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Controllability Results for Nondensely Defined Semilinear Functional Differential Equations

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Abstract. In this paper we investigate the controllability of first order semilinear functional and neutral functional differential equations in Banach spaces.

Keywords. Controllability, functional semilinear differential equations, nondensely defined operator, fixed point, semigroup, measurable, Banach space

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1. Introduction

In this paper, we shall be concerned with the controllability of first order semilinear functional and neutral functional differential equations in Banach spaces. Firstly, in Section 3 we will consider the following first order semilinear functional differential equations of the form

$$y'(t) - Ay(t) = f(t, y_t) + (Bu)(t), \quad \text{a.e. } t \in J = [0, T] \tag{1}$$

$$y(t) = \phi(t), \qquad t \in [-r, 0], \tag{2}$$

where $f\colon [0,T] \times C([-r,0],E) \to E$ is a given function, $(0<r<\infty)$, $A\colon D(A) \subset E \to E$ is a nondensely defined closed linear operator on E, $\phi \in C([-r,0],E)$, the control function given in $u(\cdot) \in L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space and $\overline{D(A)}$ a real Banach space with norm $|\cdot|$. Finally B is a bounded linear operator from U to $\overline{D(A)}$. We denote by y_t the element of $C([-r,0],E)$ defined by $y_t(\theta) = y(t+\theta)$, $\theta \in [-r,0]$. Here $y_t(\cdot)$ represents the history of the state from time $t-r$ up to the present time t.

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For the problem (1)-(2) we prove a controllability result, by using Leray–Schauder alternative.

In Section 4, we study the first order semilinear neutral functional differential equations of the form

$$\frac{d}{dt}\big[y(t) - g(t, y_t)\big] - Ay(t) = f(t, y_t) + (Bu)(t), \quad \text{a.e. } t \in J := [0, T] \tag{3}$$

$$y(t) = \phi(t), \qquad t \in [-r, 0], \tag{4}$$

where f, I_k, A and ϕ are as in problem (1)–(2) and $g : [0, T] \times C([-r, 0], E) \to E$ is a given function. In the case where A is a densely linear operator generating a semigroup has been investigated by several authors, (see, for instance, [2, 3, 4, 6, 7, 8]). In Section 5 we present an example while in the final Section 6 we discuss some possible extensions and generalizations.

The main theorems of this paper extend to a nondensely defined operators similar problems considered in the above listed papers. For more details and examples on nondensely defined operators we refer to the survey paper by Da Prato and Sinestrari [13] and Ezzinbi and Liu [16]. Our approach is based on the Leray–Schauder alternative (see [15]). Related exact controllability and approximate controllability problems have been studied by authors including Balachandran and Dauer [5], Dauer and Mahmudov [14], Gatsori [17] and McKibben [21].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

$C([-r, T], E)$ is the Banach space of all continuous functions from $[0, T]$ into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : -r \le t \le T\}.$$

Also $C([-r, 0], E)$ is the Banach space of all continuous functions from $[-r, 0]$ into E with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \le \theta \le 0\}.$$

$B(E)$ is the Banach space of all linear bounded operator from E into E with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral, see for instance, Yosida [23].) $L^1(J,E)$ denotes the Banach space of functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^T |y(t)|\, dt.$$

Definition 1 ([1]). We say that a family $\{S(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is an *integrated semigroup family* if

(1) $S(0) = 0$;

(2) $t \to S(t)$ is strongly continuous;

(3) $S(s)S(t) = \int_0^s \big(S(t+r) - S(r)\big)\, dr$ for all $t, s \geq 0$.

Definition 2 ([18]). An operator A is called a *generator of an integrated semigroup* if there exists an $\omega \in \mathbb{R}$ such that $(\omega,\infty) \subset \rho(A)$ ($\rho(A)$ is the resolvent set of A), and if there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of bounded operators such that $S(0) = 0$ and $(\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t)dt$ exists for all λ with $\lambda > \omega$.

Lemma 1 ([1]). *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $x \in E$ and $t \geq 0$,*

$$\int_0^t S(s)x\, ds \in D(A) \quad \textit{and} \quad S(t)x = A\int_0^t S(s)x\, ds + tx.$$

Definition 3. We say that a linear operator A satisfies the "Hille–Yosida condition" if there exist $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega,\infty) \subset \rho(A)$ and

$$\sup\big\{(\lambda-\omega)^n |(\lambda I - A)^{-n}| : n \in \mathbb{N}, \lambda > \omega\big\} \leq M.$$

An important special case is when the integrated semigroup is locally Lipschitz continuous (with respect to time).

Definition 4. An integrated semigroup $(S(t))_{t\geq 0}$ is *called locally Lipschitz continuous*, if for all $\tau > 0$ there exists a constant $k(\tau) > 0$ such that

$$\|S(t) - S(\tau)\| \leq k(\tau)|t-\tau|, \quad \text{for all } t, s \in [0,\tau].$$

If A is the generator of an integrated semigroup $(S(t))_{t\geq 0}$ which is locally Lipschitz, then from [1], $S(\cdot)x$ is continuously differentiable if and only if $x \in \overline{D(A)}$ and $(S'(t))_{t\geq 0}$ is a C_0 semigroup on $\overline{D(A)}$. The following theorem shows that the Hille–Yosida condition characterizes generators of locally Lipschitz continuous integrated semigroup.

Theorem 1 ([18]). *The following assertions are equivalent:*

(HY) *A satisfies the Hille–Yosida condition.*

(H1) *A is the generator of locally Lipschitz continuous integrated semigroup.*

3. Controllability of semilinear functional differential equations

The main result of this section concerns the IVP (1)–(2). Before stating and proving this one, we give first the definition of its integral solution.

Definition 5. A function $y \in C([-r,T],E)$ is said to be an *integral solution* of (1)–(2) if y is the solution of the integral equation

$$y(t) = \phi(0) + A\int_0^t y(s)\,ds + \int_0^t f(s,y_s)\,ds + \int_0^t (Bu)(s)\,ds$$
$$\int_0^t y(s)\,ds \in D(A), \quad t\in[0,T] \quad \text{and} \quad y(t)=\phi(t), \quad t\in[-r,0].$$

From the definition it follows that $y(t)\in\overline{D(A)}$, $t\geq 0$. Moreover, y satisfies the following variation of constants formula:

$$y(t) = S'(t)y_0 + \frac{d}{dt}\int_0^t S(t-s)f(s,y_s)\,ds + \frac{d}{dt}\int_0^t S(t-s)(Bu)(s)\,ds, \quad t\geq 0. \quad (5)$$

Let $B_\lambda = \lambda R(\lambda,A) := \lambda(\lambda I - A)^{-1}$, then for all $x\in\overline{D(A)}$, $B_\lambda x\to x$ as $\lambda\to\infty$. As a consequence, if y satisfies (5), then

$$y(t) = S'(t)y_0 + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda\big[f(s,y_s)+(Bu)(s)\big]\,ds, \quad t\geq 0.$$

Definition 6. The system (1)–(2) is said to be *controllable* on the interval $[-r,T]$, if for every continuous initial function $\phi\in C([-r,0],\overline{D(A)})$ and every $y_1\in E$, there exists a control $u\in L^2(J,U)$, such that the integral solution $y(t)$ of (1)–(2) satisfies $y(T)=y_1$.

Let us introduce the following hypotheses:

(H2) For each $t\in J$, the function $f(t,\cdot)$ is continuous and for each y, the function $f(\cdot,y)$ is measurable.

(H3) $S'(t)$ is compact semigroup in $\overline{D(A)}$ wherever $t>0$.

(H4) The linear operator $W: L^2(J,U)\to E$, defined by

$$Wu = \int_0^T S'(T-s)Bu(s)\,ds,$$

has an invertible operator W^{-1} which takes values in $L^2(J,U)\setminus\ker W$ and there exist positive constants M_1, M_2 such that $\|B\|\leq M_1$ and $\|W^{-1}\|\leq M_2$.

(H5) There exist a continuous non-decreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$, $\bar{p} \in L^1([0,T],\mathbb{R}_+)$ such that $|f(t,u)| \leq \bar{p}(t)\psi(\|u\|)$ for each $(t,u) \in [0,T]\times C([-r,0],E)$ and there exists a constant $M_* > 0$ with

$$\frac{M_*}{Q + Me^{\omega T}\left[1 + MM_1M_2Te^{\omega T}\right]\psi(M_*)\int_0^T e^{-\omega s}\bar{p}(s)\,ds} > 1,$$

where $Q = e^{\omega T}M\left[\|\phi\| + M_1M_2T(|y_1| + Me^{\omega T}\|\phi\|)\right]$.

Theorem 2. *Assume that hypotheses* (HY)–(H5) *hold. Then the IVP* (1)–(2) *is controllable on* $[-r,T]$.

Proof. Using hypothesis (H4) we define the control

$$u_y(t) = W^{-1}\left[y_1 - S'(T)\phi(0) - \lim_{\lambda\to+\infty}\int_0^T S'(T-s)B_\lambda f(s,y_s)\,ds\right](t).$$

Transform the problem (1)–(2) into a fixed point problem. Consider the operator $N : C([-r,T],\overline{D(A)}) \longrightarrow C([-r,T],\overline{D(A)})$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0] \\ S'(t)\phi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)f(s,y_s)\,ds & \\ + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)(Bu_y)(s)\,ds\,, & \text{if } t \in [0,T]. \end{cases}$$

Remark. Clearly the fixed points of N are integral solutions to problem (1)–(2).

We shall show that N is completely continuous. The proof will be given in several steps.

`Step 1`: *N is continuous.* Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([-r,T],\overline{D(A)})$. Then

$$\begin{aligned} |N(y_n)(t) - N(y)(t)| &\leq \left|\frac{d}{dt}\int_0^t S(t-s)\big[f(s,(y_n)_s) - f(s,y_s)\big]\,ds\right| \\ &\quad + \left|\frac{d}{dt}\int_0^t S(t-s)\big|(Bu_{y_n})(s) - (Bu_y)(s)\big|\,ds\right| \\ &\leq Me^{\omega T}\int_0^T e^{-\omega s}\big|f(s,(y_n)_s) - f(s,y_s)\big|\,ds \\ &\quad + Me^{\omega T}\int_0^T e^{-\omega s}\big|(Bu_{y_n})(s) - (Bu_y)(s)\big|\,ds \\ &\leq Me^{\omega T}\big(1 + M_1M_2TMe^{\omega T}\big) \\ &\quad \times \int_0^T e^{-\omega s}\big|f(s,(y_n)_s) - f(s,y_s)\big|\,ds. \end{aligned}$$

Since f is continuous, we have by the Lebesgue dominated convergence theorem

$$\|N(y_n) - N(y)\|_\infty \le Me^{\omega T}\big(1 + M_1M_2TMe^{\omega T}\big)\|f(\cdot, y_n) - f(\cdot, y)\|_{L^1} \to 0$$

as $n \to \infty$. Thus N is continuous.

`Step 2:` *N maps bounded sets into bounded sets in $C([-r, T], \overline{D(A)})$.* Indeed, it is enough to show that for any $q > 0$ there exists a positive constant ℓ such that for each $y \in \mathcal{B}_q = \{y \in C([-r, T], \overline{D(A)}) : \|y\|_\infty \le q\}$ we have $\|N(y)\|_\infty \le \ell$. Then we have for each $t \in [0, T]$

$$\begin{aligned}
|N(y)(t)| &= \left|S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)f(s, y_s)\,ds + \frac{d}{dt}\int_0^t S(t-s)(Bu_y)(s)\,ds\right| \\
&\le Me^{\omega T}\|\phi\| + Me^{\omega T}\int_0^T e^{-\omega s}p(s)\psi(q)\,ds \\
&\quad + Me^{\omega T}\left[M_1M_2T\left(|y_1| + Me^{\omega T}\|\phi\| + Me^{\omega T}\int_0^T e^{-\omega s}p(s)\psi(q)\,ds\right)\right].
\end{aligned}$$

Thus

$$\begin{aligned}
&\|N(y)\|_\infty \\
&\le Me^{\omega T}\big[q + \psi(q)\|p\|_{L^1} + M_1M_2T(|y_1| + Me^{\omega T}q + Me^{\omega T}\psi(q)\|p\|_{L^1})\big] := \ell.
\end{aligned}$$

`Step 3:` *N maps bounded sets into equicontinuous sets of $C([-r, T], \overline{D(A)})$.* Let $0 < \tau_1 < \tau_2 \in J$, $\tau_1 < \tau_2$ and $\mathcal{B}_q$ be a bounded set of $C([-r, T], \overline{D(A)})$ as in Step 2. Let $y \in \mathcal{B}_q$ then for each $t \in J$ we have

$$N(y)(t) = S'(t)\phi(0) + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda\big[f(s, y_s) + (Bu_y)(s)\big]\,ds.$$

Then

$$\begin{aligned}
\big|N(y)(\tau_2) - N(y)(\tau_1)\big| \le\ & \big|[S'(\tau_2) - S'(\tau_1)]\phi(0)\big| \\
& + \left|\lim_{\lambda\to\infty}\int_{\tau_1}^{\tau_2} S'(\tau_2 - s)B_\lambda f(s, y_s)\,ds\right| \\
& + \left|\lim_{\lambda\to\infty}\int_0^{\tau_1} [S'(\tau_2 - s) - S'(\tau_1 - s)]B_\lambda f(s, y_s)\,ds\right| \\
& + \left|\lim_{\lambda\to\infty}\int_{\tau_1}^{\tau_2} S'(\tau_2 - s)B_\lambda (Bu_y)(s)\,ds\right| \\
& + \left|\lim_{\lambda\to\infty}\int_0^{\tau_1} [S'(\tau_2 - s) - S'(\tau_1 - s)]B_\lambda (Bu_y)(s)\,ds\right|.
\end{aligned}$$

The right-hand side tends to zero as $\tau_1 \to \tau_2$, since $S'(t)$ is strongly continuous and the compactness of $S'(t)$, $t > 0$ implies the continuity in the uniform operator topology. The cases $\tau_1 < \tau_2 < 0$ and $\tau_1 < 0 < \tau_2$ are obvious.

As a consequence of Steps 1 to 3 together with the Arzelá–Ascoli Theorem it suffices to show that the operator N maps $\mathcal{B}_q$ into a precompact set in $\overline{D(A)}$. Let $0 < t \leq T$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in \mathcal{B}_q$ we define

$$N_\epsilon(y)(t) = S'(t)\phi(0) + S'(\epsilon) \lim_{\lambda\to\infty} \int_0^{t-\epsilon} S'(t-s-\epsilon)B_\lambda\big[f(s,y_s) + (Bu_y)(s)\big]\, ds.$$

Since $S'(t)$ is a compact operator, the set $H_\epsilon(t) = \{N_\epsilon(y)(t) : y \in \mathcal{B}_q\}$ is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover, for every $y \in \mathcal{B}_q$ we have

$$|N_\epsilon(y)(t) - N(y)(t)| \leq \left| \lim_{\lambda\to\infty} \int_{t-\epsilon}^{t} S'(t-s)B_\lambda\big[f(s,y_s) + (Bu_y)(s)\big]\, ds \right|.$$

Therefore there are precompact sets arbitrarily close to the set $\{N(y)(t) : y \in \mathcal{B}_q\}$. Hence the set $\{N(y)(t) : y \in \mathcal{B}_q\}$ is precompact in $\overline{D(A)}$. Thus we can conclude that $N : C([-r,T], \overline{D(A)}) \to C([-r,T], \overline{D(A)})$ is a completely continuous operator.

`Step 4:` *A priori bounds on solutions.* Let y by a possible solutions of the problem (1)–(2), then we get

$$\begin{aligned} |y(t)| &\leq Me^{\omega t}\|\phi\| + Me^{\omega t}\int_0^t e^{-\omega s}p(s)\psi(\|y_s\|)\, ds + Me^{\omega t}\int_0^t e^{-\omega s}|(Bu_y)(s)|\, ds \\ &\leq Me^{\omega t}\|\phi\| + Me^{\omega t}\int_0^t e^{-\omega s}\bar{p}(s)\psi(\|y_s\|)\, ds \\ &\quad + Me^{\omega t}M_1M_2T\left(|y_1| + Me^{\omega T}\|\phi\| + Me^{\omega T}\int_0^T e^{-\omega s}\bar{p}(s)\psi(\|y_s\|)\, ds\right). \end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq T.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0,T]$, by the previous inequality we have for $t \in [0,T]$

$$\begin{aligned} \mu(t) &\leq e^{\omega t}M\|\phi\| + Me^{\omega t}\int_0^t e^{-\omega s}\bar{p}(s)\psi\big(\mu(s)\big)\, ds \\ &\quad + MM_1M_2Te^{\omega t}\left(|y_1| + Me^{\omega T}\|\phi\| + Me^{\omega T}\int_0^T e^{-\omega s}\bar{p}(s)\psi\big(\mu(s)\big)\, ds\right). \end{aligned}$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds. Then we have

$$\begin{aligned} \mu(t) &\leq e^{\omega T}M\|\phi\| + Me^{\omega T}\int_0^T e^{-\omega s}\bar{p}(s)\psi\big(\mu(s)\big)\, ds \\ &\quad + MM_1M_2Te^{\omega T}\left(|y_1| + Me^{\omega T}\|\phi\| + Me^{\omega T}\int_0^T e^{-\omega s}\bar{p}(s)\psi\big(\mu(s)\big)\, ds\right). \end{aligned}$$

Consequently,

$$\frac{\|y\|_\infty}{Q + Me^{\omega T}\left[1 + MM_1M_2Te^{\omega T}\right]\psi(\|y\|_\infty)\int_0^T e^{-\omega s}\bar{p}(s)\,ds} \leq 1.$$

Then by (H5), there exists an M_* such that $\|y\|_\infty \neq M_*$. Set

$$U = \big\{y \in C\big([-r,T], \overline{D(A)}\big) : \|y\|_\infty < M_*\big\}.$$

The operator $N : \overline{U} \to C([-r,T],E)$ is continuous and completely continuous. From the choice of U there is no $y \in \partial U$ such that $y = \sigma N(y)$ for some $\sigma \in (0,1)$. As a consequence of the nonlinear alternative of Leray–Schauder type ([15]) we deduce that N has a fixed point y in $\overline{U}$, which is an integral solution of the problem (1)–(2). Thus the system (1)–(2) is controllable on $[-r,T]$. □

4. Controllability of semilinear neutral functional differential equations

In this section we study the problem (3)–(4). We give first the definition of integral solution of the problem (3)–(4).

Definition 7. A function $y \in C([-r,T],E)$ is said *integral solution* of (3)–(4) if y is the solution of the integral equation

$$y(t) = \phi(0) - g(0,\phi(0)) + g(t,y_t) + A\int_0^t y(s)\,ds + \int_0^t f(s,y_s)\,ds + \int_0^t (Bu)(s)\,ds$$
$$\int_0^t y(s)\,ds \in D(A), \quad t \in [0,T] \qquad \text{and} \qquad y(t) = \phi(t), \quad t \in [-r,0].$$

Definition 8. The system (3)–(4) is said to be *controllable* on the interval $[-r,T]$, if for every continuous initial function $\phi \in C([-r,0], \overline{D(A)})$ and every $y_1 \in E$, there exists a control $u \in L^2(J,U)$, such that the integral solution $y(t)$ of (3)–(4) satisfies $y(T) = y_1$.

Theorem 3. *Assume* (HY)–(H4) *and the conditions*

(A1) *There exist constants* $0 \leq c_1 < 1, c_2 \geq 0$ *such that*

$$|g(t,u)| \leq c_1\|u\| + c_2, \quad t \in [0,T],\ u \in C\big([-r,0], \overline{D(A)}\big).$$

(A2) *The function g is completely continuous and for any bounded set B in $C([-r,T], \overline{D(A)})$, the set $\{t \to g(t,y_t) : y \in B\}$ is equicontinuous in $C([0,T], \overline{D(A)})$.*

(A3) *There exist a continuous non-decreasing function* $\psi : [0, \infty) \to (0, \infty)$ *and* $p_* \in L^1([0, T], \mathbb{R}_+)$ *such that*

$$|f(t, u)| \le p_*(t)\psi(\|u\|) \quad \text{for each } (t, u) \in [0, T] \times C([-r, 0], E),$$

and there exists a constant $M_{**} > 0$ *with*

$$\frac{M_{**}}{\frac{1}{1-c_1}\left[Q' + Me^{\omega T}\left[1 + MM_1M_2Te^{\omega T}\right]\psi(M_{**})\int_0^T e^{-\omega s}p_*(s)\,ds\right]} > 1,$$

where $Q' = Q + Me^{\omega T}(c_1\|\phi\| + c_2) + c_2$.

are satisfied. Then the IVP (3)–(4) *is controllable on* $[-r, T]$.

Proof. Transform the problem (3)–(4) into a fixed point problem. Consider the operator $\overline{N} : C([-r, T], \overline{D(A)}) \to C([-r, T], \overline{D(A)})$ defined by

$$\overline{N}(y)(t) := \begin{cases} \phi(t), & \text{if } t \in [-r, 0] \\ S'(t)\big[\phi(0) - g(0, \phi(0))\big] \\ + g(t, y_t) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)f(s, y_s)\,ds & \text{if } t \in [0, T]. \\ + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)(Bu_y)(s)\,ds, \end{cases}$$

where u_y is the control defined in Theorem 2. Let $\widetilde{N} : C([-r, T], \overline{D(A)}) \to C([-r, T], \overline{D(A)})$ defined by

$$\widetilde{N}(y)(t) := \begin{cases} \phi(t), & \text{if } t \in [-r, 0] \\ S'(t)\phi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)f(s, y_s)\,ds & \text{if } t \in [0, T]. \\ + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)(Bu_y)(s)\,ds, \end{cases}$$

As in the proof of Theorem 2 we can prove that $\widetilde{N}$ is completely continuous and by using $(A2)$ $\overline{N}$ is completely continuous.

Now we prove the existence of a priori bounds on solutions. Let $y \in \mathcal{E}(\overline{N}) := \{y \in C([-r, T], E) : y = \sigma\overline{N}(y) \text{ for some } 0 < \sigma < 1\}$, then $\sigma\overline{N}(y) = y$ for some $0 < \sigma < 1$ and

$$y(t) = \sigma\left[S'(t)\big[(\phi(0) - g(0, \phi(0))\big] + g(t, y_t) + \frac{d}{dt}\int_0^t S(t-s)f(s, y_s)\,ds \right. \\ \left. + \frac{d}{dt}\int_0^t S(t-s)(Bu_y)(s)\,ds\right], \quad t \in [0, T].$$

This implies by (H4), (HY) and (A3) that for each $t \in [0,T]$ we have

$$\begin{aligned}|y(t)| \le{}& Me^{\omega t}\big[(1+c_1)\|\phi\| + c_2\big] + c_1\|y_t\| + c_2\\ &+ Me^{\omega t}\int_0^t e^{-\omega s}p_*(s)\psi(\|y_s\|)\,ds\\ &+ Me^{\omega t}M_1M_2T\left(|y_1| + Me^{\omega T}\|\phi\| + Me^{\omega T}\int_0^T e^{-\omega s}p_*(s)\psi(\|y_s\|)\,ds\right).\end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le T.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0,T]$, by the previous inequality we have for $t \in [0,T]$

$$\begin{aligned}(1-c_1)\mu(t) \le{}& Me^{\omega t}[(1+c_1)\|\phi\| + c_2] + c_2\\ &+ Me^{\omega t}\int_0^t e^{-\omega s}p_*(s)\psi(\mu(s))\,ds\\ &+ Me^{\omega t}M_1M_2T\left(|y_1| + Me^{\omega T}\|\phi\| + Me^{\omega T}\int_0^T e^{-\omega s}p_*(s)\psi(\mu(s))\,ds\right),\end{aligned}$$

or

$$\begin{aligned}\mu(t) \le{}& \frac{1}{1-c_1}\big[Me^{\omega t}[(1+c_1)\|\phi\| + c_2] + c_2\big]\\ &+ \frac{1}{1-c_1}Me^{\omega t}\int_0^T e^{-\omega s}p_*(s)\psi(\mu(s))\,ds\\ &+ \frac{1}{1-c_1}Me^{\omega t}M_1M_2T\left(|y_1| + Me^{\omega T}\|\phi\| + Me^{\omega T}\int_0^T e^{-\omega s}p_*(s)\psi(\mu(s))\,ds\right).\end{aligned}$$

Consequently

$$\frac{\|y\|_\infty}{\frac{1}{1-c_1}\left[Q' + Me^{\omega T}\left[1 + MM_1M_2Te^{\omega T}\right]\psi(\|y\|_\infty)\int_0^T e^{-\omega s}p_*(s)\,ds\right]} \le 1.$$

Then by $(A3)$, there exists an M_* such that $\|y\|_\infty \ne M_*$. Set

$$U = \{y \in C([-r,T], \overline{D(A)}) : \|y\|_\infty < M_*\}.$$

The operator $\overline{N} : \overline{U} \to C([-r,T],E)$ is continuous and completely continuous. From the choice of U there is no $y \in \partial U$ such that $y = \sigma\overline{N}(y)$ for some $\sigma \in (0,1)$. As a consequence of the nonlinear alternative of Leray–Schauder type ([15]) we deduce that $\overline{N}$ has a fixed point y in $\overline{U}$, which is an integral solution of the problem (3)–(4). Thus the system (3)–(4) is controllable on $[-r,T]$. □

5. An example

To apply the previous result, we consider the following partial differential equation:

$$\frac{\partial}{\partial t}v(t,x) = \triangle v(t,x) + f(t, v(t,x)) + (Bu)(t), \quad 0 \le t \le T, \quad x \in \Omega \tag{6}$$
$$v(t,x) = 0, \quad 0 \le t \le T, \quad x \in \partial\Omega \tag{7}$$
$$v(0,x) = v_0(x) \quad x \in \Omega, \tag{8}$$

where Ω is a bounded open set of $\mathbb{R}^n$ with regular boundary $\partial\Omega$, $v_0 \in C(\Omega, \mathbb{R}^n)$, f a single function, B as in (1) and $\triangle = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$. Consider $E = C(\overline{\Omega})$, the Banach space of continuous function on $\overline{\Omega}$ with values in $\mathbb{R}$. Define the linear operator A on E by

$$Az = \triangle z, \quad \text{in} \quad D(A) = \left\{ z \in C(\overline{\Omega}) : z = 0 \text{ on } \partial\Omega,\ \triangle z \in C(\overline{\Omega}) \right\}$$

Now, we have

$$\overline{D(A)} = C_0(\overline{\Omega}) = \left\{ v \in C(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega \right\} \neq C(\overline{\Omega}).$$

It is well known from [13] that A is sectorial, $(0, +\infty) \subseteq \rho(A)$ and for $\lambda > 0$ $|R(\lambda, A)| \le \frac{1}{\lambda}$. It follows that A generates an integrated semigroup $(S(t))_{t \ge 0}$ and that $|S'(t)| \le e^{-\mu t}$ for $t \ge 0$ for some constant $\mu > 0$, and A satisfied the Hille–Yosida condition. The partial differential equations (6)–(8) can be reformulated as the abstract semilinear differential equations (1)–(2) in E, where $F : [0,T] \times D(A) \to E$ is the Nemyskii operator given by $F(t,y)(x) = f(t, y(t,x))$. If we assume that f satisfies the hypotheses (H2)–(H5), then the integral solution of (6)–(8) exists by Theorem 2.

6. Concluding remarks

In this section we will discuss possible extensions and generalizations of the results of the previous sections.

6.1. Non-local functional differential equations. An obvious extension of the results of Section 3 concerns controllability results for functional differential equations with non-local initial conditions of the form

$$y'(t) - Ay(t) = f(t, y_t) + (Bu)(t), \quad \text{a.e. } t \in J = [0,T] \tag{9}$$
$$y(t) + h_t(y) = \phi(t), \quad t \in [-r, 0], \tag{10}$$

where $h_t : C([-r,0],D(A)) \to D(A)$ is a given function. The non-local condition can be applied in physics with better effect than the classical initial condition $y(0) = y_0$. For example, $h_t(y)$ may be given by

$$h_t(y) = \sum_{i=1}^{p} c_i y(t_i + t), \quad t \in [-r,0],$$

where $c_i, i = 1,\ldots,p$, are given constants and $0 < t_1 < \ldots < t_p \le T$. At time $t = 0$, we have

$$h_0(y) = \sum_{i=1}^{p} c_i y(t_i).$$

Nonlocal conditions were initiated by Byszewski [12] to which we refer for motivation and other references.

Definition 9. A function $y \in C([-r,T],E)$ is said to be an *integral solution* of (9)–(10) if y is the solution of integral equation

$$y(t) = \phi(0) - h_0(y) + A\int_0^t y(s)\,ds + \int_0^t f(s,y_s)\,ds + \int_0^t (Bu)(s)\,ds$$
$$\int_0^t y(s)\,ds \in D(A), \quad t \in [0,T] \quad \text{and} \quad y(t) + h_t(y) = \phi(t), \quad t \in [-r,0],$$

Definition 10. The system (9)–(10) is said to be *non-locally controllable* on the interval $[-r,T]$, if for every continuous initial function $\phi \in C([-r,0],\overline{D(A)})$ and every $y_1 \in E$, there exists a control $u \in L^2(J,U)$, such that the integral solution $y(t)$ of (9)–(10) satisfies $y(T) + h_T(y) = y_1$.

Theorem 4. *Assume that hypotheses* (HY)–(H5) *hold and moreover*

(H6) *The function h is continuous with respect to t, and there exists a constant $\beta > 0$ such that $\|h_t(u)\| \le \beta$, $u \in C([-r,0],E)$, and for each $k > 0$ the set $\{\phi(0) - h_0(y) : y \in C([-r,0],E),\ \|y\| \le k\}$ is precompact in E.*

Then the IVP (9)–(10) *is nonlocally controllable on* $[-r,T]$.

We omit the proof, since its steps are parallel to that of Theorem 2.

6.2. Impulsive functional differential equations. In this subsection we discuss another generalization of the results of Section 3 to first order impulsive functional differential equations of the form

$$y'(t) - Ay(t) = f(t,y_t) + (Bu)(t), \quad \text{a.e. } t \in J := [0,T],\ t \ne t_k, k = 1,\ldots,m \quad (11)$$
$$\Delta y|_{t=t_k} := y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1,\ldots,m \quad (12)$$
$$y(t) = \phi(t), \quad t \in [-r,0], \quad (13)$$

where $f : J \times \mathcal{D} \longrightarrow E$ is a given function,

$$\mathcal{D} = \left\{ \psi : [-r, 0] \to E : \begin{array}{l} \psi \text{ is continuous everywhere except for a finite} \\ \text{number of points } s \text{ at which } \psi(s) \text{ and the} \\ \text{right limit } \psi(s^+) \text{ exist and } \psi(s^-) = \psi(s) \end{array} \right\},$$

$\phi \in \mathcal{D}$, $(0 < r < \infty)$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, $I_k \in C(E, E)$ $(k = 1, 2, \ldots, m)$, and E a real Banach space with norm $|\cdot|$.

For any continuous function y defined on the interval $[-r, T] \setminus \{t_1, \ldots, t_m\}$ and any $t \in J$, we denote by y_t the element of $\mathcal{D}$ defined by $y_t(\theta) = y(t+\theta)$, $\theta \in [-r, 0]$. For $\psi \in \mathcal{D}$ the norm of ψ is defined by

$$\|\psi\|_{\mathcal{D}} = \sup\{|\psi(\theta)|,\ \theta \in [-r, 0]\}.$$

In order to the define the solutions of the above problem, we shall consider the space

$$PC\big([-r, T], E\big) = \left\{ y : [-r, T] \to E : \begin{array}{l} y(t) \text{ is continuous everywhere} \\ \text{except for some } t_k \text{ at which} \\ y(t_k^-) \text{ and } y(t_k^+),\ k = 1, \ldots, m, \\ \text{exist and } y(t_k^-) = y(t_k) \end{array} \right\}.$$

Obviously, for any $t \in [0, T]$ and $y \in PC([-r, T], E)$, we have $y_t \in \mathcal{D}$ and $PC([-r, T], E)$ is a Banach space with the norm

$$\|y\| = \sup\{|y(t)| : t \in [-r, T]\}.$$

The theory of impulsive differential equations is emerging as an important area of investigation since it is much richer than the corresponding theory of differential equations; see the monograph of Lakshmikantham et al. [19].

Let us start by defining what we mean by a solution of problem (11)–(13).

Definition 11. A function $y \in PC([-r, T], E)$ is said *integral solution* of (11)–(13) if y is the solution of integral equation

$$y(t) = \phi(0) + A\int_0^t y(s)\,ds + \int_0^t f(s, y_s)\,ds + \int_0^t (Bu)(s)\,ds + \sum_{0<t_k<t} I_k\big(y(t_k^-)\big)$$

$$\int_0^t y(s)\,ds \in D(A), \quad t \in [0, T], \quad \text{and} \quad y(t) = \phi(t), \quad t \in [-r, 0].$$

Definition 12. The system (11)–(13) is said to be *controllable* on the interval $[-r, T]$, if for every continuous initial function $\phi \in \mathcal{D}$ and every $y_1 \in E$, there exists a control $u \in L^2(J, U)$, such that the integral solution $y(t)$ of (11)–(13) satisfies $y(T) = y_1$.

We can prove, modifying the steps of Theorem 2, that the IVP (11)–(13) is controllable on the interval $[-r, T]$, if the assumptions (HY)–(H5) are satisfied and moreover

(H7) $I_k : E \to \overline{D(A)}, k = 1, \ldots m$ are continuous and there exist constants $d_k, k = 1, \ldots, m$ such that $|I_k(x)| \le d_k$, $x \in E$.

6.3. Non-local impulsive functional differential equations. Finally we can combine the above results to get controllability results for first order impulsive functional differential equations with non-local conditions of the form

$$y' - Ay = f(t, y_t) + (Bu)(t), \quad \text{a.e. } t \in J := [0, T], t \neq t_k, k = 1, \dots, m \quad (14)$$

$$\Delta y|_{t=t_k} := y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \dots, m \quad (15)$$

$$y(t) + h_t(y) = \phi(t), \quad t \in [-r, 0]. \quad (16)$$

Similar remarks hold also for neutral functional differential equations. For recent results related to the subject we refer to [9, 10, 11].

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