\sum_{n \leq x} (\omega(n) - \log\log x)^2 &= \sum_{n \leq x} \omega(n)^2 - 2(\log\log x) \sum_{n \leq x} \omega(n) + (\log\log x)^2 \sum_{n \leq x} 1 \\ &= O(x \log\log x).\end{aligned}$$

(iv) First we relate the summand $(\omega(n) - \log\log n)^2$ to the summand $(\omega(n) - \log\log x)^2$ and then we use (iii):

$$\begin{aligned}\sum_{n \leq x} (\omega(n) - \log\log n)^2 &= \sum_{n \leq x} (\omega(n) - \log\log x + \log\log x - \log\log n)^2 \\ &\ll \sum_{n \leq x} (\omega(n) - \log\log x)^2 + \sum_{n \leq x} \left(\log \frac{\log x}{\log n}\right)^2 \\ &\ll x \log\log x + \sum_{n \leq x} \left(\log \frac{\log x}{\log n}\right)^2.\end{aligned}$$

We split the remaining sum over $n \leq x$ into sums over $n \leq \sqrt{x}$ and $\sqrt{x} < n \leq x$, which we bound trivially.

(v) Let $\varepsilon > 0$ and, for any $y < x$, denote by $N(y, x)$ the number of natural numbers $y < n \leq x$ for which $|\omega(n) - \log\log n| \geq \varepsilon \log\log n$. Using elementary observations and part (iv), we obtain

$$\begin{aligned}N(1, x) &\leq \sqrt{x} + N(\sqrt{x}, x) \leq \sqrt{x} + \sum_{n \leq x} \left(\frac{\omega(n) - \log\log n}{\varepsilon \log\log n}\right)^2 \\ &\ll \sqrt{x} + \frac{x \log\log x}{\varepsilon^2 (\log\log x)^2} = o(x).\end{aligned}$$

We leave the proof of (vi) as a challenge to the reader.

Also solved by Mihaly Bencze (Romania) and Socratis Varelogiannis (France).

220. Using Chebyshev's Theorem, prove that for any integer M there exists an even integer $2k$ such that there are at least M primes p with $p+2k$ also prime. *Unfortunately $2k$ will depend on M. If it did not, we would have solved the Twin Prime Conjecture, namely, there are infinitely many primes p such that $p+2$ is also prime.*

(Steven J. Miller, Department of Mathematics & Statistics, Williams College, Massachusetts, USA)

Solution by the proposer. By Chebyshev's theorem, there exist explicit positive constants A and B such that, for $x > 30$:

$$\frac{Ax}{\log x} \le \pi(x) \le \frac{Bx}{\log x}.$$

Ignoring the lone even prime 2, the number of positive differences between the odd primes at most x is $\binom{\pi(x)-1}{2}$, or

$$(\pi(x)-1)(\pi(x)-2)/2\,.$$

Looking at the lower and upper bounds for $\pi(x)$, we get that the number of these differences is essentially at least $A^2x^2/\log^2 x$ and basically at most $B^2x^2/\log^2 x$; however, there are only about $x/2$ odd numbers which can be these differences.

Thus by the pigeonhole principle, at least one of these positive odd differences must occur at least the average number of times, and thus there is a difference that occurs essentially at least

$$(A^2x^2/\log^2 x)/(x/2) = 2A^2x/\log x\,.$$

The proof is completed by choosing x sufficiently large so that this exceeds M.

Also solved by Mihaly Bencze (Romania), Efstathios S. Louridas (Athens, Greece), George Miliakos (Sparta, Greece).

221. For any three integers a, b, c, with $\gcd(a,b,c) = 1$, prove that there exists an integer m such that

$$0 \le m \le 2^{2^{2002}} c^{\frac{1}{1000}} \text{ and } \gcd(a+mb, c) = 1.$$

(Abhishek Saha, School of Mathematical Sciences, Queen Mary University of London, UK)

Solution by the proposer. We begin with an elementary lemma.

Lemma 1 For positive integers N, k, we have

$$\prod_{p|N} 2 \le 2^{2^k} N^{\frac{1}{k}}.$$

Proof. We have

$$\frac{\prod_{p|N} 2}{N^{\frac{1}{k}}} \le \prod_{p|N} \frac{2}{p^{\frac{1}{k}}} \le \prod_{p|N, p\le 2^k} \frac{2}{p^{\frac{1}{k}}} \le \prod_{p|N, p\le 2^k} 2 \le 2^{2^k} \qquad \square$$

We now begin the solution proper. Let a, b, c as in the problem. *We may assume without loss of generality that* $\gcd(a,b) = 1$ *and that c is squarefree.* Indeed, if these conditions are not met, we can replace a by $\frac{a}{\gcd(a,b)}$, replace b by $\frac{b}{\gcd(a,b)}$ and replace c by its largest squarefree divisor, so that this modified setup does satisfy the conditions. Any integer m that satisfies the required conditions in this modified setup will automatically satisfy it in the original setup.

Next, define $Q = c/\gcd(b,c)$; note that Q is squarefree and hence $\prod_{p|Q} 2 = \sum_{d|Q} 1$. Let $X = 2^{2^{2002}} Q^{\frac{1}{1000}}$. Using well-known properties of the Mobius function μ and the Euler totient function ϕ, we have

$$\sum_{\substack{1\le m\le X\\ \gcd(a+mb,Q)=1}} 1 = \sum_{1\le m\le X} \sum_{\substack{d|Q\\ d|a+mb}} \mu(d) = \sum_{d|Q} \mu(d) \sum_{\substack{1\le m\le X\\ m\equiv -\bar{b}a \mod (d)}} 1$$
$$= \sum_{d|Q} \mu(d)\left(\frac{X}{d} + r_d\right),$$

where $|r_d| \le 1$ for all d, leading to

$$\sum_{\substack{1\le m\le X\\ \gcd(a+mb,Q)=1}} 1 \ge X\sum_{d|Q}\frac{\mu(d)}{d} - \sum_{d|Q} 1 = X\frac{\phi(Q)}{Q} - \prod_{p|Q} 2$$
$$\ge X\prod_{p|Q}\left(1-\frac{1}{p}\right) - \prod_{p|Q} 2.$$

Now, using $1-\frac{1}{p} \ge \frac{1}{2}$, putting in the bounds from the lemma (with $k = 2000$), and substituting the value of X, we obtain that

$$\sum_{\substack{1\le m\le X\\ \gcd(a+mb,Q)=1}} 1 \ge \left(2^{3\cdot 2^{2000}} - 2^{2^{2000}}\right) Q^{\frac{1}{2000}} > 1.$$

It follows that there exists some m (between 1 and X), such that

$$\gcd(a+mb, c/\gcd(b,c)) = 1.$$

However, since $\gcd(a,b) = 1$, this implies that $\gcd(a+mb, c) = 1$. This completes the solution of the problem.

Also solved by Socratis Varelogiannis (France).

222. Show that

$$\sum_{n=1}^{\infty} \frac{\sin^2(\pi\delta n)}{n^2} = \tfrac{1}{2}\pi^2\delta(1-\delta) \quad \text{for } 0 \le \delta \le 1,$$
$$\sum_{n=1}^{\infty} \frac{\sin^3(\pi\delta n)}{n^3} = \tfrac{1}{2}\pi^3\delta^2(\tfrac{3}{4}-\delta) \quad \text{for } 0 \le \delta \le 1/2,$$
$$\sum_{n=1}^{\infty} \frac{\sin^4(\pi\delta n)}{n^4} = \tfrac{1}{2}\pi^4\delta^3(\tfrac{2}{3}-\delta) \quad \text{for } 0 \le \delta \le 1/2.$$

Setting $\delta = 1/2$, deduce the values of $\zeta(2)$ and $\zeta(4)$.

(Olof Sisask, Department of Mathematics, Stockholm University, Sweden)

Solution by the proposer. Let $f = 1_{[-\delta/2,\delta/2]}$ be the indicator function of the interval $[-\frac{\delta}{2}, \frac{\delta}{2}]$ in

$$\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong [-\tfrac{1}{2}, \tfrac{1}{2}].$$

We use Fourier analysis on $\mathbb{T}$; in particular, for $n \in \mathbb{Z}\setminus\{0\}$, by definition and computation we have

$$\widehat{f}(n) = \int_{\mathbb{T}} f(x)e^{2\pi inx}\,dx = \int_{-\delta/2}^{\delta/2} e^{2\pi inx}\,dx = \frac{1}{2\pi in}\left(e^{\pi in\delta} - e^{-\pi in\delta}\right)$$
$$= \frac{\sin(\pi\delta n)}{\pi n}.$$

Parseval's identity $\sum_{n\in\mathbb{Z}}|\widehat{f}(n)|^2 = \int_{\mathbb{T}}|f(x)|^2\,dx$, which is easily verified, then yields

$$\delta^2 + 2\sum_{n=1}^{\infty}\frac{\sin^2(\pi\delta n)}{\pi^2 n^2} = \delta,$$

which is a rearrangement of our first identity.

For the third identity, we use the convolution identity $\widehat{f*f}(n) = \widehat{f}(n)^2$ and Parseval to write

$$\sum_{n\in\mathbb{Z}}|\widehat{f}(n)|^4 = \sum_{n\in\mathbb{Z}}|\widehat{f*f}(n)|^2 = \int_{\mathbb{T}}|(f*f)(x)|^2\,dx, \qquad (\star)$$

where, by definition of convolution followed by a simple computation (where we use that $\delta \le 1/2$),

$$f*f(x) = \int_{\mathbb{T}} f(y)f(x-y)\,dx = \begin{cases}\delta - |x| & \text{for } |x| \le \delta\\ 0 & \text{elsewhere}\end{cases}.$$

Thus
$$\int_{\mathbb{T}} |(f * f)(x)|^2\,dx = 2\int_0^{\delta} (\delta - x)^2\,dx = \tfrac{2}{3}\delta^3,$$
and so (★) gives
$$\delta^4 + 2\sum_{n=1}^{\infty} \frac{\sin^4(\pi\delta n)}{\pi^4 n^4} = \tfrac{2}{3}\delta^3.$$
Rearranging, we have the third identity.

For the second identity, note similarly that, by the convolution identity and Fourier inversion formula $g(x) = \sum_n \widehat{g}(n)e^{-2\pi i n x}$,
$$\sum_{n\in\mathbb{Z}} \widehat{f}(n)^3 = f * f * f(0) = \int_{\mathbb{T}} (f * f)(x) f(x)\,dx = \int_{-\delta/2}^{\delta/2} (\delta - x)\,dx$$
$$= \tfrac{3}{4}\delta^2$$
for $0 \le \delta \le 1/2$, whence
$$\sum_{n=1}^{\infty} \frac{\sin^3(\pi\delta n)}{n^3} = \tfrac{1}{2}\pi^3\delta^2(\tfrac{3}{4} - \delta).$$
To deduce the ζ-values, we take $\delta = 1/2$ and note that
$$\sin\left(\tfrac{1}{2}\pi n\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$
In particular, the first of our identities yields
$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \tfrac{1}{8}\pi^2.$$
Since
$$\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \tfrac{1}{4}\zeta(2),$$
we see that
$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \tfrac{1}{8}\pi^2 + \tfrac{1}{4}\zeta(2)$$
and, rearranging,
$$\zeta(2) = \frac{\pi^2}{6}.$$
In exactly the same manner we obtain
$$\zeta(4) = \frac{\pi^4}{90}.$$

Comment. The integrals computed above all have an additive combinatorial interpretation: in the case of the fourth-power denominators, the integral measures how many solutions there are to
$$a_1 + a_2 = a_3 + a_4$$
in the interval $[-\frac{\delta}{2}, \frac{\delta}{2}]$. In the case of third-powers, the equation is
$$a_1 + a_2 + a_3 = 0,$$
and in the case of squares it is simply
$$a_1 = a_2.$$
The reason for restricting to $\delta \le 1/2$ in two of the cases is that these solutions are easy to measure provided there is no 'wrap-around' when adding two variables.

Also solved by Mihaly Bencze (Romania) and Rudolf Rupp (Nuremberg, Bavaria, Germany).

223. Fix a prime number p, and an integer $\beta \ge 2$. Consider the function defined on $x \in \mathbf{R}$ by $e(x) = \exp(2\pi i x)$. Given a coprime residue class $r \bmod p^\beta$, consider the additive character defined on integers $m \in \mathbf{Z}$ by $m \mapsto e\left(\frac{mr}{p^\beta}\right)$. Given a complex parameter $s \in \mathbf{C}$ with $\Re(s) > 1$, consider the Dirichlet series defined by
$$D(s, r, p^\beta) = \sum_{m\ge1} e\left(\frac{rm}{p^\beta}\right) m^{-s}.$$
Show that this series has an analytic continuation to all $s \in \mathbf{C}$, and moreover that it satisfies a functional equation relating values at s to $1 - s$.

(Jeanine Van Order, Fakultät für Mathematik, Universität Bielefeld, Germany)

Solution by the proposer. The key idea is to express the additive character as a certain linear combination of multiplicative (Dirichlet) characters, to reduce to the well-known classical setting of Dirichlet L-series and their functional equations. To be more precise, let χ denote a primitive Dirichlet character modulo p^β, and let
$$\tau(\chi) = \sum_{\substack{x \bmod p^\beta \\ (x,p^\beta)=1}} \chi(x) e\left(\frac{x}{p^\beta}\right)$$
denote the corresponding Gauss sum. We claim that for any integer $n \ge 1$, we have
$$\sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \overline{\chi}(r)\tau(\chi)^n = \varphi(p^\beta)\,\mathrm{Kl}_n(r, p^\beta), \tag{1}$$
where the sum runs over primitive Dirichlet characters $\chi \bmod p^\beta$, $\varphi(p^\beta)$ denotes the Euler-phi function evaluated at p^β, and
$$\mathrm{Kl}_n(r, p^\beta) = \sum_{\substack{x_1,\dots,x_n \bmod p^\beta \\ x_1\cdots x_n \equiv r \bmod p^\beta}} e\left(\frac{x_1 + \cdots + x_n}{p^\beta}\right)$$
denotes the hyper-Kloosterman sum of dimension n and modulus p^β evaluated at a coprime residue class $r \bmod p^\beta$. Indeed, let us write
$$\varphi^\star(p^\beta) = \varphi(p^\beta) - \varphi(p^{\beta-1})$$
to denote the number of primitive Dirichlet characters $\chi \bmod p^\beta$. Note that by the Möbius inversion formula, we have for any integer $m \ge 1$ prime to p the relation
$$\sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \chi(m) = \sum_{\substack{0\le x\le\beta \\ p^x \mid (m-1,p^\beta)}} \varphi(p^x)\mu\left(\frac{p^\beta}{p^x}\right),$$
from which it is easy to derive the corresponding orthogonality relation
$$\sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \chi(m) = \begin{cases} \varphi^\star(p^\beta) & \text{if } m \equiv 1 \bmod p^\beta, \\ -\varphi(p^{\beta-1}) & \text{if } m \equiv 1 \bmod p^{\beta-1} \text{ but } m \not\equiv 1 \bmod p^\beta, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$
Now to show (1), we open up sums and switch the order of summation to find
$$\sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \overline{\chi}(r)\tau(\chi)^n = \sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \sum_{x_1,\dots,x_n \bmod p^\beta} \chi(\overline{r}x_1\cdots x_n) e\left(\frac{x_1 + \cdots + x_n}{p^\beta}\right)$$
$$= \varphi^\star(p^\beta) \sum_{\substack{x_1,\dots,x_n \bmod p^\beta \\ x_1\cdots x_n \equiv r \bmod p^\beta}} e\left(\frac{x_1 + \cdots + x_n}{p^\beta}\right) - \varphi(p^{\beta-1})$$
$$\sum_{\substack{x_1,\dots,x_n \bmod p^\beta \\ x_1\cdots x_n \equiv r \bmod p^{\beta-1} \\ x_1\cdots x_n \not\equiv r \bmod p^\beta}} e\left(\frac{x_1 + \cdots + x_n}{p^\beta}\right).$$

Here, we write $\bar{r}$ to denote the multiplicative inverse of the class $r \bmod p^\beta$, so that $r\bar{r} \equiv 1 \bmod p^\beta$. Let us now consider the second sum in this latter expression, which after putting $y := x_1 \cdots x_{n-1}\bar{r}$ is the same as

$$\sum_{\substack{x_1,\dots,x_n \bmod p^\beta \\ x_1\cdots x_n \equiv r \bmod p^{\beta-1} \\ x_1\cdots x_n \not\equiv r \bmod p^\beta}} e\left(\frac{x_1+\cdots+x_n}{p^\beta}\right)$$

$$= \sum_{x_1,\cdots,x_{n-1} \bmod p^\beta} e\left(\frac{x_1+\cdots+x_{n-1}}{p^\beta}\right) \sum_{\substack{x_n \bmod p^\beta \\ x_n \equiv y \bmod p^{\beta-1} \\ x_n \not\equiv y \bmod p^\beta}} e\left(\frac{x_n}{p^\beta}\right)$$

Notice that the inner sum in this latter expression can be parametrised equivalently by $x_n = y + lp^{\beta-1}$ for l ranging over integers $1 \leq l \leq p-1$, so that we have

$$\sum_{x_1,\cdots,x_{n-1} \bmod p^\beta} e\left(\frac{x_1+\cdots+x_{n-1}}{p^\beta}\right) \sum_{\substack{x_n \bmod p^\beta \\ x_n \equiv y \bmod p^{\beta-1} \\ x_n \not\equiv y \bmod p^\beta}} e\left(\frac{x_n}{p^\beta}\right)$$

$$= \sum_{x_1,\cdots x_{n-1} \bmod p^\beta} e\left(\frac{x_1+\cdots+x_{n-1}+y}{p^\beta}\right) \sum_{l=1}^{p-1} e\left(\frac{lp^{\beta-1}}{p^\beta}\right)$$

$$= \sum_{\substack{x_1\cdots x_n \bmod p^\beta \\ x_1\cdots x_n \equiv r \bmod p^\beta}} e\left(\frac{x_1+\cdots+x_n}{p^\beta}\right) \sum_{l=1}^{p-1} e\left(\frac{l}{p}\right) = -\operatorname{Kl}_n(r, p^\beta)$$

via the well-known elementary identity $\sum_{l=1}^{p-1} e\left(\frac{l}{p}\right) = -1$. Hence, see that

$$\sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \bar{\chi}(r)\tau(\chi)^n = \left(\varphi^\star(p^\beta) + \varphi(p^{\beta-1})\right) \operatorname{Kl}_n(r, p^\beta)$$

$$= \varphi(p^\beta) \operatorname{Kl}_n(r, p^\beta).$$

Let us now consider the series in question, which after using (1) is equivalent to a sum over Dirichlet series $L(s,\chi)$ attached to each Dirichlet character $\chi \bmod p^\beta$,

$$\sum_{m\geq 1} e\left(\frac{rm}{p^\beta}\right) m^{-s} = \sum_{m\geq 1} \operatorname{Kl}_1(rm, p^\beta) m^{-s}$$

$$= \varphi(p^\beta) \sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \bar{\chi}(r)\tau(\chi) \sum_{m\geq 1} \bar{\chi}(m) m^{-s}$$

$$= \varphi(p^\beta) \sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \bar{\chi}(r)\tau(\chi) L(s,\bar{\chi}).$$

Observe that we have for each $L(s,\chi)$ the asymmetric functional equation

$$L(s,\chi) = p^{-\beta s}\tau(\chi) \left(\frac{\Gamma_{\mathbf{R}}(\frac{1-s}{2})}{\Gamma_{\mathbf{R}}\left(\frac{s}{2}\right)}\right) L(1-s,\bar{\chi}), \tag{3}$$

i.e., after bringing gamma factors $\Gamma_{\mathbf{R}}(s) = \pi^{-s}\Gamma(s)$ over to the right-hand side. Substituting this functional equation (3) for each $L(s,\bar{\chi})$ then gives us

$$\sum_{m\geq 1} e\left(\frac{rm}{p^\beta}\right) m^{-s}$$

$$= \varphi(p^\beta) p^{-\beta s} \left(\frac{\Gamma_{\mathbf{R}}\left(\frac{1-s}{2}\right)}{\Gamma_{\mathbf{R}}\left(\frac{s}{2}\right)}\right) \sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \bar{\chi}(r)\tau(\chi)\tau(\bar{\chi}) L(1-s,\chi)$$

$$= \varphi(p^\beta) p^{\beta(1-s)} \left(\frac{\Gamma_{\mathbf{R}}\left(\frac{1-s}{2}\right)}{\Gamma_{\mathbf{R}}\left(\frac{s}{2}\right)}\right) \sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \bar{\chi}(r) \sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} L(1-s,\chi).$$

Here, we use that

$$\tau(\chi)\tau(\bar{\chi}) = \tau(\chi)\overline{\tau(\chi)} = |\tau(\chi)|^2 = p^\beta$$

for each primitive Dirichlet character $\chi \bmod p^\beta$. Hence, we derive the relation

$$\Gamma_{\mathbf{R}}\left(\frac{s}{2}\right) D(s,r,p^\beta) = \varphi(p^\beta) p^{\beta(1-s)} \Gamma_{\mathbf{R}}\left(\frac{1-s}{2}\right) \sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \bar{\chi}(r) L(1-s,\chi).$$

Observe that when $\Re(s) < 0$, we can open up the absolutely convergent Dirichlet series on the right-hand side and use the orthogonality relation (2) to see that

$$\Gamma_{\mathbf{R}}\left(\frac{s}{2}\right) D(s,r,p^\beta)$$

$$= \varphi(p^\beta) p^{\beta(1-s)} \Gamma_{\mathbf{R}}\left(\frac{1-s}{2}\right)$$

$$\left(\varphi^\star(p^\beta) \sum_{\substack{m\geq 1 \\ m\equiv r \bmod p^\beta}} \frac{1}{m^{1-s}} - \varphi(p^{\beta-1}) \sum_{\substack{m\geq 1 \\ m\equiv r \bmod p^{\beta-1} \\ m\not\equiv r \bmod p^\beta}} \frac{1}{m^{1-s}}\right).$$

Note that a similar discussion carries over to the more general setting of Dirichlet series defined with respect to $\operatorname{Kl}_n(mr, p^\beta)$ instead of $\operatorname{Kl}_1(mr, p^\beta)$, as well as to additive twists of $\mathrm{GL}_n(\mathbf{A}_\mathbf{Q})$-automorphic L-functions (see [1] for more details).

Reference

[1] J. Van Order, Dirichlet twists of GL_n-automorphic L-functions and hyper-Kloosterman Dirichlet series preprint (2017). To appear in *Ann. Fac. Sci. Toulouse Math. (6).*

We would like for you to submit solutions to the proposed problems and ideas on the open problems. Send your solutions by email to Michael Th. Rassias, Institute of Mathematics, University of Zürich, Switzerland, michail.rassias@math.uzh.ch.

We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *Probability Theory*.