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Bost–Connes type systems for function fields

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Abstract. We describe a construction which associates to any function field k and any place ∞ of k a C*-dynamical system $(C_{k,\infty}, \sigma_t)$ that is analogous to the Bost–Connes system associated to $\mathbb{Q}$ and its archimedean place. Our construction relies on Hayes' explicit class field theory in terms of sign-normalized rank one Drinfel'd modules. We show that $C_{k,\infty}$ has a faithful continuous action of $\mathrm{Gal}(K/k)$, where K is a certain field constructed by Hayes such that $k^{\mathrm{ab},\infty} \subset K \subset k^{\mathrm{ab}}$. Here $k^{\mathrm{ab},\infty}$ is the maximal abelian extension of k that is totally split at ∞. We classify the extremal KMS_β states of $(C_{k,\infty}, \sigma_t)$ at any temperature $0 < 1/\beta < \infty$ and show that a phase transition with spontaneous symmetry breaking occurs at temperature $1/\beta = 1$. At high temperature $1/\beta \geqslant 1$, there is a unique KMS_β state, of type $\mathrm{III}_{q^{-\beta}}$, where q is the cardinal of the constant subfield of k. At low temperature $1/\beta < 1$, the space of extremal KMS_β states is principal homogeneous under $\mathrm{Gal}(K/k)$. Each such state is of type I_∞ and the partition function is the Dedekind zeta function $\zeta_{k,\infty}$. Moreover, we construct a $*$-subalgebra $\mathcal{H}$, we give a presentation of $\mathcal{H}$ and of $C_{k,\infty}$, and we show that the values of the low-temperature extremal KMS_β states at certain elements of $\mathcal{H}$ are related to special values of partial zeta functions.

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Introduction

Statement of the main results. Let k be any global function field. Let ∞ be any place of k. In this paper, we shall associate to the pair (k,∞) a C*-dynamical system $(C_{k,\infty},(\sigma_t))$.

Our system aims to be an analog of the Bost–Connes (BC for short) system associated to $\mathbb{Q}$, cf. Bost and Connes [3]. The partition function of the BC system is the Riemann zeta function without the Γ-factor at infinity. Similarly, we shall check (Lemma 4.3.3) that the partition function of our system is the zeta function of the field k without the factor corresponding to the place ∞ of k.

The BC system admits $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ as symmetry group. Similarly, we shall check (Proposition 3.4.2) that our system has $\mathrm{Gal}(K/k)$ as symmetry group (meaning that $\mathrm{Gal}(K/k)$ acts continuously and faithfully on $C_{k,\infty}$, commuting with the flow σ_t), where K is a field having the following property:

$$k^{\mathrm{ab},\infty}\subset K\subset k^{\mathrm{ab}},$$

where $k^{\mathrm{ab},\infty}$ is the maximal abelian extension of k that is totally split at ∞. The field K is generated over k by coefficients and torsion points of certain rank one Drinfel'd modules; this is part of David R. Hayes' explicit class field theory for function fields, cf. Hayes [18], [19] and [20], which we shall quickly review. If ∞' is any place of k other than ∞, we have (cf. [18], Theorem 7.2)

$$k^{\mathrm{ab},\infty'}\cdot k^{\mathrm{ab},\infty}=k^{\mathrm{ab}}.$$

We shall construct our C*-algebra $C_{k,\infty}$ as the maximal C*-algebra of a certain groupoid $\mathcal{G}$. We shall also give (Proposition 3.3.6) a presentation of $C_{k,\infty}$ as a C*-algebra.

For any temperature $1/\beta\in\mathbb{R}_+^*$, let K_β be space of KMS_β states of $\big(C_{k,\infty},(\sigma_t)\big)$, endowed with the weak* topology. By Bratteli and Robinson [4], II, Theorem 5.3.30, the space K_β is a compact simplex (in particular, it is convex). Let $\mathcal{E}(K_\beta)$ denote the subspace of extreme points of K_β. The elements of $\mathcal{E}(K_\beta)$ are called the *extremal* KMS_β states. By loc. cit., a KMS_β state is extremal if, and only if it is a factor state. Thus, $\mathcal{E}(K_\beta)$ is equal to the space of KMS_β factor states.

We shall classify the KMS_β states of our system for any temperature $1/\beta\in\mathbb{R}_+^*$: At low temperature $1/\beta<1$, we shall prove (Theorem 4.3.10) that $\mathcal{E}(K_\beta)$ is principal homogeneous[1] under $\mathrm{Gal}(K/k)$. The states in $\mathcal{E}(K_\beta)$ are of type I_∞ (Proposition 4.3.8). At high temperature $1/\beta\geqslant 1$, we shall prove (Theorem 4.4.15) that there exists a unique KMS_β state. It is of type $\mathrm{III}_{q^{-\beta}}$ (Theorem 4.5.8), where q is the cardinal of the constant subfield of k.

[1] Let G be topological group acting on a topological space X. One says that X is *principal homogeneous* under G if, for any $x\in X$, the map $g\mapsto gx$ is a homeomorphism $G\to X$.

We shall construct a dense $*$-subalgebra $\mathcal{H}$ which gives an arithmetic structure to our dynamical system, as in [3]. For example, we shall show (Theorem 4.3.12) that evaluating low-temperature extremal KMS_β states on certain elements of the subalgebra $\mathcal{H}$ gives rise to formulas involving special values of partial zeta functions.

Many of our proofs are adapted from [3], and we have also borrowed several ideas from Harari and Leichtnam [17].

Outline. This paper is divided into four sections. In Section 1 we first review definitions and results in the arithmetic of function fields and in the analytic theory of Drinfel'd modules. We review Hayes' explicit class field theory for function fields, in terms of sign-normalized rank one Drinfel'd modules. We choose once and for all a sign-function sgn, and Hayes' theory provides us with a finite set $H(\mathrm{sgn})$ of Drinfel'd modules with special arithmetic properties. In particular, their coefficients and torsion points generate the extension K/k which we mentioned above. In the rest of this paper, the only Drinfel'd modules which we consider are the elements of $H(\mathrm{sgn})$.

In Section 2 we do the actual construction of the C*-dynamical system $(C_{k,\infty}, (\sigma_t))$. From the finite set $H(\mathrm{sgn})$ provided by Hayes' theory, we construct a compact topological space X in the following way: for any $\phi \in H(\mathrm{sgn})$, let X_ϕ denote the dual group of the discrete group of torsion points of the Drinfel'd module ϕ. Let X be the disjoint union of the X_ϕ, where ϕ runs over $H(\mathrm{sgn})$. The compact space X is endowed with a natural action of the semigroup $\mathfrak{I}_\mathcal{O}$ of ideals. This gives rise to a groupoid $\mathcal{G}$, and the C*-algebra $C_{k,\infty}$ is obtained as the maximal groupoid C*-algebra of $\mathcal{G}$. The flow (σ_t) is then easy to define.

In Section 3 we prove a number of results about the algebraic structure of $(C_{k,\infty}, (\sigma_t))$. We introduce a $*$-subalgebra $\mathcal{H}$ which plays the rôle of the algebra $\mathcal{H}$ in the paper [3]. We prove that $\mathcal{H}$ is dense in $C_{k,\infty}$, and we give a presentation of $\mathcal{H}$ as a $*$-algebra and of $C_{k,\infty}$ as a C*-algebra. We then study an action of $\mathrm{Gal}(K/k)$ on $C_{k,\infty}$ and compute the fixed-point subalgebra C_1. The rest of this section is devoted to miscellaneous arithmetical results which we use in the last section.

In Section 4 for any temperature $1/\beta \in \mathbb{R}_+^*$, we describe the space $\mathcal{E}(K_\beta)$ of extremal KMS_β states (endowed with the weak* topology), and we compute the type of all such states. We first construct a KMS_β state φ_β and show that it is the unique $\mathrm{Gal}(K/k)$-invariant KMS_β state. We then show that the action of $\mathrm{Gal}(K/k)$ on $\mathcal{E}(K_\beta)$ is transitive and continuous. Thus, in order to describe $\mathcal{E}(K_\beta)$, it is enough to find an element of $\mathcal{E}(K_\beta)$ and to describe its orbit under $\mathrm{Gal}(K/k)$. At low temperature $1/\beta < 1$, we associate to any admissible character χ a Gibbs state $\varphi_{\beta,\chi}$ in the regular representation at χ. We prove that the map $\chi \mapsto \varphi_{\beta,\chi}$ is a homeomorphism from the space X^{adm} of admissible characters to $\mathcal{E}(K_\beta)$. We also prove that both spaces are principal homogeneous under $\mathrm{Gal}(K/k)$. We check that the states in $\mathcal{E}(K_\beta)$ are of type I_∞, that the partition function is the Dedekind zeta function $\zeta_{k,\infty}$, and we compute the values of the $\varphi_{\beta,\chi}$ at some points of $\mathcal{H}$ in terms of special values at β of partial zeta functions of k. At high temperature $1/\beta \geqslant 1$, we prove that $\mathcal{E}(K_\beta) = \{\varphi_\beta\}$ and that the type of φ_β is $\mathrm{III}_{q^{-\beta}}$, where q is the cardinal of the constant subfield of k.

Literature on Bost–Connes type constructions. The 1995 paper [3] has inspired many mathematicians. Unfortunately, it would be impossible to mention all of them here; we refer to Section 1.4 of Connes and Marcolli [11] for a more complete summary. M. Laca, N. Larsen, I. Raeburn and others have investigated in a number of papers (see for instance [2], [25], [26], [27], [28]) the semigroup crossed product and Hecke algebra aspects of the BC construction and generalizations of it. In 1997, D. Harari and E. Leichtnam have obtained in [17] a system with spontaneous symmetry breaking for any global field. In 1999, P. Cohen has obtained in [6] a system for number fields whose partition function is the Dedekind zeta function. In 2002, S. Neshveyev has given in [29] a new proof of the uniqueness of the KMS_β state at high temperature. In 2004, A. Connes and M. Marcolli have introduced in [11] the noncommutative space of $\mathbb{Q}$-lattices up to scaling and commensurability, allowing for a comprehensive reformulation of the BC construction, and have studied the case of rank 2. In 2005, A. Connes, M. Marcolli and N. Ramachandran have obtained in [12], [13] the "good" system for quadratic imaginary number fields and have studied its relation to complex multiplication of elliptic curves. The same year, E. Ha and F. Paugam have extended in [16] the Connes–Marcolli setting to arbitrary Shimura varieties. Finally, in the paper [10], A. Connes, C. Consani and M. Marcolli have introduced the notion of an *endomotive*, putting the BC construction into a much wider perspective which also includes A. Connes' spectral realization [9] of the zeroes of the Riemann zeta function.

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Here are two interesting remarks that people made at the end of my MPI talk.

1. As Alain Connes pointed out, our system lacks one feature of the BC system: fabulous states. The reason for that is obvious: values of states are elements of $\mathbb{C}$, so the symmetry group $\mathrm{Gal}(K/k)$ does not act naturally on them. Obtaining fabulous states would require to have a theory of dynamical systems of positive characteristic, where states would take values in some field of positive characteristic. Note that even

though the low temperature extremal KMS_β states of our system do not have the fabulous property, they have interesting special values (Theorem 4.3.12).

2. Arkady Kholodenko mentioned that it might be possible to adapt his work on $2+1$ gravity [23] in order to obtain zeta functions of function fields as partition functions, and that Drinfel'd modules should play a rôle.

Notations. In this paper, $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{N}^*$ denotes the set of positive integers, and $\mathbb{R}_+^*$ denotes the set of positive real numbers. Thus $0 \in \mathbb{N}$, $0 \notin \mathbb{N}^*$, and $0 \notin \mathbb{R}_+^*$. For any Hilbert space H, we let $B(H)$ denote the algebra of all bounded linear operators on H. For any set X, we write $B\ell^2(X)$ for $B(\ell^2(X))$. For any $x \in \mathbb{R}$, we set

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leqslant x\}.$$

For any predicate P, we define 1_P to be equal to 1 if P is true, and 0 if P is false. Thus, we have for any two predicates P and Q:

$$1_{P \text{ and } Q} = 1_P 1_Q.$$

1. Function fields, Drinfel'd modules, and Hayes' explicit class field theory

1.1. Function fields. Here are three equivalent definitions of a *function field*:

- A field which is a finite extension of $\mathbb{F}_p(T)$, for some prime number p.
- A global field of positive characteristic.
- The field $K(C)$ of rational functions on a projective curve C over a finite field. The curve C can always be chosen to be smooth.

Thus, global fields fall into two categories: those of characteristic 0 are the number fields, and those of positive characteristic are the function fields.

Recall that at the beginning of this paper, we chose a function field k and a place ∞ of k.

Function fields have many similarities with number fields. An important part of algebraic number theory works in the same way for all global fields.

The analog of the Dedekind ring of integers is defined as follows. According to the third definition of a function field, view k as the field $K(C)$ of rational functions on a smooth projective curve C over a finite field. View ∞ as a closed point of C. Let $\mathcal{O}$ be the subring of k of all functions having no pole away from ∞. In other words, $\mathcal{O}$ is the ring of regular functions on the affine curve $C - \{\infty\}$. Note that $k = K(C)$ is the field of fractions of $\mathcal{O}$.

Example. $k = \mathbb{F}_p(T)$ and ∞ is the place corresponding to an absolute value $|\cdot|$ such that $|T| > 1$. The subring $\mathcal{O}$ is then the polynomial ring $\mathbb{F}_p[T]$.

Call *finite* the places of k other than ∞. We have a natural bijection

$$\text{finite places of } k \longleftrightarrow \text{ maximal ideals of } \mathcal{O}.$$

Let p denote the characteristic of k. The range of the unique unital ring morphism $\mathbb{Z} \to k$ is a finite field with p elements; we denote it by $\mathbb{F}_p$. The algebraic closure of $\mathbb{F}_p$ in k is called the *constant subfield* of k. Let q denote its cardinal. Of course, q is a power of p. We let $\mathbb{F}_q$ denote the constant subfield of k. An element of k is said to be *constant* if it belongs to $\mathbb{F}_q$.

For any place $\mathfrak{p}$ of k, we let $\mathbf{N}\mathfrak{p}$ denote the cardinal of the residue field of $\mathfrak{p}$. Thus, $\mathbf{N}\mathfrak{p} = q^{n_\mathfrak{p}}$ for some positive integer $n_\mathfrak{p}$ called the *degree* of $\mathfrak{p}$. Note that if $\mathfrak{p}$ is finite, then the residue field is the quotient $\mathcal{O}/\mathfrak{p}$.

The rest of this subsection is a review of a few well-known theorems about function fields, which will be used in the proofs of our classification of KMS_β states. These theorems are: the strong approximation theorem, Weil's "Riemann Hypothesis for curves", and the abelian case of the Čebotarev density theorem for the natural density. The first one will be used in Subsection 3.6, which in turn will be used in the classification of KMS_β states at low temperature. The two other ones will be used in the classification of KMS_β states at high temperature.

Let A_f denote the ring of finite adèles of k. This is the restricted product of the $k_\mathfrak{p}$ with respect to the $\mathcal{O}_\mathfrak{p}$, where $\mathfrak{p}$ runs over all finite places of k. Let $\iota_f : k \hookrightarrow A_f$ be the diagonal embedding.

Theorem 1.1.1 (Strong approximation theorem). *The field $\iota_f(k)$ is dense in A_f.*

Proof. See Cassels and Fröhlich [5], Chapter II, §15, p. 67. □

This is contrasted with the fact that if $\iota : k \hookrightarrow A$ is the diagonal embedding into the full ring of adèles, then $\iota(k)$ is discrete in A. Note that

$$A = A_f \times k_\infty,$$

where k_∞ is the completion of k at ∞.

Let us now recall Weil's "Riemann hypothesis for curves" theorem. The *genus* of a function field is the genus of any projective smooth curve of which it is the function field. For the statement of the following theorem, we temporarily forget that we already chose a function field k and defined q as the cardinal of its constant subfield.

Theorem 1.1.2 (A. Weil, the Riemann Hypothesis for curves). *Let k be a function field of genus g. Let q be the cardinal of its constant subfield. Let N be the number of places of k with norm q (i.e. with degree* 1*). Then*

$$q - 2g\sqrt{q} + 1 \leqslant N \leqslant q + 2g\sqrt{q} + 1.$$

Proof. Weil's original proof is published in [34]. □

Let us now come back to the function field k that we fixed at the beginning of this paper. Let g denote the genus of k.

Given an integer $n \geqslant 1$, one may ask how to obtain a result similar to Theorem 1.1.2 for places of k with norm q^n (i.e. with degree n). Note that one cannot replace q by q^n in Theorem 1.1.2. Here one has to be wary of the distinction between closed points, which correspond to places of k, and geometric points, which correspond to places of suitable extensions of k. The following corollary will be used in Subsections 4.4 and 4.5.

Corollary 1.1.3. *For any $n \geqslant 1$, let $Q(k,q^n)$ denote the number of places of k with norm q^n, and let $P(k,q^n)$ denote the number of places k with norm $\leqslant q^n$. The following estimates hold when $n \to \infty$:*

$$Q(k,q^n) = \frac{q^n}{n} + O\big(q^{n/2}\big), \tag{1}$$

$$P(k,q^n) \sim \frac{q}{q-1} \cdot \frac{q^n}{n}. \tag{2}$$

Proof. For any $n \geqslant 1$, let $k_n = k \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$. Note that the constant subfield of k_n is $\mathbb{F}_{q^n}$. Let N_n denote the number of places of k_n with norm q^n. By Theorem 1.1.2 applied to the function field k_n, we have

$$q^n - 2gq^{n/2} + 1 \leqslant N_n \leqslant q^n + 2gq^{n/2} + 1. \tag{3}$$

Let $n \geqslant 1$. One easily checks that for any $m \,|\, n$ there is a bijection

places of k with norm $q^m \longleftrightarrow \mathrm{Gal}(k_n/k)$-orbits with cardinal m
of places of k_n with norm q^n.

Thus we have

$$N_n = \sum_{m|n} mQ(k,q^m). \tag{4}$$

This gives

$$nQ(k,q^n) = N_n - \sum_{m|n,\, m\leqslant n/2} mQ(k,q^m).$$

By equation (4), we have $mQ(k,q^m) \leqslant N_m$, so we find

$$\begin{aligned} N_n \geqslant nQ(k,q^n) &\geqslant N_n - \sum_{m|n,\, m\leqslant n/2} N_m. \\ &\geqslant N_n - (n/2)N_{\lfloor n/2\rfloor}. \end{aligned}$$

Applying the inequality (3), we get

$$q^n + 2gq^{n/2} + 1 \geqslant nQ(k,q^n) \geqslant q^n - 2gq^{n/2} + 1 - (n/2)(q^{n/2} + 2gq^{n/4} + 1),$$

and the estimate (1) follows. From the estimate (1), using the equality

$$P(k,q^n) = \sum_{m=1}^{n} Q(k,q^m),$$

one can obtain the estimate (2) by an elementary computation. □

For any $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$, put

$$\zeta_k(s) = \prod_{\mathfrak{p}} \frac{1}{1 - \mathbf{N}\mathfrak{p}^{-s}}$$

where the product is taken over all places of k. One shows that ζ_k can be continued to a meromorphic function on $\mathbb{C}$. Note that ζ_k is periodic, with period $2\pi i/\log q$. The inequality (3) for all $n \geqslant 1$ is then equivalent to the statement that all zeroes of ζ_k have real part $1/2$. One defines the *zeta function without the factor at* ∞, denoted by $\zeta_{k,\infty}$, to be the meromorphic continuation of the function defined when $\operatorname{Re} s > 1$ by

$$\zeta_{k,\infty}(s) = \prod_{\mathfrak{p}\neq\infty} \frac{1}{1 - \mathbf{N}\mathfrak{p}^{-s}} = (1 - \mathbf{N}\infty^{-s})\zeta_k(s).$$

Note that when $\operatorname{Re} s > 1$, we have

$$\zeta_{k,\infty}(s) = \sum_{\mathfrak{a}\in\mathfrak{I}_{\mathcal{O}}} \frac{1}{\mathbf{N}\mathfrak{a}^s}.$$

Let us now recall a version of the Čebotarev density theorem.

Let S denote the set of all places of k. A set P of places of k is said to *have a Dirichlet density* if the following limit exists in $\mathbb{R}$:

$$d(P) = \lim_{s\to 1_+} \frac{\sum_{\mathfrak{p}\in P} \mathbf{N}\mathfrak{p}^{-s}}{\sum_{\mathfrak{p}\in S} \mathbf{N}\mathfrak{p}^{-s}}.$$

Moreover, P is said to *have a natural density* if the following limit exists in $\mathbb{R}$:

$$\delta(P) = \lim_{N\to+\infty} \frac{\operatorname{Card}\{\mathfrak{p} \in P \mid \mathbf{N}\mathfrak{p} \leqslant N\}}{\operatorname{Card}\{\mathfrak{p} \in S \mid \mathbf{N}\mathfrak{p} \leqslant N\}}.$$

If a set P has a natural density, then it also has a Dirichlet density, and $d(P) = \delta(P)$.

Theorem 1.1.4 (Čebotarev density theorem, abelian case, for the natural density). *Let L be a finite abelian extension of k. Let $\sigma \in \mathrm{Gal}(L/k)$. Let P denote the set of all places $\mathfrak{p}$ of k unramified in L and such that $\sigma_{\mathfrak{p}} = \sigma$, where $\sigma_{\mathfrak{p}} = (\mathfrak{p}, L/k) \in \mathrm{Gal}(L/k)$ is the Artin automorphism of L associated to $\mathfrak{p}$. Then P has natural density $\delta(P) = 1/[L:k]$. Therefore, it also has Dirichlet density $d(P) = 1/[L:k]$.*

Proof. Combine [5], Chapter VIII, Theorem 4 with the Artin reciprocity law. □

We shall use the Čebotarev density theorem in Subsection 4.4, and we shall also use the following corollary in Subsection 4.5.

Corollary 1.1.5. *Let L be a finite abelian extension of k. Let $\sigma \in \mathrm{Gal}(L/k)$. For any $n \geqslant 1$, let $P(L/k, q^n, \sigma)$ denote the number of places of $\mathfrak{p}$ of k unramified in L such that $\mathbf{N}\mathfrak{p} \leqslant q^n$ and $\sigma_{\mathfrak{p}} = \sigma$, where $\sigma_{\mathfrak{p}} = (\mathfrak{p}, L/k) \in \mathrm{Gal}(L/k)$ is the Artin automorphism of L associated to $\mathfrak{p}$. Let $Q(L/k, q^n, \sigma)$ denote the number of places of $\mathfrak{p}$ of k unramified in L such that $\mathbf{N}\mathfrak{p} = q^n$ and $\sigma_{\mathfrak{p}} = \sigma$. The following estimates hold when $n \to \infty$:*

$$P(L/k, q^n, \sigma) \sim \frac{q}{(q-1)\,[L:k]} \cdot \frac{q^n}{n}, \tag{5}$$

$$Q(L/k, q^n, \sigma) \sim \frac{1}{[L:k]} \cdot \frac{q^n}{n}. \tag{6}$$

Proof. The estimate (5) follows from Theorem 1.1.4 and the estimate (2). We have

$$Q(L/k, q^n, \sigma) = P(L/k, q^n, \sigma) - P(L/k, q^{n-1}, \sigma),$$

so

$$\frac{Q(L/k, q^n, \sigma)}{P(k, q^n)} = \frac{P(L/k, q^n, \sigma)}{P(k, q^n)} - \frac{P(L/k, q^{n-1}, \sigma)}{P(k, q^{n-1})} \cdot \frac{P(k, q^{n-1})}{P(k, q^n)}.$$

Hence

$$\frac{Q(L/k, q^n, \sigma)}{P(k, q^n)} \xrightarrow{n\to\infty} \frac{1}{[L:k]} - \frac{1}{[L:k]} \cdot \frac{1}{q} = \frac{q-1}{q[L:k]}.$$

Applying the estimate (2) to that, we get the estimate (6). □

1.2. Drinfel'd modules over $\mathbf{C}_\infty$. Our references in this subsection are [20] and Chapter IV of Goss [15].

Recall that the maximal abelian extension of a quadratic imaginary number field is generated by the j-invariant and the torsion points of a suitable elliptic curve over $\mathbb{C}$. One wishes to develop a similar theory for function fields. Thus, one looks for good analogs of $\mathbb{C}$ and of the notion of an elliptic curve over $\mathbb{C}$. The analog of the field $\mathbb{C}$ has been well known for a long time and is what we shall denote $\mathbf{C}_\infty$. The analog of the notion of an elliptic curve over $\mathbb{C}$ is going to be the notion of a Drinfel'd module over $\mathbf{C}_\infty$.

We begin with describing the analog of $\mathbb{C}$. Let k_∞ be the completion of k at ∞. The problem is that k_∞ is not algebraically closed. Take an algebraic closure $k_\infty^{\text{alg}}/k_\infty$. One shows that ∞ extends uniquely to a place of k_∞^{alg}. Then the problem is that k_∞^{alg} is not complete. So let $\mathbf{C}_\infty$ denote the completion of k_∞^{alg} at ∞. The field $\mathbf{C}_\infty$ is both complete and algebraically closed.

Let us choose once and for all an imbedding $\iota\colon k \hookrightarrow \mathbf{C}_\infty$, and use it to view k as a subfield of $\mathbf{C}_\infty$.

Lattices. We are now ready to introduce Drinfel'd modules. The most concrete way to introduce elliptic curves over $\mathbb{C}$ is to first define *lattices* in $\mathbb{C}$. Similarly, we are going to first define lattices in $\mathbf{C}_\infty$.

Recall that $\mathcal{O}$ is the subring of integers of k, defined in the previous subsection. A subgroup $L \subset \mathbf{C}_\infty$ is said to be *discrete* if there exists a neighborhood U of 0 in $\mathbf{C}_\infty$ such that $U \cap L = \{0\}$.

Definition 1.2.1. An *$\mathcal{O}$-lattice* in $\mathbf{C}_\infty$ is a discrete, finitely generated $\mathcal{O}$-submodule of $\mathbf{C}_\infty$.

We shall say "lattice" instead of "$\mathcal{O}$-lattice in $\mathbf{C}_\infty$".

This is an abstract definition, but in this paper we shall only have to deal with a special case of lattices, rank one lattices, for which there is a very concrete definition. Let us first define the *rank* of a lattice.

Let L be a lattice. As $\mathbf{C}_\infty$ is a field containing $\mathcal{O}$, it is obviously a torsion-free $\mathcal{O}$-module. Hence L is also torsion-free. As $\mathcal{O}$ is a Dedekind ring, the $\mathcal{O}$-module L, being finitely generated and torsion-free, is automatically projective, so there exist an integer $r \geqslant 1$ and ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_r \in \mathfrak{I}_\mathcal{O}$ such that L is isomorphic as an $\mathcal{O}$-module to $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$.

Definition 1.2.2. The integer r above is called the *rank* of L.

Let $\mathfrak{I}_\mathcal{O}$ be the semigroup of all nonzero ideals of $\mathcal{O}$, under the usual multiplication law of ideals. For rank one lattices, we have the following result:

> *A subset of $\mathbf{C}_\infty$ is a rank one lattice if, and only if it is of the form $\xi\mathfrak{a}$ with $\xi \in \mathbf{C}_\infty^*$ and $\mathfrak{a} \in \mathfrak{I}_\mathcal{O}$.*

The Drinfel'd module associated to a lattice. Let L be a lattice (of any rank). Remember the following product formula:

$$\sin z = z \prod_{t \in \pi\mathbb{Z} - \{0\}} (1 - z/t) \quad \text{for all } z \in \mathbb{C}.$$

Similarly, let us define a function $e_L\colon \mathbf{C}_\infty \to \mathbf{C}_\infty$ by the following formula:

$$e_L(x) = x \prod_{\ell \in L - \{0\}} (1 - x/\ell) \quad \text{for all } x \in \mathbf{C}_\infty.$$

One shows that this product converges for all x. The function e_L should be called the "sinus function associated to L", but authors have decided to call it the "exponential function associated to L". We have

$$e_L(x + y) = e_L(x) + e_L(y) \quad \text{for all } x, y \in \mathbf{C}_\infty, \tag{7}$$

and

$$e_L(ax) = \phi_a^L(e_L(x)) \quad \text{for all } a \in \mathcal{O},\ x \in \mathbf{C}_\infty, \tag{8}$$

where $\phi_a^L \in \mathbf{C}_\infty[X]$ is the polynomial given by the following formula if $a \neq 0$:

$$\phi_a^L = aX \prod_{0 \neq \ell \in a^{-1}L/L} (1 - X/e_L(\ell)),$$

and $\phi_0^L = 0$. Note that if a is a nonzero constant (that is, $a \in \mathbb{F}_q^*$), then it is invertible in $\mathcal{O}$ and hence $a^{-1}L = L$. Thus, one has

$$\phi_a = aX \quad \text{for all } a \in \mathbb{F}_q. \tag{9}$$

As we shall shortly see, this allows to check that for any $a \in \mathcal{O}$ the polynomial ϕ_a^L is $\mathbb{F}_q$-linear, which means that it can have nonzero coefficients only in degrees that are powers of q.

Equation (7) is an analog of the classical formula for $\sin(x+y)$, not of the formula for $\exp(x + y)$. The fact that e_L is additive, while sin is not, is a phenomenon typical of characteristic p algebra, just like the additivity of the Frobenius map $x \mapsto x^p$. The polynomials ϕ_a^L can be viewed as analogs of the classical Chebycheff polynomials of trigonometry.

One shows, by analytic means, that e_L induces a bijection

$$e_L\colon \mathbf{C}_\infty/L \to \mathbf{C}_\infty.$$

So this is a group isomorphism. Use it to transport the $\mathcal{O}$-module structure of $\mathbf{C}_\infty/L$ to a new $\mathcal{O}$-module structure on $\mathbf{C}_\infty$, which we denote $\phi^L(\mathbf{C}_\infty)$. Thus, $\phi^L(\mathbf{C}_\infty)$ is the $\mathcal{O}$-module that is equal to $\mathbf{C}_\infty$ as an additive group and whose $\mathcal{O}$-module structure is given by

$$(a, x) \mapsto \phi_a^L(x).$$

Thus, by definition, the map e_L is an isomorphism of $\mathcal{O}$-modules

$$e_L\colon \mathbf{C}_\infty/L \;\to\; \phi^L(\mathbf{C}_\infty). \tag{10}$$

The *Drinfel'd module associated to L* is the map

$$\begin{aligned} \phi^L\colon \mathcal{O} &\to \mathbf{C}_\infty[X], \\ a &\mapsto \phi_a^L. \end{aligned}$$

Definition of a Drinfel'd module over $\mathbf{C}_\infty$. The map ϕ^L that we have just defined satisfies

$$\phi^L_{a+b} = \phi^L_a + \phi^L_b \quad \text{for all } a, b \in \mathcal{O}, \tag{11}$$
$$\phi^L_{ab} = \phi^L_a \circ \phi^L_b = \phi^L_b \circ \phi^L_a \quad \text{for all } a, b \in \mathcal{O}. \tag{12}$$

Let $\tau = X^q$ and, for $n \geqslant 0$, $\tau^n = X^{q^n}$. In particular, $\tau^0 = X$. Let $\mathbf{C}_\infty\{\tau\}$ denote the (noncommutative) $\mathbf{C}_\infty$-algebra whose underlying vector space is the $\mathbf{C}_\infty$-linear span of the τ^n, for $n \geqslant 0$, and where the "multiplication" law is the composition law $\circ$. Note that $\mathbf{C}_\infty\{\tau\}$ consists exactly of those polynomials that are $\mathbb{F}_q$-linear. Combining equations (9) and (12), one obtains that the polynomial ϕ^L_a is $\mathbb{F}_q$-linear,

$$\phi^L_a \in \mathbf{C}_\infty\{\tau\} \quad \text{for all } a \in \mathcal{O},$$

and that the map $\mathcal{O} \to \mathbf{C}_\infty\{\tau\}$, $a \mapsto \phi^L_a$, is $\mathbb{F}_q$-linear as well. Thus, it is a morphism of $\mathbb{F}_q$-algebras

$$\begin{aligned} \phi^L \colon \mathcal{O} &\to \mathbf{C}_\infty\{\tau\}, \\ a &\mapsto \phi^L_a. \end{aligned}$$

Let

$$D \colon \mathbf{C}_\infty\{\tau\} \to \mathbf{C}_\infty$$

be the derivative-at-0 map. In other words, D is the $\mathbf{C}_\infty$-linear map defined by $D(\tau^0) = 1$ and $D(\tau^n) = 0$ for any $n \geqslant 1$. We have

$$D(\phi^L_a) = a \quad \text{for all } a \in \mathcal{O}.$$

This leads to the general definition of a Drinfel'd module over $\mathbf{C}_\infty$:

Definition 1.2.3. Let $\phi \colon \mathcal{O} \to \mathbf{C}_\infty\{\tau\}$, $a \mapsto \phi_a$, be a morphism of $\mathbb{F}_q$-algebras. Then ϕ is a *Drinfel'd module over* $\mathbf{C}_\infty$ if and only if

(1) for all $a \in \mathcal{O}$, $D(\phi_a) = a$,

(2) ϕ is non-trivial, i.e. ϕ is not the map $a \mapsto a\tau^0$.

To any lattice L of any rank we have associated a Drinfel'd module over $\mathbf{C}_\infty$, which we denoted by ϕ^L. The uniformization theorem states that any Drinfel'd module over $\mathbf{C}_\infty$ comes from a unique lattice. Thus, the map $L \mapsto \phi^L$ is a bijection between lattices and Drinfel'd modules over $\mathbf{C}_\infty$.

The rank of a Drinfel'd module over $\mathbf{C}_\infty$ is the rank of the associated lattice.

Action of the ideals. For any Drinfel'd module ϕ over $\mathbf{C}_\infty$ and any $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, we define the polynomial $\phi_{\mathfrak{a}} \in \mathbf{C}_\infty\{\tau\}$ as follows. Let $I_{\mathfrak{a},\phi}$ be the left ideal of $\mathbf{C}_\infty\{\tau\}$ generated by the ϕ_a, for $a \in \mathfrak{a}$. One can show that every left ideal of $\mathbf{C}_\infty\{\tau\}$ is principal, so there exists a unique monic $\phi_{\mathfrak{a}} \in \mathbf{C}_\infty\{\tau\}$ such that $I_{\mathfrak{a},\phi} = \mathbf{C}_\infty\{\tau\}\phi_{\mathfrak{a}}$.

For any Drinfel'd module ϕ over $\mathbf{C}_\infty$ and any $a \in \mathcal{O}$, we define an element $\mu_\phi(a) \in \mathbf{C}_\infty^*$ by

$$\mu_\phi(a) = \text{leading (highest-degree) coefficient of the polynomial } \phi_a.$$

Note that if $\mathfrak{a}$ is a principal ideal of $\mathcal{O}$, for any $a \in \mathcal{O}$ such that $\mathfrak{a} = a\mathcal{O}$, we have

$$\phi_{\mathfrak{a}} = \mu_\phi(a)^{-1}\phi_a.$$

It is easy to see that for any $b \in \mathcal{O}$, we have $I_{\mathfrak{a},\phi}\phi_b \subset I_{\mathfrak{a},\phi}$. Thus, for any $b \in \mathcal{O}$ we have $\phi_{\mathfrak{a}}\phi_b \in I_{\mathfrak{a},\phi}$, so there is a unique $\phi'_b \in \mathbf{C}_\infty\{\tau\}$ such that

$$\phi_{\mathfrak{a}}\phi_b = \phi'_b\phi_{\mathfrak{a}}.$$

One shows that the map $b \mapsto \phi'_b$ is a Drinfel'd module over $\mathbf{C}_\infty$. We denote it by $\mathfrak{a} * \phi$. For any two $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$, we have

$$\mathfrak{a} * (\mathfrak{b} * \phi) = (\mathfrak{a}\mathfrak{b}) * \phi.$$

Thus, $(\mathfrak{a}, \phi) \mapsto \mathfrak{a} * \phi$ is an action of $\mathfrak{I}_{\mathcal{O}}$ on the set of all Drinfel'd modules over $\mathbf{C}_\infty$.

Let $\mathfrak{F}_{\mathcal{O}}$ be the enveloping ("Grothendieck") group of the abelian semigroup $\mathfrak{I}_{\mathcal{O}}$. The abelian group $\mathfrak{F}_{\mathcal{O}}$ may be realized concretely as the group of fractional ideals of k with respect to the Dedekind ring $\mathcal{O}$. One shows that the action of $\mathfrak{I}_{\mathcal{O}}$ on the set of Drinfel'd modules over $\mathbf{C}_\infty$ extends to an action of $\mathfrak{F}_{\mathcal{O}}$. One also has the equality

$$\phi_{\mathfrak{a}\mathfrak{b}} = (\mathfrak{b} * \phi)_{\mathfrak{a}}\phi_{\mathfrak{b}}. \tag{13}$$

Torsion points. Let $\phi\colon \mathcal{O} \to \mathbf{C}_\infty\{\tau\}$, $a \mapsto \phi_a$, be a Drinfel'd module over $\mathbf{C}_\infty$. Remember that $\phi(\mathbf{C}_\infty)$ is the $\mathcal{O}$-module that is equal to $\mathbf{C}_\infty$ as an abelian group and whose $\mathcal{O}$-module structure is given by

$$(a, x) \mapsto \phi_a(x).$$

Let $\phi(\mathbf{C}_\infty)^{\text{tor}}$ denote the $\mathcal{O}$-torsion submodule of $\phi(\mathbf{C}_\infty)$. In other words, an element $x \in \phi(\mathbf{C}_\infty)$ is in $\phi(\mathbf{C}_\infty)^{\text{tor}}$ if and only if $\phi_a(x) = 0$ for some nonzero $a \in \mathcal{O}$.

For any $a \in \mathcal{O}$, let $\phi[a] = \ker \phi_a$. For any $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, let $\phi[\mathfrak{a}] = \ker \phi_{\mathfrak{a}}$. Under the bijection given by equation (10) the sets $\phi(\mathbf{C}_\infty)^{\text{tor}}$, $\phi[a]$ and $\phi[\mathfrak{a}]$ are identified with the following subsets of $\mathbf{C}_\infty/L$:

$$\begin{aligned} e_L^{-1}(\phi(\mathbf{C}_\infty)^{\text{tor}}) &= kL/L, \\ e_L^{-1}(\phi[a]) &= a^{-1}L/L \quad \text{for all } a \in \mathcal{O} - \{0\}, \\ e_L^{-1}(\phi[\mathfrak{a}]) &= \mathfrak{a}^{-1}L/L \quad \text{for all } \mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}. \end{aligned}$$

Here $\mathfrak{a}^{-1}$ is the inverse of $\mathfrak{a}$ as a fractional ideal with respect to $\mathcal{O}$, i.e.

$$\mathfrak{a}^{-1} = \{x \in k \mid x\mathfrak{a} \subset \mathcal{O}\}.$$

The following equalities follow from the definitions:

$$\begin{aligned} \phi[a] &= \phi[a\mathcal{O}] \quad \text{for all } a \in \mathcal{O}, \\ \phi[\mathfrak{a}] &= \bigcap_{a \in \mathfrak{a}} \phi[a] \quad \text{for all } \mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}, \\ \phi(\mathbf{C}_\infty)^{\mathrm{tor}} &= \bigcup_{a \in \mathcal{O}} \phi[a], \\ \phi(\mathbf{C}_\infty)^{\mathrm{tor}} &= \bigcup_{\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}} \phi[\mathfrak{a}], \end{aligned}$$

and

$$\mathfrak{a} \mid \mathfrak{b} \iff \phi[\mathfrak{a}] \subset \phi[\mathfrak{b}] \quad \text{for all } \mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}. \tag{14}$$

One also checks that, for all $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$,

$$\phi[\mathfrak{a}] \cap \phi[\mathfrak{b}] = \phi[\mathfrak{a} + \mathfrak{b}], \tag{15}$$

$$\phi[\mathfrak{a}] + \phi[\mathfrak{b}] = \phi[\mathfrak{a} \cap \mathfrak{b}]. \tag{16}$$

We have

$$\operatorname{Card} \phi[\mathfrak{a}] = (\mathbf{N}\mathfrak{a})^r \quad \text{for all } \mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}, \tag{17}$$

where r is the rank of ϕ and $\mathbf{N}\mathfrak{a}$ is the absolute norm of $\mathfrak{a}$, i.e., $\mathbf{N}\mathfrak{a}$ is the cardinal of $\mathcal{O}/\mathfrak{a}$.

Let $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$. By construction, $\phi_{\mathfrak{a}}$ is an $\mathcal{O}$-module morphism

$$\phi_{\mathfrak{a}} \colon \phi(\mathbf{C}_\infty) \to (\mathfrak{a} * \phi)(\mathbf{C}_\infty).$$

For any $\mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$, let $\phi_{\mathfrak{a}}|_{\phi[\mathfrak{b}]}$ denote the restriction of $\phi_{\mathfrak{a}}$ to $\phi[\mathfrak{b}]$.

Lemma 1.2.4. *Let ϕ be a Drinfel'd module over $\mathbf{C}_\infty$. Let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$. Let $\mathfrak{d} = \mathfrak{a} + \mathfrak{b}$ be the gcd of $\mathfrak{a}$ and $\mathfrak{b}$. We have*

$$\begin{aligned} \operatorname{Ker}(\phi_{\mathfrak{a}}|_{\phi[\mathfrak{b}]}) &= \phi[\mathfrak{d}], \\ \operatorname{Im}(\phi_{\mathfrak{a}}|_{\phi[\mathfrak{b}]}) &= (\mathfrak{a} * \phi)[\mathfrak{d}^{-1}\mathfrak{b}]. \end{aligned}$$

Proof. First equality: we have $\operatorname{Ker}(\phi_{\mathfrak{a}}|_{\phi[\mathfrak{b}]}) = \phi[\mathfrak{a}] \bigcap \phi[\mathfrak{b}]$, so the result follows from equation (15).

Second equality: let r denote the rank of ϕ. We have

$$\begin{aligned} \operatorname{Card}(\operatorname{Im}(\phi_{\mathfrak{a}}|_{\phi[\mathfrak{b}]})) &= \operatorname{Card}(\phi[\mathfrak{b}]) / \operatorname{Card}(\operatorname{Ker}(\phi_{\mathfrak{a}}|_{\phi[\mathfrak{b}]})) \\ &= \operatorname{Card}(\phi[\mathfrak{b}]) / \operatorname{Card}(\phi[\mathfrak{d}]) \\ &= (\mathbf{N}\mathfrak{b})^r / (\mathbf{N}\mathfrak{d})^r \end{aligned}$$

and

$$\mathrm{Card}((\mathfrak{a} * \phi)[\mathfrak{d}^{-1}\mathfrak{b}]) = \mathbf{N}(\mathfrak{d}^{-1}\mathfrak{b})^r,$$

so the two cardinals are equal, so it is enough to show one inclusion. Let $x \in \mathrm{Im}(\phi_{\mathfrak{a}}|_{\phi[\mathfrak{b}]})$. It is enough to show that $(\mathfrak{a} * \phi)_{\mathfrak{d}^{-1}\mathfrak{b}}(x) = 0$. Let $y \in \phi[\mathfrak{b}]$ such that $\phi_{\mathfrak{a}}(y) = x$. Let $\mathfrak{c} = \mathfrak{a} \cap \mathfrak{b}$ be the lcm. We have $\mathfrak{d}^{-1}\mathfrak{b} = \mathfrak{a}^{-1}\mathfrak{c}$. But

$$(\mathfrak{a} * \phi)_{\mathfrak{a}^{-1}\mathfrak{c}}(x) = (\mathfrak{a} * \phi)_{\mathfrak{a}^{-1}\mathfrak{c}}(\phi_{\mathfrak{a}}(y)),$$

so,by equation (13), we get

$$(\mathfrak{a} * \phi)_{\mathfrak{a}^{-1}\mathfrak{c}}(x) = \phi_{\mathfrak{c}}(y).$$

But $\mathfrak{b}|\mathfrak{c}$ and $y \in \phi[\mathfrak{b}]$, so $y \in \phi[\mathfrak{c}]$, so

$$(\mathfrak{a} * \phi)_{\mathfrak{a}^{-1}\mathfrak{c}}(x) = 0. \qquad \square$$

Corollary 1.2.5. *Let ϕ be a Drinfel'd module over $\mathbf{C}_\infty$. Let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$. For all $\lambda \in \phi[\mathfrak{b}]$, there exists $\mu \in (\mathfrak{a}^{-1} * \phi)[\mathfrak{a}\mathfrak{b}]$ such that*

$$(\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}}(\mu) = \lambda.$$

Proof. Let $\psi = \mathfrak{a}^{-1} * \phi$. Let $\mathfrak{b}_2 = \mathfrak{a}\mathfrak{b}$. Let $\mathfrak{d}_2 = \mathfrak{a}$, so that $\mathfrak{d}_2$ is the gcd of $\mathfrak{a}$ and $\mathfrak{b}_2$. By Lemma 1.2.4, we have

$$\mathrm{Im}(\psi_{\mathfrak{a}}|_{\psi[\mathfrak{b}_2]}) = \phi[\mathfrak{d}_2^{-1}\mathfrak{b}_2],$$

so

$$\mathrm{Im}(\psi_{\mathfrak{a}}|_{\psi[\mathfrak{a}\mathfrak{b}]}) = \phi[\mathfrak{b}]. \qquad \square$$

Corollary 1.2.6. *Let ϕ be a Drinfel'd module over $\mathbf{C}_\infty$. For all $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, the map*

$$(\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}} \colon (\mathfrak{a}^{-1} * \phi)(\mathbf{C}_\infty)^{\mathrm{tor}} \to \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$$

is surjective.

1.3. Hayes' explicit class field theory. In this subsection we review D. R. Hayes' explicit class field theory for function fields, in terms of sign-normalized rank one Drinfel'd modules. We follow [20], Part II, and [15], Chapter VII. Recall that k_∞ is the completion of k at ∞. Let $\mathbb{F}_\infty$ denote the constant subfield of k_∞. The field $\mathbb{F}_\infty$ is a finite extension of $\mathbb{F}_q$, and its degree is equal to the degree of the place ∞.

Definition 1.3.1. A *sign function* on k_∞^* is a group morphism $\mathrm{sgn}\colon k_\infty^* \to \mathbb{F}_\infty^*$ which induces the identity map on $\mathbb{F}_\infty^*$.

Let us choose once and for all a sign-function sgn (by [20], Corollary 12.2, the number of possible choices is equal to the cardinal of $\mathbb{F}_\infty^*$). We let $\mathrm{sgn}(0) = 0$ so that sgn becomes a function $k_\infty \to \mathbb{F}_\infty$.

Definition 1.3.2. A Drinfel'd module ϕ over $\mathbf{C}_\infty$ is said to be sgn-*normalized* if there exists an element $\sigma \in \mathrm{Gal}(\mathbb{F}_\infty/\mathbb{F}_q)$ such that

$$\mu_\phi(a) = \sigma(\mathrm{sgn}(a)) \quad \text{for all } a \in \mathcal{O}.$$

Let us now focus on the case of Drinfel'd modules of rank one.

Definition 1.3.3. Let $H(\mathrm{sgn})$ denote the set of sgn-normalized rank one Drinfel'd modules over $\mathbf{C}_\infty$. The elements of $H(\mathrm{sgn})$ are also called *Hayes modules* (for the triple $(k, \infty, \mathrm{sgn})$).

Proposition 1.3.4. $H(\mathrm{sgn})$ *is a finite set, and its cardinal* $h(\mathrm{sgn})$ *is given by*

$$h(\mathrm{sgn}) = \frac{\mathrm{Card}\,\mathbb{F}_\infty^*}{\mathrm{Card}\,\mathbb{F}_q^*} \cdot h(\mathcal{O}),$$

where $h(\mathcal{O})$ *is the class number of the Dedekind ring* $\mathcal{O}$.

Proof. See [20], Corollary 13.4. □

Proposition 1.3.5. *For any* $\phi \in H(\mathrm{sgn})$ *and any* $\mathfrak{a} \in \mathfrak{I}_\mathcal{O}$, *we have* $\mathfrak{a} * \phi \in H(\mathrm{sgn})$. *Thus,* $\mathfrak{I}_\mathcal{O}$ *acts on* $H(\mathrm{sgn})$.

Proof. See [20], p. 22. □

Definition 1.3.6. Let $\phi \in H(\mathrm{sgn})$, and let $y \in \mathcal{O} - \mathbb{F}_q$ (recall that $\mathbb{F}_q$ denotes the constant subfield of k). Let H^+ be the field generated over k by the coefficients of ϕ_y.

One shows (see [20], p. 23) that H^+ does not depend on the choice of ϕ and y.

Proposition 1.3.7. *The extension* H^+/k *is finite, abelian, and unramified away from* ∞.

Proof. See [20], Propositions 14.1 and 14.4. □

One shows (see [20], §15) that H^+ contains a subfield H which plays the rôle of the Hilbert class field for the pair (k, ∞).

Here is a concrete picture of the Galois group $\mathrm{Gal}(H^+/k)$. First, let $\mathcal{P}_\mathcal{O}^+$ be the following subgroup of $\mathfrak{I}_\mathcal{O}$:

$$\mathcal{P}_\mathcal{O}^+ = \{x\mathcal{O} \mid x \in k,\ \mathrm{sgn}(x) = 1\}.$$

We then have the following proposition.

Proposition 1.3.8. *The Artin map* $(\cdot, H^+/k)$ *induces an isomorphism from* $\mathfrak{F}_{\mathcal{O}}/\mathcal{P}_{\mathcal{O}}^+$ *to* $\mathrm{Gal}(H^+/k)$.

Proof. See [20], Theorem 14.7. □

The Galois group $\mathrm{Gal}(H^+/k)$ acts on $H(\mathrm{sgn})$ by $(\sigma, \phi) \mapsto \sigma\phi$, where $\sigma\phi$ is defined by $(\sigma\phi)_a = \sigma(\phi_a)$ for all $a \in \mathcal{O}$ (one checks that $\sigma\psi \in H(\mathrm{sgn})$).

Theorem 1.3.9. *For any* $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, *if* $\sigma_{\mathfrak{a}} = (\mathfrak{a}, H^+/k) \in \mathrm{Gal}(H^+/k)$ *denotes the Artin automorphism of* H^+ *associated to* $\mathfrak{a}$, *then we have*

$$\sigma_{\mathfrak{a}}\phi = \mathfrak{a} * \phi \quad \textit{for all } \phi \in H(\mathrm{sgn}).$$

The set $H(\mathrm{sgn})$ *is principal homogeneous under the action of* $\mathrm{Gal}(H^+/k)$.

Proof. See [20], Theorems 13.8 and 14.7. □

Definition 1.3.10. For any $\phi \in H(\mathrm{sgn})$, let K denote the field generated over H^+ by the elements of $\phi(\mathbf{C}_\infty)^{\mathrm{tor}}$. For any $\mathfrak{c} \in \mathfrak{I}_{\mathcal{O}}$, let $K_{\mathfrak{c}}$ denote the field generated over H^+ by the elements of $\phi[\mathfrak{c}]$.

One shows (see [20], p. 28) that K and $K_{\mathfrak{c}}$ are independent of the choice of ϕ. The extension $K_{\mathfrak{c}}/k$ is called the *narrow ray class extension modulo* $\mathfrak{c}$. By construction, we have

$$K = \bigcup_{\mathfrak{c} \in \mathfrak{I}_{\mathcal{O}}} K_{\mathfrak{c}}.$$

Theorem 1.3.11. *For any* $\mathfrak{c} \in \mathfrak{I}_{\mathcal{O}}$, *the extension* $K_{\mathfrak{c}}/k$ *is finite, abelian, and unramified away from* ∞ *and the prime divisors of* $\mathfrak{c}$. *Moreover,* $K_{\mathfrak{c}}$ *contains the ray class field of* k *of conductor* $\mathfrak{c}$ *totally split at* ∞. *For any* $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$ *prime to* $\mathfrak{c}$, *if* $\sigma_{\mathfrak{a}} = (\mathfrak{a}, K_{\mathfrak{c}}/k) \in \mathrm{Gal}(K_{\mathfrak{c}}/k)$ *denotes the Artin automorphism of* $K_{\mathfrak{c}}$ *associated to* $\mathfrak{a}$, *then we have*

$$\sigma_{\mathfrak{a}}\lambda = \phi_{\mathfrak{a}}(\lambda) \quad \textit{for all } \phi \in H(\mathrm{sgn}),\ \lambda \in \phi[\mathfrak{c}].$$

Proof. See [20], p. 28, or [19], Section 8. □

In particular, this shows that

$$k^{\mathrm{ab},\infty} \subset K \subset k^{\mathrm{ab}},$$

where $k^{\mathrm{ab},\infty}$ is the maximal abelian extension of k that is totally split at ∞.

Let us give a concrete picture of the Galois group $\mathrm{Gal}(K_{\mathfrak{c}}/k)$, for $\mathfrak{c} \in \mathfrak{I}_{\mathcal{O}}$. Let $\mathfrak{F}_{\mathcal{O}}(\mathfrak{c})$ denote the subgroup of $\mathfrak{F}_{\mathcal{O}}$ of all fractional ideals that are prime to $\mathfrak{c}$, and let

$$\mathcal{P}_{\mathcal{O}}^+(\mathfrak{c}) = \{x\mathcal{O} \mid x \in k,\ \mathrm{sgn}(x) = 1,\ x \equiv 1 \bmod \mathfrak{c}\}.$$

We then have the following proposition.

Proposition 1.3.12. *The Artin map* $(\cdot, K_{\mathfrak{c}}/k)$ *induces an isomorphism from* $\mathfrak{F}_{\mathcal{O}}(\mathfrak{c})/\mathcal{P}^{+}_{\mathcal{O}}(\mathfrak{c})$ *to* $\mathrm{Gal}(K_{\mathfrak{c}}/k)$.

Proof. See [20], p. 28. □

Moreover, the Galois group $\mathrm{Gal}(K_{\mathfrak{c}}/H^{+})$ has an even simpler description: one can check (loc. cit.) that it is isomorphic to the group of invertible elements in $\mathcal{O}/\mathfrak{c}$.

2. Construction of the C*-dynamical system $(C_{k,\infty}, (\sigma_t))$

2.1. The space X of characters. For any $\phi \in H(\mathrm{sgn})$, let X_ϕ be the dual group of the discrete abelian torsion group $\phi(\mathbf{C}_\infty)^{\mathrm{tor}}$. Thus, an element of X_ϕ is a character of $\phi(\mathbf{C}_\infty)^{\mathrm{tor}}$. The group X_ϕ is profinite,

$$X_\phi = \lim_{\leftarrow \mathfrak{a}} \widehat{\phi[\mathfrak{a}]},$$

where $\mathfrak{a}$ runs over $\mathfrak{I}_{\mathcal{O}}$ ordered by divisibility. Let X be the (disjoint) union of the X_ϕ,

$$X = \bigcup_{\phi \in H(\mathrm{sgn})} X_\phi.$$

Note that the elements of X are reminiscent of characters in [17] and of $\mathbb{Q}$-lattices (or k-lattices) in [11] and [12], [13].

Lemma 2.1.1. *For any character $\chi \in X$, we have*

$$\mathrm{Im}\, \chi \subset \mathbb{U}_p,$$

where $\mathbb{U}_p$ is the group of p-th roots of unity in $\mathbb{C}$.

Proof. Recall that for any $\phi \in H(\mathrm{sgn})$, as a group, $\phi(\mathbf{C}_\infty)$ is equal to $\mathbf{C}_\infty$, which is a field of characteristic p. Thus, for all $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$, we have $\chi(\lambda)^p = \chi(p\lambda) = \chi(0) = 1$. □

Lemma 2.1.2. *X is compact (and Hausdorff).*

Proof. For any $\phi \in H(\mathrm{sgn})$, the group X_ϕ is profinite, hence compact. As $H(\mathrm{sgn})$ is finite, X is compact. □

We define an action of $\mathfrak{I}_{\mathcal{O}}$ on X by

$$\chi^{\mathfrak{a}} = \chi \circ (\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}} \quad \text{for all } \mathfrak{a} \in \mathfrak{I}_{\mathcal{O}},\ \phi \in H(\mathrm{sgn}),\ \chi \in X_\phi. \tag{18}$$

Recall that $(\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}}$ is a map from $(\mathfrak{a}^{-1} * \phi)(\mathbf{C}_\infty)$ to $\phi(\mathbf{C}_\infty)$. Thus, if $\chi \in X_\phi$ then $\chi^{\mathfrak{a}} \in X_{\mathfrak{a}^{-1}*\phi}$. Note that equation (13) guarantees that this is a semigroup action of $\mathfrak{I}_{\mathcal{O}}$.

The exponent notation ($\chi^{\mathfrak{a}}$) is inspired by what happens with characters of $\mathbb{Q}/\mathbb{Z}$. These characters may be composed with the map $\phi_n : x \mapsto nx$, for any $n \in \mathbb{N}^*$. By definition of a character, we have $\chi \circ \phi_n = \chi^n$. In our case $\mathbb{N}^*$ is replaced by $\mathfrak{I}_{\mathcal{O}}$ and the maps ϕ_n are replaced by the $\phi_{\mathfrak{a}}$.

We define an action of $\mathrm{Gal}(K/k)$ on X by

$$\sigma\chi = \chi \circ \sigma \quad \text{for all } \sigma \in \mathrm{Gal}(K/k),\ \chi \in X. \tag{19}$$

One checks that the actions of $\mathrm{Gal}(K/k)$ and of $\mathfrak{I}_{\mathcal{O}}$ on X commute with one another.

Lemma 2.1.3. *For all $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, the map $X \to X$, $\chi \mapsto \chi^{\mathfrak{a}}$, is injective.*

Proof. Let $\chi_1, \chi_2 \in X$ such that $\chi_1^{\mathfrak{a}} = \chi_2^{\mathfrak{a}}$. For $i = 1, 2$ let ϕ^i be such that $\chi_i \in X_{\phi^i}$. By definition, we have $\chi_i^{\mathfrak{a}} \in X_{\mathfrak{a}^{-1}*\phi^i}$, so $\mathfrak{a}^{-1} * \phi^1 = \mathfrak{a}^{-1} * \phi^2$, so $\phi^1 = \phi^2$. Let $\phi = \phi^1 = \phi^2$. We have

$$\chi_1 \circ (\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}} = \chi_2 \circ (\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}}.$$

Corollary 1.2.6 then shows that $\chi_1 = \chi_2$. □

Corollary 2.1.4. *Let $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2 \in \mathfrak{I}_{\mathcal{O}}$ be such that $\mathfrak{a}_1^{-1}\mathfrak{a}_2 = \mathfrak{b}_1^{-1}\mathfrak{b}_2$.*

(1) *Let $\chi_1, \chi_2 \in X$. We have*

$$\chi_1^{\mathfrak{a}_1} = \chi_2^{\mathfrak{a}_2} \iff \chi_1^{\mathfrak{b}_1} = \chi_2^{\mathfrak{b}_2}.$$

(2) *Let $\chi_1, \chi_2, \chi_3 \in X$. We have*

$$\chi_1^{\mathfrak{a}_1} = \chi_2^{\mathfrak{a}_2} \text{ and } \chi_3^{\mathfrak{b}_1} = \chi_2^{\mathfrak{b}_2} \implies \chi_1 = \chi_3.$$

Proof. Let us first prove (1). Suppose that $\chi_1^{\mathfrak{a}_1} = \chi_2^{\mathfrak{a}_2}$. We have $\chi_1^{\mathfrak{a}_1\mathfrak{b}_2} = \chi_2^{\mathfrak{a}_2\mathfrak{b}_2}$. But $\mathfrak{a}_2\mathfrak{b}_1 = \mathfrak{a}_1\mathfrak{b}_2$, so $\chi_1^{\mathfrak{a}_2\mathfrak{b}_1} = \chi_2^{\mathfrak{a}_2\mathfrak{b}_2}$, so

$$(\chi_1^{\mathfrak{b}_1})^{\mathfrak{a}_2} = (\chi_2^{\mathfrak{b}_2})^{\mathfrak{a}_2},$$

so Lemma 2.1.3 gives $\chi_1^{\mathfrak{b}_1} = \chi_2^{\mathfrak{b}_2}$, which proves one implication, and the other implication follows by swapping $\mathfrak{a}_i$ with $\mathfrak{b}_i$ for $i = 1, 2$.

Let us now prove (2). We have

$$\chi_1^{\mathfrak{a}_1\mathfrak{b}_1} = \chi_2^{\mathfrak{a}_2\mathfrak{b}_1} = \chi_2^{\mathfrak{a}_1\mathfrak{b}_2} = \chi_3^{\mathfrak{a}_1\mathfrak{b}_1},$$

so Lemma 2.1.3 gives $\chi_1 = \chi_3$. □

Corollary 2.1.4 allows to extend the action of $\mathfrak{I}_{\mathcal{O}}$ on X to a partially defined action of $\mathfrak{F}_{\mathcal{O}}$ as follows.

Definition 2.1.5. For any $\chi \in X$, let $\mathfrak{F}_\chi$ denote the set of all $\mathfrak{c} \in \mathfrak{F}_{\mathcal{O}}$ such that there exists $\chi_1 \in X$ satisfying

$$\chi_1^{\mathfrak{a}_1} = \chi^{\mathfrak{a}_2} \tag{20}$$

for some $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{I}_{\mathcal{O}}$ with $\mathfrak{c} = \mathfrak{a}_1^{-1}\mathfrak{a}_2$. By Corollary 2.1.4 (1), the existence of χ_1 only depends on χ and $\mathfrak{c}$, and does not depend on the choice of $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{I}_{\mathcal{O}}$ such that $\mathfrak{c} = \mathfrak{a}_1^{-1}\mathfrak{a}_2$. By Corollary 2.1.4 (2), the character χ_1, when it exists, is uniquely determined by χ and $\mathfrak{c}$. When $\mathfrak{c} \in \mathfrak{F}_\chi$, we define a character $\chi^{\mathfrak{c}}$ by

$$\chi^{\mathfrak{c}} = \chi_1.$$

The partially defined map $\mathfrak{F}_{\mathcal{O}} \times X \to X$, $(\mathfrak{c}, \chi) \mapsto \chi^{\mathfrak{c}}$, should be regarded as a *partially defined* group action of $\mathfrak{F}_{\mathcal{O}}$ on X. For any $(\mathfrak{c}, \chi)$, the character $\chi^{\mathfrak{c}}$ is defined if and only if $\mathfrak{c} \in \mathfrak{F}_\chi$. For any $\mathfrak{c}_1, \mathfrak{c}_2$ in $\mathfrak{F}_\chi$, if $\mathfrak{c}_1\mathfrak{c}_2 \in \mathfrak{F}_\chi$ one checks that $\chi^{\mathfrak{c}_1\mathfrak{c}_2} = (\chi^{\mathfrak{c}_1})^{\mathfrak{c}_2}$. Of course, when $\mathfrak{c} \in \mathfrak{I}_{\mathcal{O}}$ the character $\chi^{\mathfrak{c}}$ is just the one that was defined in equation (18).

For any χ, we have $\mathfrak{I}_{\mathcal{O}} \subset \mathfrak{F}_\chi$. Characters $\chi \in X$ for which this inclusion is an equality ($\mathfrak{F}_\chi = \mathfrak{I}_{\mathcal{O}}$) will be called *admissible*, and will play an important rôle later (see Subsection 3.6).

Note that we obviously have

$$\mathfrak{F}_{\chi^{\mathfrak{a}}} = \mathfrak{a}^{-1}\mathfrak{F}_\chi \quad \text{for all } \chi \in X,\ \mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}. \tag{21}$$

Lemma 2.1.6. *Let $\chi \in X$. Let $\phi \in H(\mathrm{sgn})$ such that $\chi \in X_\phi$. For any $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, we have*

$$\mathfrak{a}^{-1} \in \mathfrak{F}_\chi \iff \chi(\lambda) = 1 \quad \textit{for all } \lambda \in \phi[\mathfrak{a}].$$

When this is the case, the character $\chi^{\mathfrak{a}^{-1}}$ is given by

$$\chi^{\mathfrak{a}^{-1}}(\lambda) = (\mathbf{N}\mathfrak{a})^{-1} \sum_{\phi_{\mathfrak{a}}(\mu)=\lambda} \chi(\mu) \quad \textit{for all } \lambda \in (\mathfrak{a} * \phi)(\mathbf{C}_\infty)^{\mathrm{tor}}.$$

Proof. If $\mathfrak{a}^{-1} \in \mathfrak{F}_\chi$, then there exists $\chi_1 \in X$ such that $\chi = \chi_1^{\mathfrak{a}}$. Thus, for all $\lambda \in \phi[\mathfrak{a}]$, we have $\chi(\lambda) = \chi_1(\phi_{\mathfrak{a}}(\lambda))$, but $\phi_{\mathfrak{a}}(\lambda) = 0$, so $\chi(\lambda) = 1$.

Now suppose that for all $\lambda \in \phi[\mathfrak{a}]$, $\chi(\lambda) = 1$. For all $\lambda \in (\mathfrak{a} * \phi)(\mathbf{C}_\infty)^{\mathrm{tor}}$, set

$$\chi_1(\lambda) = (\mathbf{N}\mathfrak{a})^{-1} \sum_{\phi_{\mathfrak{a}}(\mu)=\lambda} \chi(\mu).$$

Let us show that this defines a character χ_1 of $(\mathfrak{a} * \phi)(\mathbf{C}_\infty)^{\mathrm{tor}}$. Let $\lambda \in (\mathfrak{a} * \phi)(\mathbf{C}_\infty)^{\mathrm{tor}}$. By Lemma 1.2.6, there exists $\mu_1 \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$ such that $\phi_{\mathfrak{a}}(\mu_1) = \lambda$. We have

$$\chi_1(\lambda) = (\mathbf{N}\mathfrak{a})^{-1} \sum_{\mu_0 \in \phi[\mathfrak{a}]} \chi(\mu_0 + \mu_1) = (\mathbf{N}\mathfrak{a})^{-1}\Big(\sum_{\mu_0 \in \phi[\mathfrak{a}]} \chi(\mu_0)\Big)\chi(\mu_1).$$

But we have $\chi(\mu_0) = 1$ for all $\mu_0 \in \phi[\mathfrak{a}]$, and by equation (17) we have that $\mathrm{Card}(\phi[\mathfrak{a}]) = \mathbf{N}\mathfrak{a}$. Thus, we get

$$\chi_1(\lambda) = \chi(\mu_1) \quad \text{for all } \lambda \in (\mathfrak{a} * \phi)(\mathbf{C}_\infty)^{\mathrm{tor}} \text{ and for all } \mu_1 \text{ with } \phi_{\mathfrak{a}}(\mu_1) = \lambda.$$

Now let $\lambda' \in (\mathfrak{a} * \phi)(\mathbf{C}_\infty)^{\mathrm{tor}}$ and μ_1' such that $\phi_{\mathfrak{a}}(\mu_1') = \lambda'$. We have

$$\lambda + \lambda' = \phi_{\mathfrak{a}}(\mu_1) + \phi_{\mathfrak{a}}(\mu_1') = \phi_{\mathfrak{a}}(\mu_1 + \mu_1'),$$

hence

$$\chi_1(\lambda + \lambda') = \chi_1(\phi_{\mathfrak{a}}(\mu_1 + \mu_1')) = \chi(\mu_1 + \mu_1') = \chi(\mu_1)\chi(\mu_1'),$$

so

$$\chi_1(\lambda + \lambda') = \chi_1(\lambda)\chi_1(\lambda'),$$

which implies $\chi_1(\lambda)^p = \chi_1(p\lambda) = \chi_1(0) = 1$, i.e.,

$$\chi_1(\lambda) \in \mathbb{U}_p \quad \text{for all } \lambda \in (\mathfrak{a} * \phi)(\mathbf{C}_\infty)^{\mathrm{tor}}.$$

Hence χ_1 is a group morphism $(\mathfrak{a} * \phi)(\mathbf{C}_\infty)^{\mathrm{tor}} \to \mathbb{U}_p$, so $\chi_1 \in X$, and we have by construction $\chi_1^{\mathfrak{a}} = \chi$. Thus, we have $\mathfrak{a}^{-1} \in \mathfrak{F}_\chi$ and $\chi^{\mathfrak{a}^{-1}} = \chi_1$. □

Lemma 2.1.7. *For all $\chi \in X$, for all $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$ relatively prime, we have*

$$\mathfrak{a}^{-1}\mathfrak{b} \in \mathfrak{F}_\chi \iff \mathfrak{a}^{-1} \in \mathfrak{F}_\chi.$$

Proof. Let ϕ be such that $\chi \in X_\phi$. We have $\mathfrak{a}^{-1}\mathfrak{b} \in \mathfrak{F}_\chi \Leftrightarrow \mathfrak{a}^{-1} \in \mathfrak{F}_{\chi^{\mathfrak{b}}}$. Lemma 2.1.6 applied to $\chi^{\mathfrak{b}}$ thus gives

$$\mathfrak{a}^{-1}\mathfrak{b} \in \mathfrak{F}_\chi \iff \chi((\mathfrak{b}^{-1} * \phi)_{\mathfrak{b}}(\lambda)) = 1 \quad \text{for all } \lambda \in (\mathfrak{b}^{-1} * \phi)[\mathfrak{a}].$$

But, as $\mathfrak{a}$ and $\mathfrak{b}$ are relatively prime, by Lemma 1.2.4, the map $\lambda \mapsto \mathfrak{b}\lambda$ is a bijection from $(\mathfrak{b}^{-1} * \phi)[\mathfrak{a}]$ onto $\phi[\mathfrak{a}]$. Thus we get

$$\mathfrak{a}^{-1}\mathfrak{b} \in \mathfrak{F}_\chi \iff \chi(\lambda) = 1 \quad \text{for all } \lambda \in \phi[\mathfrak{a}],$$

and, by Lemma 2.1.6, this is equivalent to $\mathfrak{a}^{-1} \in \mathfrak{F}_\chi$. □

Lemma 2.1.8. *For all $\chi \in X$, for all $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$ relatively prime, we have*

$$(\mathfrak{a}\mathfrak{b})^{-1} \in \mathfrak{F}_\chi \iff \mathfrak{a}^{-1} \in \mathfrak{F}_\chi \text{ and } \mathfrak{b}^{-1} \in \mathfrak{F}_\chi.$$

Proof. Let ϕ be such that $\chi \in X_\phi$. By Lemma 2.1.6, the statement that we want to prove is equivalent to the following:

$$\chi(\lambda) = 1 \quad \text{for all } \lambda \in \phi[\mathfrak{a}\mathfrak{b}] \iff \begin{array}{ll} \chi(\lambda) = 1 & \text{for all } \lambda \in \phi[\mathfrak{a}], \\ \chi(\lambda) = 1 & \text{for all } \lambda \in \phi[\mathfrak{b}]. \end{array} \tag{22}$$

By equations (15) and (16), as $\mathfrak{a}$ and $\mathfrak{b}$ are relatively prime, we have

$$\phi[\mathfrak{a}\mathfrak{b}] = \phi[\mathfrak{a}] \oplus \phi[\mathfrak{b}],$$

so, for any $\lambda \in \phi[\mathfrak{a}\mathfrak{b}]$, there exists a unique pair $(\lambda_1, \lambda_2) \in \phi[\mathfrak{a}] \times \phi[\mathfrak{b}]$ such that $\lambda = \lambda_1 + \lambda_2$. We have $\chi(\lambda) = \chi(\lambda_1)\chi(\lambda_2)$, so equation (22) follows. □

2.2. Construction of the groupoid $\mathscr{G}$ and of the dynamical system $(C_{k,\infty}, (\sigma_t))$. Let $\mathscr{G}$ be the following subset of $X \times \mathfrak{F}_{\mathcal{O}}$:

$$\mathscr{G} = \{(\chi, \mathfrak{c}) \in X \times \mathfrak{F}_{\mathcal{O}} \mid \mathfrak{c} \in \mathfrak{F}_{\chi}\}.$$

We turn $\mathscr{G}$ into a groupoid by endowing it with the groupoid law

$$(\chi_1, \mathfrak{c}_1) \circ (\chi_2, \mathfrak{c}_2) = (\chi_2, \mathfrak{c}_1\mathfrak{c}_2) \quad \text{if } \chi_1 = \chi_2^{\mathfrak{c}_2}$$

and the inverse map

$$(\chi, \mathfrak{c})^{-1} = (\chi^{\mathfrak{c}}, \mathfrak{c}^{-1}).$$

One checks that, under the identification $\mathscr{G}^{(0)} = X \times \{1\} \simeq X$, the range and source maps r and s are respectively given by $r(\chi, \mathfrak{c}) = \chi^{\mathfrak{c}}$ and $s(\chi, \mathfrak{c}) = \chi$.

The abelian group $\mathfrak{F}_{\mathcal{O}}$ is endowed with the discrete topology. The groupoid $\mathscr{G}$ is endowed with its topology as a subset of $X \times \mathfrak{F}_{\mathcal{O}}$.

Lemma 2.2.1. *$\mathscr{G}$ is a locally compact groupoid.*

Proof. $X \times \mathfrak{F}_{\mathcal{O}}$ is locally compact by Lemma 2.1.2 and $\mathscr{G}$ is a closed subset of it, so it is also locally compact. It is clear that the composition and inverse maps are continuous, so this is a locally compact groupoid. □

The C*-algebra $C_{k,\infty}$ that was advertised in the introduction of this paper is the maximal[2] C*-algebra of the groupoid $\mathscr{G}$. Let us quickly explain what that means.

For $\chi \in X$, let $\mathscr{G}_\chi$ denote the fiber of s above χ, that is,

$$\mathscr{G}_\chi = \{\chi\} \times \mathfrak{F}_\chi,$$

so $\mathscr{G}_\chi$ is discrete and is in bijection with $\mathfrak{F}_\chi$.

Let $C_c(\mathscr{G})$ denote the convolution algebra of continuous maps $\mathscr{G} \to \mathbb{C}$ with compact support, where the convolution product is given by

$$(f_1 f_2)(g) = \sum_{g_1 \circ g_2 = g} f_1(g_1) f_2(g_2). \tag{23}$$

[2]Actually, it coincides with the reduced C*-algebra because $\mathfrak{F}_{\mathcal{O}}$ is an abelian group, but we shall not need that in this paper.

$C_c(\mathcal{G})$ is endowed with the involution $f \mapsto f^*$ defined by

$$f^*(g) = \overline{f(g^{-1})}.$$

For any $\chi \in X$, we define a $*$-representation of $C_c(\mathcal{G})$ on the Hilbert space $\ell^2(\mathcal{G}_\chi)$ by

$$(\pi_\chi(f)\xi)(g) = \sum_{g_1 \circ g_2 = g} f(g_1)\xi(g_2) \quad \text{for all } f \in C_c(\mathcal{G}),\ \xi \in \ell^2(\mathcal{G}_\chi). \tag{24}$$

In other words, π_χ is the left regular representation on $\ell^2(\mathcal{G}_\chi)$. Let us define a C*-norm $\|\cdot\|$ on $C_c(\mathcal{G})$ by

$$\|f\| = \sup_\pi \|\pi(f)\|,$$

where π runs over all $*$-representations of $C_c(\mathcal{G})$. The completion $C^*(\mathcal{G})$ of $C_c(\mathcal{G})$ under $\|\cdot\|$ is a C*-algebra, called the *maximal C*-algebra* of the groupoid $\mathcal{G}$. For more details about groupoid C*-algebras, see Renault [31], Khoshkam and Skandalis [24], or Connes [8], Chapter II, §5.

Definition 2.2.2. We define the C*-algebra $C_{k,\infty}$ by letting

$$C_{k,\infty} = C^*(\mathcal{G}).$$

By definition, any $*$-representation π of $C_c(\mathcal{G})$ extends uniquely to a representation of $C_{k,\infty}$, which we still denote π.

Lemma 2.2.3. *For any $*$-automorphism σ of $C_c(\mathcal{G})$, there exists an unique extension of σ to a $*$-automorphism of $C_{k,\infty}$.*

Proof. For any $*$-automorphism σ of $C_c(\mathcal{G})$ and any $*$-representation π of $C_c(\mathcal{G})$, note that $\pi \circ \sigma$ is a $*$-representation of $C_c(\mathcal{G})$. Thus, by definition of the norm $\|\cdot\|$, σ is an isometry: for all $f \in C_c(\mathcal{G})$ we have $\|\sigma(f)\| = \|f\|$. The result then follows easily. □

For any $g = (\chi, \mathfrak{c}) \in \mathcal{G}$, put $\mathbf{N}g = \mathbf{N}\mathfrak{c}$, where $\mathbf{N}\mathfrak{c}$ is the absolute norm of the fractional ideal $\mathfrak{c}$, defined by $\mathbf{N}\mathfrak{c} = (\mathbf{N}\mathfrak{a})^{-1}\mathbf{N}\mathfrak{b}$ for any $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_\mathcal{O}$ such that $\mathfrak{c} = \mathfrak{a}^{-1}\mathfrak{b}$.

Let us define a one parameter $*$-automorphism group $(\sigma_t)_{t\in\mathbb{R}}$ of $C_c(\mathcal{G})$ by

$$(\sigma_t(f))(g) = (\mathbf{N}g)^{it} f(g) \quad \text{for all } t \in \mathbb{R},\ f \in C_c(\mathcal{G}),\ g \in \mathcal{G}.$$

Definition 2.2.4. We still denote σ_t the unique extension (given by Lemma 2.2.3) of σ_t to an automorphism of $C_{k,\infty}$.

It remains to check that the pair $(C_{k,\infty}, (\sigma_t))$ is a C*-dynamical system in the sense of [4], i.e. that the flow (σ_t) is strongly continuous, which means that for any $f \in C_{k,\infty}$, the map $t \mapsto \sigma_t(f)$ is continuous.

Lemma 2.2.5. *The flow (σ_t) on $C_{k,\infty}$ is strongly continuous.*

Proof. Let $f \in C_{k,\infty}$. Let us show that the map $t \mapsto \sigma_t(f)$ is continuous. Let $\varepsilon > 0$. It is enough to show that when $|t|$ is small enough, we have $\|f - \sigma_t(f)\| < \varepsilon$. Let $f' \in C_c(\mathcal{G})$ be such that $\|f - f'\| < \varepsilon/3$. Like any $*$-automorphism, σ_t is an isometry, so we have $\|\sigma_t(f) - \sigma_t(f')\| = \|\sigma_t(f - f')\| = \|f - f'\| < \varepsilon/3$, so it is enough to show that when $|t|$ is small enough, we have $\|f' - \sigma_t(f')\| < \varepsilon/3$. For any $\mathfrak{d} \in \mathfrak{F}_{\mathcal{O}}$, define a function $f'_{\mathfrak{d}} \in C_c(\mathcal{G})$ by

$$f'_{\mathfrak{d}}(\chi, \mathfrak{c}) = \begin{cases} f'(\chi, \mathfrak{c}) & \text{if } \mathfrak{c} = \mathfrak{d}, \\ 0 & \text{if } \mathfrak{c} \neq \mathfrak{d}, \end{cases} \quad \text{for all } (\chi, \mathfrak{c}) \in \mathcal{G}.$$

Note that, as f' has compact support, the set $\{\mathfrak{d} \in \mathfrak{F}_{\mathcal{O}} \mid f'_{\mathfrak{d}} \neq 0\}$ is finite, and we have $f' = \sum_{\mathfrak{d}} f'_{\mathfrak{d}}$. For any $\mathfrak{d}$ we have $\sigma_t(f'_{\mathfrak{d}}) = \mathbf{N}\mathfrak{d}^{it} f'_{\mathfrak{d}}$, so

$$\|f' - \sigma_t(f')\| \leqslant \sum_{\mathfrak{d}} \|f'_{\mathfrak{d}} - \sigma_t(f'_{\mathfrak{d}})\| \leqslant \sum_{\mathfrak{d}} |1 - \mathbf{N}\mathfrak{d}^{it}| \|f'_{\mathfrak{d}}\|.$$

It is now obvious that when $|t|$ is small enough, this is smaller than $\varepsilon/3$. □

The resulting C*-dynamical system $(C_{k,\infty}, (\sigma_t))$ is the one that was announced in the introduction of this paper.

3. Algebraic structure of $(C_{k,\infty}, (\sigma_t))$

3.1. The $*$-subalgebra $\mathcal{H}$. In this subsection, we construct a $*$-subalgebra $\mathcal{H}$ which will play the rôle of the algebra $\mathcal{H}$ in the Bost–Connes construction.

For any $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, let $\mu_{\mathfrak{a}} \in C_c(\mathcal{G})$ be defined by

$$\mu_{\mathfrak{a}}(\chi, \mathfrak{c}) = 1_{\mathfrak{c}=\mathfrak{a}} \quad \text{for all } (\chi, \mathfrak{c}) \in \mathcal{G}.$$

For any $\phi \in H(\mathrm{sgn})$ and for any $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$, let us define a function $e(\phi, \lambda) \in C_c(\mathcal{G})$ by

$$e(\phi, \lambda)(\chi, \mathfrak{c}) = 1_{\mathfrak{c}=1}\, 1_{\chi \in X_\phi}\, \chi(\lambda) \quad \text{for all } (\chi, \mathfrak{c}) \in \mathcal{G}.$$

Definition 3.1.1. Let $\mathcal{H}$ denote the $*$-subalgebra of $C_c(\mathcal{G})$ generated by the $\mu_{\mathfrak{a}}$, for all $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, and the $e(\phi, \lambda)$, for all $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$ and for all $\phi \in H(\mathrm{sgn})$.

We shall later show (Proposition 3.3.5) that $\mathcal{H}$ is dense in $C_{k,\infty}$. For now we concentrate on checking several algebraic relations between the generators $\mu_{\mathfrak{a}}$ and $e(\phi, \lambda)$ (see Proposition 3.1.2). We shall later see (Proposition 3.2.3) that the relations of Proposition 3.1.2 define a presentation of $\mathcal{H}$.

Recall that the inverse map in $\mathcal{G}$ is given by

$$(\chi, \mathfrak{c})^{-1} = (\chi^{\mathfrak{c}}, \mathfrak{c}^{-1}). \tag{25}$$

The product law in $C_c(\mathcal{G})$, defined by equation (23), can be rewritten as

$$(fg)(\chi, \mathfrak{c}) = \sum_{\mathfrak{c}_2 \in \mathfrak{F}_\chi} f(\chi^{\mathfrak{c}_2}, \mathfrak{c}\mathfrak{c}_2^{-1})\, g(\chi, \mathfrak{c}_2) \quad \text{for all } f, g \in C_c(\mathcal{G}),\ (\chi, \mathfrak{c}) \in \mathcal{G}. \tag{26}$$

From equation (25), we check that for any $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, the adjoint $\mu_{\mathfrak{a}}^*$ is given by

$$\mu_{\mathfrak{a}}^*(\chi, \mathfrak{c}) = 1_{\mathfrak{c}=\mathfrak{a}^{-1}} \quad \text{for all } (\chi, \mathfrak{c}) \in \mathcal{G}.$$

Using formula (26), we then check that, for all $f \in C_c(\mathcal{G})$ and all $(\chi, \mathfrak{c}) \in \mathcal{G}$, we have

$$(\mu_{\mathfrak{a}} f)(\chi, \mathfrak{c}) = 1_{\mathfrak{c}\mathfrak{a}^{-1} \in \mathfrak{F}_\chi} f(\chi, \mathfrak{c}\mathfrak{a}^{-1}), \tag{27}$$

$$(f\mu_{\mathfrak{a}})(\chi, \mathfrak{c}) = f(\chi^{\mathfrak{a}}, \mathfrak{c}\mathfrak{a}^{-1}), \tag{28}$$

$$(\mu_{\mathfrak{a}}^* f)(\chi, \mathfrak{c}) = f(\chi, \mathfrak{c}\mathfrak{a}), \tag{29}$$

$$(f\mu_{\mathfrak{a}}^*)(\chi, \mathfrak{c}) = 1_{\mathfrak{a}^{-1} \in \mathfrak{F}_\chi} f(\chi^{\mathfrak{a}^{-1}}, \mathfrak{c}\mathfrak{a}).$$

From that we deduce that $C_c(\mathcal{G})$ is unital, with unit μ_1 (where, as usual, 1 denotes the principal ideal $(1) = \mathcal{O}$)

$$\mu_1 = 1,$$

and we also deduce the formulas

$$(\mu_{\mathfrak{a}} f \mu_{\mathfrak{b}}^*)(\chi, \mathfrak{c}) = 1_{\mathfrak{c}\mathfrak{a}^{-1} \in \mathfrak{F}_\chi} 1_{\mathfrak{b}^{-1} \in \mathfrak{F}_\chi} f(\chi^{\mathfrak{b}^{-1}}, \mathfrak{c}\mathfrak{a}^{-1}\mathfrak{b}), \tag{30}$$

$$(\mu_{\mathfrak{b}}^* f \mu_{\mathfrak{a}})(\chi, \mathfrak{c}) = f(\chi^{\mathfrak{a}}, \mathfrak{c}\mathfrak{a}^{-1}\mathfrak{b}),$$

$$(\mu_{\mathfrak{a}} \mu_{\mathfrak{b}}^*)(\chi, \mathfrak{c}) = 1_{\mathfrak{b}^{-1} \in \mathfrak{F}_\chi} 1_{\mathfrak{c}=\mathfrak{a}\mathfrak{b}^{-1}}, \tag{31}$$

$$(\mu_{\mathfrak{b}}^* \mu_{\mathfrak{a}})(\chi, \mathfrak{c}) = 1_{\mathfrak{c}=\mathfrak{a}\mathfrak{b}^{-1}}. \tag{32}$$

In particular, for $\mathfrak{b} = \mathfrak{a}$, equation (31) gives

$$(\mu_{\mathfrak{a}} \mu_{\mathfrak{a}}^*)(\chi, \mathfrak{c}) = 1_{\mathfrak{a}^{-1} \in \mathfrak{F}_\chi} 1_{\mathfrak{c}=1}. \tag{33}$$

The next proposition establishes some relations between the generators $\mu_{\mathfrak{a}}$ and $e(\phi, \lambda)$. As we said above, it will later turn out that these relations really define a presentation of $\mathcal{H}$ as a $*$-algebra (Proposition 3.2.3) and also a presentation of $C_{k,\infty}$ as a C*-algebra (Proposition 3.3.6).

Proposition 3.1.2. *The functions $\mu_{\mathfrak{a}}$, for $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, and $e(\phi, \lambda)$, for $\phi \in H(\mathrm{sgn})$ and $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$, satisfy the following relations*:

(a_1) $\mu_{\mathfrak{a}}^* \mu_{\mathfrak{a}} = \mu_1$ *for all* $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$.

(a_2) $\sum_\phi e(\phi, 0) = \mu_1$ *where* ϕ *runs over* $H(\mathrm{sgn})$.

(b) $\mu_{\mathfrak{a}} \mu_{\mathfrak{b}} = \mu_{\mathfrak{a}\mathfrak{b}}$ *for all* $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$.

(c) $\mu_{\mathfrak{a}} \mu_{\mathfrak{b}}^* = \mu_{\mathfrak{b}}^* \mu_{\mathfrak{a}}$ *for all* $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$ *relatively prime.*

(d_1) $e(\phi, \lambda)^* = e(\phi, -\lambda)$ *for all* $\phi \in H(\mathrm{sgn})$, $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$.

(d_2) $e(\phi, \lambda_1) e(\phi, \lambda_2) = e(\phi, \lambda_1 + \lambda_2)$ *for all* $\phi \in H(\mathrm{sgn})$, $\lambda_1, \lambda_2 \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$.

(d_3) $e(\phi^1, \lambda_1) e(\phi^2, \lambda_2) = 0$ *for all* $\phi^1 \neq \phi^2 \in H(\mathrm{sgn})$, $\lambda_i \in \phi^i(\mathbf{C}_\infty)^{\mathrm{tor}}$.

(e) $e(\phi, \lambda) \mu_{\mathfrak{a}} = \mu_{\mathfrak{a}} e(\mathfrak{a} * \phi, \phi_{\mathfrak{a}}(\lambda))$ *for all* $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, $\phi \in H(\mathrm{sgn})$, $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$.

(f) $\mu_{\mathfrak{a}} e(\phi, \lambda) \mu_{\mathfrak{a}}^* = \frac{1}{\mathrm{N}\mathfrak{a}} \sum_{(\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}}(\mu) = \lambda} e(\mathfrak{a}^{-1} * \phi, \mu)$ *for all* $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, $\phi \in H(\mathrm{sgn})$, $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$.

Proof. (a_1): Equation (32) applied with $\mathfrak{b} = \mathfrak{a}$ gives

$$(\mu_{\mathfrak{a}}^* \mu_{\mathfrak{a}})(\chi, \mathfrak{c}) = 1_{c=1} = \mu_1(\chi, \mathfrak{c}).$$

(a_2): One checks directly that $\sum_\phi e(\phi, 0) = \mu_1$.

(b): Equation (27) applied with $f = \mu_{\mathfrak{b}}$ gives

$$(\mu_{\mathfrak{a}} \mu_{\mathfrak{b}})(\chi, \mathfrak{c}) = 1_{\mathfrak{c}\mathfrak{a}^{-1} \in \mathfrak{F}_\chi} 1_{\mathfrak{b} = \mathfrak{c}\mathfrak{a}^{-1}} = 1_{\mathfrak{b} \in \mathfrak{F}_\chi} 1_{\mathfrak{b} = \mathfrak{c}\mathfrak{a}^{-1}}.$$

As $\mathfrak{b}$ is in $\mathfrak{I}_{\mathcal{O}}$, we always have $\mathfrak{b} \in \mathfrak{F}_\chi$, so we find

$$(\mu_{\mathfrak{a}} \mu_{\mathfrak{b}})(\chi, \mathfrak{c}) = 1_{\mathfrak{b} = \mathfrak{c}\mathfrak{a}^{-1}} = 1_{\mathfrak{a}\mathfrak{b} = \mathfrak{c}}.$$

Thus, $\mu_{\mathfrak{a}} \mu_{\mathfrak{b}} = \mu_{\mathfrak{a}\mathfrak{b}}$.

(c): By equations (31), (32) it is enough to show that for all $(\chi, \mathfrak{c}) \in \mathcal{G}$, we have

$$1_{\mathfrak{b}^{-1} \in \mathfrak{F}_\chi} 1_{\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}} = 1_{\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}}.$$

If $\mathfrak{c} \neq \mathfrak{a}\mathfrak{b}^{-1}$, then both sides are zero, so the equality holds. If $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$, then we have $\mathfrak{a}\mathfrak{b}^{-1} \in \mathfrak{F}_\chi$. As $\mathfrak{a}$ and $\mathfrak{b}$ are relatively prime, Lemma 2.1.7 then shows that $\mathfrak{b}^{-1} \in \mathfrak{F}_\chi$, so the equality holds.

(d_1): For all $\chi \in X$, as χ is a character we have

$$\chi(-\lambda) = \overline{\chi(\lambda)} \quad \text{for all } \lambda \in \psi(\mathbf{C}_\infty)^{\mathrm{tor}}.$$

Relation (d_1) follows.

(d_2) and (d_3): From equation (26) and the formula $\chi(\lambda_1 + \lambda_2) = \chi(\lambda_1)\chi(\lambda_2)$, one checks directly that for all $(\chi, \mathfrak{c}) \in \mathcal{G}$, letting ψ be such that $\chi \in X_\psi$, we have

$$(e(\phi^1, \lambda_1) e(\phi^2, \lambda_2))(\chi, \mathfrak{c}) = 1_{\mathfrak{c}=1} 1_{\phi^1 = \phi^2 = \psi} \chi(\lambda_1 + \lambda_2),$$

which proves (d_2) and (d_3).

(e): By equation (27) and the definition of $e(\phi, \lambda)$, we have, for any $\phi \in H(\mathrm{sgn})$, $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$ and $(\chi, \mathfrak{c}) \in \mathcal{G}$,

$$\begin{aligned}(\mu_{\mathfrak{a}} e(\mathfrak{a} * \phi, \phi_{\mathfrak{a}}(\lambda)))(\chi, \mathfrak{c}) &= 1_{\mathfrak{c}\mathfrak{a}^{-1} \in \mathfrak{F}_\chi} 1_{\mathfrak{c}\mathfrak{a}^{-1} = 1} 1_{\chi \in X_{\mathfrak{a} * \phi}} \chi(\phi_{\mathfrak{a}}(\lambda)) \\ &= 1_{\mathfrak{c}\mathfrak{a}^{-1} = 1} 1_{\chi \in X_{\mathfrak{a} * \phi}} \chi^{\mathfrak{a}}(\lambda).\end{aligned}$$

By equation (28), this is equal to $(e(\phi, \lambda)\mu_{\mathfrak{a}})(\chi, \mathfrak{c})$.

(f): For any $\phi \in H(\mathrm{sgn})$, $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$ and $(\chi, \mathfrak{c}) \in \mathcal{G}$, we have

$$\begin{aligned}(\mu_{\mathfrak{a}} e(\phi, \lambda)\mu_{\mathfrak{a}}^*)(\chi, \mathfrak{c}) &= 1_{\mathfrak{c}\mathfrak{a}^{-1} \in \mathfrak{F}_\chi} 1_{\mathfrak{a}^{-1} \in \mathfrak{F}_\chi} e(\phi, \lambda)(\chi^{\mathfrak{a}^{-1}}, \mathfrak{c}) \quad \text{by equation (30)} \\ &= 1_{\mathfrak{c}\mathfrak{a}^{-1} \in \mathfrak{F}_\chi} 1_{\mathfrak{a}^{-1} \in \mathfrak{F}_\chi} 1_{\mathfrak{c} = 1} 1_{\chi \in X_{\mathfrak{a}^{-1} * \phi}} \chi^{\mathfrak{a}^{-1}}(\lambda) \\ &= 1_{\mathfrak{a}^{-1} \in \mathfrak{F}_\chi} 1_{\mathfrak{c} = 1} 1_{\chi \in X_{\mathfrak{a}^{-1} * \phi}} \chi^{\mathfrak{a}^{-1}}(\lambda) \\ &= 1_{\mathfrak{a}^{-1} \in \mathfrak{F}_\chi} 1_{\mathfrak{c} = 1} 1_{\chi \in X_{\mathfrak{a}^{-1} * \phi}} (\mathbf{N}\mathfrak{a})^{-1} \sum_{(\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}}(\mu) = \lambda} \chi(\mu),\end{aligned}$$

where the last equality follows from Lemma 2.1.6. Let us first suppose that $\mathfrak{a}^{-1} \in \mathfrak{F}_\chi$. We then have

$$\begin{aligned}(\mu_{\mathfrak{a}} e(\phi, \lambda)\mu_{\mathfrak{a}}^*)(\chi, \mathfrak{c}) &= 1_{\mathfrak{c} = 1} 1_{\chi \in X_{\mathfrak{a}^{-1} * \phi}} (\mathbf{N}\mathfrak{a})^{-1} \sum_{(\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}}(\mu) = \lambda} \chi(\mu) \\ &= (\mathbf{N}\mathfrak{a})^{-1} \sum_{(\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}}(\mu) = \lambda} e(\mathfrak{a}^{-1} * \phi, \mu)(\chi, \mathfrak{c}),\end{aligned}$$

so we are done.

Let us now suppose that $\mathfrak{a}^{-1} \notin \mathfrak{F}_\chi$. We then have $1_{\mathfrak{a}^{-1} \in \mathfrak{F}_\chi} = 0$ and hence $(\mu_{\mathfrak{a}} e(\phi, \lambda)\mu_{\mathfrak{a}}^*)(\chi, \mathfrak{c}) = 0$. Thus, it is enough to show that $\sum_{(\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}}(\mu) = \lambda} e(\mathfrak{a}^{-1} * \phi, \mu)(\chi, \mathfrak{c}) = 0$. We have

$$\sum_{(\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}}(\mu) = \lambda} e(\mathfrak{a}^{-1} * \phi, \mu)(\chi, \mathfrak{c}) = 1_{\mathfrak{c} = 1} 1_{\chi \in X_{\mathfrak{a}^{-1} * \phi}} \sum_{(\mathfrak{a}^{-1} * \phi)_{\mathfrak{a}}(\mu) = \lambda} \chi(\mu),$$

so it is enough to show that if $\chi \in X_{\mathfrak{a}^{-1} * \phi}$, then $\sum_{\psi_{\mathfrak{a}}(\mu) = \lambda} \chi(\mu) = 0$, where we have set $\psi = \mathfrak{a}^{-1} * \phi$. Let $\mu_1 \in \psi(\mathbf{C}_\infty)^{\mathrm{tor}}$ such that $\psi_{\mathfrak{a}}(\mu_1) = \lambda$ (see Lemma 1.2.6). We then have

$$\sum_{\psi_{\mathfrak{a}}(\mu) = \lambda} \chi(\mu) = \sum_{\psi_{\mathfrak{a}}(\mu_0) = 0} \chi(\mu_0 + \mu_1),$$

so

$$\sum_{\psi_{\mathfrak{a}}(\mu) = \lambda} \chi(\mu) = \Big(\sum_{\mu_0 \in \psi[\mathfrak{a}]} \chi(\mu_0) \Big) \chi(\mu_1).$$

But by Lemma 2.1.6, since $\mathfrak{a}^{-1} \notin \mathfrak{F}_\chi$, the restriction of χ to $\psi[\mathfrak{a}]$ is a non-trivial character of $\psi[\mathfrak{a}]$, so

$$\sum_{\mu_0 \in \psi[\mathfrak{a}]} \chi(\mu_0) = 0,$$

so $\sum_{\psi_{\mathfrak{a}}(\mu)=\lambda} \chi(\mu) = 0$, which completes the proof. □

3.2. Presentation of $\mathcal{H}$. The goal of this subsection is to show (Proposition 3.2.3) that the relations (a)–(f) of Proposition 3.1.2 define a presentation of $\mathcal{H}$ as a $*$-algebra.

The proof of the next lemma follows that of Proposition 18 in [3].

Lemma 3.2.1. *Let $\widetilde{\mathcal{H}}$ be a $*$-algebra with elements $\tilde{\mu}_{\mathfrak{a}}$, for $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, and $\tilde{e}(\phi, \lambda)$, for $\phi \in H(\mathrm{sgn})$ and $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$, satisfying the relations* (a)–(f) *of Proposition* 3.1.2. *Let S be the following subset of $\widetilde{\mathcal{H}}$*:

$$S = \{\tilde{\mu}_{\mathfrak{a}} \tilde{e}(\phi, \lambda) \tilde{\mu}_{\mathfrak{b}}^* \mid \mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}} \text{ relatively prime},\ \phi \in H(\mathrm{sgn}),\ \lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}\}.$$

Then:

(1) *The elements $\tilde{\mu}_{\mathfrak{a}}$, for $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, and $\tilde{e}(\phi, \lambda)$, for $\phi \in H(\mathrm{sgn})$ and $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$, belong to the linear span of S. More specifically*:

$$\tilde{\mu}_{\mathfrak{a}} = \sum_{\phi \in H(\mathrm{sgn})} \tilde{\mu}_{\mathfrak{a}} \tilde{e}(\phi, 0) \tilde{\mu}_1^* \quad \text{and} \quad \tilde{e}(\phi, \lambda) = \tilde{\mu}_1 \tilde{e}(\phi, \lambda) \tilde{\mu}_1^*.$$

(2) *Let $x_1, x_2 \in S$. For $i = 1, 2$ write $x_i = \mu_{\mathfrak{a}_i} e(\phi^i, \lambda_i) \mu_{\mathfrak{b}_i}^*$. Let $\mathfrak{d} = \mathfrak{a}_2 + \mathfrak{b}_1$ be the gcd of $\mathfrak{a}_2$ and $\mathfrak{b}_1$. Let $\mathfrak{c}$ be the gcd of $\mathfrak{d}^{-1}\mathfrak{a}_1\mathfrak{a}_2$ and $\mathfrak{d}^{-1}\mathfrak{b}_1\mathfrak{b}_2$. Set $\psi = \mathfrak{c}^{-1}\mathfrak{d}^{-1}\mathfrak{a}_2 * \phi^1$ and $\lambda' = \phi^1_{\mathfrak{d}^{-1}\mathfrak{a}_2}(\lambda_1) + \phi^2_{\mathfrak{d}^{-1}\mathfrak{b}_1}(\lambda_2)$. Then*:

$$x_1 x_2 = 1_{\mathfrak{a}_2 * \phi^1 = \mathfrak{b}_1 * \phi^2} \tilde{\mu}_{\mathfrak{d}^{-1}\mathfrak{a}_1\mathfrak{a}_2} \tilde{e}(\mathfrak{d}^{-1}\mathfrak{a}_2 * \phi^1, \lambda') \tilde{\mu}^*_{\mathfrak{d}^{-1}\mathfrak{b}_1\mathfrak{b}_2} \tag{34}$$

$$= 1_{\mathfrak{a}_2 * \phi^1 = \mathfrak{b}_1 * \phi^2} \sum_{\psi_{\mathfrak{c}}(\gamma) = \lambda'} \tilde{\mu}_{\mathfrak{c}^{-1}\mathfrak{d}^{-1}\mathfrak{a}_1\mathfrak{a}_2} \tilde{e}(\psi, \gamma) \tilde{\mu}^*_{\mathfrak{c}^{-1}\mathfrak{d}^{-1}\mathfrak{b}_1\mathfrak{b}_2}. \tag{35}$$

In particular, equation (35) *shows that $x_1 x_2$ belongs to the $\mathbb{C}$-linear span of S.*

(3) *If the elements $\tilde{\mu}_{\mathfrak{a}}$ and $\tilde{e}(\phi, \lambda)$ generate $\widetilde{\mathcal{H}}$ as a $*$-algebra, then the set S generates $\widetilde{\mathcal{H}}$ as a $\mathbb{C}$-vector space.*

Proof. (1) easily follows from relations (a_1), (a_2) of Proposition 3.1.2.

(2): We have

$$x_1 x_2 = \mu_{\mathfrak{a}_1} e(\phi^1, \lambda_1) \mu^*_{\mathfrak{b}_1} \mu_{\mathfrak{a}_2} e(\phi^2, \lambda_2) \mu^*_{\mathfrak{b}_2}.$$

Using relations (a_1), (b) and (c) of Proposition 3.1.2, we find

$$\tilde{\mu}^*_{\mathfrak{b}_1} \tilde{\mu}_{\mathfrak{a}_2} = \tilde{\mu}^*_{\mathfrak{d}^{-1}\mathfrak{b}_1} \tilde{\mu}^*_{\mathfrak{d}} \tilde{\mu}_{\mathfrak{d}} \tilde{\mu}_{\mathfrak{d}^{-1}\mathfrak{a}_2} = \tilde{\mu}^*_{\mathfrak{d}^{-1}\mathfrak{b}_1} \tilde{\mu}_{\mathfrak{d}^{-1}\mathfrak{a}_2}.$$

Hence we get

$$x_1x_2 = \tilde{\mu}_{\mathfrak{a}_1}\tilde{e}(\phi^1,\lambda_1)\tilde{\mu}_{\mathfrak{d}^{-1}\mathfrak{a}_2}\tilde{\mu}^*_{\mathfrak{d}^{-1}\mathfrak{b}_1}\tilde{e}(\phi^2,\lambda_2)\tilde{\mu}^*_{\mathfrak{b}_2}.$$

Using relations (e) and (d_1) of Proposition 3.1.2, we get

$$x_1x_2 = \tilde{\mu}_{\mathfrak{a}_1}\tilde{\mu}_{\mathfrak{d}^{-1}\mathfrak{a}_2}\tilde{e}(\mathfrak{d}^{-1}\mathfrak{a}_2*\phi^1,\phi^1_{\mathfrak{d}^{-1}\mathfrak{a}_2}(\lambda_1))\tilde{e}(\mathfrak{d}^{-1}\mathfrak{b}_1*\phi^2,\phi^2_{\mathfrak{d}^{-1}\mathfrak{b}_1}(\lambda_2))\tilde{\mu}^*_{\mathfrak{d}^{-1}\mathfrak{b}_1}\tilde{\mu}^*_{\mathfrak{b}_2}.$$

Hence relation (b) of Proposition 3.1.2 gives

$$x_1x_2 = \tilde{\mu}_{\mathfrak{d}^{-1}\mathfrak{a}_1\mathfrak{a}_2}\tilde{e}(\mathfrak{d}^{-1}\mathfrak{a}_2*\phi^1,\phi^1_{\mathfrak{d}^{-1}\mathfrak{a}_2}(\lambda_1))\tilde{e}(\mathfrak{d}^{-1}\mathfrak{b}_1*\phi^2,\phi^2_{\mathfrak{d}^{-1}\mathfrak{b}_1}(\lambda_2))\tilde{\mu}^*_{\mathfrak{d}^{-1}\mathfrak{b}_1\mathfrak{b}_2}.$$

Thus, using relations (d_2) and (d_3) of Proposition 3.1.2, we get

$$x_1x_2 = 1_{\mathfrak{a}_2*\phi^1=\mathfrak{b}_1*\phi^2}\,\tilde{\mu}_{\mathfrak{d}^{-1}\mathfrak{a}_1\mathfrak{a}_2}\tilde{e}(\mathfrak{d}^{-1}\mathfrak{a}_2*\phi^1,\phi^1_{\mathfrak{d}^{-1}\mathfrak{a}_2}(\lambda_1)+\phi^2_{\mathfrak{d}^{-1}\mathfrak{b}_1}(\lambda_2))\tilde{\mu}^*_{\mathfrak{d}^{-1}\mathfrak{b}_1\mathfrak{b}_2}.$$

By definition of ψ, λ' and $\mathfrak{c}$, and using relation (b) of Proposition 3.1.2, we obtain

$$x_1x_2 = 1_{\mathfrak{a}_2*\phi^1=\mathfrak{b}_1*\phi^2}\,\tilde{\mu}_{\mathfrak{c}^{-1}\mathfrak{d}^{-1}\mathfrak{a}_1\mathfrak{a}_2}(\tilde{\mu}_{\mathfrak{c}}\tilde{e}(\mathfrak{c}*\psi,\lambda')\tilde{\mu}^*_{\mathfrak{c}})\tilde{\mu}^*_{\mathfrak{c}^{-1}\mathfrak{d}^{-1}\mathfrak{b}_1\mathfrak{b}_2}.$$

Relation (f) of Proposition 3.1.2 then gives the result.

(3): The $\mathbb{C}$-linear span of S contains the generators $\tilde{\mu}_{\mathfrak{a}}$ and $\tilde{e}(\phi,\lambda)$ by (1) and is stable under multiplication by (2). Moreover, it is obviously stable under the involution. Hence it is equal to $\widetilde{\mathcal{H}}$. □

Lemma 3.2.2. *The functions $\mu_{\mathfrak{a}}e(\phi,\lambda)\mu^*_{\mathfrak{b}}$, for $\mathfrak{a},\mathfrak{b}\in\mathfrak{I}_{\mathcal{O}}$ relatively prime, $\phi\in H(\mathrm{sgn})$ and $\lambda\in\phi(\mathbf{C}_\infty)^{\mathrm{tor}}$, form a basis of $\mathcal{H}$ as a $\mathbb{C}$-vector space.*

Proof. By Lemma 3.2.1 (3), they generate $\mathcal{H}$ as a $\mathbb{C}$-vector space. Thus we only have to prove that they are linearly independent. Let us suppose that there exist $\alpha_1,\dots,\alpha_n\in\mathbb{C}$, $\mathfrak{a}_0,\dots,\mathfrak{a}_n,\mathfrak{b}_0,\dots,\mathfrak{b}_n\in\mathfrak{I}_{\mathcal{O}}$ with $\mathfrak{a}_i$ relatively prime to $\mathfrak{b}_i$ for each i, $\phi^1,\dots,\phi^n\in H(\mathrm{sgn})$ and, for each i, $\lambda_i\in\phi(\mathbf{C}_\infty)^{\mathrm{tor}}$, such that

$$\mu_{\mathfrak{a}_0}e(\phi^0,\lambda_0)\mu^*_{\mathfrak{b}_0} = \sum_{i=1}^n \alpha_i\mu_{\mathfrak{a}_i}e(\phi^i,\lambda_i)\mu^*_{\mathfrak{b}_i}.$$

By equation (30) and the definition of $e(\phi^i,\lambda_i)$, we have

$$\begin{aligned}(\mu_{\mathfrak{a}_i}&e(\phi^i,\lambda_i)\mu^*_{\mathfrak{b}_i})(\chi,\mathfrak{c})\\ &= 1_{\mathfrak{c}\mathfrak{a}_i^{-1}\in\mathfrak{F}_\chi}1_{\mathfrak{b}_i^{-1}\in\mathfrak{F}_\chi}1_{\mathfrak{c}\mathfrak{a}_i^{-1}\mathfrak{b}_i=1}1_{\chi\in X_{\mathfrak{b}_i^{-1}*\phi^i}}\,\chi\mathfrak{b}_i^{-1}(\lambda_i)\quad\text{for all }(\chi,\mathfrak{c})\in\mathcal{G}.\end{aligned}$$

Thus, the support of $\mu_{\mathfrak{a}_i}e(\phi^i,\lambda_i)\mu^*_{\mathfrak{b}_i}$ is included in

$$\{g=(\chi,\mathfrak{c})\in\mathcal{G}\mid\chi\in X_{\mathfrak{b}_i^{-1}*\phi^i}\text{ and }\mathfrak{c}=\mathfrak{a}_i\mathfrak{b}_i^{-1}\}.$$

Let I denote the set of all $i \neq 0$ such that $\mathfrak{a}_i \mathfrak{b}_i{}^{-1} = \mathfrak{a}_0 \mathfrak{b}_0{}^{-1}$ and $\mathfrak{b}_i^{-1} * \phi^i = \mathfrak{b}_0^{-1} * \phi^0$. We thus have

$$\mu_{\mathfrak{a}_0} e(\phi^0, \lambda_0) \mu_{\mathfrak{b}_0}^* = \sum_{i \in I} \alpha_i \mu_{\mathfrak{a}_i} e(\phi^i, \lambda_i) \mu_{\mathfrak{b}_i}^*.$$

As $\mathfrak{a}_i$ is relatively prime to $\mathfrak{b}_i$, we see that for all $i \in I$, we have $\mathfrak{a}_i = \mathfrak{a}_0$ and $\mathfrak{b}_i = \mathfrak{b}_0$, so $\phi^i = \phi^0$. Hence we get

$$\mu_{\mathfrak{a}_0} e(\phi^0, \lambda_0) \mu_{\mathfrak{b}_0}^* = \sum_{i \in I} \alpha_i \mu_{\mathfrak{a}_0} e(\phi^i, \lambda_i) \mu_{\mathfrak{b}_0}^*.$$

Hence, multiplying by $\mu_{\mathfrak{a}_0}^*$ on the left and by $\mu_{\mathfrak{b}_0}$ on the right, and using relation (a) of Proposition 3.1.2, we get

$$e(\phi^0, \lambda_0) = \sum_{i \in I} \alpha_i e(\phi^0, \lambda_i).$$

But the $e(\phi^0, \lambda)$, for $\lambda \in \phi^0(\mathbf{C}_\infty)^{\mathrm{tor}}$, are linearly independent (use e.g. the isomorphism $C(X_{\phi^0} \times \{1\}) \simeq C^*(\phi^0(\mathbf{C}_\infty)^{\mathrm{tor}})$ as in Lemma 3.3.2), so this is absurd. □

Proposition 3.2.3. *The relations* (a)–(f) *of Proposition* 3.1.2 *define a presentation of* $\mathcal{H}$ *as a* $*$*-algebra.*

Proof. Let $\widetilde{\mathcal{H}}$ be another $*$-algebra having elements $\tilde{\mu}_{\mathfrak{a}}$, for $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, and $\tilde{e}(\phi, \lambda)$, for $\phi \in H(\mathrm{sgn})$ and $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$, satisfying the relations (a)–(f) of Proposition 3.1.2. We want to show that there exists a unique morphism $\sigma\colon \mathcal{H} \to \widetilde{\mathcal{H}}$ such that $\sigma\mu_{\mathfrak{a}} = \tilde{\mu}_{\mathfrak{a}}$ and $\sigma e(\phi, \lambda) = \tilde{e}(\phi, \lambda)$.

The uniqueness is clear by definition of $\mathcal{H}$. Let us now prove existence. By Lemma 3.2.2, we may define a $\mathbb{C}$-linear map $\sigma\colon \mathcal{H} \to \widetilde{\mathcal{H}}$ by letting

$$\sigma(\mu_{\mathfrak{a}} e(\phi, \lambda) \mu_{\mathfrak{b}}^*) = \tilde{\mu}_{\mathfrak{a}} \tilde{e}(\phi, \lambda) \tilde{\mu}_{\mathfrak{b}}^*$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$ relatively prime, $\phi \in H(\mathrm{sgn})$ and $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$. Clearly, $\sigma(f^*) = \sigma(f)^*$. Moreover, Lemma 3.2.1 shows that $\sigma\mu_{\mathfrak{a}} = \tilde{\mu}_{\mathfrak{a}}$, $\sigma e(\phi, \lambda) = \tilde{e}(\phi, \lambda)$ and

$$\sigma(f_1 f_2) = \sigma(f_1)\sigma(f_2),$$

which completes the proof. □

3.3. Presentation of $C_{k,\infty}$. The goal of this subsection is to show (Proposition 3.3.6) that the relations (a)–(f) of Proposition 3.1.2 define a presentation of $C_{k,\infty}$ as a C*-algebra.

Let $\phi \in H(\mathrm{sgn})$. Let C_ϕ denote the subset of $C_c(\mathcal{G})$ of all functions whose support is a subset of $X_\phi \times \{1\}$.

Lemma 3.3.1. *Let $f_1, f_2 \in C_\phi$. For all $g \in \mathcal{G}$, we have*

$$(f_1 f_2)(g) = f_1(g) f_2(g).$$

Proof. Let $g = (\chi, \mathrm{c}) \in \mathcal{G}$. By equation (26), we have

$$(f_1 f_2)(\chi, \mathrm{c}) = \sum_{\mathrm{c}_2 \in \mathfrak{F}_\chi} f_1(\chi^{\mathrm{c}_2}, \mathrm{c}\mathrm{c}_2^{-1}) f_2(\chi, \mathrm{c}_2).$$

Thus, since $f_1, f_2 \in C_\phi$, we can only have a nonzero term when $\mathrm{c}\mathrm{c}_2^{-1} = 1$ and $\mathrm{c}_2 = 1$. If $\mathrm{c} \neq 1$ then we get $(f_1 f_2)(\chi, \mathrm{c}) = 0$, as expected. If $\mathrm{c} = 1$ we obtain

$$(f_1 f_2)(\chi, 1) = f_1(\chi, 1) f_2(\chi, 1),$$

as expected. □

In particular, we see that for any $f_1, f_2 \in C_\phi$, we have $f_1 f_2 \in C_\phi$. We also have $f_1^* \in C_\phi$. Thus C_ϕ is a $*$-subalgebra of $C_c(\mathcal{G})$.

Let us define a norm $\|\cdot\|_\phi$ on C_ϕ by:

$$\|f\|_\phi = \sup_{g \in \mathcal{G}} |f(g)| \quad \text{for all } f \in C_\phi.$$

Lemma 3.3.2. *C_ϕ is a C*-algebra for the norm $\|\cdot\|_\phi$. We have isomorphisms of C*-algebras*

$$C_\phi \simeq C(X_\phi) \simeq C^*(\phi(\mathbf{C}_\infty)^{\mathrm{tor}}).$$

Proof. The identification $X_\phi \times \{1\} \simeq X_\phi$ gives a bijection $C_\phi \simeq C(X_\phi)$. By Lemma 3.3.1, this is a $*$-isomorphism. By definition of $\|\cdot\|_\phi$, this is an isometry, so $\|\cdot\|_\phi$ is a C*-norm on C_ϕ. The isomorphism $C(X_\phi) \simeq C^*(\phi(\mathbf{C}_\infty)^{\mathrm{tor}})$ is a classical result, see Davidson [14], Proposition VII.1.1. □

Corollary 3.3.3. *C_ϕ is a C*-subalgebra of $C_{k,\infty}$.*

Proof. It is a classical result that any injective $*$-morphism between two C*-algebras is an isometry; see [14], Theorem 1.5.5. Apply this to the inclusion map $\iota\colon C_\phi \to C_{k,\infty}$. □

Lemma 3.3.4. *The $e(\phi, \lambda)$, for $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$, generate a norm-dense $*$-subalgebra of C_ϕ.*

Proof. By definition of the $e(\phi, \lambda)$, the isomorphism $C_\phi \simeq C^*(\phi(\mathbf{C}_\infty)^{\mathrm{tor}})$ given by Lemma 3.3.2 identifies $e(\phi, \lambda)$ with λ. But, by definition of $C^*(\phi(\mathbf{C}_\infty)^{\mathrm{tor}})$, the λ generate a dense $*$-subalgebra of $C^*(\phi(\mathbf{C}_\infty)^{\mathrm{tor}})$, so the result follows. □

Proposition 3.3.5. *$\mathcal{H}$ is dense in $C_{k,\infty}$ and any $*$-representation of $\mathcal{H}$ extends uniquely to a representation of $C_{k,\infty}$.*

Proof. Let us first prove density. Since $C_c(\mathcal{G})$ is dense in $C_{k,\infty}$, it is enough to show that any $f \in C_c(\mathcal{G})$ can be approached by elements of $\mathcal{H}$. Let $f \in C_c(\mathcal{G})$. As f has compact support, there is a finite subset $\{\mathfrak{c}_1, \dots, \mathfrak{c}_n\} \subset \mathfrak{F}_{\mathcal{O}}$ such that for all $(\chi, \mathfrak{c}) \in \mathcal{G}$, if $\mathfrak{c} \notin \{\mathfrak{c}_1, \dots, \mathfrak{c}_n\}$, then $f(\chi, \mathfrak{c}) = 0$. Let f_i be defined by

$$f_i(\chi, \mathfrak{c}) = 1_{\mathfrak{c}=\mathfrak{c}_i}\ f(\chi, \mathfrak{c}) \quad \text{for all } (\chi, \mathfrak{c}) \in \mathcal{G}.$$

We have

$$f = f_1 + \cdots + f_n.$$

It is thus enough to show that each of the f_i can be approached by elements of $\mathcal{H}$. Let $i \in \mathbb{N}$ such that $1 \leqslant i \leqslant n$. Write $\mathfrak{c}_i = \mathfrak{a}_i^{-1}\mathfrak{b}_i$, with $\mathfrak{a}_i, \mathfrak{b}_i \in \mathfrak{I}_{\mathcal{O}}$ relatively prime. Let $f_i' = \mu_{\mathfrak{a}_i} f_i \mu_{\mathfrak{b}_i}^*$. We have $f_i = \mu_{\mathfrak{a}_i}^* f_i' \mu_{\mathfrak{b}_i}$, so it is enough to show that each of the f_i' can be approached by elements of $\mathcal{H}$. By equation (30), we have, for all $(\chi, \mathfrak{c}) \in \mathcal{G}$,

$$f_i'(\chi, \mathfrak{c}) = 1_{\mathfrak{c}\mathfrak{a}_i^{-1} \in \mathfrak{F}_\chi}\ 1_{\mathfrak{b}_i^{-1} \in \mathfrak{F}_\chi}\ f_i(\chi^{\mathfrak{b}_i^{-1}}, \mathfrak{c}\mathfrak{c}_i).$$

Thus, the support of f_i' is a subset of $X \times \{1\}$. For $\phi \in H(\mathrm{sgn})$, let $f_{i,\phi}'$ be defined by

$$f_{i,\phi}'(\chi, \mathfrak{c}) = 1_{\chi \in X_\phi} f_i'(\chi, \mathfrak{c}) \quad \text{for all } (\chi, \mathfrak{c}) \in \mathcal{G}.$$

We have

$$f_i' = \sum_{\phi \in H(\mathrm{sgn})} f_{i,\phi}',$$

so it is enough to show that each of the $f_{i,\phi}'$ can be approached by elements of $\mathcal{H}$. We have $f_{i,\phi}' \in C_\phi$, so the result follows from Lemma 3.3.4.

Now let us prove that any $*$-representation of $\mathcal{H}$ extends uniquely to a representation of $C_{k,\infty}$. Uniqueness follows from the density of $\mathcal{H}$ in $C_{k,\infty}$. Let us show existence. Let π be a $*$-representation of $\mathcal{H}$. By definition of $C_{k,\infty}$, it is enough to show that π extends to a $*$-representation of $C_c(\mathcal{G})$. The construction we just made with the f_i, f_i' and $f_{i,\phi}'$ shows that as a $*$-algebra, $C_c(\mathcal{G})$ is generated by the C_ϕ, for $\phi \in H(\mathrm{sgn})$, and the $\mu_\mathfrak{a}$, for $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$. It is thus enough to show that the restriction of π to the group algebra $\mathbb{C}[\phi(\mathbf{C}_\infty)^{\mathrm{tor}}]$ extends to a representation of C_ϕ. But this follows from Lemma 3.3.2 □

Proposition 3.3.6. *The relations* (a)–(f) *of Proposition* 3.1.2 *define a presentation of $C_{k,\infty}$ as a C*-algebra.*

Proof. Let $\widetilde{C}$ be another C*-algebra having elements $\tilde{\mu}_\mathfrak{a}$, for $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, and $\tilde{e}(\phi, \lambda)$, for $\phi \in H(\mathrm{sgn})$ and $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$, satisfying the relations (a)–(f) of Proposition 3.1.2.

We want to show that there exists a unique morphism $\sigma\colon C_{k,\infty} \to \widetilde{C}$ such that $\sigma\mu_{\mathfrak{a}} = \tilde{\mu}_{\mathfrak{a}}$ and $\sigma e(\phi,\lambda) = \tilde{e}(\phi,\lambda)$.

Uniqueness follows from the density of $\mathcal{H}$ in $C_{k,\infty}$, see Proposition 3.3.5. Let us prove existence.

Let $\widetilde{\mathcal{H}}$ denote the $*$-algebra generated by the $\tilde{\mu}_{\mathfrak{a}}$ and the $\tilde{e}(\phi,\lambda)$. By the universal property of $\mathcal{H}$ (Proposition 3.2.3), there exists a $*$-morphism $\sigma\colon \mathcal{H} \to \widetilde{\mathcal{H}}$ such that $\sigma\mu_{\mathfrak{a}} = \tilde{\mu}_{\mathfrak{a}}$ and $\sigma e(\phi,\lambda) = \tilde{e}(\phi,\lambda)$. Composing it with the inclusion $\widetilde{\mathcal{H}} \to \widetilde{C}$ gives a $*$-representation of $\mathcal{H}$. By Proposition 3.3.5, this representation extends to a $*$-morphism from $C_{k,\infty}$ into $\widetilde{C}$, so we are done. □

The flow (σ_t) has a simple expression for this presentation: one checks directly that

$$\sigma_t(\mu_{\mathfrak{a}}) = \mathbf{N}\mathfrak{a}^{it}\mu_{\mathfrak{a}} \quad \text{for all } t \in \mathbb{R},\ \mathfrak{a} \in \mathfrak{I}_{\mathcal{O}} \tag{36}$$

and

$$\sigma_t e(\phi,\lambda) = e(\phi,\lambda) \quad \text{for all } t \in \mathbb{R},\ \phi \in H(\mathrm{sgn}),\ \lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}. \tag{37}$$

3.4. Galois symmetry of $(C_{k,\infty}, (\sigma_t))$. Recall that an action of $\mathrm{Gal}(K/k)$ on X has been defined by equation (19).

Let $\mathrm{Gal}(K/k)$ act by $*$-automorphisms on $C_c(\mathcal{G})$ by

$$(\sigma f)(\chi,\mathrm{c}) = f(\sigma\chi,\mathrm{c}) \quad \text{for all } \sigma \in \mathrm{Gal}(K/k),\ f \in C_c(\mathcal{G}),\ (\chi,\mathrm{c}) \in \mathcal{G}.$$

Definition 3.4.1. We still denote $(\sigma, f) \mapsto \sigma f$ the unique extension (given by Lemma 2.2.3) of this action to an action of $\mathrm{Gal}(K/k)$ on $C_{k,\infty}$.

One checks directly that the action of $\mathrm{Gal}(K/k)$ on the generators is given by

$$\sigma\mu_{\mathfrak{a}} = \mu_{\mathfrak{a}} \quad \text{for all } \sigma \in \mathrm{Gal}(K/k),\ \mathfrak{a} \in \mathfrak{I}_{\mathcal{O}} \tag{38}$$

and

$$\sigma(e(\phi,\lambda)) = e(\sigma\phi,\sigma\lambda) \ \text{ for all } \sigma \in \mathrm{Gal}(K/k),\ \phi \in H(\mathrm{sgn}),\ \lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}. \tag{39}$$

Proposition 3.4.2. *The group* $\mathrm{Gal}(K/k)$*, endowed with its profinite topology, is a topological symmetry group of* $(C_{k,\infty}, (\sigma_t))$*. In other words, the action of* $\mathrm{Gal}(K/k)$ *on* $C_{k,\infty}$ *is faithful, continuous, and commutes with the flow* (σ_t)*, i.e.,*

$$\sigma(\sigma_t(f)) = \sigma_t(\sigma f) \quad \textit{for all } \sigma \in \mathrm{Gal}(K/k),\ t \in \mathbb{R},\ f \in C_{k,\infty}. \tag{40}$$

Proof. By Lemma 2.2.3, it is enough to check equation (40) for $f \in C_c(\mathcal{G})$, which is easily done by going back to the definitions.

Let us check that the action of $\mathrm{Gal}(K/k)$ on $C_{k,\infty}$ is faithful. Let $\sigma \in \mathrm{Gal}(K/k)$ with $\sigma \neq 1$. Let $\phi \in H(\mathrm{sgn})$. If $\sigma\phi \neq \phi$ then it is clear that σ acts non-trivially on $C_{k,\infty}$. If $\sigma\phi = \phi$ then, by definition of H^+, we have $\sigma \in \mathrm{Gal}(K/H^+)$. By definition of K, the action of $\mathrm{Gal}(K/H^+)$ on $\phi(\mathbf{C}_\infty)^{\mathrm{tor}}$ is faithful. Thus there exists $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$ such that $\sigma\lambda \neq \lambda$, so $e(\phi, \sigma\lambda) \neq e(\phi, \lambda)$. Thus, by equation (39), $\sigma e(\phi, \lambda) \neq e(\phi, \lambda)$, so the action of $\mathrm{Gal}(K/k)$ on $C_{k,\infty}$ is faithful.

Let us check that the action of $\mathrm{Gal}(K/k)$ on $C_{k,\infty}$ is continuous. Let $f \in C_{k,\infty}$ and $\varepsilon > 0$. By Proposition 3.3.5, the subalgebra $\mathcal{H}$ is dense in $C_{k,\infty}$, so there exists $f_0 \in \mathcal{H}$ with $\|f - f_0\| < \varepsilon/3$. Write f_0 in the basis provided by Lemma 3.2.2,

$$f_0 = \sum_{i \in I} c_i \mu_{\mathfrak{a}_i} e(\phi^i, \lambda_i) \mu_{\mathfrak{b}_i}^*,$$

where I is a finite set and where, for all $i \in I$, we have $c_i \in \mathbb{C}$, $\mathfrak{a}_i, \mathfrak{b}_i \in \mathfrak{I}_{\mathcal{O}}$ relatively prime, $\phi^i \in H(\mathrm{sgn})$, and $\lambda_i \in \phi^i(\mathbf{C}_\infty)^{\mathrm{tor}}$. Let K_0 be the extension of k generated by the λ_i and all their conjugates under $\mathrm{Gal}(K/k)$. Thus, K_0/k is a finite Galois subextension of K/k. Let $V = \mathrm{Gal}(K/K_0)$. By definition of the profinite topology, V is a neighborhood of 1 in $\mathrm{Gal}(K/k)$. For all $\sigma \in V$, we have $\sigma f_0 = f_0$. We have $\|\sigma f - f_0\| = \|\sigma(f - f_0)\| = \|f - f_0\| < \varepsilon/3$, so we find $\|\sigma f - f\| < 2\varepsilon/3$. Let W denote the open ball of radius $\varepsilon/3$ centered at f. For all $f' \in W$, we have $\|\sigma f' - \sigma f\| = \|\sigma(f' - f)\| = \|f' - f\| < \varepsilon/3$, whence $\|\sigma f' - f\| < \varepsilon$, which completes the proof of the continuity. □

3.5. The Galois-fixed subalgebra. In this subsection, we introduce two C*-subalgebras of $C_{k,\infty}$, and it will turn out (Lemma 3.5.2) that they are the same one.

The first one, denoted by $C^*(\mathfrak{I}_{\mathcal{O}})$, is the C*-subalgebra of $C_{k,\infty}$ generated by the $\mu_{\mathfrak{a}}$, for all $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$. The second one, denoted by $C_{k,\infty}^{\mathrm{Gal}(K/k)}$, is the subset of $C_{k,\infty}$ of all fixed points under the action of $\mathrm{Gal}(K/k)$. This is a C*-subalgebra of $C_{k,\infty}$.

Let

$$\Phi \colon \mathfrak{I}_{\mathcal{O}} \to \mathbb{N}$$

and

$$\mathbf{M} \colon \mathfrak{I}_{\mathcal{O}} \to \mathbb{Z}$$

denote the Euler totient and Möbius inversion functions respectively, i.e., Φ and $\mathbf{M}$ are the multiplicative functions defined, for all primes $\mathfrak{p}$ and for all $n \geqslant 0$, by

$$\Phi(\mathfrak{p}^n) = \mathbf{N}\mathfrak{p}^n - 1_{n \geqslant 1}\mathbf{N}\mathfrak{p}^{n-1}$$

and

$$\mathbf{M}(\mathfrak{p}^n) = 1_{n=0} - 1_{n=1}.$$

Note that we have, for all $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$,

$$\Phi(\mathfrak{a}) = \sum_{\mathfrak{b} \mid \mathfrak{a}} \mathbf{M}(\mathfrak{b}^{-1}\mathfrak{a})\mathbf{N}\mathfrak{b}.$$

Lemma 3.5.1. *For all* $\phi \in H(\mathrm{sgn})$, *for all* $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, *the* $\mathcal{O}$*-module* $\phi[\mathfrak{a}]$ *has exactly* $\Phi(\mathfrak{a})$ *generators.*

Proof. Let $\mathfrak{a} = \prod_i \mathfrak{p}_i^{n_i}$ be the factorization of $\mathfrak{a}$. Since the $\mathfrak{p}_i$ are relatively prime, we have $\mathfrak{a} = \bigcap_i \mathfrak{p}_i^{n_i}$, so by equations (15) and (16), we have

$$\phi[\mathfrak{a}] = \bigoplus_i \phi[\mathfrak{p}_i^{n_i}],$$

so it is enough to do the proof when $\mathfrak{a}$ is a prime power, which is then easy. □

The proof of the next lemma has been inspired by that of Proposition 21 (b) in [3] and of Proposition 4.1 (3) in [17].

Lemma 3.5.2. *The two subalgebras* $C^*(\mathfrak{I}_{\mathcal{O}})$ *and* $C_{k,\infty}^{\mathrm{Gal}(K/k)}$ *of* $C_{k,\infty}$ *are the same*:

$$C^*(\mathfrak{I}_{\mathcal{O}}) = C_{k,\infty}^{\mathrm{Gal}(K/k)}.$$

Definition 3.5.3. We let C_1 denote this C*-algebra:

$$C_1 = C^*(\mathfrak{I}_{\mathcal{O}}) = C_{k,\infty}^{\mathrm{Gal}(K/k)}.$$

This notation will be justified in Subsection 4.4, where C_1 will be viewed as a spectral subspace of $C_{k,\infty}$ for the action of $\mathrm{Gal}(K/k)$.

Proof. One inclusion is clear: $C_{k,\infty}^{\mathrm{Gal}(K/k)}$ contains $C^*(\mathfrak{I}_{\mathcal{O}})$. Let us check the other inclusion. The Galois group $\mathrm{Gal}(K/k)$ is endowed with its profinite topology, so it is a compact abelian group. Let $d\sigma$ be the normalized Haar measure on it. Let us consider the map $\mathbf{E}$ defined by

$$\begin{aligned} \mathbf{E}\colon C_{k,\infty} &\to C_{k,\infty}^{\mathrm{Gal}(K/k)}, \\ x &\mapsto \int_{\mathrm{Gal}(K/k)} \sigma(x)d\sigma. \end{aligned} \tag{41}$$

By Proposition 3.3.5, $\mathcal{H}$ is dense in $C_{k,\infty}$, so $\mathbf{E}(\mathcal{H})$ is dense in $C_{k,\infty}^{\mathrm{Gal}(K/k)}$. But, by Lemma 3.2.1, $\mathcal{H}$ is the linear span of the $\mu_{\mathfrak{a}} e(\phi,\lambda)\mu_{\mathfrak{b}}^*$, for $\mathfrak{a},\mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$, $\phi \in H(\mathrm{sgn})$, and $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$. Thus $\mathbf{E}(\mathcal{H})$ is the linear span of the $\mu_{\mathfrak{a}}\mathbf{E}(e(\phi,\lambda))\mu_{\mathfrak{b}}^*$. Hence, it is enough to show that for all $\phi \in H(\mathrm{sgn})$ and for all $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$, the element $\mathbf{E}(e(\phi,\lambda))$ belongs to $C^*(\mathfrak{I}_{\mathcal{O}})$.

So let $\phi \in H(\mathrm{sgn})$ and $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$. Let us first assume that $\lambda = 0$. The group $\mathrm{Gal}(H^+/k)$ acts transitively on $H(\mathrm{sgn})$ (see Theorem 1.3.9). By Galois theory, the restriction map $\mathrm{Gal}(K/k) \to \mathrm{Gal}(H^+/k)$ is surjective. Hence $\mathrm{Gal}(K/k)$ acts transitively on $H(\mathrm{sgn})$. Thus, by relation (a_2) in Proposition 3.1.2, we get

$$\mathbf{E}(e(\phi, \lambda)) = 1/h(\mathrm{sgn}), \tag{42}$$

where $h(\mathrm{sgn})$ is the cardinal of $H(\mathrm{sgn})$. So the proof is complete.

Let us now assume that $\lambda \neq 0$. Let

$$\mathfrak{a} = \mathrm{ann}_{\mathcal{O}}(\lambda) = \{a \in \mathcal{O} \mid \phi_a(\lambda) = 0\}.$$

We have $\lambda \in \phi[\mathfrak{a}]$ and, for all $\mathfrak{b} \neq \mathfrak{a}$ such that $\mathfrak{b} | \mathfrak{a}$, $\lambda \notin \phi[\mathfrak{b}]$. So λ is a generator of the $\mathcal{O}$-module $\phi[\mathfrak{a}]$. Let $K_\mathfrak{a}$ denote the extension of H^+ generated by the elements of $\phi[\mathfrak{a}]$. By [20], Theorem 16.2, $\mathrm{Gal}(K_\mathfrak{a}/k)$ acts transitively on the set $\mathfrak{X}_\mathfrak{a}$ defined by

$$\mathfrak{X}_\mathfrak{a} = \{(\psi, \mu) \mid \psi \in H(\mathrm{sgn}),\ \mu \text{ is a generator of } \psi[\mathfrak{a}]\}.$$

By Galois theory, the map $\mathrm{Gal}(K/k) \to \mathrm{Gal}(K_\mathfrak{a}/k)$ is surjective, so $\mathrm{Gal}(K/k)$ also acts transitively on $\mathfrak{X}_\mathfrak{a}$. Thus $\mathbf{E}(e(\phi, \lambda))$ only depends on $\mathfrak{a}$. We therefore note

$$\mathbf{E}(e(\mathfrak{a}^{-1})) = \mathbf{E}(e(\phi, \lambda)).$$

Relation (f) of Proposition 3.1.2 gives, for all $\psi \in H(\mathrm{sgn})$ and $\mathfrak{b} \in \mathfrak{I}_\mathcal{O}$,

$$\mu_\mathfrak{b} e(\psi, 0) \mu_\mathfrak{b}^* = \frac{1}{\mathbf{N}\mathfrak{b}} \sum_{\mu \in (\mathfrak{b}^{-1} * \psi)[\mathfrak{b}]} e(\mathfrak{b}^{-1} * \psi, \mu).$$

Thus equation (a_2) of Proposition 3.1.2 gives

$$\mu_\mathfrak{b} \mu_\mathfrak{b}^* = \frac{1}{\mathbf{N}\mathfrak{b}} \sum_{\psi \in H(\mathrm{sgn})} \sum_{\mu \in (\mathfrak{b}^{-1} * \psi)[\mathfrak{b}]} e(\mathfrak{b}^{-1} * \psi, \mu).$$

Applying $\mathbf{E}$ to this equality and using Lemma 3.5.1, we get

$$\mathbf{N}\mathfrak{b} \mu_\mathfrak{b} \mu_\mathfrak{b}^* = h(\mathrm{sgn}) \sum_{\mathfrak{c} | \mathfrak{b}} \Phi(\mathfrak{c}) \mathbf{E}(e(\mathfrak{c}^{-1})),$$

where $h(\mathrm{sgn})$ is the cardinal of $H(\mathrm{sgn})$. Doing a Möbius inversion, we then find

$$h(\mathrm{sgn})\, \Phi(\mathfrak{b}) \mathbf{E}(e(\mathfrak{b}^{-1})) = \sum_{\mathfrak{c} | \mathfrak{b}} \mathbf{M}(\mathfrak{c}^{-1}\mathfrak{b})\, \mathbf{N}\mathfrak{c}\, \mu_\mathfrak{c} \mu_\mathfrak{c}^*.$$

Thus, for all $\mathfrak{b} \in \mathfrak{I}_\mathcal{O}$, we get the following explicit expression of $\mathbf{E}(e(\mathfrak{b}^{-1}))$ as an element of $C^*(\mathfrak{I}_\mathcal{O})$:

$$\mathbf{E}(e(\mathfrak{b}^{-1})) = \frac{\sum_{\mathfrak{c} | \mathfrak{b}} \mathbf{M}(\mathfrak{c}^{-1}\mathfrak{b})\, \mathbf{N}\mathfrak{c}\, \mu_\mathfrak{c} \mu_\mathfrak{c}^*}{h(\mathrm{sgn}) \Phi(\mathfrak{b})} = \frac{\sum_{\mathfrak{c} | \mathfrak{b}} \mathbf{M}(\mathfrak{c}^{-1}\mathfrak{b})\, \mathbf{N}\mathfrak{c}\, \mu_\mathfrak{c} \mu_\mathfrak{c}^*}{h(\mathrm{sgn}) \sum_{\mathfrak{c} | \mathfrak{b}} \mathbf{M}(\mathfrak{c}^{-1}\mathfrak{b})\, \mathbf{N}\mathfrak{c}}. \tag{43}$$

□

Proposition 3.5.4. *C_1 is isomorphic to the universal C*-algebra generated by elements $\tilde{\mu}_{\mathfrak{a}}$, for $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, subject to the relations* (a_1), (b) *and* (c) *of Proposition* 3.1.2.

Proof. This follows directly from Proposition 3.3.6 and Lemma 3.5.2. □

3.6. Admissible characters. Some ideas in this subsection have been inspired by [17], §5. Our main goal here is to prove Proposition 3.6.9, which will be useful for the classification of extremal KMS_β states at low temperature.

Lemma 3.6.1. *Let $\chi \in X$. Let $\phi \in H(\mathrm{sgn})$ be such that $\chi \in X_\phi$. The following conditions are equivalent*:

(1) *For any maximal ideal $\mathfrak{p} \in \mathfrak{I}_{\mathcal{O}}$, the restriction of χ to $\phi[\mathfrak{p}]$ is non-trivial.*

(2) *For any $\mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$ different from 1, the restriction of χ to $\phi[\mathfrak{b}]$ is non-trivial.*

(3) $\mathfrak{F}_\chi = \mathfrak{I}_{\mathcal{O}}$.

Proof. (2) $\Rightarrow$ (1) is trivial. (1) $\Rightarrow$ (2): Since $\mathfrak{b} \neq 1$ there exists a maximal ideal $\mathfrak{p}$ dividing $\mathfrak{b}$. By equation (14), we then have $\phi[\mathfrak{p}] \subset \phi[\mathfrak{b}]$, so the result follows. (2) $\Rightarrow$ (3): Let $\mathfrak{c} \in \mathfrak{F}_\chi$. Write $\mathfrak{c} = \mathfrak{b}^{-1}\mathfrak{a}$ with $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$ relatively prime. By Lemma 2.1.7, we have $\mathfrak{b}^{-1} \in \mathfrak{F}_\chi$. Thus, by Lemma 2.1.6, the restriction of χ to $\phi[\mathfrak{b}]$ is trivial, so $\mathfrak{b} = 1$, so $\mathfrak{c} \in \mathfrak{I}_{\mathcal{O}}$. (3) $\Rightarrow$ (2): Let $\mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$ with $\mathfrak{b} \neq 1$. We have $\mathfrak{b}^{-1} \notin \mathfrak{F}_\chi$, so the result follows by Lemma 2.1.6. □

Definition 3.6.2. A character $\chi \in X$ is said to be *admissible* if it satisfies the above equivalent conditions. Let X^{adm} denote the topological subspace of X of admissible elements.

Recall that A_f is the ring of finite adèles of k with respect to $\mathcal{O}$. Thus, A_f is the restricted product of the $k_{\mathfrak{p}}$ with respect to the $\mathcal{O}_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over all finite places of k.

The following lemma is well known.

Lemma 3.6.3. *Let $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$. The diagonal map $\iota \colon k \hookrightarrow A_f$ induces an $\mathcal{O}$-module isomorphism*

$$k/\mathfrak{a} \xrightarrow{\sim} \bigoplus_{\mathfrak{p}} k_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}},$$

where $\mathfrak{p}$ runs over all finite places of k, $k_{\mathfrak{p}}$ is the completion of k at $\mathfrak{p}$, and $\mathfrak{a}_{\mathfrak{p}}$ is the closure of $\mathfrak{a}$ in $k_{\mathfrak{p}}$.

Proof. Let $R = \prod_{\mathfrak{p}} \mathfrak{a}_{\mathfrak{p}} \subset A_f$. This contains $\iota(\mathfrak{a})$. Hence ι induces a map

$$k/\mathfrak{a} \to A_f/R.$$

This map is an $\mathcal{O}$-module morphism. It is injective because $\iota^{-1}(R) = \mathfrak{a}$. By the strong approximation theorem (Theorem 1.1.1), the range of ι is dense in A_f. But by definition of the restricted product, R is an open subset of A_f. Hence ι induces a surjection modulo R. Thus ι induces an isomorphism of $\mathcal{O}$-modules $k/\mathfrak{a} \simeq A_f/R$. But $A_f/R = \bigoplus_{\mathfrak{p}} k_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$, so the result follows. □

Lemma 3.6.4. *For any ideal $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$ and for any finite place $\mathfrak{p}$ of k, there exists a character χ of $k_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$ whose restriction to $\mathfrak{p}^{-1}\mathfrak{a}_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$ is non-trivial.*

Proof. Let $\mathbb{F}_{\mathfrak{p}}$ denote the residue field of $\mathcal{O}_{\mathfrak{p}}$. This is a finite extension of $\mathbb{F}_p$. The ring $\mathcal{O}_{\mathfrak{p}}$ is principal (as is any local ring of a Dedekind ring), so its maximal ideal $\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$ is equal to $u\mathcal{O}_{\mathfrak{p}}$ for some $u \in \mathcal{O}_{\mathfrak{p}}$. Now $\mathfrak{a}_{\mathfrak{p}}$ is also an ideal of $\mathcal{O}_{\mathfrak{p}}$, so it is equal to $u^v\mathcal{O}_{\mathfrak{p}}$ for some $v \geqslant 0$. Hence we have $\mathfrak{p}^{-1}\mathfrak{a}_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}} = u^{v-1}\mathcal{O}_{\mathfrak{p}}/u^v\mathcal{O}_{\mathfrak{p}}$. But we have $k_{\mathfrak{p}} = \mathbb{F}_{\mathfrak{p}}((u))$ and $\mathcal{O}_{\mathfrak{p}} = \mathbb{F}_{\mathfrak{p}}[[u]]$, so we can define a character χ on $k_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$ by letting

$$\chi\Big(\sum_{k\in\mathbb{Z}} a_k u^k\Big) = \exp\big(2i\pi \operatorname{Tr}_{\mathbb{F}_p}^{\mathbb{F}_{\mathfrak{p}}}(a_{v-1})/p\big).$$

The restriction of χ to $\mathfrak{p}^{-1}\mathfrak{a}_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$ is non-trivial since we have $\chi(u^{v-1}) = \exp(2i\pi/p)$. □

Lemma 3.6.5. *For any ideal $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, there exists a character χ of $k/\mathfrak{a}$ whose restriction to $\mathfrak{p}^{-1}\mathfrak{a}_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$, for any finite place $\mathfrak{p}$ of k, is non-trivial.*

Proof. Use Lemma 3.6.3 to identify $k/\mathfrak{a}$ with $\bigoplus_{\mathfrak{p}} k_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$. For all $\mathfrak{p}$, let $\chi_{\mathfrak{p}}$ be a character of $k_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$ as given by the preceding lemma. Let $\chi = \prod_{\mathfrak{p}} \chi_{\mathfrak{p}}$. Then χ is a character of $k/\mathfrak{a}$ which has the required property. □

Lemma 3.6.6. *For any $\phi \in H(\mathrm{sgn})$, there exists an admissible character $\chi \in X_\phi$. In particular, X^{adm} is non-empty.*

Proof. Let L denote the lattice corresponding to ϕ. Write $L = \xi\mathfrak{a}$ with $\xi \in \mathbf{C}_\infty^*$ and $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$. Let χ_0 be a character of $k/\mathfrak{a}$ as given by Lemma 3.6.5. Define a character χ of $\phi(\mathbf{C}_\infty)^{\mathrm{tor}}$ by

$$\chi(\lambda) = \chi_0(e_L^{-1}(\lambda)/\xi).$$

Then χ is admissible. □

Lemma 3.6.7. *For any $\chi \in X^{\mathrm{adm}}$, the map $\mathfrak{I}_{\mathcal{O}} \to X$, $\mathfrak{a} \mapsto \chi^{\mathfrak{a}}$, is injective.*

Proof. By definition of admissibility and equation (21), we have $\mathfrak{F}_{\chi^{\mathfrak{a}}} = \mathfrak{a}^{-1}\mathfrak{I}_{\mathcal{O}}$, so the result follows. □

Lemma 3.6.8. *For any $\chi \in X^{\mathrm{adm}}$ and for any $\sigma \in \mathrm{Gal}(K/k)$, we have $\sigma\chi \in X^{\mathrm{adm}}$.*

Proof. The actions of $\mathrm{Gal}(K/k)$ and of $\mathfrak{I}_{\mathcal{O}}$ on X commute with one another. Hence, $\mathfrak{F}_{\sigma\chi} = \mathfrak{F}_{\chi} = \mathfrak{I}_{\mathcal{O}}$. Hence $\sigma\chi$ is admissible. □

Proposition 3.6.9. *For any* $\chi \in X^{\mathrm{adm}}$, *the map* $\mathrm{Gal}(K/k) \to X^{\mathrm{adm}}$, $\sigma \mapsto \sigma\chi$, *is injective.*

Proof. Let $\phi \in H(\mathrm{sgn})$ such that $\chi \in X_{\phi}$. Let $1 \neq \sigma \in \mathrm{Gal}(K/k)$. Suppose that $\sigma\chi = \chi$. We have $\sigma\chi \in X_{\sigma^{-1}\phi}$, so $\sigma^{-1}\phi = \phi$. Thus $\sigma\phi = \phi$, so by definition of H^{+}, we see that $\sigma \in \mathrm{Gal}(K/H^{+})$. Also σ induces a map

$$\sigma \colon \phi(\mathbf{C}_{\infty})^{\mathrm{tor}} \to \phi(\mathbf{C}_{\infty})^{\mathrm{tor}}.$$

For any $\lambda \in \phi(\mathbf{C}_{\infty})^{\mathrm{tor}}$, for any $a \in \mathcal{O}$, we have $\phi_a(\sigma\lambda) = (\sigma\phi_a)(\sigma\lambda) = \sigma(\phi_a(\lambda))$, so σ is an $\mathcal{O}$-module automorphism of $\phi(\mathbf{C}_{\infty})^{\mathrm{tor}}$. Let L denote the lattice corresponding to ϕ. Write $L = \xi\mathfrak{a}$ with $\xi \in \mathbf{C}_{\infty}^{*}$ and $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$. Thus, we have $\mathcal{O}$-module isomorphisms

$$k/\mathfrak{a} \xrightarrow{\xi} kL/L \xrightarrow{e_L} \phi(\mathbf{C}_{\infty})^{\mathrm{tor}}, \tag{44}$$

which we use to identify $k/\mathfrak{a}$ with $\phi(\mathbf{C}_{\infty})^{\mathrm{tor}}$ as $\mathcal{O}$-modules. Thus, σ is seen as an $\mathcal{O}$-module automorphism of $k/\mathfrak{a}$. Use Lemma 3.6.3 to identify $k/\mathfrak{a}$ with $\bigoplus_{\mathfrak{p}} k_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$. For any finite place $\mathfrak{p}$ of k, writing $k_{\mathfrak{p}}$ as a field of Laurent series as in the proof of Lemma 3.6.4, one sees that $k_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}} \simeq k_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}$ as $\mathcal{O}_{\mathfrak{p}}$-modules, hence as $\mathcal{O}$-modules. Hence $\mathrm{End}_{\mathcal{O}}(k_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}) = \mathcal{O}_{\mathfrak{p}}$, acting by multiplication. Thus

$$\mathrm{End}_{\mathcal{O}}(k/\mathfrak{a}) = \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}.$$

View σ as an element of $\mathrm{End}_{\mathcal{O}}(k/\mathfrak{a})$ and write $\sigma = \prod_{\mathfrak{p}} \sigma_{\mathfrak{p}}$ with $\sigma_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}$ for all $\mathfrak{p}$.

By definition of K, the action of $\mathrm{Gal}(K/H^{+})$ on $\phi(\mathbf{C}_{\infty})^{\mathrm{tor}}$ is faithful. Thus, as an $\mathcal{O}$-module automorphism of $\phi(\mathbf{C}_{\infty})^{\mathrm{tor}}$, we have $\sigma \neq 1$. Thus, there exists a $\mathfrak{p}$ such that $\sigma_{\mathfrak{p}} \neq 1$, so $\sigma_{\mathfrak{p}} - 1 \in \mathcal{O}_{\mathfrak{p}} - \{0\}$. Since χ is admissible, there exists $\lambda \in \phi[\mathfrak{p}]$ such that $\chi(\lambda) \neq 1$. View λ as an element of $\mathfrak{p}^{-1}\mathfrak{a}_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$. Let $\tilde{\lambda} \in \mathfrak{p}^{-1}\mathfrak{a}_{\mathfrak{p}} \subset k_{\mathfrak{p}}$ be a representative of λ. Let $\tilde{\mu} = (\sigma_{\mathfrak{p}} - 1)^{-1}\tilde{\lambda} \in k_{\mathfrak{p}}$. Let μ denote the class of $\tilde{\mu}$ in $k_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$. We have $(\sigma_{\mathfrak{p}} - 1)\mu = \lambda$, so $(\sigma - 1)\mu = \lambda$, so

$$\chi((\sigma - 1)\mu) \neq 1$$

and so

$$\chi(\sigma\mu) \neq \chi(\mu),$$

which is absurd since $\sigma\chi = \chi$. □

3.7. Irreducibility of regular representations at admissible characters. The goal of this subsection is to show that the regular representations of $\mathcal{G}$ associated to admissible characters are irreducible. This will be used to classify extremal KMS states at low temperature.

Recall that for any $\chi \in X$ we defined the regular representation π_χ of $C_c(\mathcal{G})$ by equation (24). By definition of $C_{k,\infty}$, π_χ extends uniquely to a representation of $C_{k,\infty}$.

Recall that X^{adm} is the subset of X of admissible elements.

Lemma 3.7.1. *For all $\chi \in X^{\mathrm{adm}}$, the regular representation π_χ of $C_{k,\infty}$ is irreducible.*

Proof. Let $\chi \in X^{\mathrm{adm}}$. Let ϕ be such that $\chi \in X_\phi$. The representation π_χ is a map $C_{k,\infty} \to B\ell^2(\mathcal{G}_\chi)$. Identify $\mathfrak{F}_\chi$ with $\mathcal{G}_\chi$ through the map $\mathfrak{c} \mapsto (\chi, \mathfrak{c})$. As χ is admissible, we have $\mathfrak{F}_\chi = \mathfrak{I}_\mathcal{O}$. Thus $\mathcal{G}_\chi$ is identified with $\mathfrak{I}_\mathcal{O}$. Let $A \in B\ell^2(\mathfrak{I}_\mathcal{O})$ such that

$$\pi_\chi(f)A = A\pi_\chi(f) \quad \text{for all } f \in C_{k,\infty}.$$

Let us show that A is a scalar multiple of the identity. For that let us first prove that A is diagonal. Let $(\varepsilon_\mathfrak{c})_{\mathfrak{c}\in\mathfrak{I}_\mathcal{O}}$ be the standard orthonormal basis of $\ell^2(\mathfrak{I}_\mathcal{O})$: in other words, for all $\mathfrak{c}, \mathfrak{a} \in \mathfrak{I}_\mathcal{O}$, $\varepsilon_\mathfrak{c}(\mathfrak{a}) = 1_{\mathfrak{a}=\mathfrak{c}}$. Let $(a_{\mathfrak{c},\mathfrak{b}})$ be the matrix representing A in this basis. Thus we have

$$A\varepsilon_\mathfrak{b} = \sum_\mathfrak{c} a_{\mathfrak{c},\mathfrak{b}}\varepsilon_\mathfrak{c} \quad \text{for all } \mathfrak{c} \in \mathfrak{I}_\mathcal{O}.$$

Using equation (24), we check that

$$\pi_\chi(\mu_\mathfrak{a})\varepsilon_\mathfrak{b} = \varepsilon_{\mathfrak{a}\mathfrak{b}} \quad \text{for all } \mathfrak{a} \in \mathfrak{I}_\mathcal{O},\ \mathfrak{b} \in \mathfrak{I}_\mathcal{O} \tag{45}$$

and

$$\pi_\chi(e(\psi,\lambda))\varepsilon_\mathfrak{b} = 1_{\psi=\mathfrak{b}^{-1}*\phi}\chi^\mathfrak{b}(\lambda)\varepsilon_\mathfrak{b} \quad \text{for all } \psi \in H(\mathrm{sgn}),\ \lambda \in \psi(\mathbf{C}_\infty)^{\mathrm{tor}},\ \mathfrak{b} \in \mathfrak{I}_\mathcal{O}.$$

Now let $\underline{\lambda} = (\lambda_\psi)_{\psi\in H(\mathrm{sgn})}$ be a family with $\lambda_\psi \in \psi(\mathbf{C}_\infty)^{\mathrm{tor}}$ for all $\psi \in H(\mathrm{sgn})$. Let

$$e(\underline{\lambda}) = \sum_{\psi\in H(\mathrm{sgn})} e(\psi,\lambda_\psi).$$

We have

$$\pi_\chi(e(\underline{\lambda}))\varepsilon_\mathfrak{b} = \chi^\mathfrak{b}(\lambda_{\mathfrak{b}^{-1}*\phi})\varepsilon_\mathfrak{b} \quad \text{for all } \mathfrak{b} \in \mathfrak{I}_\mathcal{O}.$$

Thus, for all $\mathfrak{b} \in \mathfrak{I}_\mathcal{O}$, we get

$$A\pi_\chi(e(\underline{\lambda}))\varepsilon_\mathfrak{b} = \sum_{\mathfrak{a}\in\mathfrak{I}_\mathcal{O}} a_{\mathfrak{a},\mathfrak{b}}\chi^\mathfrak{b}(\lambda_{\mathfrak{b}^{-1}*\phi})\varepsilon_\mathfrak{a},$$

$$\pi_\chi(e(\underline{\lambda}))A\varepsilon_\mathfrak{b} = \sum_{\mathfrak{a}\in\mathfrak{I}_\mathcal{O}} a_{\mathfrak{a},\mathfrak{b}}\chi^\mathfrak{a}(\lambda_{\mathfrak{a}^{-1}*\phi})\varepsilon_\mathfrak{a}.$$

Thus, for all $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$ with $a_{\mathfrak{a},\mathfrak{b}} \neq 0$ and for all $\underline{\lambda}$, we get

$$\chi^{\mathfrak{b}}(\lambda_{\mathfrak{b}^{-1}*\phi}) = \chi^{\mathfrak{a}}(\lambda_{\mathfrak{a}^{-1}*\phi}). \tag{46}$$

If $\mathfrak{b}^{-1} * \phi \neq \mathfrak{a}^{-1} * \phi$, since χ is admissible, we can obviously choose $\underline{\lambda}$ to make equation (46) fail. Thus we have $\mathfrak{b}^{-1} * \phi = \mathfrak{a}^{-1} * \phi$. By letting $\underline{\lambda}$ vary, we see that $\chi^{\mathfrak{b}}$ and $\chi^{\mathfrak{a}}$ are the same character of $(\mathfrak{a}^{-1} * \phi)(\mathbf{C}_\infty)^{\mathrm{tor}}$. Thus $\chi^{\mathfrak{a}} = \chi^{\mathfrak{b}}$. Thus, as χ is admissible, by Lemma 3.6.7, we find $\mathfrak{a} = \mathfrak{b}$. Thus $(a_{\mathfrak{c},\mathfrak{d}})$ is a diagonal matrix. Finally, using the equality $A\pi_\chi(\mu_{\mathfrak{a}}) = \pi_\chi(\mu_{\mathfrak{a}})A$ for all $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, one sees that the diagonal entries $(a_{\mathfrak{c},\mathfrak{c}})$ are all equal, so that $(a_{\mathfrak{c},\mathfrak{d}})$ is a scalar multiple of the identity matrix. Thus π_χ is an irreducible representation. □

3.8. A lemma on the action of Gal(K/k) on $\mathcal{H}$. In this subsection we prove an important lemma which we shall use in Subsections 4.4 and 4.5.

Definition 3.8.1. Let F be a set of finite places of k. An ideal $\mathfrak{c} \in \mathfrak{I}_{\mathcal{O}}$ is said to be *F-localized* if all its prime divisors belong to F.

Definition 3.8.2. Let $\mathfrak{d} \in \mathfrak{I}_{\mathcal{O}}$. Let $F_{\mathfrak{d}}$ be the set of all places of k dividing $\mathfrak{d}$. We define $\mathcal{H}[\mathfrak{d}]$ to be the $*$-algebra generated by the $\mu_{\mathfrak{a}}$, for all $F_{\mathfrak{d}}$-localized ideals $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, and the $e(\phi, \lambda)$, for all $\phi \in H(\mathrm{sgn})$ and $\lambda \in \phi[\mathfrak{d}]$.

Note that for any $\mathfrak{d} \in \mathfrak{I}_{\mathcal{O}}$, $\mathrm{Gal}(K/K_{\mathfrak{d}})$ acts trivially on $\mathcal{H}[\mathfrak{d}]$. Thus the action of $\mathrm{Gal}(K/k)$ on $\mathcal{H}[\mathfrak{d}]$ gives an action of the quotient group $\mathrm{Gal}(K_{\mathfrak{d}}/k) = \mathrm{Gal}(K/k)/\mathrm{Gal}(K/K_{\mathfrak{d}})$ on $\mathcal{H}[\mathfrak{d}]$ (remember that the field $K_{\mathfrak{d}}$ was defined in Definition 1.3.10).

Lemma 3.8.3. *Let $\mathfrak{d} \in \mathfrak{I}_{\mathcal{O}}$. Let $\mathfrak{p}$ be a maximal ideal of $\mathfrak{I}_{\mathcal{O}}$ not dividing $\mathfrak{d}$. Let $\sigma_{\mathfrak{p}} = (\mathfrak{p}, K_{\mathfrak{d}}/k) \in \mathrm{Gal}(K_{\mathfrak{d}}/k)$ be the Artin automorphism of $K_{\mathfrak{d}}$ associated to $\mathfrak{p}$. For all $x \in \mathcal{H}[\mathfrak{d}]$, we have*

$$x\mu_{\mathfrak{p}} = \mu_{\mathfrak{p}}\sigma_{\mathfrak{p}}(x). \tag{47}$$

Proof. Let A denote the subset of $\mathcal{H}[\mathfrak{d}]$ of all elements x such that equation (47) holds. Obviously, A is a $\mathbb{C}$-subalgebra of $\mathcal{H}[\mathfrak{d}]$. But $\mathcal{H}[\mathfrak{d}]$ is generated as a $\mathbb{C}$-algebra by the $\mu_{\mathfrak{a}}$, the $\mu_{\mathfrak{a}}^*$ and the $e(\phi, \lambda)$, for all $F_{\mathfrak{d}}$-localized ideals $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, all $\phi \in H(\mathrm{sgn})$ and all $\lambda \in \phi[\mathfrak{d}]$. Indeed, by relation ($\mathrm{d}_1$) of Proposition 3.1.2, we have $e(\phi, \lambda)^* = e(\phi, -\lambda)$. Hence in order to prove that $A = \mathcal{H}[\mathfrak{d}]$, it is enough to check that $\mu_{\mathfrak{a}} \in A$, $\mu_{\mathfrak{a}}^* \in A$ and $e(\phi, \lambda) \in A$ for any $F_{\mathfrak{d}}$-localized ideal $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$, any $\phi \in H(\mathrm{sgn})$ and any $\lambda \in \phi[\mathfrak{d}]$.

Let $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$ be a $F_{\mathfrak{d}}$-localized ideal. By relation (b) of Proposition 3.1.2, we have $\mu_{\mathfrak{a}}\mu_{\mathfrak{p}} = \mu_{\mathfrak{p}}\mu_{\mathfrak{a}} = \mu_{\mathfrak{p}}\sigma_{\mathfrak{p}}(\mu_{\mathfrak{a}})$, so $\mu_{\mathfrak{a}} \in A$. As $\mathfrak{a}$ is $F_{\mathfrak{d}}$-localized and $\mathfrak{p}$ does not divide $\mathfrak{d}$, relation (c) of Proposition 3.1.2 gives $\mu_{\mathfrak{a}}^*\mu_{\mathfrak{p}} = \mu_{\mathfrak{p}}\mu_{\mathfrak{a}}^* = \mu_{\mathfrak{p}}\sigma_{\mathfrak{p}}(\mu_{\mathfrak{a}}^*)$, so $\mu_{\mathfrak{a}}^* \in A$.

Now let $\phi \in H(\mathrm{sgn})$ and $\lambda \in \phi[\mathfrak{d}]$. We have

$$\begin{aligned} e(\phi,\lambda)\mu_{\mathfrak{p}} &= \mu_{\mathfrak{p}} e(\mathfrak{p} * \phi, \phi_{\mathfrak{p}}(\lambda)) && \text{by relation (e) of Proposition 3.1.2} \\ &= \mu_{\mathfrak{p}} e(\sigma_{\mathfrak{p}}\phi, \phi_{\mathfrak{p}}(\lambda)) && \text{by Theorem 1.3.9} \\ &= \mu_{\mathfrak{p}} e(\sigma_{\mathfrak{p}}\phi, \sigma_{\mathfrak{p}}(\lambda)) && \text{by Theorem 1.3.11, as } \mathfrak{p} \nmid \mathfrak{d} \\ &= \mu_{\mathfrak{p}}\, \sigma_{\mathfrak{p}}(e(\phi,\lambda)). \end{aligned}$$

Thus $e(\phi,\lambda) \in A$, which completes the proof. □

4. KMS$_\beta$ equilibrium states of $(C_{k,\infty}, (\sigma_t))$

4.1. The Galois-invariant KMS$_\beta$ state at any temperature. The goal of this subsection is to construct (Proposition 4.1.2), for any $\beta \in \mathbb{R}_+^*$, a Galois-invariant KMS$_\beta$ state φ_β of $(C_{k,\infty}, (\sigma_t))$. We shall also show (Proposition 4.1.3) that φ_β is the only Galois-invariant KMS$_\beta$ state of $(C_{k,\infty}, (\sigma_t))$.

Proposition 3.5.4 shows that C_1 is isomorphic to the infinite tensor product

$$C_1 = \bigotimes_{\mathfrak{p}} \tau_{\mathfrak{p}},$$

where $\mathfrak{p}$ runs over the finite places of k and where, for each $\mathfrak{p}$, $\tau_{\mathfrak{p}}$ is the (Toeplitz) C*-algebra generated by $\mu_{\mathfrak{p}}$. Note that the $\tau_{\mathfrak{p}}$ are nuclear.

Let $\beta \in \mathbb{R}_+^*$. For each $\mathfrak{p}$, define a state $\varphi_{\beta,\mathfrak{p}}$ on $\tau_{\mathfrak{p}}$ by

$$\varphi_{\beta,\mathfrak{p}}(\mu_{\mathfrak{p}}^n \mu_{\mathfrak{p}}^{*m}) = 1_{n=m} \mathbf{N}\mathfrak{p}^{-n\beta} \quad \text{for all } n, m \geqslant 0.$$

Define a state φ_β on C_1 by

$$\varphi_\beta = \bigotimes_{\mathfrak{p}} \varphi_{\beta,\mathfrak{p}}.$$

Note that we have

$$\varphi_\beta(\mu_{\mathfrak{a}}\mu_{\mathfrak{b}}^*) = 1_{\mathfrak{a}=\mathfrak{b}} \mathbf{N}\mathfrak{a}^{-\beta} \quad \text{for all } \mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}. \tag{48}$$

Recall that the map $\mathbf{E}\colon C_{k,\infty} \to C_1$ was defined in equation (41).

Definition 4.1.1. We extend φ_β to a state on $C_{k,\infty}$ by letting

$$\varphi_\beta(f) = \varphi_\beta(\mathbf{E}(f)) \quad \text{for all } f \in C_{k,\infty}.$$

Proposition 4.1.2. *For any $\beta \in \mathbb{R}_+^*$, the state φ_β on $C_{k,\infty}$ is a KMS$_\beta$ state of $(C_{k,\infty}, (\sigma_t))$. In particular, the state φ_β on C_1 is a KMS$_\beta$ state of $(C_1, (\sigma_t))$.*

Proof. For any $f_1, f_2 \in C_{k,\infty}$, we look for a bounded holomorphic function F_{β,f_1,f_2} on the strip $0 < \operatorname{Im} z < \beta$ realizing the KMS_β property for the state φ_β and the pair (f_1, f_2).

Since $\mathcal{H}$ is a dense (σ_t)-invariant $*$-subalgebra of $C_{k,\infty}$, by [4], §5.3.1, it is enough to do that for $f_1, f_2 \in \mathcal{H}$. In Lemma 3.2.2, we found a basis of $\mathcal{H}$ as a $\mathbb{C}$-vector space. Obviously, it is enough to check the KMS_β condition in the case when f_1 and f_2 are elements of that basis. Thus, write $f_1 = \mu_{\mathfrak{a}_1} e(\psi^1, \lambda_1)\mu^*_{\mathfrak{b}_1}$ and $f_2 = \mu_{\mathfrak{a}_2} e(\psi^2, \lambda_2)\mu^*_{\mathfrak{b}_2}$ with $\mathfrak{a}_i, \mathfrak{b}_i \in \mathfrak{I}_\mathcal{O}$ relatively prime, with $\psi^i \in H(\mathrm{sgn})$ and with $\lambda_i \in \psi^i(\mathbf{C}_\infty)^{\mathrm{tor}}$. By Lemma 3.2.1 (2),

$$f_1 f_2 = 1_{\mathfrak{a}_2 * \psi^1 = \mathfrak{b}_1 * \psi^2}\, \mu_{\mathfrak{d}^{-1}\mathfrak{a}_1\mathfrak{a}_2}\, e(\mathfrak{d}^{-1}\mathfrak{a}_2 * \psi^1, \lambda')\, \mu^*_{\mathfrak{d}^{-1}\mathfrak{b}_1\mathfrak{b}_2}$$

where $\mathfrak{d}$ is the gcd of $\mathfrak{a}_2$ and $\mathfrak{b}_1$ and $\lambda' = \psi^1_{\mathfrak{d}^{-1}\mathfrak{a}_2}(\lambda_1) + \psi^2_{\mathfrak{d}^{-1}\mathfrak{b}_1}(\lambda_2)$. We thus have

$$\mathbf{E}(f_1 f_2) = 1_{\mathfrak{a}_2 * \psi^1 = \mathfrak{b}_1 * \psi^2}\, \mu_{\mathfrak{d}^{-1}\mathfrak{a}_1\mathfrak{a}_2}\, \mathbf{E}(e(\mathfrak{d}^{-1}\mathfrak{a}_2 * \psi^1, \lambda'))\, \mu^*_{\mathfrak{d}^{-1}\mathfrak{b}_1\mathfrak{b}_2}.$$

Let $\mathfrak{c} = \mathrm{ann}_\mathcal{O}(\lambda')$. Using equation (43), we deduce

$$\mathbf{E}(f_1 f_2) = 1_{\mathfrak{a}_2 * \psi^1 = \mathfrak{b}_1 * \psi^2}\, \mu_{\mathfrak{d}^{-1}\mathfrak{a}_1\mathfrak{a}_2}\, \frac{\sum_{\mathfrak{f}|\mathfrak{c}} \mathbf{M}(\mathfrak{f}^{-1}\mathfrak{c})\, \mathbf{N}\mathfrak{f}\, \mu_\mathfrak{f}\mu^*_\mathfrak{f}}{h(\mathrm{sgn})\Phi(\mathfrak{c})}\, \mu^*_{\mathfrak{d}^{-1}\mathfrak{b}_1\mathfrak{b}_2},$$

where $h(\mathrm{sgn})$ is the cardinal of $H(\mathrm{sgn})$. Using the formula for $\varphi_\beta(\mu_\mathfrak{a}\mu^*_\mathfrak{b})$ given in equation (48), we then get

$$\varphi_\beta(f_1 f_2) = \frac{1_{\mathfrak{a}_2 * \psi^1 = \mathfrak{b}_1 * \psi^2}}{h(\mathrm{sgn})\Phi(\mathfrak{c})} \sum_{\mathfrak{f}|\mathfrak{c}} \mathbf{M}(\mathfrak{f}^{-1}\mathfrak{c})\mathbf{N}\mathfrak{f}\, 1_{\mathfrak{d}^{-1}\mathfrak{a}_1\mathfrak{a}_2\mathfrak{f} = \mathfrak{d}^{-1}\mathfrak{b}_1\mathfrak{b}_2\mathfrak{f}} \mathbf{N}(\mathfrak{d}^{-1}\mathfrak{a}_1\mathfrak{a}_2\mathfrak{f})^{-\beta}.$$

Now the condition $\mathfrak{d}^{-1}\mathfrak{a}_1\mathfrak{a}_2\mathfrak{f} = \mathfrak{d}^{-1}\mathfrak{b}_1\mathfrak{b}_2\mathfrak{f}$ is equivalent to $\mathfrak{a}_1\mathfrak{a}_2 = \mathfrak{b}_1\mathfrak{b}_2$ and, as $\mathfrak{a}_i$ is relatively prime to $\mathfrak{b}_i$, this is equivalent to $\mathfrak{a}_1 = \mathfrak{b}_2$ and $\mathfrak{a}_2 = \mathfrak{b}_1$. We thus get

$$\varphi_\beta(f_1 f_2) = \frac{1_{\mathfrak{a}_1 = \mathfrak{b}_2} 1_{\mathfrak{a}_2 = \mathfrak{b}_1} 1_{\psi^1 = \psi^2}}{h(\mathrm{sgn})\Phi(\mathfrak{c})} \sum_{\mathfrak{f}|\mathfrak{c}} \mathbf{M}(\mathfrak{f}^{-1}\mathfrak{c})\, \mathbf{N}\mathfrak{f}\, \mathbf{N}(\mathfrak{d}^{-1}\mathfrak{a}_1\mathfrak{a}_2\mathfrak{f})^{-\beta}.$$

Now if $\mathfrak{a}_2 = \mathfrak{b}_1$, then $\mathfrak{d} = \mathfrak{a}_2 = \mathfrak{b}_1$, and so $\mathfrak{c} = \mathrm{ann}_\mathcal{O}(\lambda_1 + \lambda_2)$. Summing this up, we have

$$\varphi_\beta(f_1 f_2) = \frac{1_{\mathfrak{a}_1 = \mathfrak{b}_2}\, 1_{\mathfrak{a}_2 = \mathfrak{b}_1}\, 1_{\psi^1 = \psi^2}}{h(\mathrm{sgn})\Phi(\mathfrak{c})} \sum_{\mathfrak{f}|\mathfrak{c}} \mathbf{M}(\mathfrak{f}^{-1}\mathfrak{c})\, \mathbf{N}\mathfrak{f}\, \mathbf{N}(\mathfrak{a}_1\mathfrak{f})^{-\beta} \tag{49}$$

where $\mathfrak{c} = \mathrm{ann}_\mathcal{O}(\lambda_1 + \lambda_2)$. Swapping f_1 with f_2 amounts to swapping 1 with 2 in the indices, so we get

$$\varphi_\beta(f_2 f_1) = \frac{1_{\mathfrak{a}_1 = \mathfrak{b}_2}\, 1_{\mathfrak{a}_2 = \mathfrak{b}_1}\, 1_{\psi^1 = \psi^2}}{h(\mathrm{sgn})\Phi(\mathfrak{c})} \sum_{\mathfrak{f}|\mathfrak{c}} \mathbf{M}(\mathfrak{f}^{-1}\mathfrak{c})\, \mathbf{N}\mathfrak{f}\, \mathbf{N}(\mathfrak{a}_2\mathfrak{f})^{-\beta}$$

where $\mathfrak{c} = \operatorname{ann}_{\mathcal{O}}(\lambda_1 + \lambda_2)$. Thus we find

$$\varphi_\beta(f_2 f_1) = \left(\frac{\mathbf{N}\mathfrak{a}_2}{\mathbf{N}\mathfrak{a}_1}\right)^{-\beta} \varphi_\beta(f_1 f_2).$$

We already know that both sides vanish unless $\mathfrak{a}_1 = \mathfrak{b}_2$, so we get

$$\varphi_\beta(f_2 f_1) = \left(\frac{\mathbf{N}\mathfrak{a}_2}{\mathbf{N}\mathfrak{b}_2}\right)^{-\beta} \varphi_\beta(f_1 f_2).$$

Now we have for all $t \in \mathbb{R}$,

$$\sigma_t(f_2) = \sigma_t(\mu_{\mathfrak{a}_2} e(\psi^2, \lambda_2)\mu_{\mathfrak{b}_2}^*) = \mathbf{N}\mathfrak{a}_2^{it}\mu_{\mathfrak{a}_2} e(\psi^2, \lambda_2)\mathbf{N}\mathfrak{b}_2^{-it}\mu_{\mathfrak{b}_2}^* = \left(\frac{\mathbf{N}\mathfrak{a}_2}{\mathbf{N}\mathfrak{b}_2}\right)^{it} f_2.$$

Thus, letting

$$F_{\beta, f_1, f_2}(z) = \left(\frac{\mathbf{N}\mathfrak{a}_2}{\mathbf{N}\mathfrak{b}_2}\right)^{iz} \varphi_\beta(f_1 f_2)$$

defines a bounded holomorphic function F_{β, f_1, f_2} on the strip, realizing the KMS$_\beta$ property for the state φ_β and the pair (f_1, f_2). □

Proposition 4.1.3. *Let $\beta \in \mathbb{R}_+^*$.*

(1) *The state φ_β on C_1 is the only KMS$_\beta$ state of $(C_1, (\sigma_t))$.*

(2) *The state φ_β on $C_{k,\infty}$ is the only Galois-invariant KMS$_\beta$ state of $(C_{k,\infty}, (\sigma_t))$.*

Proof. Clearly, the two statements are equivalent. Let us prove (1). Let φ be a KMS$_\beta$ state of $(C_1, (\sigma_t))$. Let us show that

$$\varphi = \varphi_\beta.$$

Let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$. We have

$$\varphi(\mu_\mathfrak{a}\mu_\mathfrak{b}^*) = \varphi(\mu_\mathfrak{b}^*\sigma_{i\beta}(\mu_\mathfrak{a})) = \mathbf{N}\mathfrak{a}^{-\beta}\varphi(\mu_\mathfrak{b}^*\mu_\mathfrak{a}). \tag{50}$$

Let us first work in the case when $\mathfrak{a} \neq \mathfrak{b}$. Let us prove that $\varphi(\mu_\mathfrak{a}\mu_\mathfrak{b}^*) = 0$. Since $\varphi(\mu_\mathfrak{a}\mu_\mathfrak{b}^*) = \overline{\varphi(\mu_\mathfrak{b}\mu_\mathfrak{a}^*)}$, we may swap $\mathfrak{a}$ and $\mathfrak{b}$, and therefore we may assume without loss of generality that $\mathfrak{a} \nmid \mathfrak{b}$. Let $\mathfrak{d} = \mathfrak{a} + \mathfrak{b}$ denote the gcd of $\mathfrak{a}$ and $\mathfrak{b}$. We have

$$\mu_\mathfrak{b}^*\mu_\mathfrak{a} = \mu_{\mathfrak{d}^{-1}\mathfrak{b}}^*\mu_\mathfrak{d}^*\mu_\mathfrak{d}\mu_{\mathfrak{d}^{-1}\mathfrak{a}} = \mu_{\mathfrak{d}^{-1}\mathfrak{b}}^*\mu_{\mathfrak{d}^{-1}\mathfrak{a}}. \tag{51}$$

Since $\mathfrak{d}^{-1}\mathfrak{a}$ and $\mathfrak{d}^{-1}\mathfrak{b}$ are relatively prime, we have

$$\mu_{\mathfrak{d}^{-1}\mathfrak{b}}^*\mu_{\mathfrak{d}^{-1}\mathfrak{a}} = \mu_{\mathfrak{d}^{-1}\mathfrak{a}}\mu_{\mathfrak{d}^{-1}\mathfrak{b}}^*. \tag{52}$$

Thus, equation (50) applied to $\mathfrak{d}^{-1}\mathfrak{a}$ and $\mathfrak{d}^{-1}\mathfrak{b}$ gives

$$\varphi(\mu_{\mathfrak{d}^{-1}\mathfrak{a}}\mu^*_{\mathfrak{d}^{-1}\mathfrak{b}}) = \mathbf{N}(\mathfrak{d}^{-1}\mathfrak{a})^{-\beta}\varphi_\beta(\mu_{\mathfrak{d}^{-1}\mathfrak{a}}\mu^*_{\mathfrak{d}^{-1}\mathfrak{b}}). \tag{53}$$

As $\mathfrak{a} \nmid \mathfrak{b}$, we have $\mathfrak{d}^{-1}\mathfrak{a} \neq 1$, so equation (53) gives

$$\varphi(\mu_{\mathfrak{d}^{-1}\mathfrak{a}}\mu^*_{\mathfrak{d}^{-1}\mathfrak{b}}) = 0.$$

Hence equation (52) gives $\varphi(\mu^*_{\mathfrak{d}^{-1}\mathfrak{b}}\mu_{\mathfrak{d}^{-1}\mathfrak{a}}) = 0$, so equation (51) gives $\varphi(\mu^*_{\mathfrak{b}}\mu_{\mathfrak{a}}) = 0$, so equation (50) gives $\varphi(\mu_{\mathfrak{a}}\mu^*_{\mathfrak{b}}) = 0$. Hence we have proven that

$$\mathfrak{a} \neq \mathfrak{b} \implies \varphi(\mu_{\mathfrak{a}}\mu^*_{\mathfrak{b}}) = 0 = \varphi_\beta(\mu_{\mathfrak{a}}\mu^*_{\mathfrak{b}}).$$

In the case when $\mathfrak{a} = \mathfrak{b}$ equation (50) gives

$$\varphi(\mu_{\mathfrak{a}}\mu^*_{\mathfrak{a}}) = \mathbf{N}\mathfrak{a}^{-\beta}\varphi(\mu^*_{\mathfrak{a}}\mu_{\mathfrak{a}}) = \mathbf{N}\mathfrak{a}^{-\beta} = \varphi_\beta(\mu_{\mathfrak{a}}\mu^*_{\mathfrak{a}}).$$

Thus we have proven that

$$\varphi(\mu_{\mathfrak{a}}\mu^*_{\mathfrak{b}}) = \varphi_\beta(\mu_{\mathfrak{a}}\mu^*_{\mathfrak{b}}) \quad \text{for all } \mathfrak{a}, \mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}.$$

As the linear span of the $\mu_{\mathfrak{a}}\mu^*_{\mathfrak{b}}$ is the $*$-algebra generated by the $\mu_{\mathfrak{a}}$, it is dense in C_1 (by Definition 3.5.3), so we get $\varphi = \varphi_\beta$. □

4.2. Action of Gal(K/k) on extremal KMS$_\beta$ states. As usual $\mathrm{Gal}(K/k)$ is endowed with its profinite topology. It acts on the set of states by $(\sigma, \varphi) \mapsto \varphi \circ \sigma$. Obviously the KMS$_\beta$ condition and factoriality are preserved by this action. Hence, the sets K_β and $\mathcal{E}(K_\beta)$ are invariant under the action of $\mathrm{Gal}(K/k)$.

The proof of the next proposition comes from that of Theorem 25 in [3].

Proposition 4.2.1. *For any $\beta \in \mathbb{R}^*_+$, the action of* $\mathrm{Gal}(K/k)$ *on $\mathcal{E}(K_\beta)$ is transitive.*

Proof. The main ingredient is that the Galois-fixed subalgebra has a unique KMS$_\beta$ state (cf. Proposition 4.1.3). As in the proof of Lemma 3.5.2, let $d\sigma$ be the normalized Haar measure on $\mathrm{Gal}(K/k)$, and let $\mathbf{E}$ denote the map defined in equation (41).

Let $\varphi_1, \varphi_2 \in \mathcal{E}(K_\beta)$. Then $\varphi_1 \circ \mathbf{E}$ and $\varphi_2 \circ \mathbf{E}$ are Galois-invariant elements of K_β. Thus, by Proposition 4.1.3, they are equal:

$$\varphi_1 \circ \mathbf{E} = \varphi_2 \circ \mathbf{E}.$$

But we have, for $i = 1, 2$,

$$\varphi_i \circ \mathbf{E} = \int_{\mathrm{Gal}(K/k)} \varphi_i \circ \sigma \, d\sigma. \tag{54}$$

Equation (54) gives two decompositions of the same state as a barycenter of extremal KMS$_\beta$ states, but such a decomposition is unique (cf. [4], II, Theorem 5.3.30), so the orbits of φ_1 and of φ_2 under $\mathrm{Gal}(K/k)$ are the same one. □

Let S denote the space of all states of $C_{k,\infty}$, endowed with the weak* topology. Recall that the weak* topology on S is the one for which a basis of open neighborhoods of a state φ_0 is given by the

$$B(\varphi_0; x_1, \dots, x_n; \varepsilon) = \{\varphi \in S \mid |\varphi(x_i) - \varphi_0(x_i)| < \varepsilon \text{ for all } i\} \tag{55}$$

for all $n \geqslant 1$, $x_1, \dots, x_n \in C_{k,\infty}$ and $\varepsilon > 0$.

Lemma 4.2.2. *The action of* $\mathrm{Gal}(K/k)$ *on* S*, given by* $(\sigma, \varphi) \mapsto \varphi \circ \sigma$*, is continuous.*

Proof. Let $\varphi_0 \in S$, $n \leqslant 1$, let $x_1, \dots, x_n \in C_{k,\infty}$, and let $\varepsilon > 0$. Let $U = B(\varphi_0; x_1, \dots, x_n; \varepsilon)$, as defined in equation (55). Let us find an open set $V \subset \mathrm{Gal}(K/k)$ and an open set $W \subset S$ such that

$$\varphi \circ \sigma \in U \quad \text{for all } \sigma \in V,\ \varphi \in W. \tag{56}$$

Let us take $W = B(\varphi_0 \in S; x_1, \dots, x_n; \varepsilon/2)$. By Proposition 3.4.2, for any i, $1 \leqslant i \leqslant n$, the map

$$\begin{aligned} \mathrm{Gal}(K/k) &\to C_{k,\infty}, \\ \sigma &\mapsto \sigma(x_i) \end{aligned}$$

is continuous, so the finite intersection

$$V = \bigcap_{i=1}^{n} \{\sigma \in \mathrm{Gal}(K/k) \mid \|\sigma(x_i) - x_i\| < \varepsilon/2\}$$

is an open neighborhood of 1 in $\mathrm{Gal}(K/k)$. Hence, all we have to do is to check equation (56). Let $\sigma \in V$ and $\varphi \in W$. Let $1 \leqslant i \leqslant n$. We have $\|\sigma(x_i) - x_i\| < \varepsilon/2$, so, as φ is a state, $|\varphi(\sigma(x_i)) - \varphi(x_i)| < \varepsilon/2$. On the other hand, as $\varphi \in W$, we have $|\varphi(x_i) - \varphi_0(x_i)| < \varepsilon/2$. Thus $|\varphi(\sigma(x_i)) - \varphi_0(x_i)| < \varepsilon$, so $\varphi \circ \sigma \in U$. □

4.3. Extremal KMS$_\beta$ states at low temperature $1/\beta < 1$ and special values. Recall that X^{adm} is the subspace of X of admissible elements, that $\mathcal{E}(K_\beta)$ is endowed with the weak* topology, and that $\mathrm{Gal}(K/k)$ is endowed with its profinite topology. In this subsection, for any $\beta > 1$, we shall construct a homeomorphism $X^{\mathrm{adm}} \to \mathcal{E}(K_\beta)$, $\chi \mapsto \varphi_{\beta,\chi}$, commuting with the actions of $\mathrm{Gal}(K/k)$, and we shall show that both $\mathcal{E}(K_\beta)$ (for $\beta > 1$) and X^{adm} are principal homogeneous spaces under $\mathrm{Gal}(K/k)$. Moreover, we shall compute the values of $\varphi_{\beta,\chi}$ at certain elements of $\mathcal{H}$ and relate them to special values of partial zeta functions of k.

For any $\chi \in X^{\mathrm{adm}}$, as at the beginning of the proof of Lemma 3.7.1, let us make the identification $\mathcal{G}_\chi = \mathfrak{F}_\chi = \mathfrak{I}_{\mathcal{O}}$ so that π_χ is seen as a representation in $\ell^2(\mathfrak{I}_{\mathcal{O}})$. Let $(\varepsilon_{\mathfrak{a}})_{\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}}$ be the standard orthonormal basis of $\ell^2(\mathfrak{I}_{\mathcal{O}})$.

Definition 4.3.1. Let H be the unbounded operator on $\ell^2(\mathfrak{I}_{\mathcal{O}})$ defined by

$$H\varepsilon_{\mathfrak{a}} = (\log \mathbf{N}\mathfrak{a})\varepsilon_{\mathfrak{a}} \quad \text{for all } \mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}.$$

Lemma 4.3.2. *For any $\chi \in X^{\mathrm{adm}}$, for all $t \in \mathbb{R}$ and for all $f \in C_{k,\infty}$, we have*

$$\pi_\chi(\sigma_t(f)) = e^{itH}\pi_\chi(f)e^{-itH}.$$

Proof. By Lemma 2.2.3, it is enough to do the proof in the case when $f \in C_c(\mathcal{G})$. It is then a straightforward computation. □

The function $\beta \mapsto \mathrm{Tr}(e^{-\beta H})$ is trivially computed:

Lemma 4.3.3. *For all $\beta > 1$, we have* $\mathrm{Tr}(e^{-\beta H}) = \zeta_{k,\infty}(\beta)$.

Proof. $\mathrm{Tr}(e^{-\beta H}) = \sum_{\mathfrak{a}\in\mathfrak{I}_{\mathcal{O}}} e^{-\beta \log \mathbf{N}\mathfrak{a}} = \sum_{\mathfrak{a}\in\mathfrak{I}_{\mathcal{O}}} \mathbf{N}\mathfrak{a}^{-\beta} = \zeta_{k,\infty}(\beta)$. □

Definition 4.3.4. For any $\chi \in X^{\mathrm{adm}}$, for any $\beta > 1$, we define a linear functional $\varphi_{\beta,\chi}$ on $C_{k,\infty}$ by

$$\varphi_{\beta,\chi}(f) = \zeta_{k,\infty}(\beta)^{-1}\,\mathrm{Tr}(\pi_\chi(f)e^{-\beta H}).$$

Let $(\varepsilon_{\mathfrak{a}})_{\mathfrak{a}\in\mathfrak{I}_{\mathcal{O}}}$ denote the standard basis of $\ell^2(\mathfrak{I}_{\mathcal{O}})$.

Lemma 4.3.5. *For any $\chi \in X^{\mathrm{adm}}$ and for any $\beta > 1$, $\varphi_{\beta,\chi}$ is a KMS$_\beta$ state of the C^*-dynamical system $(C_{k,\infty}, (\sigma_t))$.*

Proof. By Lemma 4.3.3, we have $\varphi_{\beta,\chi}(1) = 1$. We also have, for any $f \in C_{k,\infty}$,

$$\varphi_{\beta,\chi}(ff^*) = \zeta_{k,\infty}(\beta)^{-1}\,\mathrm{Tr}(\pi_\chi(f^*)e^{-\beta H}\pi_\chi(f)) \geqslant 0,$$

so $\varphi_{\beta,\chi}$ is a state on $C_{k,\infty}$. For any $f, f' \in C_{k,\infty}$, let us define a bounded continuous function $F_{\beta,\chi,f,f'}$ on the strip $\{z \in \mathbb{C} \mid 0 \leqslant \mathrm{Im}\, z \leqslant \beta\}$ by

$$F_{\beta,\chi,f,f'}(z) = \zeta_{k,\infty}(\beta)^{-1}\,\mathrm{Tr}(e^{-\beta H}\pi_\chi(f)e^{izH}\pi_\chi(f')e^{-izH}).$$

One checks that the restriction of $F_{\beta,\chi,f,f'}$ to $\{z \in \mathbb{C} \mid 0 < \mathrm{Im}\, z < \beta\}$ is holomorphic. By Lemma 4.3.2, we have, for all $t \in \mathbb{R}$,

$$F_{\beta,\chi,f,f'}(t) = \varphi_{\beta,\chi}(f\sigma_t(f')) \quad \text{and} \quad F_{\beta,\chi,f,f'}(t+i\beta) = \varphi_{\beta,\chi}(\sigma_t(f')f).$$

So $\varphi_{\beta,\chi}$ is a KMS$_\beta$ state of $(C_{k,\infty}, (\sigma_t))$. □

Lemma 4.3.6. *For any $\chi \in X^{\mathrm{adm}}$, for any $\beta > 1$, for any $\sigma \in \mathrm{Gal}(K/k)$, we have*

$$\varphi_{\beta,\sigma\chi} = \varphi_{\beta,\chi} \circ \sigma.$$

Proof. By definition of $\varphi_{\beta,\sigma\chi}$, it is enough to check that $\pi_{\sigma\chi}(f) = \pi_\chi(\sigma f)$. By Proposition 3.3.5, it is enough to prove it when f is one of the $e(\psi,\lambda)$ or one of the $\mu_{\mathfrak{a}}$. The result then follows from equations (38), (39). □

Lemma 4.3.7. *For any $\chi \in X^{\mathrm{adm}}$ and for any $\beta > 1$, the GNS representation of $\varphi_{\beta,\chi}$ is $(\pi_{\beta,\chi}, \Omega_{\beta,\chi})$, where $\pi_{\beta,\chi}\colon C_{k,\infty} \to B(\ell^2(\mathfrak{I}_{\mathcal{O}}) \otimes \ell^2(\mathfrak{I}_{\mathcal{O}}))$ is given by*

$$\pi_{\beta,\chi}(f)(\xi \otimes \eta) = \pi_\chi(f)\xi \otimes \eta,$$

and the cyclic vector $\Omega_{\beta,\chi} \in \ell^2(\mathfrak{I}_{\mathcal{O}}) \otimes \ell^2(\mathfrak{I}_{\mathcal{O}})$ is given by

$$\Omega_{\beta,\chi} = \zeta_{k,\infty}(\beta)^{-1/2} \sum_{\mathfrak{a}\in\mathfrak{I}_{\mathcal{O}}} \mathbf{N}\mathfrak{a}^{-\beta/2} \varepsilon_{\mathfrak{a}} \otimes \varepsilon_{\mathfrak{a}}.$$

Proof. We obviously have

$$\varphi_{\beta,\chi}(f) = \langle \pi_{\beta,\chi}(f)\Omega_{\beta,\chi}, \Omega_{\beta,\chi}\rangle.$$

Hence, we only have to show that $\Omega_{\beta,\chi}$ is a cyclic vector for $\pi_{\beta,\chi}$. For any maximal ideal $\mathfrak{p}$ of $\mathcal{O}$ and any $n \geqslant 0$, using equation(45), we find

$$\pi_\chi(\mu^*_{\mathfrak{p}^n})\varepsilon_{\mathfrak{a}} = 1_{\mathfrak{p}^n|\mathfrak{a}} \varepsilon_{\mathfrak{p}^{-n}\mathfrak{a}} \quad \text{for all } \mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$$

and hence

$$\pi_\chi(\mu_{\mathfrak{p}^n}\mu^*_{\mathfrak{p}^n})\varepsilon_{\mathfrak{a}} = 1_{\mathfrak{p}^n|\mathfrak{a}} \varepsilon_{\mathfrak{a}} \quad \text{for all } \mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}.$$

Thus, if we let $\nu_{\mathfrak{p}^n} = \mu_{\mathfrak{p}^n}\mu^*_{\mathfrak{p}^n} - \mu_{\mathfrak{p}^{n+1}}\mu^*_{\mathfrak{p}^{n+1}}$ we get

$$\pi_\chi(\nu_{\mathfrak{p}^n})\varepsilon_{\mathfrak{a}} = 1_{\mathfrak{p}^n|\mathfrak{a} \text{ and } \mathfrak{p}^{n+1}\nmid\mathfrak{a}} \varepsilon_{\mathfrak{a}} \quad \text{for all } \mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}.$$

Now let $\mathfrak{b} \in \mathfrak{I}_{\mathcal{O}}$. Let us show that $\varepsilon_{\mathfrak{b}} \otimes \varepsilon_{\mathfrak{b}}$ is in the closure of $\pi_{\beta,\chi}(C_{k,\infty})(\Omega_{\beta,\chi})$. Write $\mathfrak{b} = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}}$ with $n_{\mathfrak{p}} \geqslant 0$. For $T > 0$, let P_T denote the set of all maximal ideals $\mathfrak{p}$ with $\mathbf{N}\mathfrak{p} < T$. The family $(P_T)_T$ is a growing family of finite sets whose union is the set of all maximal ideals of $\mathcal{O}$. For all T, let $\nu_T \in C_{k,\infty}$ be defined by

$$\nu_T = \prod_{\mathfrak{p}\in P_T} \nu_{\mathfrak{p}^{n_{\mathfrak{p}}}}.$$

We have

$$\pi_{\beta,\chi}(\nu_T)(\Omega_{\beta,\chi}) = \zeta_{k,\infty}(\beta)^{-1/2} \sum_{\mathfrak{a}\in Q_T} \mathbf{N}\mathfrak{a}^{-\beta/2} \varepsilon_{\mathfrak{a}} \otimes \varepsilon_{\mathfrak{a}},$$

where Q_T is the set of all $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$ such that for all $\mathfrak{p} \in P_T$, the $\mathfrak{p}$-adic valuations of $\mathfrak{a}$ and $\mathfrak{b}$ are equal. Since the series $\sum_{\mathfrak{a}} \mathbf{N}\mathfrak{a}^{-\beta}$ is convergent, we see that

$$\pi_{\beta,\chi}(\nu_T)(\Omega_{\beta,\chi}) \xrightarrow{T\to+\infty} \zeta_{k,\infty}(\beta)^{-1/2}\mathbf{N}\mathfrak{b}^{-\beta/2}\varepsilon_{\mathfrak{b}} \otimes \varepsilon_{\mathfrak{b}}.$$

Thus, we have shown that $\varepsilon_{\mathfrak{b}} \otimes \varepsilon_{\mathfrak{b}}$ is in the closure of $\pi_{\beta,\chi}(C_{k,\infty})(\Omega_{\beta,\chi})$. Applying the $\pi_{\beta,\chi}(\mu_{\mathfrak{a}})$ and the $\pi_{\beta,\chi}(\mu^*_{\mathfrak{a}})$ to that shows that for all $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathfrak{I}_{\mathcal{O}}$, the element $\varepsilon_{\mathfrak{b}_1} \otimes \varepsilon_{\mathfrak{b}_2}$ is in the closure of $\pi_{\beta,\chi}(C_{k,\infty})(\Omega_{\beta,\chi})$. □

Proposition 4.3.8. *For any $\chi \in X^{\mathrm{adm}}$ and for any $\beta > 1$, the state $\varphi_{\beta,\chi}$ is factorial* (*hence extremal*) *of type* I_∞.

Proof. Let A denote the weak closure of $\pi_{\beta,\chi}(C_{k,\infty})$ in $B(\ell^2(\mathfrak{I}_{\mathcal{O}}) \otimes \ell^2(\mathfrak{I}_{\mathcal{O}}))$. By Lemma 3.7.1, the representation π_χ is irreducible. Thus inside $B\ell^2(\mathfrak{I}_{\mathcal{O}})$, we have $\pi_\chi(C_{k,\infty})' = \mathbb{C}$. Using Takesaki [33], I, Chapter IV, Proposition 1.6 (i), we deduce that inside $B(\ell^2(\mathfrak{I}_{\mathcal{O}}) \otimes \ell^2(\mathfrak{I}_{\mathcal{O}}))$, we have

$$\pi_{\beta,\chi}(C_{k,\infty})' = \mathbb{C} \otimes B\ell^2(\mathfrak{I}_{\mathcal{O}}).$$

Thus, using [33], I, Chapter IV, Proposition 1.6 (ii), we deduce

$$A = \pi_{\beta,\chi}(C_{k,\infty})'' = B\ell^2(\mathfrak{I}_{\mathcal{O}}) \otimes \mathbb{C}.$$

In particular, we have $A \simeq B\ell^2(\mathfrak{I}_{\mathcal{O}})$, so A is a factor of type I_∞. □

Lemma 4.3.9. *For any $\beta > 1$, the map $X^{\mathrm{adm}} \to \mathcal{E}(K_\beta)$, $\chi \mapsto \varphi_{\beta,\chi}$, is injective.*

Proof. We reuse the notations of the proof of the previous lemma. Let us extend $\varphi_{\beta,\chi}$ to a state $\widetilde{\varphi}_{\beta,\chi}$ on the von Neumann algebra $A = B\ell^2(\mathfrak{I}_{\mathcal{O}}) \otimes \mathbb{C}$ by

$$\widetilde{\varphi}_{\beta,\chi}(a \otimes 1) = \langle a(\Omega_{\beta,\chi}), \Omega_{\beta,\chi}\rangle \quad \text{for all } a \in B\ell^2(\mathfrak{I}_{\mathcal{O}}).$$

For any $\hat{\beta} > 0$, we have $e^{-\hat{\beta}H} \in B\ell^2(\mathfrak{I}_{\mathcal{O}})$. We have, for all $\psi \in H(\mathrm{sgn})$ and for all $\lambda \in \psi(\mathbf{C}_\infty)^{\mathrm{tor}}$:

$$\zeta_{k,\infty}(\beta) \lim_{\hat{\beta}\to+\infty} \widetilde{\varphi}_{\beta,\chi}(\pi_\chi(e(\psi,\lambda))e^{-\hat{\beta}H} \otimes 1) = \langle \pi_\chi(e(\psi,\lambda))(\varepsilon_1), \varepsilon_1\rangle = 1_{\chi\in X_\psi}\,\chi(\lambda).$$

Thus, χ is uniquely determined. □

We can now prove the main result classifying extremal KMS_β states at low temperature. Recall that $\mathrm{Gal}(K/k)$ is endowed with its profinite topology, and $\mathcal{E}(K_\beta)$ is endowed with the weak* topology.

Theorem 4.3.10. *For any $\beta > 1$, the topological space $\mathcal{E}(K_\beta)$ is principal homogeneous under* $\mathrm{Gal}(K/k)$.

Proof. We must show that for any $\varphi \in \mathcal{E}(K_\beta)$, the map $\mathrm{Gal}(K/k) \to \mathcal{E}(K_\beta)$, $\sigma \mapsto \varphi \circ \sigma$, is a homeomorphism. We already know that it is surjective (Proposition 4.2.1) and continuous (Lemma 4.2.2). Thus, as $\mathrm{Gal}(K/k)$ is compact, it only remains to show that it is injective. Let $\varphi \in \mathcal{E}(K_\beta)$ and $\sigma \in \mathrm{Gal}(K/k)$ such that $\varphi \circ \sigma = \varphi$. We have to show that $\sigma = 1$. Let $\chi \in X^{\mathrm{adm}}$. By Proposition 4.3.8 we have $\varphi_{\beta,\chi} \in \mathcal{E}(K_\beta)$. By Proposition 4.2.1, there exists $\tau \in \mathrm{Gal}(K/k)$ such that $\varphi = \varphi_{\beta,\chi} \circ \tau$. By Lemma 4.3.6 we have $\varphi = \varphi_{\beta,\tau\chi}$ and $\varphi \circ \sigma = \varphi_{\beta,\sigma\tau\chi}$, so $\varphi_{\beta,\sigma\tau\chi} = \varphi_{\beta,\tau\chi}$. By Lemma 4.3.9, we deduce $\sigma\tau\chi = \tau\chi$. By Proposition 3.6.9 we find $\sigma\tau = \tau$, so $\sigma = 1$. □

Theorem 4.3.11. *For any $\beta > 1$, the map $X^{\mathrm{adm}} \to \mathcal{E}(K_\beta)$, $\chi \mapsto \varphi_{\beta,\chi}$, is a homeomorphism.*

Proof. It is injective by Lemma 4.3.9. Let us check surjectivity. Let $\varphi \in \mathcal{E}(K_\beta)$ and let $\chi_0 \in X^{\mathrm{adm}}$. By Proposition 4.2.1 and Lemma 4.3.6, there exists $\sigma \in \mathrm{Gal}(K/k)$ such that $\varphi = \varphi_{\beta,\sigma\chi_0}$. Thus, the map $\chi \mapsto \varphi_{\beta,\chi}$ is bijective. One checks that it is continuous. By definition of an admissible character, X^{adm} is a closed subspace of X. Thus X^{adm} is compact, so the considered map is a homeomorphism. □

Relations between certain special values of KMS$_\beta$ states and of partial zeta functions. Let us now compute the values of the states $\varphi_{\beta,\chi}$ on some of the generators $e(\phi,\lambda)$. Let A_+ denote the subset of $\mathfrak{I}_\mathcal{O}$ of all ideals $\mathfrak{a}$ such that $\sigma_\mathfrak{a} = 1$, where $\sigma_\mathfrak{a} = (\mathfrak{a}, H^+/k) \in \mathrm{Gal}(H^+/k)$ is the Artin automorphism of H^+ associated to $\mathfrak{a}$. For any $\mathfrak{c} \in \mathfrak{I}_\mathcal{O}$ and any $\sigma \in \mathrm{Gal}(K_\mathfrak{c}/k)$, let $A_{\mathfrak{c},\sigma}$ denote the subset of A_+ of all ideals $\mathfrak{a}$ prime to $\mathfrak{c}$ and such that $\sigma_\mathfrak{a} = \sigma$, where $\sigma_\mathfrak{a} = (\mathfrak{a}, K_\mathfrak{c}/k) \in \mathrm{Gal}(K_\mathfrak{c}/k)$ is the Artin automorphism of $K_\mathfrak{c}$ associated to $\mathfrak{a}$. Note that A_+ and the $A_{\mathfrak{c},\sigma}$ are generalized ideal classes of $\mathcal{O}$.

Let $\zeta^+_{k,\infty}$ and $\zeta^{\mathfrak{c},\sigma}_{k,\infty}$ (for any $\mathfrak{c} \in \mathfrak{I}_\mathcal{O}$ and $\sigma \in \mathrm{Gal}(K_\mathfrak{c}/H^+)$) be the partial zeta functions associated to A_+ and $A_{\mathfrak{c},\sigma}$, respectively:

$$\zeta^+_{k,\infty}(\beta) = \sum_{\mathfrak{a}\in A_+} \mathbf{N}\mathfrak{a}^{-\beta},$$
$$\zeta^{\mathfrak{c},\sigma}_{k,\infty}(\beta) = \sum_{\mathfrak{a}\in A_{\mathfrak{c},\sigma}} \mathbf{N}\mathfrak{a}^{-\beta}.$$

Theorem 4.3.12. *Let $\beta > 1$, $\phi \in H(\mathrm{sgn})$, and $\chi \in X^{\mathrm{adm}} \cap X_\phi$.*

(1) *We have*

$$\varphi_{\beta,\chi}(e(\phi,0)) = \frac{\zeta^+_{k,\infty}(\beta)}{\zeta_{k,\infty}(\beta)}.$$

(2) *For any maximal ideal $\mathfrak{p}$ of $\mathcal{O}$, for any $\lambda \in \phi[\mathfrak{p}]$, we have*

$$\varphi_{\beta,\chi}(e(\phi,\lambda)) = \zeta_{k,\infty}(\beta)^{-1}\Big(\mathbf{N}\mathfrak{p}^{-\beta}\zeta^+_{k,\infty}(\beta) + \sum_{\sigma\in\mathrm{Gal}(K_\mathfrak{p}/H^+)} \chi(\sigma\lambda)\zeta^{\mathfrak{p},\sigma}_{k,\infty}(\beta)\Big).$$

Proof. Let us first prove (1). By definition, A_+ is the subset of $\mathfrak{I}_\mathcal{O}$ of all ideals $\mathfrak{a}$ such that $\sigma_\mathfrak{a} = 1$, where $\sigma_\mathfrak{a} = (\mathfrak{a}, H^+/k) \in \mathrm{Gal}(H^+/k)$ is the Artin automorphism of H^+ associated to $\mathfrak{a}$. Hence, by Theorem 1.3.9, we have

$$A_+ = \{\mathfrak{a} \in \mathfrak{I}_\mathcal{O} \mid \mathfrak{a} * \phi = \phi\} = \{\mathfrak{a} \in \mathfrak{I}_\mathcal{O} \mid \mathfrak{a}^{-1} * \phi = \phi\} = \{\mathfrak{a} \in \mathfrak{I}_\mathcal{O} \mid \chi^\mathfrak{a} \in X_\phi\}.$$

Thus, by definition of $\varphi_{\beta,\chi}$, for any $\lambda \in \phi(\mathbf{C}_\infty)^{\mathrm{tor}}$, we have

$$\begin{aligned}\varphi_{\beta,\chi}(e(\phi,\lambda)) &= \zeta_{k,\infty}(\beta)^{-1} \sum_{\mathfrak{a}\in\mathfrak{I}_{\mathcal{O}}} 1_{\chi^{\mathfrak{a}}\in X_\phi}\, \chi^{\mathfrak{a}}(\lambda)\, \mathbf{N}\mathfrak{a}^{-\beta} \\ &= \zeta_{k,\infty}(\beta)^{-1} \sum_{\mathfrak{a}\in A_+} \chi(\phi_{\mathfrak{a}}(\lambda))\, \mathbf{N}\mathfrak{a}^{-\beta}.\end{aligned} \tag{57}$$

Applying this equality to $\lambda = 0$ we get (1).

Let us now prove (2). Let $\mathfrak{a} \in A_+$. In the case when $\mathfrak{p}\,|\,\mathfrak{a}$, we have $\phi_{\mathfrak{a}}(\lambda) = 0$, so

$$\chi(\phi_{\mathfrak{a}}(\lambda)) = 1.$$

In the case when $\mathfrak{p} \nmid \mathfrak{a}$, by Theorem 1.3.11, we have $\phi_{\mathfrak{a}}(\lambda) = \sigma_{\mathfrak{a}}(\lambda)$. Hence, equation (57) gives

$$\begin{aligned}\zeta_{k,\infty}(\beta)\varphi_{\beta,\chi}(e(\phi,\lambda)) &= \sum_{\mathfrak{a}\in A_+,\,\mathfrak{p}|\mathfrak{a}} \chi(\phi_{\mathfrak{a}}(\lambda))\, \mathbf{N}\mathfrak{a}^{-\beta} + \sum_{\mathfrak{a}\in A_+,\,\mathfrak{p}|\mathfrak{a}} \chi(\phi_{\mathfrak{a}}(\lambda))\, \mathbf{N}\mathfrak{a}^{-\beta} \\ &= \sum_{\mathfrak{a}\in A_+,\,\mathfrak{p}|\mathfrak{a}} \mathbf{N}\mathfrak{a}^{-\beta} + \sum_{\mathfrak{a}\in A_+,\,\mathfrak{p}|\mathfrak{a}} \chi(\sigma_{\mathfrak{a}}(\lambda))\, \mathbf{N}\mathfrak{a}^{-\beta} \\ &= \sum_{\mathfrak{a}\in A_+} \mathbf{N}(\mathfrak{p}\mathfrak{a})^{-\beta} + \sum_{\sigma\in\mathrm{Gal}(K_{\mathfrak{p}}/H^+)} \sum_{\mathfrak{a}\in A_{\mathfrak{p},\sigma}} \chi(\sigma\lambda)\, \mathbf{N}\mathfrak{a}^{-\beta} \\ &= \mathbf{N}\mathfrak{p}^{-\beta}\zeta_{k,\infty}^{+}(\beta) + \sum_{\sigma\in\mathrm{Gal}(K_{\mathfrak{p}}/H^+)} \chi(\sigma\lambda)\, \zeta_{k,\infty}^{\mathfrak{p},\sigma}(\beta),\end{aligned}$$

which proves (2). □

4.4. Uniqueness of the KMS$_\beta$ state at high temperature $1/\beta \geqslant 1$. Recall that in Proposition 4.1.2, for any $\beta \in \mathbb{R}_+^*$, we found a Galois-invariant KMS$_\beta$ state φ_β of $\big(C_{k,\infty}, (\sigma_t)\big)$.

In this subsection we shall prove (Theorem 4.4.15) that when $\beta \leqslant 1$, there is no other KMS$_\beta$ state of $(C_{k,\infty}, (\sigma_t))$. In other words,

$$\beta \leqslant 1 \implies K_\beta = \{\varphi_\beta\}.$$

Most of the ideas here come from [3], §7.

Let $\beta \in \mathbb{R}_+^*$ be such that $\beta \leqslant 1$, and let ψ be a KMS$_\beta$ state of $(C_{k,\infty}, (\sigma_t))$. We must show that $\psi = \varphi_\beta$.

Let $\widehat{\mathrm{Gal}(K/k)}$ be the dual group of $\mathrm{Gal}(K/k)$. Since $\mathrm{Gal}(K/k)$ is profinite, $\widehat{\mathrm{Gal}(K/k)}$ is discrete.

Let F be a non-empty finite set of finite places of k. Recall from Definition 3.8.1 that an ideal $\mathfrak{a} \in \mathfrak{I}_{\mathcal{O}}$ is F-localized if all its prime divisors belong to F. We also need to define what it means to be *F-localized* for an element of $\mathrm{Gal}(K/k)$ and for an element of $C(X)$.

Let us first define what it means to be F-localized for an element of $\mathrm{Gal}(K/k)$. We have

$$K = \lim_{\mathfrak{c}\to} K_{\mathfrak{c}},$$

so

$$\mathrm{Gal}(K/k) = \lim_{\leftarrow \mathfrak{c}} \mathrm{Gal}(K_{\mathfrak{c}}/k),$$

so

$$\widehat{\mathrm{Gal}(K/k)} = \lim_{\mathfrak{c}\to} \widehat{\mathrm{Gal}(K_{\mathfrak{c}}/k)}.$$

This means that for any character ν of $\mathrm{Gal}(K/k)$, there exists $\mathfrak{c} \in \mathfrak{I}_{\mathcal{O}}$ such that ν factors through the projection $\mathrm{Gal}(K/k) \to \mathrm{Gal}(K_{\mathfrak{c}}/k)$.

Definition 4.4.1. A character ν of $\mathrm{Gal}(K/k)$ is said to be *F-localized* if there exists an F-localized ideal $\mathfrak{c} \in \mathfrak{I}_{\mathcal{O}}$ such that ν factors through the projection $\mathrm{Gal}(K/k) \to \mathrm{Gal}(K_{\mathfrak{c}}/k)$.

Thus any $\nu \in \mathrm{Gal}(K/k)$ is F-localized for some F.

Let K_F denote the extension of H^+ generated by the elements of the $\phi[F]$, for $\phi \in H(\mathrm{sgn})$. In other words,

$$K_F = \lim_{\mathfrak{c}\to} K_{\mathfrak{c}},$$

where $\mathfrak{c}$ runs over $\mathfrak{I}_{\mathcal{O}}$. Thus a character ν of $\mathrm{Gal}(K/k)$ is F-localized if and only if it factors through the surjection

$$\mathrm{Gal}(K/k) \to \mathrm{Gal}(K_F/k).$$

Let us now define what it means to be F-localized for an element of $C(X)$. For any $\phi \in H(\mathrm{sgn})$, let $\phi[F]$ denote the following subgroup of $\phi(\mathbf{C}_\infty)^{\mathrm{tor}}$:

$$\phi[F] = \bigcup_{\mathfrak{c} \text{ is } F\text{-loc.}} \phi[\mathfrak{c}].$$

Here $\mathfrak{c}$ runs over all the F-localized ideals in $\mathfrak{I}_{\mathcal{O}}$. Let $X_{\phi,F}$ denote the dual group of $\phi[F]$. The restriction-to-$\phi[F]$ map is a surjective morphism

$$X_\phi \to X_{\phi,F}.$$

Let X_F denote the (disjoint) union of the $X_{\phi,F}$, for all $\phi \in H(\mathrm{sgn})$. The restriction maps $X_\phi \to X_{\phi,F}$ give a surjection

$$X \to X_F.$$

This gives an injective morphism of C*-algebras

$$C(X_F) \hookrightarrow C(X).$$

Thus, we regard $C(X_F)$ as a C*-subalgebra of $C(X)$.

Definition 4.4.2. An element $f \in C(X)$ is said to be F-*localized* if it belongs to $C(X_F)$. In other words, f is F-localized if, seen as a function $f : X \to \mathbb{C}$, it factors through the map $X \to X_F$.

Lemma 4.4.3. *The C^*-algebra $C(X_F)$ is generated by the $e(\phi, \lambda)$, for all $\phi \in H(\mathrm{sgn})$ and all $\lambda \in \phi[F]$.*

Proof. This can be checked like Lemma 3.3.4. □

For any character ν of $\mathrm{Gal}(K/k)$, let C_ν be the following spectral subspace of $C_{k,\infty}$:

$$C_\nu = \{f \in C_{k,\infty} \mid \sigma f = \nu(\sigma) f \text{ for all } \sigma \in \mathrm{Gal}(K/k)\}.$$

Thus, when $\nu = 1$ is the trivial character, the corresponding subspace C_1 is the Galois-fixed subalgebra computed in 3.5.2.

Lemma 4.4.4. *The following subspace is dense in $C_{k,\infty}$:*

$$\bigoplus_{\nu \in \widehat{\mathrm{Gal}(K/k)}} C_\nu .$$

Proof. Since $\mathrm{Gal}(K/k)$ is a compact abelian group of $*$-automorphisms of $C_{k,\infty}$, this follows from a result found in Pedersen [30], §§8.1.4 and 8.1.10, p. 349. □

Lemma 4.4.5. *The states ψ and φ_β agree on C_1.*

Proof. We saw in Proposition 4.1.3 that (C_1, σ_t) has only one KMS_β state. Thus, as ψ and φ_β are KMS_β, they must agree on C_1. □

Lemma 4.4.6. *Suppose that for any $\nu \in \widehat{\mathrm{Gal}(K/k)}$ with $\nu \neq 1$ the state ψ vanishes on the spectral subspace C_ν. Then $\psi = \varphi_\beta$.*

Proof. By Lemma 4.4.4, in order to show that ψ and φ_β are equal, it is enough to show that they agree on C_ν for all ν. We already know that ψ and φ_β agree on C_1. As φ_β is $\mathrm{Gal}(K/k)$-invariant, it is easy to see that it vanishes on C_ν for any non-trivial ν, so we deduce that $\psi = \varphi_\beta$. □

Thus, in order to prove that $\psi = \varphi_\beta$, it is enough to prove that ψ vanishes on each of the spectral subspaces C_ν for $\nu \neq 1$. The following lemma, which is inspired by Lemma 27 (c) in [3], will be useful to show this.

Lemma 4.4.7. *Let* $\nu \in \widehat{\mathrm{Gal}(K/k)}$ *with* $\nu \neq 1$. *Let* F *be a non-empty finite set of finite places of* k *such that* ν *is* F*-localized. Suppose that for any* F*-localized partial isometry* $V \in C(X) \cap C_\nu$, *we have*

$$\psi(Vx) = 0 \quad \textit{for all } x \in C_1.$$

Then ψ *vanishes on the spectral subspace* C_ν.

Proof. From Theorems 4.3.10 and 4.3.11 we know that X^{adm} is principal homogeneous under $\mathrm{Gal}(K/k)$. Thus, by choosing a base point $\chi_0 \in X^{\mathrm{adm}}$, we can identify $\mathrm{Gal}(K/k)$ with X^{adm} through the map $\sigma \mapsto \sigma\chi_0$. Let $\mathfrak{f} \in \mathfrak{I}_{\mathcal{O}}$ be defined by

$$\mathfrak{f} = \prod_{\mathfrak{p} \in F} \mathfrak{p}.$$

For any $n \geqslant 1$, let $V_n \in C(X)$ be defined as follows. Let $\chi \in X$. If $\mathfrak{F}_\chi$ is of the form $\mathfrak{a}^{-1}\mathfrak{I}_{\mathcal{O}}$ with $\mathfrak{a} \,|\, \mathfrak{f}^n$, write $\chi = \sigma\chi_0^{\mathfrak{a}}$ with $\sigma \in \mathrm{Gal}(K/k)$ and put $V_n(\chi) = \nu(\sigma)$. Otherwise, put $V_n(\chi) = 0$. Note that V_n is a partial isometry and belongs to C_ν. Moreover, V_n is F-localized because ν is.

For any $\chi \in X$ we have $|V_n(\chi)| = 1$ if $\mathfrak{F}_\chi$ is of the form $\mathfrak{a}^{-1}\mathfrak{I}_{\mathcal{O}}$ for some $\mathfrak{a}|\mathfrak{f}^n$ and $|V_n(\chi)| = 0$ otherwise. As $|V_n|$ takes values in $\{0, 1\}$, we have $|V_n| = |V_n|^2 = V_n V_n^*$. Thus, by Lemmas 2.1.7 and 2.1.8, for any $\chi \in X$, we have

$$V_n V_n^*(\chi) = 1_{\mathfrak{F}_\chi \subset \mathfrak{f}^{-n}\mathfrak{I}_{\mathcal{O}}} = \prod_{\mathfrak{p} \in F} 1_{\mathfrak{f}^{-n}\mathfrak{p}^{-1} \notin \mathfrak{F}_\chi}.$$

Hence, by equation (33), we get

$$V_n V_n^* = \prod_{\mathfrak{p} \in F} (1 - \mu_{\mathfrak{f}^n\mathfrak{p}}\mu_{\mathfrak{f}^n\mathfrak{p}}^*).$$

Since ψ and φ_β agree on C_1 (Lemma 4.4.5) and F is finite, we obtain

$$\psi(V_n V_n^*) = \varphi_\beta\Big(\prod_{\mathfrak{p} \in F} (1 - \mu_{\mathfrak{f}^n\mathfrak{p}}\mu_{\mathfrak{f}^n\mathfrak{p}}^*)\Big) \xrightarrow{n\to\infty} 1. \tag{58}$$

Now let $x \in C_\nu$. We want to prove that $\psi(x) = 0$. For any $n \geqslant 1$, let $P_n = 1 - V_n V_n^*$. The Schwarz inequality gives

$$|\psi(P_n x)|^2 \leqslant \psi(P_n)\psi(xx^*). \tag{59}$$

By equation (58), we have $\psi(P_n) \xrightarrow{n\to\infty} 0$, so equation (59) gives $\psi(P_n x) \xrightarrow{n\to\infty} 0$, so

$$\psi(V_n V_n^* x) \xrightarrow{n\to\infty} \psi(x). \tag{60}$$

For any $n \geqslant 1$, as $x \in C_\nu$ and $V_n^* \in C_{\nu^{-1}}$, we have $V_n^* x \in C_1$. Hence, by assumption, we have

$$\psi(V_n V_n^* x) = 0.$$

Together with equation (60), this gives $\psi(x) = 0$, which completes the proof of Lemma 4.4.7. □

Thus in order to prove that $\psi = \varphi_\beta$ it is enough to prove the following lemma.

Lemma 4.4.8. *Let $\nu \in \widehat{\mathrm{Gal}(K/k)}$ with $\nu \neq 1$. Let F be a non-empty finite set of finite places of k such that ν is F-localized. For any F-localized partial isometry $V \in C(X) \cap C_\nu$, we have*

$$\psi(Vx) = 0 \quad \textit{for all } x \in C_1.$$

Proof. This proof is directly inspired by the proof of Lemma 27 (b) of [3]. It will make use of Lemmas 4.4.9, 4.4.10, 4.4.11, 4.4.12, 4.4.13 and 4.4.14, and will only be completed on p. 202.

Let $V \in C(X)$ be an F-localized partial isometry such that $V \in C_\nu$.

Let $E = V^*V = VV^*$ (the algebra $C(X)$ is commutative). Note that E is a projection and belongs to C_1. Let

$$C_{1,E} = EC_1E = \{f \in C_1 \mid f = fE = Ef\}$$

denote the reduced algebra. As V is fixed by the flow (σ_t) and ψ and φ_β are KMS_β states for the flow (σ_t), we see that V belongs to the centralizer of ψ and of φ_β.

Let α denote the following automorphism of $C_{1,E}$:

$$\alpha(f) = VfV^* \quad \text{for all } f \in C_{1,E}.$$

Let M be the weak closure of $C_{k,\infty}$ in the GNS representation of φ_β. Let us extend the state φ_β to a normal state $\widetilde{\varphi}_\beta$ on M. Let $M_1 \subset M$ denote the weak closure of C_1 in the GNS representation of φ_β.

Since V belongs to the centralizer of φ_β for all $f \in C_{1,E}$, we have $\varphi_\beta(\alpha(f)) = \varphi_\beta(f)$. Thus α preserves φ_β, so it extends to an automorphism of the reduced algebra $M_{1,E}$ preserving $\widetilde{\varphi}_\beta$.

Let $\mathfrak{c} \in \mathfrak{I}_\mathcal{O}$ be an F-localized ideal such that ν factors through $\mathrm{Gal}(K_\mathfrak{c}/k)$.

Lemma 4.4.9. *Let $\mathfrak{p}$ be a finite place of k with $\mathfrak{p} \notin F$. We have*

$$E\mu_\mathfrak{p} \in C_{1,E} \quad \textit{and} \quad \alpha(E\mu_\mathfrak{p}) = \nu(\sigma_\mathfrak{p})E\mu_\mathfrak{p} \quad \textit{for all } \mathfrak{p} \notin F, \tag{61}$$

where $\sigma_\mathfrak{p} = (\mathfrak{p}, K_F/k) \in \mathrm{Gal}(K_F/k)$ is the Artin automorphism of K_F associated to $\mathfrak{p}$.

Proof. Let $\mathcal{H}_F$ denote the $*$-algebra generated by the $e(\phi, \lambda)$, for all $\phi \in H(\mathrm{sgn})$ and all $\lambda \in \phi[F]$. By Proposition 4.4.3, we know that $\mathcal{H}_F$ is norm-dense in $C(X_F)$. So let $(V_n)_{n \in \mathbb{N}}$ be a sequence of elements of $\mathcal{H}_F$ converging to V in the norm topology. Obviously we have

$$\mathcal{H}_F \subset \bigcup_{\mathfrak{d} \in \mathfrak{I}_{\mathcal{O}},\, \mathfrak{d}\, F\text{-loc.}} \mathcal{H}[\mathfrak{d}],$$

where $\mathfrak{d}$ runs over the F-localized elements of $\mathfrak{I}_{\mathcal{O}}$. Thus, for any $n \in \mathbb{N}$, there exists an F-localized $\mathfrak{d}_n \in \mathfrak{I}_{\mathcal{O}}$ such that $V_n \in \mathcal{H}[\mathfrak{d}_n]$. Since $\mathfrak{p} \notin F$ and $\mathfrak{d}_n$ is F-localized, we have $\mathfrak{p} \nmid \mathfrak{d}_n$, so Lemma 3.8.3 gives $V_n \mu_{\mathfrak{p}} = \mu_{\mathfrak{p}} \sigma_{\mathfrak{p}}(V_n)$. Now view $\sigma_{\mathfrak{p}}$ as an automorphism of the C*-algebra $C(X_F)$. In particular it is continuous. Hence we obtain

$$V\mu_{\mathfrak{p}} = \mu_{\mathfrak{p}}\sigma_{\mathfrak{p}}(V) = \nu(\sigma_{\mathfrak{p}})\mu_{\mathfrak{p}}V \quad \text{for all } \mathfrak{p} \notin F, \tag{62}$$

and the result follows. □

The ITPFI structure of M_1. For any $\mathfrak{p}$, recall that $\tau_{\mathfrak{p}}$ is the (Toeplitz) C*-algebra generated by $\mu_{\mathfrak{p}}$, and that $\varphi_{\beta,\mathfrak{p}}$ is the restriction of φ_β to $\tau_{\mathfrak{p}}$. Let $(\varepsilon_n)_{n \geqslant 0}$ denote the standard orthonormal basis of $\ell^2(\mathbb{N})$. Let $\pi_{\beta,\mathfrak{p}}$ be the following representation of $\tau_{\mathfrak{p}}$:

$$\begin{aligned} \pi_{\beta,\mathfrak{p}} \colon \tau_{\mathfrak{p}} &\to B(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})), \\ \mu_{\mathfrak{p}} &\mapsto (\varepsilon_n \otimes \varepsilon_m \mapsto \varepsilon_{n+1} \otimes \varepsilon_m). \end{aligned}$$

Let $\Omega_{\beta,\mathfrak{p}} \in \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ be the following vector:

$$\Omega_{\beta,\mathfrak{p}} = \sqrt{1 - \mathbf{N}\mathfrak{p}^{-\beta}} \sum_{n \geqslant 0} \mathbf{N}\mathfrak{p}^{-n\beta/2} \varepsilon_n \otimes \varepsilon_n.$$

It is easy to check that the pair $(\pi_{\beta,\mathfrak{p}}, \Omega_{\beta,\mathfrak{p}})$ is the GNS representation of $\varphi_{\beta,\mathfrak{p}}$. Let $M_{1,\mathfrak{p}}$ denote the weak closure of $\tau_{\mathfrak{p}}$ in the representation $\pi_{\beta,\mathfrak{p}}$. One checks that

$$M_{1,\mathfrak{p}} = B\ell^2(\mathbb{N}) \otimes \mathbb{C}. \tag{63}$$

In particular $M_{1,\mathfrak{p}}$ is a type I_∞ factor. Let $\widetilde{\varphi}_{\beta,\mathfrak{p}}$ be the unique extension of $\varphi_{\beta,\mathfrak{p}}$ to a normal linear functional on $M_{1,\mathfrak{p}}$. Alternatively, $\widetilde{\varphi}_{\beta,\mathfrak{p}}$ is the restriction of $\widetilde{\varphi}_\beta$ to $M_{1,\mathfrak{p}}$. Note that the eigenvalue list of $\widetilde{\varphi}_{\beta,\mathfrak{p}}$ is the sequence

$$((1 - \mathbf{N}\mathfrak{p}^{-\beta})\mathbf{N}\mathfrak{p}^{-n\beta})_{n \geqslant 0}.$$

We have

$$(M_1, \widetilde{\varphi}_\beta) = \bigotimes_{\mathfrak{p}} (M_{1,\mathfrak{p}}, \widetilde{\varphi}_{\beta,\mathfrak{p}}), \tag{64}$$

where $\mathfrak{p}$ runs over the finite places of k. Recall from [33], III, Chapter XIV, Corollary 1.10, that any ITPFI is a factor. In particular, M_1 is a factor. We shall check later (Lemma 4.5.1) that it is of type $\mathrm{III}_{q^{-\beta}}$.

For any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, let $\rho_{\mathfrak{p},\lambda}$ denote the $*$-automorphism of $\tau_{\mathfrak{p}}$ such that $\rho_{\mathfrak{p},\lambda}(\mu_{\mathfrak{p}}) = \lambda\mu_{\mathfrak{p}}$. As $\rho_{\mathfrak{p},\lambda}$ preserves $\varphi_{\beta,\mathfrak{p}}$, it extends to an automorphism of $M_{1,\mathfrak{p}}$. Let $\theta_{F,\nu}$ be the following automorphism of M_1:

$$\theta_{F,\nu} = (\otimes_{\mathfrak{p}\in F} \mathrm{id}_{M_{1,\mathfrak{p}}}) \otimes (\otimes_{\mathfrak{p}\notin F} \rho_{\mathfrak{p},\nu(\sigma_{\mathfrak{p}})}).$$

Lemma 4.4.10. *$\theta_{F,\nu}$ is an outer automorphism of M_1.*

Proof. Suppose that $\theta_{F,\nu}$ is inner. Lemma 1.3.8 (b) from Connes [7] states that there exists a sequence $(u_{\mathfrak{p}})$ where, for any finite place $\mathfrak{p}$ of k, $u_{\mathfrak{p}}$ is an unitary of $M_{1,\mathfrak{p}}$ with

$$\theta_{F,\nu}(x) = u_{\mathfrak{p}} x u_{\mathfrak{p}}^* \quad \text{for all } x \in M_{1,\mathfrak{p}} \tag{65}$$

and such that

$$\sum_{\mathfrak{p}} (1 - |\varphi_{\beta,\mathfrak{p}}(u_{\mathfrak{p}})|) < \infty. \tag{66}$$

Since $M_{1,\mathfrak{p}}$ is a factor, equation (65) determines $u_{\mathfrak{p}}$ up to multiplication by a $z \in \mathbb{C}$ with $|z| = 1$. By definition of $\theta_{F,\nu}$, when $\mathfrak{p} \in F$ one can take $u_{\mathfrak{p}} = 1$. When $\mathfrak{p} \notin F$ one can take $u_{\mathfrak{p}} \in M_{1,\mathfrak{p}} = B\ell^2(\mathbb{N})$ to be the diagonal matrix with eigenvalue list $(\nu(\sigma_{\mathfrak{p}}^n))_{n\in\mathbb{N}}$. Using the expression of the GNS representation of $\varphi_{\beta,\mathfrak{p}}$ that we saw above, we get

$$\varphi_{\beta,\mathfrak{p}}(u_{\mathfrak{p}}) = (1 - \mathbf{N}\mathfrak{p}^{-\beta}) \sum_{n\in\mathbb{N}} \nu(\sigma_{\mathfrak{p}})^n \mathbf{N}\mathfrak{p}^{-n\beta} = \frac{1 - \mathbf{N}\mathfrak{p}^{-\beta}}{1 - \nu(\sigma_{\mathfrak{p}})\mathbf{N}\mathfrak{p}^{-\beta}}.$$

This is equal to 1 whenever $\nu(\sigma_{\mathfrak{p}}) = 1$; so, letting

$$Y = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a finite place of } k,\ \mathfrak{p} \notin F, \text{ and } \nu(\sigma_{\mathfrak{p}}) \neq 1\},$$

equation (66) gives

$$\sum_{\mathfrak{p}\in Y} \left(1 - \left|\frac{1 - \mathbf{N}\mathfrak{p}^{-\beta}}{1 - \nu(\sigma_{\mathfrak{p}})\mathbf{N}\mathfrak{p}^{-\beta}}\right|\right) < \infty. \tag{67}$$

Recall that we let $\mathfrak{c}$ be an F-localized ideal such that ν factors through $\mathrm{Gal}(K_{\mathfrak{c}}/k)$. This is a finite group, so the range of ν is finite, so there exists a γ with $0 < \gamma < 1$ such that for any $\sigma \in \mathrm{Gal}(K/k)$ with $\nu(\sigma) \neq 1$, we have $\mathrm{Re}\,\nu(\sigma) \leqslant \gamma$. Let $\mathfrak{p} \in Y$. We have $|1 - \nu(\sigma_{\mathfrak{p}})\mathbf{N}\mathfrak{p}^{-\beta}| \geqslant 1 - \gamma\mathbf{N}\mathfrak{p}^{-\beta}$. Thus we find

$$1 - \left|\frac{1 - \mathbf{N}\mathfrak{p}^{-\beta}}{1 - \nu(\sigma_{\mathfrak{p}})\mathbf{N}\mathfrak{p}^{-\beta}}\right| \geqslant 1 - \frac{1 - \mathbf{N}\mathfrak{p}^{-\beta}}{1 - \gamma\mathbf{N}\mathfrak{p}^{-\beta}} = \frac{(1-\gamma)\mathbf{N}\mathfrak{p}^{-\beta}}{1 - \gamma\mathbf{N}\mathfrak{p}^{-\beta}} \geqslant (1-\gamma)\mathbf{N}\mathfrak{p}^{-\beta}.$$

Since $1 - \gamma > 0$, together with equation (67) this gives

$$\sum_{\mathfrak{p}\in Y} \mathbf{N}\mathfrak{p}^{-\beta} < \infty. \tag{68}$$

Since $\beta \leqslant 1$, this implies that for all $s \geqslant 1$ we have

$$\sum_{\mathfrak{p}\in Y} \mathbf{N}\mathfrak{p}^{-s} \leqslant \sum_{\mathfrak{p}\in Y} \mathbf{N}\mathfrak{p}^{-1} < \infty. \tag{69}$$

The Čebotarev density theorem (Theorem 1.1.4) states that for any $\sigma \in \mathrm{Gal}(K_{\mathfrak{c}}/k)$ the following set P_σ of places of k,

$$P_\sigma = \{\mathfrak{p} \mid \mathfrak{p} \text{ does not ramify in } K_{\mathfrak{c}} \text{ and } \sigma_{\mathfrak{p}} = \sigma\},$$

has a positive Dirichlet density

$$d(P_\sigma) > 0.$$

Up to a finite set of places of k, we have

$$Y = \bigcup_{\nu(\sigma)\neq 1} P_\sigma,$$

where σ runs over $\mathrm{Gal}(K_{\mathfrak{c}}/k)$. Hence, as $\nu \neq 1$, we have $d(Y) > 0$, so

$$\lim_{s\to 1_+} \sum_{\mathfrak{p}\in Y} \mathbf{N}\mathfrak{p}^{-s} = \infty,$$

contradicting equation (69). □

Define two subfactors $M_1^{F\pm}$ of M_1,

$$M_1^{F+} = \bigotimes_{\mathfrak{p}\in F} M_{1,\mathfrak{p}},$$

$$M_1^{F-} = \bigotimes_{\mathfrak{p}\notin F} (M_{1,\mathfrak{p}}, \widetilde{\varphi}_{\beta,\mathfrak{p}}),$$

where $\mathfrak{p}$ runs over the finite places of k. We thus have

$$M_1 = M_1^{F+} \otimes M_1^{F-}.$$

As the projection E is F-localized, we have $E \in M_1^{F+}$, so letting

$$N = \left(M_1^{F+}\right)_E$$

and using [33], I, Chapter IV, Proposition 1.9, we get

$$M_{1,E} = N \otimes M_1^{F-}. \tag{70}$$

Lemma 4.4.11. *α is an outer automorphism of $M_{1,E}$.*

Proof. Suppose that α is an inner automorphism of $M_{1,E}$. Let $\tau = \alpha^{-1} \circ \theta_{F,\nu} \in \operatorname{Aut}(M_{1,E})$. By construction, τ induces the identity on $\mathbb{C}E \otimes M_1^{F-}$. As $N \otimes \mathbb{C}$ is of type I_∞, the restriction of τ to $N \otimes \mathbb{C}$ is inner. By equation (70), we get that τ is an inner automorphism of $M_{1,E}$. Hence, $\theta_{F,\nu}$ restricts to an inner automorphism of $M_{1,E}$. As M_1 is a factor, using Lemma 1.5.2 of [7], we deduce that $\theta_{F,\nu}$ is an inner automorphism of M_1, contradicting Lemma 4.4.10. □

As we already noted, V belongs to the centralizer of ψ. Define a linear functional L on $C_{1,E}$ by

$$L(x) = \psi(Vx) = \psi(xV) \quad \text{for all } x \in C_{1,E}.$$

We want to prove by contradiction that L is zero. Thus suppose that L is nonzero.

The Schwarz inequality gives

$$|L(x)|^2 \leqslant \psi(E)\psi(x^*x) \quad \text{for all } x \in C_{1,E}.$$

By Lemma 4.4.5, the states ψ and φ_β agree on $C_{1,E}$, so $\psi(x^*x) = \varphi_\beta(x^*x)$. Thus L extends to a normal linear functional on $M_{1,E}$, which we still denote L.

Since φ_β is KMS_β on $C_{k,\infty}$, by [4], II, Corollary 5.3.4 there exists a unique extension of (σ_t) to an ultraweakly continuous flow $(\widetilde{\sigma}_t)$ on M for which $\widetilde{\varphi}_\beta$ is KMS_β.

Lemma 4.4.12. *The linear functional L satisfies the α-twisted KMS_β condition for the flow $(\widetilde{\sigma}_t)$ on $M_{1,E}$. In other words, for any $x, y \in M_{1,E}$, there exists a bounded continuous function $F_{x,y}$ on the strip $0 \leqslant \operatorname{Im} z \leqslant \beta$, holomorphic on the interior of the strip, such that for any $t \in \mathbb{R}$ we have*

$$F_{x,y}(t) = L(x\sigma_t(y)) \quad \textit{and} \quad F_{x,y}(t + i\beta) = L(\sigma_t(y)\alpha(x)). \tag{71}$$

Proof. In the case when $x, y \in C_{1,E}$, this can be easily checked by applying the KMS_β condition for ψ to the pair (Vx, y). As both $(\widetilde{\sigma}_t)$ and L are ultraweakly continuous, the result follows. □

Lemma 4.4.13. *There exists a nonzero $\widetilde{\sigma}_t$-invariant $w \in M_{1,E}$ such that*

$$L(x) = \widetilde{\varphi}_\beta(wx) \quad \textit{for all } x \in M_{1,E}.$$

Proof. Let $L = |L|u$ be the polar decomposition of L (see [33], I, Chapter III, Theorem 4.2). In particular, $u \in M_{1,E}$ is a partial isometry, $|L|$ is a positive normal linear functional on $M_{1,E}$, and

$$L(x) = |L|(ux) \quad \text{for all } x \in M_{1,E}.$$

We want to apply Connes' Radon–Nikodým theorem to $|L|$ and $\widetilde{\varphi}_\beta$, seen as finite normal faithful weights on $M_{1,E}$.

Since L is $\widetilde{\sigma}_t$-invariant, by uniqueness of the polar decomposition, u and $|L|$ are $\widetilde{\sigma}_t$-invariant. As $\widetilde{\varphi}_\beta$ is KMS$_\beta$ for the flow $(\widetilde{\sigma}_t)$, we deduce that $|L|$ is $\sigma_t^{\widetilde{\varphi}_\beta}$-invariant, where $(\sigma_t^{\widetilde{\varphi}_\beta})$ is the modular automorphism group associated to the finite faithful normal weight $\widetilde{\varphi}_\beta$ on $M_{1,E}$. Connes' Radon–Nikodým theorem ([7], Lemme 1.2.3 (b)) then states that there exists a positive $\widetilde{\sigma}_t$-invariant $h \in M_{1,E}$ such that

$$|L|(x) = \widetilde{\varphi}_\beta(hx) \quad \text{for all } x \in M_{1,E}.$$

Letting $w = hu$, we get

$$L(x) = \widetilde{\varphi}_\beta(wx) \quad \text{for all } x \in M_{1,E},$$

and w is nonzero by our assumption that L is nonzero. It is $\widetilde{\sigma}_t$-invariant because both u and h are. □

Lemma 4.4.14. *Let w be given by Lemma* 4.4.13. *Then*

$$\alpha(x)w = wx \quad \textit{for all } x \in M_{1,E}.$$

Proof. Let $x, y \in M_{1,E}$. Let $F^L_{x,y}$ be the function given by Lemma 4.4.12 such that for any $t \in \mathbb{R}$ we have

$$F^L_{x,y}(t) = L(x\widetilde{\sigma}_t(y)) \quad \text{and} \quad F^L_{x,y}(t + i\beta) = L(\widetilde{\sigma}_t(y)\alpha(x)).$$

By definition of w, we get

$$F^L_{x,y}(t) = \widetilde{\varphi}_\beta(wx\widetilde{\sigma}_t(y)) \quad \text{and} \quad F^L_{x,y}(t + i\beta) = \widetilde{\varphi}_\beta(\widetilde{\sigma}_t(wy)\alpha(x)).$$

Now let $F^{\widetilde{\varphi}_\beta}_{\alpha(x),wy}$ be the function given by the KMS$_\beta$ property of $\widetilde{\varphi}_\beta$ applied to the pair $(\alpha(x), wy)$ so that

$$F^{\widetilde{\varphi}_\beta}_{\alpha(x),wy}(t) = \widetilde{\varphi}_\beta(\alpha(x)\widetilde{\sigma}_t(wy)) \quad \text{and} \quad F^{\widetilde{\varphi}_\beta}_{\alpha(x),wy}(t + i\beta) = \widetilde{\varphi}_\beta(\widetilde{\sigma}_t(wy)\alpha(x)).$$

Let $G = F^{\widetilde{\varphi}_\beta}_{\alpha(x),wy} - F^L_{x,y}$. Note that G vanishes on $\mathbb{R} + i\beta$. Therefore, one can extend G to a holomorphic function on the broader strip $0 < \operatorname{Im} z < 2\beta$ by letting

$$G(z) = \overline{G(\bar{z} + 2i\beta)} \quad \text{for all } z \in \mathbb{C} \text{ with } \beta < \operatorname{Im} z < 2\beta.$$

As G vanishes on $\mathbb{R} + i\beta$ and is holomorphic on an open set containing $\mathbb{R} + i\beta$, it vanishes everywhere, so $F^{\widetilde{\varphi}_\beta}_{\alpha(x),wy} = F^L_{x,y}$. In particular, evaluating that at 0, we get

$$\widetilde{\varphi}_\beta(wxy) = \widetilde{\varphi}_\beta(\alpha(x)wy).$$

Since this holds for all $y \in M_{1,E}$ and the state $\widetilde{\varphi}_\beta$ is faithful on $M_{1,E}$, we get

$$wx = \alpha(x)w. \qquad \square$$

We already know by Lemma 4.4.11 that α is outer. Thus, Proposition 4.1.16 of Sunder [32] shows that

$$\{y \in M_{1,E} \mid \alpha(x)y = yx \text{ for all } x \in M_{1,E}\} = \{0\}.$$

Together with Lemma 4.4.14 this shows that $w = 0$. But w is nonzero by construction (cf. Lemma 4.4.13), so we get a contradiction. Thus our assumption that L is nonzero was false. Thus L is zero, so

$$\psi(Vx) = 0 \quad \text{for all } x \in C_{1,E}.$$

Now let $x \in C_1$. We have $ExE \in C_{1,E}$, so it follows that $\psi(VExE) = 0$. As $E = V^*V = VV^*$ is a projection and belongs to the centralizer of ψ, we get $\psi(VExE) = \psi(EVEx) = \psi(Vx)$, so $\psi(Vx) = 0$, which proves Lemma 4.4.8. □

From this we can deduce the main result of this subsection. Recall that we have assumed $0 < \beta \leqslant 1$.

Theorem 4.4.15. *The C^*-system $(C_{k,\infty}, \sigma_t)$ has exactly one KMS$_\beta$ state, φ_β.*

Proof. This follows from Lemmas 4.4.6, 4.4.7 and 4.4.8. □

Corollary 4.4.16. *The state φ_β of $C_{k,\infty}$ is a factor state, i.e., the von Neumann algebra M is a factor.*

Proof. This follows from Theorem 4.4.15 and [4], II, Theorem 5.3.30 (3). □

4.5. The type III$_{q^{-\beta}}$ of the KMS$_\beta$ state at high temperature $1/\beta \geqslant 1$. Let us go on with the notations of the preceding subsection. In particular, we assume that $\beta \leqslant 1$. The goal of this subsection is to prove (Theorem 4.5.8) that the state φ_β on $C_{k,\infty}$ is of type III$_{q^{-\beta}}$. In other words, we want to show that the factor M is of type III$_{q^{-\beta}}$. Before doing that we show that the subfactor M_1 is of type III$_{q^{-\beta}}$.

Recall from equation (64) that M_1 is the following infinite tensor product, where $\mathfrak{p}$ runs over the finite places of k:

$$(M_1, \tilde{\varphi}_\beta) = \bigotimes_{\mathfrak{p}} (M_{1,\mathfrak{p}}, \tilde{\varphi}_{\beta,\mathfrak{p}}).$$

Here each of the $M_{1,\mathfrak{p}}$ is a type I$_\infty$ factor, so the usual methods (cf. Araki and Woods [1], [8]) allowing to compute asymptotic ratio sets cannot be applied directly to M_1. Instead we first find an integer $\tau \in \mathbb{N}$ and, for each $\mathfrak{p}$, a projection $e_\mathfrak{p} \in M_{1,\mathfrak{p}}$ such that the reduced factor $M_{1,\mathfrak{p},e_\mathfrak{p}}$ is of type I$_\tau$ and the infinite tensor product $e = \otimes_\mathfrak{p} e_\mathfrak{p}$ is a nonzero projection in M_1.

Let $\tau \in \mathbb{N}$ be such that $\tau > 1/\beta$. For any finite place $\mathfrak{p}$ of k, let $e_{\mathfrak{p}} = 1 - \mu_{\mathfrak{p}}^{\tau}\mu_{\mathfrak{p}}^{*\tau} \in M_{1,\mathfrak{p}}$. Recall from equation (63) that $M_{1,\mathfrak{p}}$ is naturally identified with $B\ell^2(\mathbb{N})$. Under this identification, the projection $e_{\mathfrak{p}}$ is the diagonal matrix whose τ first diagonal entries are 1 and whose other entries are 0. Thus, the reduced subfactor $M_{1,\mathfrak{p},e_{\mathfrak{p}}}$ is of type I_{τ}. Note that

$$\widetilde{\varphi}_{\beta,\mathfrak{p}}(e_{\mathfrak{p}}) = 1 - \mathbf{N}\mathfrak{p}^{-\tau\beta}. \tag{72}$$

Any decreasing sequence of projections in a von Neumann algebra converges weakly to a projection, so we can define a projection $e \in M_1$ by

$$e = \prod_{\mathfrak{p}} e_{\mathfrak{p}} = \bigotimes_{\mathfrak{p}} e_{\mathfrak{p}}.$$

By definition of τ we have $\tau\beta > 1$, so

$$\widetilde{\varphi}_{\beta}(e) = \prod_{\mathfrak{p}} \widetilde{\varphi}_{\beta,\mathfrak{p}}(e_{\mathfrak{p}}) = \prod_{\mathfrak{p}} (1 - \mathbf{N}\mathfrak{p}^{-\tau\beta}) = \frac{1}{\zeta_{k,\infty}(\tau\beta)} \neq 0.$$

In particular, $e \neq 0$. Let us define a state $\widetilde{\varphi}_{\beta,e}$ on $M_{1,e}$ by

$$\widetilde{\varphi}_{\beta,e}(x) = \zeta_{k,\infty}(\tau\beta)\,\widetilde{\varphi}_{\beta}(x) = \frac{\widetilde{\varphi}_{\beta}(x)}{\widetilde{\varphi}_{\beta}(e)} \quad \text{for all } x \in M_{1,e}.$$

For any $\mathfrak{p}$, let us define a state $\widetilde{\varphi}_{\beta,\mathfrak{p},e_{\mathfrak{p}}}$ on $M_{1,\mathfrak{p},e_{\mathfrak{p}}}$ by

$$\widetilde{\varphi}_{\beta,\mathfrak{p},e_{\mathfrak{p}}}(x) = (1 - \mathbf{N}\mathfrak{p}^{-\tau\beta})^{-1}\widetilde{\varphi}_{\beta,\mathfrak{p}}(x) = \frac{\widetilde{\varphi}_{\beta,\mathfrak{p}}(x)}{\widetilde{\varphi}_{\beta,\mathfrak{p}}(e_{\mathfrak{p}})} \quad \text{for all } x \in M_{1,\mathfrak{p},e_{\mathfrak{p}}}.$$

Coming back to the definition of the infinite tensor product ([33], III, Chapter XIV, §1) and using the expression (64) of M_1 as the infinite tensor product of the $(M_{1,\mathfrak{p}}, \widetilde{\varphi}_{\beta,\mathfrak{p}})$, one can check that

$$(M_{1,e}, \widetilde{\varphi}_{\beta,e}) = \bigotimes_{\mathfrak{p}} (M_{1,\mathfrak{p},e_{\mathfrak{p}}}, \widetilde{\varphi}_{\beta,\mathfrak{p},e_{\mathfrak{p}}}). \tag{73}$$

Let $(\sigma_t^{\widetilde{\varphi}_{\beta}})$ denote the modular flow of $\widetilde{\varphi}_{\beta}$. Since $\widetilde{\varphi}_{\beta}$ is KMS_{β} for the flow $(\widetilde{\sigma}_t)$, we have

$$\sigma_t^{\widetilde{\varphi}_{\beta}} = \widetilde{\sigma}_{\beta t} \quad \text{for all } t \in \mathbb{R}. \tag{74}$$

Lemma 4.5.1. *The factor M_1 is of type* $\mathrm{III}_{q^{-\beta}}$.

Proof. Let us first prove that $q^{-\beta}$ belongs to the asymptotic ratio set $r_{\infty}(M_{1,e})$. We want to apply the criterion given on p. 465 of [8] to the ITPFI in equation (73). The eigenvalue list of $\widetilde{\varphi}_{\beta,\mathfrak{p},e_{\mathfrak{p}}}$ is $(\lambda_{\mathfrak{p},a})_{a=0,\dots,\tau-1}$, where

$$\lambda_{\mathfrak{p},a} = \frac{(1 - \mathbf{N}\mathfrak{p}^{-\beta})\mathbf{N}\mathfrak{p}^{-a\beta}}{1 - \mathbf{N}\mathfrak{p}^{-\tau\beta}}.$$

Let r be such that $0 < r < 1$. For any $n \in \mathbb{N}$, let

$$r(n) = \lfloor rq^n/n \rfloor.$$

By equation (1), there exists an $n_0 \geqslant 1$ such that for any $n \geqslant n_0$, there exist (at least) $r(n)$ distinct finite places

$$\mathfrak{p}_n^1, \ldots, \mathfrak{p}_n^{r(n)}$$

of k such that

$$\mathbf{N}\mathfrak{p}_n^i = q^n \quad \text{for all } i \in \{1, \ldots, r(n)\}.$$

For any $n \geqslant n_0$, let I_n be the following set of places of k:

$$I_n = \{\mathfrak{p}_{2n}^1, \ldots, \mathfrak{p}_{2n}^{r(2n)}, \mathfrak{p}_{2n+1}^1, \ldots, \mathfrak{p}_{2n+1}^{r(2n)}\}.$$

Let $X(I_n) = \{0, \ldots, \tau - 1\}^{I_n}$ be the set of all maps from I_n to $\{0, \ldots, \tau - 1\}$. Define a measure λ on $X(I_n)$ by

$$\lambda(\{f\}) = \prod_{i=1}^{r(2n)} \lambda_{\mathfrak{p}_{2n}^i, f(\mathfrak{p}_{2n}^i)} \lambda_{\mathfrak{p}_{2n+1}^i, f(\mathfrak{p}_{2n+1}^i)}.$$

For any $i \in \{1, \ldots, r(2n)\}$, define elements $k_n^{1,i}$ and $k_n^{2,i}$ of $X(I_n)$ by

$$k_n^{1,i}(\mathfrak{p}) = 1_{\mathfrak{p}=\mathfrak{p}_{2n}^i} \quad \text{and} \quad k_n^{2,i}(\mathfrak{p}) = 1_{\mathfrak{p}=\mathfrak{p}_{2n+1}^i} \quad \text{for all } \mathfrak{p} \in I_n.$$

Let $K_n^1 = \{k_n^{1,1}, \ldots, k_n^{1,r(2n)}\}$ and let $K_n^2 = \{k_n^{2,1}, \ldots, k_n^{2,r(2n)}\}$. For any $i \in \{1, \ldots, r(2n)\}$, we have $\lambda(\{k_n^{1,i}\}) = \lambda(\{k_n^{1,1}\})$, so

$$\begin{aligned} \lambda(K_n^1) &= r(2n)\,\lambda(\{k_n^{1,1}\}) \\ &= r \cdot \left(\frac{1 - q^{-2n\beta}}{1 - q^{-2n\tau\beta}}\right)^{\lfloor rq^{2n}/(2n) \rfloor} \left(\frac{1 - q^{-(2n+1)\beta}}{1 - q^{-(2n+1)\tau\beta}}\right)^{\lfloor rq^{2n}/(2n) \rfloor} \frac{(q^{1-\beta})^{2n}}{(2n)}. \end{aligned}$$

Since $\beta \leqslant 1$, one checks easily that

$$\sum_{n \geqslant n_0} \lambda(K_n^1) = \infty. \tag{75}$$

Let $\phi_n \colon K_n^1 \to K_n^2$ be the bijection defined by $\phi_n(k_n^{1,i}) = k_n^{2,i}$. For any $i \in \{1, \ldots, 2n\}$, we have

$$\frac{\lambda(\{\phi_n(k_n^{1,i})\})}{\lambda(\{k_n^{1,i}\})} = \frac{\lambda_{\mathfrak{p}_{2n}^i,0}\lambda_{\mathfrak{p}_{2n+1}^i,1}}{\lambda_{\mathfrak{p}_{2n}^i,1}\lambda_{\mathfrak{p}_{2n+1}^i,0}} = \frac{q^{-(2n+1)\beta}}{q^{-2n\beta}} = q^{-\beta}.$$

Together with equation (75), this allows to apply the criterion given on p. 465 of [8], and we get

$$q^{-\beta} \in r_\infty(M_{1,e}).$$

Hence, by [7], Théorème 3.6.1, we have $q^{-\beta} \in S(M_{1,e})$. Hence, by [7], Corollaire 3.2.8 (b), we have

$$q^{-\beta} \in S(M_1).$$

In particular, this shows that $S(M_1) \neq \{0, 1\}$, so, by [7], Théorème 3.4.1, one gets that $S(M_1) \cap \mathbb{R}_+^*$ is the orthogonal of $T(M_1)$ for the duality $(s,t) \mapsto s^{it}$. By construction, $\widetilde{\sigma}_{2\pi/\log q} = 1$, so equation (74) gives

$$\sigma^{\widetilde{\varphi}_\beta}_{2\pi/(\beta \log q)} = 1. \tag{76}$$

Thus

$$T(M_1) \supset \frac{2\pi}{\beta \log q}\,\mathbb{Z}.$$

Hence, by orthogonality, we get

$$S(M_1) \cap \mathbb{R}_+^* \subset q^{\beta\mathbb{Z}}.$$

Since we already know that $q^{-\beta} \in S(M_1)$, we obtain

$$S(M_1) \cap \mathbb{R}_+^* = q^{\beta\mathbb{Z}}.$$

Thus M_1 is of type $\mathrm{III}_{q^{-\beta}}$. □

We only use Lemma 4.5.1 for the proof the following corollary. Let $M_{1,\widetilde{\varphi}_\beta}$ denote the centralizer of $\widetilde{\varphi}_\beta$ in M_1.

Corollary 4.5.2. *The centralizer $M_{1,\widetilde{\varphi}_\beta}$ is a factor, of type* II_1.

Proof. Lemma 4.5.1 and equation (76) allow to apply [7], Théorème 4.2.6, and we obtain that $M_{1,\widetilde{\varphi}_\beta}$ is a factor. Note that $\widetilde{\varphi}_\beta$ is a finite faithful normal trace on $M_{1,\widetilde{\varphi}_\beta}$. Hence, the type of $M_{1,\widetilde{\varphi}_\beta}$ can only be either II_1 or I_n with $n \in \mathbb{N}^*$. Let $\mathfrak{p}$ be a finite place of k. For any $n \geqslant 1$, set $x_n = \mu_{\mathfrak{p}}^n \mu_{\mathfrak{p}}^{*n}$. Note that the x_n are fixed by the flow (σ_t), hence by equation (74) they belong to $M_{1,\widetilde{\varphi}_\beta}$. Equation (33) shows that the x_n are linearly independent over $\mathbb{C}$. Thus, $M_{1,\widetilde{\varphi}_\beta}$ is infinite-dimensional over $\mathbb{C}$, so its type cannot be I_n with $n \in \mathbb{N}^*$. Hence, it must be II_1. □

Our next goal is to prove (Lemma 4.5.6) that the centralizer $M_{\widetilde{\varphi}_\beta}$ of $\widetilde{\varphi}_\beta$ in M is also a factor.

Definition 4.5.3. For any $\mathfrak{d} \in \mathfrak{I}_{\mathcal{O}}$, let $M[\mathfrak{d}]$ denote the weak closure of $\mathcal{H}[\mathfrak{d}]$ in M.

Lemma 4.5.4. *Let* $\mathfrak{d} \in \mathfrak{I}_{\mathcal{O}}$. *Let* $\mathfrak{p}$ *be a maximal ideal of* $\mathfrak{I}_{\mathcal{O}}$ *not dividing* $\mathfrak{d}$. *Let* $\sigma_{\mathfrak{p}} = (\mathfrak{p}, K_{\mathfrak{d}}/k) \in \mathrm{Gal}(K_{\mathfrak{d}}/k)$ *be the Artin automorphism of* $K_{\mathfrak{d}}$ *associated to* $\mathfrak{p}$. *Then*:

(1) *The automorphism* $\sigma_{\mathfrak{p}}$ *of* $\mathcal{H}[\mathfrak{d}]$ *extends uniquely to an ultraweakly continuous automorphism of* $M[\mathfrak{d}]$.

(2) *For all* $x \in M[\mathfrak{d}]$, *we have*

$$x\mu_{\mathfrak{p}} = \mu_{\mathfrak{p}}\sigma_{\mathfrak{p}}(x). \tag{77}$$

Proof. Let us first prove (1). Uniqueness is clear because, by the von Neumann density theorem, $\mathcal{H}[\mathfrak{d}]$ is ultraweakly dense in $M[\mathfrak{d}]$. Let $\sigma \in \mathrm{Gal}(K/k)$ be such that $\sigma|_{K_{\mathfrak{d}}} = \sigma_{\mathfrak{p}}$. As $\varphi_\beta \circ \sigma = \varphi_\beta$ on $C_{k,\infty}$, we know that σ extends to an ultraweakly continuous automorphism of M, which we still note σ. The required extension of $\sigma_{\mathfrak{p}}$ is then obtained by taking the restriction of σ to $M[\mathfrak{d}]$.

Let us now check (2). By density, it is enough to check equation (77) when $x \in \mathcal{H}[\mathfrak{d}]$. It then follows from Lemma 3.8.3. □

Lemma 4.5.5. *Let* $\mathfrak{d} \in \mathfrak{I}_{\mathcal{O}}$. *Let* $M[\mathfrak{d}]_{\widetilde{\varphi}_\beta}$ *denote the centralizer of* $\widetilde{\varphi}_\beta$ *in* $M[\mathfrak{d}]$. *Let* $Z(M[\mathfrak{d}]_{\widetilde{\varphi}_\beta})$ *denote the center of* $M[\mathfrak{d}]_{\widetilde{\varphi}_\beta}$. *Then*:

$$Z(M[\mathfrak{d}]_{\widetilde{\varphi}_\beta}) \subset M_1.$$

Proof. Let x belong to $Z(M[\mathfrak{d}]_{\widetilde{\varphi}_\beta})$. As x belongs to $M[\mathfrak{d}]$, it is fixed by $\mathrm{Gal}(K/K_{\mathfrak{d}})$. Let $\sigma \in \mathrm{Gal}(K_{\mathfrak{d}}/k) = \mathrm{Gal}(K/k)/\mathrm{Gal}(K/K_{\mathfrak{d}})$. By Corollary 1.1.5, there exist finite places $\mathfrak{p}, \mathfrak{q}$ of k not dividing $\mathfrak{d}$ such that $\mathbf{N}\mathfrak{p} = \mathbf{N}\mathfrak{q}$, $\sigma_{\mathfrak{p}} = \sigma$ and $\sigma_{\mathfrak{q}} = 1$. Since $\mathbf{N}\mathfrak{p} = \mathbf{N}\mathfrak{q}$, we have

$$\sigma_t(\mu_{\mathfrak{p}}\mu_{\mathfrak{q}}^*) = \mathbf{N}\mathfrak{p}^{it}\mathbf{N}\mathfrak{q}^{-it}\mu_{\mathfrak{p}}\mu_{\mathfrak{q}}^* = \mu_{\mathfrak{p}}\mu_{\mathfrak{q}}^* \quad \text{for all } t \in \mathbb{R}.$$

Hence, by equation (74), $\mu_{\mathfrak{p}}\mu_{\mathfrak{q}}^* \in M[\mathfrak{d}]_{\widetilde{\varphi}_\beta}$. Thus, as x belongs to the center of $M[\mathfrak{d}]_{\widetilde{\varphi}_\beta}$, we have

$$x\mu_{\mathfrak{p}}\mu_{\mathfrak{q}}^* = \mu_{\mathfrak{p}}\mu_{\mathfrak{q}}^* x. \tag{78}$$

On the other hand, by Lemma 4.5.4 (2), we have

$$x\mu_{\mathfrak{p}}\mu_{\mathfrak{q}}^* = \mu_{\mathfrak{p}}\sigma_{\mathfrak{p}}(x)\mu_{\mathfrak{q}}^*, \tag{79}$$

and we also compute

$$\begin{aligned} \mu_{\mathfrak{p}}\mu_{\mathfrak{q}}^* x &= \mu_{\mathfrak{p}}(x^*\mu_{\mathfrak{q}})^* \\ &= \mu_{\mathfrak{p}}(\mu_{\mathfrak{q}}\sigma_{\mathfrak{q}}(x^*))^* \quad \text{by Lemma 4.5.4 (2)} \\ &= \mu_{\mathfrak{p}}\sigma_{\mathfrak{q}}(x)\mu_{\mathfrak{q}}^*. \end{aligned} \tag{80}$$

Combining equations (78), (79), and (80), we get

$$\mu_{\mathfrak{p}}\sigma_{\mathfrak{p}}(x)\mu_{\mathfrak{q}}^* = \mu_{\mathfrak{p}}\sigma_{\mathfrak{q}}(x)\mu_{\mathfrak{q}}^*. \tag{81}$$

Multiplying both sides of equation (81) by $\mu_{\mathfrak{p}}^*$ on the left and by $\mu_{\mathfrak{q}}$ on the right, and applying relation (a_1) of Proposition 3.1.2, we get

$$\sigma_{\mathfrak{p}}(x) = \sigma_{\mathfrak{q}}(x).$$

Since $\sigma_{\mathfrak{p}} = \sigma$ and $\sigma_{\mathfrak{q}} = 1$, we get

$$\sigma(x) = x.$$

Thus, $x \in M_1$. □

Let $M_{\widetilde{\varphi}_\beta}$ denote the centralizer of $\widetilde{\varphi}_\beta$ in M.

Lemma 4.5.6. *The centralizer $M_{\widetilde{\varphi}_\beta}$ is a factor of type* II_1.

Proof. Note that $\widetilde{\varphi}_\beta$ is a finite, faithful, normal, positive, normalized trace on $M_{\widetilde{\varphi}_\beta}$. Let tr be another such trace on $M_{\widetilde{\varphi}_\beta}$. Let us prove that $\mathrm{tr} = \widetilde{\varphi}_\beta$. Let $\mathfrak{d} \in \mathfrak{I}_{\mathcal{O}}$. By Connes' Radon–Nikodým theorem, [7], Lemme 1.2.3 (b), there exists a positive element h of $M[\mathfrak{d}]_{\widetilde{\varphi}_\beta}$ such that

$$\mathrm{tr}(x) = \widetilde{\varphi}_\beta(hx) \quad \text{for all } x \in M[\mathfrak{d}]_{\widetilde{\varphi}_\beta}.$$

Since $\widetilde{\varphi}_\beta$ and tr are faithful traces, one easily checks that h belongs to the center $Z(M[\mathfrak{d}]_{\widetilde{\varphi}_\beta})$. Thus, by Lemma 4.5.5, $h \in M_1$. Hence, the restriction of tr to $M[\mathfrak{d}]_{\widetilde{\varphi}_\beta}$ is $\mathrm{Gal}(K/k)$-invariant, so

$$\mathrm{tr}(x) = \mathrm{tr}(\mathbf{E}(x)) \quad \text{for all } x \in M[\mathfrak{d}]_{\widetilde{\varphi}_\beta}. \tag{82}$$

As $(\sigma_t^{\widetilde{\varphi}_\beta})$ is $(2\pi/\log q)$-periodic, we have a normal conditional expectation

$$\begin{aligned} E_{\widetilde{\varphi}_\beta}\colon M &\to M_{\widetilde{\varphi}_\beta}, \\ x &\mapsto \frac{\log q}{2\pi}\int_0^{2\pi/\log q} \sigma_t^{\widetilde{\varphi}_\beta}(x)\,dt. \end{aligned}$$

Since $\mathcal{H}$ is norm-dense in $C_{k,\infty}$ (see Proposition 3.3.5), it is ultraweakly dense in M, and it follows that $E_{\widetilde{\varphi}_\beta}(\mathcal{H})$ is ultraweakly dense in $E_{\widetilde{\varphi}_\beta}(M)$. We have

$$\begin{aligned} E_{\widetilde{\varphi}_\beta}(\mathcal{H}) = E_{\widetilde{\varphi}_\beta}\Big(\bigcup_{\mathfrak{d}\in\mathfrak{I}_{\mathcal{O}}} \mathcal{H}[\mathfrak{d}]\Big) &\subset E_{\widetilde{\varphi}_\beta}\Big(\bigcup_{\mathfrak{d}\in\mathfrak{I}_{\mathcal{O}}} M[\mathfrak{d}]\Big) \\ &\subset \bigcup_{\mathfrak{d}\in\mathfrak{I}_{\mathcal{O}}} E_{\widetilde{\varphi}_\beta}(M[\mathfrak{d}]) \subset \bigcup_{\mathfrak{d}\in\mathfrak{I}_{\mathcal{O}}} M[\mathfrak{d}]_{\widetilde{\varphi}_\beta}. \end{aligned}$$

Thus, $\bigcup_{\mathfrak{d}\in\mathfrak{I}_{\mathcal{O}}} M[\mathfrak{d}]_{\widetilde{\varphi}_\beta}$ is ultraweakly dense in $E_{\widetilde{\varphi}_\beta}(M) = M_{\widetilde{\varphi}_\beta}$. Thus equation (82) gives

$$\operatorname{tr}(x) = \operatorname{tr}(\mathbf{E}(x)) \quad \text{for all } x \in M_{\widetilde{\varphi}_\beta}. \tag{83}$$

We know by Corollary 4.5.2 that $M_{1,\widetilde{\varphi}_\beta}$ is a type II_1 factor. Hence, by Jones [22], Corollary 7.1.19, we know that tr and $\widetilde{\varphi}_\beta$ agree on $M_{1,\widetilde{\varphi}_\beta}$. Thus, by equation (83), we deduce that tr and $\widetilde{\varphi}_\beta$ agree on $M_{\widetilde{\varphi}_\beta}$. Hence, by [22], Corollary 7.1.20, we deduce that $M_{\widetilde{\varphi}_\beta}$ is a factor, and the same argument that we made for $M_{1,\widetilde{\varphi}_\beta}$ shows that $M_{\widetilde{\varphi}_\beta}$ is also of type II_1. □

Corollary 4.5.7. *We have $S(M) \neq \{0, 1\}$. In other words, the factor M is not of type* III_0.

Proof. Suppose that $S(M) = \{0, 1\}$. Then, by [7], Corollaire 3.2.7 (b), the center of $M_{\widetilde{\varphi}_\beta}$ has no minimal nonzero projection. Hence, by Lemma 4.5.6, one deduces that $\mathbb{C}$ has no minimal nonzero projection, which is absurd. □

Finally we can prove the main result of this subsection. Recall that we have assumed $0 < \beta \leqslant 1$.

Theorem 4.5.8. *The state φ_β on $C_{k,\infty}$ is of type $\mathrm{III}_{q^{-\beta}}$. In other words, the factor M is of type $\mathrm{III}_{q^{-\beta}}$.*

Proof. By Corollary 4.5.7 and [7], Théorème 3.4.1, the set $S(M) \cap \mathbb{R}_+^*$ is the orthogonal of $T(M)$ for the duality $(s, t) \mapsto s^{it}$. Hence, it is enough to prove that

$$T(M) = \frac{2\pi}{\beta \log q} \cdot \mathbb{Z}.$$

Since $\widetilde{\sigma}_{2\pi/\log q} = 1$, equation (74) gives

$$\frac{2\pi}{\beta \log q} \in T(M),$$

which proves one inclusion. Let us prove the other one. Let $t_0 \in \mathbb{R}$ be such that $t_0/\beta \in T(M)$. Thus, by equation (74), $\widetilde{\sigma}_{t_0}$ is an inner automorphism of M. Let u be an unitary of M such that

$$\widetilde{\sigma}_{t_0}(x) = uxu^* \quad \text{for all } x \in M.$$

For any $t \in \mathbb{R}$ and $x \in M$, we have

$$\widetilde{\sigma}_t(u)\widetilde{\sigma}_t(x)\widetilde{\sigma}_t(u)^* = \widetilde{\sigma}_{t_0+t}(uxu^*) = u\widetilde{\sigma}_t(x)u^*,$$

so the unitaries u and $\sigma_t(u)$ implement the same inner automorphism of the factor M, so there exists some $z_t \in \mathbb{C}$ with $|z_t| = 1$ and $\sigma_t(u) = z_t u$. The map $t \mapsto z_t$ is a character of $\mathbb{R}$, so there exists $\theta \in \mathbb{R}$ such that

$$z_t = e^{i\theta t} \quad \text{for all } t \in \mathbb{R}.$$

The KMS$_\beta$ property of the state $\widetilde{\varphi}_\beta$ for the flow $(\widetilde{\sigma}_t)$ applied to the pair (u^*, u) gives a bounded continuous function F on the strip $0 \leqslant \operatorname{Im} z \leqslant \beta$, holomorphic on the interior of the strip, such that

$$F(t) = \widetilde{\varphi}_\beta(u^* \sigma_t(u)) \quad \text{and} \quad F(t + i\beta) = \widetilde{\varphi}_\beta(\sigma_t(u)u^*) \quad \text{for all } t \in \mathbb{R}.$$

Thus,

$$F(t) = e^{i\theta t} = F(t + i\beta) \quad \text{for all } t \in \mathbb{R}. \tag{84}$$

Hence F is the holomorphic function $z \mapsto e^{i\theta z}$ and, evaluating equation (84) at $t = 0$, one gets

$$e^{-\theta\beta} = 1.$$

Thus $\theta = 0$, so u is fixed by the flow $(\widetilde{\sigma}_t)$. Hence, by equation (74), the unitary u belongs to the centralizer $M_{\widetilde{\varphi}_\beta}$ of $\widetilde{\varphi}_\beta$. Moreover, by equation (74), any element of $M_{\widetilde{\varphi}_\beta}$ is fixed by the flow $(\widetilde{\sigma}_t)$ and so commutes with u, by definition of u. Hence u belongs to the center of $M_{\widetilde{\varphi}_\beta}$. Thus, by Lemma 4.5.6, one deduces that $u \in \mathbb{C}$, so, as an automorphism of M,

$$\widetilde{\sigma}_{t_0} = 1. \tag{85}$$

By equation (1), for any sufficiently large n, there exist finite places $\mathfrak{p}$ and $\mathfrak{q}$ of k such that $\mathbf{N}\mathfrak{p} = q^n$ and $\mathbf{N}\mathfrak{q} = q^{n+1}$. We then have $\sigma_{t_0}(\mu_\mathfrak{q}\mu_\mathfrak{p}^*) = q^{(n+1)it_0 - nit_0}\mu_\mathfrak{q}\mu_\mathfrak{p}^* = q^{it_0}\mu_\mathfrak{q}\mu_\mathfrak{p}^*$. On the other hand, equation (85) gives $\sigma_{t_0}(\mu_\mathfrak{q}\mu_\mathfrak{p}^*) = \mu_\mathfrak{q}\mu_\mathfrak{p}^*$. Thus, we get $1 = q^{it_0}$, so $t_0 \in 2\pi/(\log q)\mathbb{Z}$, which completes the proof. □

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