Multiplicative Functions in Short Intervals, with Applications

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The understanding of the behaviour of multiplicative functions in short intervals has significantly improved during the past decade. This has also led to several applications, in particular concerning correlations of multiplicative functions.

1 Introduction

Let us start by defining the key players. A function $f : \mathbb{N} \to \mathbb{C}$ is said to be multiplicative if $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. We define the Liouville function $\lambda : \mathbb{N} \to \{-1, 1\}$ by $\lambda(n) := (-1)^k$ when n has k prime factors (counted with multiplicity). For instance, $\lambda(45) = \lambda(3 \cdot 3 \cdot 5) = (-1)^3 = -1$. The function $\lambda(n)$ is clearly multiplicative.

It is well known that the average value of $\lambda(n)$ is 0, i.e.

$$\lim_{X\to\infty} \frac{1}{X} \sum_{n\leq X} \lambda(n) = 0. \tag{1}$$

In other words, about half of the numbers have an odd number of prime factors and half of the numbers have an even number of prime factors. The result (1) is actually equivalent to the prime number theorem, asserting that

$$\lim_{X\to\infty} \frac{|\{p \leq X : p \in \mathbb{P}\}|}{X/\log X} = 1, \tag{2}$$

where $\mathbb{P}$ denotes the set of prime numbers. In this article, the letter p will always denote a prime.

It will be very convenient for us to use $o(1)$ and $O(1)$ notations, so that $A = o(B)$ means that $|A|/B \to 0$ for $X \to \infty$ and $A = O(B)$ means that $|A| \leq CB$ for some constant $C > 0$ depending only on subscripts of O. In this notation (1) and (2) can be written as

$$\sum_{n\leq X} \lambda(n) = o(X) \tag{3}$$

and

$$\sum_{p\leq X} 1 = \frac{X}{\log X} + o\left(\frac{X}{\log X}\right). \tag{4}$$

Before discussing the Liouville function further, let us define another important object: write $\zeta : \mathbb{C} \to \mathbb{C}$ for the Riemann zeta function which is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1} \quad \text{for } \Re s > 1, \tag{5}$$

where $\Re s$ denotes the real part of s. A series of the type $\sum_{n\in\mathbb{N}} a_n n^{-s}$ with $a_n \in \mathbb{C}$ is called a Dirichlet series. The ζ-function can be analytically continued to the whole complex plane apart from a simple pole at $s = 1$.

It is easy to see that $\zeta(s)$ has no zeros with $\Re s > 1$, and furthermore for $\Re s < 0$ the only zeros are the "trivial zeros" at negative even integers. The remaining zeros with $0 \leq \Re s \leq 1$ are called the non-trivial zeros. One of the most famous open problems in mathematics, the Riemann hypothesis, asserts that all these non-trivial zeros satisfy $\Re s = 1/2$.

The zeros of the zeta function are closely related to the behaviour of the Liouville function. This relation stems from the fact that, for $\Re s > 1$

$$\frac{1}{\zeta(s)} = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\lambda(n)\mathbf{1}_{n \text{ square-free}}}{n^s}, \tag{6}$$

where $\mathbf{1}_{n \text{ square-free}}$ denotes the characteristic function of the set of integers that are not divisible by a square of a prime.

The function $\mu(n) := \lambda(n)\mathbf{1}_{n \text{ square-free}}$ is called the Möbius function and its behaviour is very similar to that of the Liouville function. Consequently, the zeros of $\zeta(s)$ correspond to the poles of the Dirichlet series $\sum_{n\in\mathbb{N}} \mu(n)n^{-s}$ that is closely related to the Liouville function.

One can show through (6) that (3) (and thus also the prime number theorem (4)) is equivalent to the fact that the Riemann zeta function has no zeros with $\Re s = 1$. The equivalence with the prime number theorem stems from the fact that, for $\Re s > 1$, one has

$$-\frac{\zeta'}{\zeta}(s) = \sum_{k=1}^{\infty} \sum_{p\in\mathbb{P}} \frac{\log p}{p^{ks}},$$

so the zeros of the zeta-function also correspond to the poles of a Dirichlet series that is closely related to the characteristic function of the primes.

In general, the Liouville function is expected to behave more or less randomly. In particular, we expect that it has so-called square-root cancellation, i.e. one has, for all $X \geq 2$,

$$\sum_{n\leq X} \lambda(n) = O_\epsilon(X^{1/2+\varepsilon}) \quad \text{for any } \varepsilon > 0. \tag{7}$$

The conjecture (7) is in fact equivalent to the Riemann hypothesis, and proving (7) even with $X^{1/2+\varepsilon}$ replaced by $X^{1-\delta}$ with a small fixed δ seems to be a distant dream which would correspond to the Riemann zeta function having no zeros with real part $\geq 1-\delta$ for some fixed $\delta > 0$. The best result currently is that

$$\sum_{n\leq X} \lambda(n) = O\left(X \exp\left(- \frac{C(\log X)^{3/5}}{(\log\log X)^{1/5}}\right)\right) \tag{8}$$

for some absolute constant $C > 0$. This follows from the Vinogradov–Korobov zero-free region for the Riemann zeta function that has been essentially unimproved for sixty years.

2 Short intervals

A natural question is whether the average of the Liouville function is still $o(1)$ if taken over short segments; one can ask how slowly H can tend to infinity with X so that we are guaranteed to have

$$\sum_{X<n\leq X+H} \lambda(n) = o(H), \tag{9}$$

so that in the segment $(X, X + H]$ roughly half of the numbers have an even and half of the numbers have an odd number of prime factors.

The bound (8) together with the triangle inequality immediately implies (9) when

$$H \geq X \exp\left(-\frac{C(\log X)^{3/5}}{2(\log\log X)^{1/5}}\right).$$

However, one can show this for much shorter intervals. In 1972, Huxley proved prime number theorem in short intervals by showing that, for any $\varepsilon > 0$,

$$\sum_{X<p\leq X+H} 1 = \frac{H}{\log X} + o\left(\frac{H}{\log X}\right) \quad \text{for } H \geq X^{7/12+\varepsilon}.$$

Subsequently, in 1976 Motohashi [9] and Ramachandra [12] independently showed that Huxley's ideas also work in the case of the Liouville function, showing that, for any $\varepsilon > 0$, (9) holds for $H \geq X^{7/12+\epsilon}$.

The proofs of these results are based on zero-density estimates for the Riemann zeta function, i.e. estimates that give an upper bound for the number of zeta zeros in the rectangle

$$\{s \in \mathbb{C} \colon \Re(s) \in [\sigma, 1] \text{ and } |\Im(s)| \leq T\}$$

for given $\sigma \in (1/2, 1]$ and $T \geq 2$.

Using the multiplicativity of $\lambda(n)$ in a crucial way, (9) was recently shown to hold for $H \geq X^{0.55+\varepsilon}$ for any $\varepsilon > 0$ by Teräväinen and the author [7]. However, this result is still very far from what is expected to be true – random models suggest that (9) holds in intervals much shorter than $H = X^\varepsilon$.

One can ask what about (9) in almost all short intervals rather than all (we say that a statement holds for almost all $x \leq X$ if the cardinality of the exceptional set is $o(X)$). In this case, the techniques based on the proof of Huxley's prime number theorem get one down to $H \geq X^{1/6+\varepsilon}$ for any $\varepsilon > 0$; whereas assuming the Riemann hypothesis, Gao has proved it (in an unpublished work) for $H \geq (\log X)^A$ for certain fixed $A > 0$.

All the results discussed so far, with the exception of the very recent work [7], have their counterparts for the primes. Given these similarities and the so-called parity phenonenom, it is natural that until recently the problems for the primes and for the Liouville function have been expected to be of equal difficulty.

However, in the recent years this expectation has turned out to be wrong; in [2], Radziwiłł and the author made a breakthrough on understanding the Liouville function in short intervals by proving the following.

Theorem 1. *Let $X \geq H \geq 2$. Assume that $H \to \infty$ with $X \to \infty$. Then*

$$\sum_{x<n\leq x+H} \lambda(n) = o(H) \tag{10}$$

for almost all $x \leq X$.

Note that H can go to infinity arbitrarily slowly here, e.g. $H = \log\log\log\log X$, so this unconditionally improves upon Gao's result that was conditional on the Riemann hypothesis.

While our method fundamentally fails for the primes, it does work for more general multiplicative functions (see [2, Theorem 1]):

Theorem 2. *Let $X \geq H \geq 2$. Let $f : \mathbb{N} \to [-1, 1]$ be multiplicative. Assume that $H \to \infty$ with $X \to \infty$. Then*

$$\left| \frac{1}{H} \sum_{x<n\leq x+H} f(n) - \frac{1}{X} \sum_{n\leq X} f(n) \right| = o(1)$$

for almost all $x \leq X$.

For many applications, it is helpful to also have a result for complex-valued functions, and such an extension can be found in the recent pre-print [3], where we also extend our result in other directions, as we will explain below.

3 Applications

Already in [2] we presented several applications of our general theorem such as

Corollary 3. *For any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that, for all large x, the interval $(x, x + C\sqrt{x}]$ contains x^ε-smooth numbers (i.e. numbers whose all prime factors are $\leq x^\varepsilon$).*

Corollary 4. *There exists a constant $\delta > 0$ such that the Liouville function has $\geq \delta X$ sign changes up to X.*

Starting from [4] our work [2] has led to several applications concerning correlations of multiplicative functions. Chowla's conjecture from the 1960s concerning correlations of the Liouville function asserts that, whenever $h_1, \ldots, h_k$ are distinct, one has

$$\sum_{n\leq X} \lambda(n + h_1) \cdots \lambda(n + h_k) = o(X).$$

This is in line with the general philosophy that additive and multiplicative structures are independent of each other.

Chowla's conjecture can be equivalently stated as saying that, for any $k \geq 1$, each sign pattern $(\varepsilon_1, \ldots, \varepsilon_k) \in \{-1, 1\}^k$ appears in the sequence $(\lambda(n+1), \ldots, \lambda(n+k))_{n\in\mathbb{N}}$ with density $1/2^k$.

Given the analogues between the primes and the Liouville function, Chowla's conjecture can be seen as a "Liouville variant" of the notoriously difficult prime k-tuple conjecture asserting an asymptotic formula for the number of prime k-tuples $(n + h_1, \ldots, n + h_k) \in \mathbb{P}^k$.

Since already the twin prime conjecture that n and $n + 2$ are both primes infinitely often is completely open, a natural starting point is to try to show that, for any $h \neq 0$, one has

$$\sum_{n\leq X} \lambda(n)\lambda(n + h) = o(X).$$

In [4] Radziwiłł, Tao and the author managed to show this for almost all shifts h from a very short range, i.e.

Theorem 5. *Let $X \geq H \geq 2$. Assume that $H \to \infty$ with $X \to \infty$. Then*

$$\sum_{|h|\leq H} \left| \sum_{n\leq X} \lambda(n)\lambda(n + h) \right| = o(HX).$$

To prove this, we extended Theorem 1 to twists by linear phases $e(\alpha n)$ where $e(x) := e^{2\pi i x}$. More precisely, we showed that

Theorem 6. *Let $\alpha \in \mathbb{R}$ and let $X \geq H \geq 2$. Assume that $H \to \infty$ with $X \to \infty$. Then*

$$\sum_{x<n\leq x+H} \lambda(n)e(\alpha n) = o(H)$$

for almost all $x \leq X$.

Theorem 5 follows from Theorem 6 through Fourier analytic techniques. Note that the case $\alpha = 0$ of Theorem 6 corresponds to Theorem 1. Subsequently, Tao [14] used Theorem 6 alongside a novel entropy decrement argument to prove a logarithmically averaged variant of Chowla's conjecture for a fixed shift in the case $k = 2$:

Theorem 7. *Let $h \neq 0$. Then*

$$\sum_{n\leq X} \frac{\lambda(n)\lambda(n + h)}{n} = o(\log X).$$

Both this and Theorems 5 and 6 have variants for much more general multiplicative functions. Remarkably, Tao [13] was able to utilise the general version of Theorem 7 to prove the long-standing Erdős discrepancy problem from combinatorics:

Theorem 8. *For any $f : \mathbb{N} \to \{-1, 1\}$, one has*

$$\sup_{k,N\in\mathbb{N}} \left| \sum_{n\leq N} f(kn) \right| = \infty.$$

Later Tao and Teräväinen [16, 17] managed to solve all the odd order cases of the logarithmically averaged Chowla conjecture (without needing Theorem 6).

Theorem 9. *Let k be odd and $h_1, \ldots, h_k \in \mathbb{Z}$. Then*

$$\sum_{n\leq X} \frac{\lambda(n + h_1) \cdots \lambda(n + h_k)}{n} = o(\log X).$$

(When k is odd, one can trivially dispose of the condition about h_j being distinct.)

The even cases $k \geq 4$ of the logarithmic Chowla conjecture remain open. Tao [15] has shown that the complete resolution is equivalent to two other conjectures, the logarithmically averaged Sarnak conjecture and the logarithmically averaged local higher order uniformity conjecture for the Liouville function.

Sarnak's conjecture roughly asserts that, for a bounded sequence $a(n)$, one has

$$\sum_{n\leq X} a(n)\lambda(n) = o(X)$$

whenever $a(n)$ is of "low complexity", whereas the higher order uniformity conjecture is a vast generalisation of Theorem 6 that allows α to depend on x and also replaces the linear phase $e(\alpha n)$ by much more general nilsequences that are the characters of the higher order Fourier analysis. The definition of nilsequence is so involved that we do not give it here, but instead we mention two special cases.

First, the higher order uniformity conjecture includes the claim that the Liouville function is locally orthogonal to polynomial phases. More precisely, for any $k \in \mathbb{N}$ it asserts that, for almost all $x \leq X$,

$$\sup_{P(y)\in\mathrm{Poly}_{\leq k}(\mathbb{R}\to\mathbb{R})} \sum_{x<n\leq x+H} \lambda(n)e(P(n)) = o(H), \qquad (11)$$

where $\mathrm{Poly}_{\leq k}$ denotes the set of polynomials of degree at most k. Secondly, the conjecture includes the claim that, for almost all $x \leq X$,

$$\sup_{\alpha,\beta} \sum_{x<n\leq x+H} \lambda(n)e(\lfloor \alpha n \rfloor \beta n) = o(H). \qquad (12)$$

The fact that the phase is allowed to depend on x makes the problem much more difficult, and indeed in the recent progress [5, 6] on this conjecture (in the range $H \geq X^{\varepsilon}$) the main ingredient is to show that if, e.g., (11) failed for many x, then the corresponding polynomials yielding the supremum must be related to each other in certain way.

In [6] Radziwiłł, Tao, Teräväinen, Ziegler and the author were able to establish the higher order uniformity conjecture for $H \geq X^{\varepsilon}$; consequently for instance (11) and (12) hold for almost all intervals of length $H \geq X^{\varepsilon}$.

Unfortunately, in order to deduce the logarithmic Chowla conjecture, one would need to establish the higher order uniformity conjecture in much shorter intervals of length $H \leq (\log X)^{\varepsilon}$.

However, the result in [6] still has some interesting applications: first, it yields a new averaged version of Chowla's conjecture (as a special case of a result for more general patterns):

Corollary 10. *Let $k \in \mathbb{N}$. For $H \geq X^{\varepsilon}$,*

$$\sum_{|h|\leq H} \left| \sum_{n\leq X} \lambda(n)\lambda(n+h)\cdots\lambda(n+(k-1)h) \right| = o(HX).$$

Secondly, we obtain that the Liouville function has super-polynomially many sign patterns. More precisely, if one writes

$$s(k) = \Big| \{(\varepsilon_1, \ldots, \varepsilon_k) \in \{-1,1\}^k : \\ \exists n \colon (\lambda(n+1), \ldots, \lambda(n+k)) = (\varepsilon_1, \ldots, \varepsilon_k)\} \Big|$$

for the number of sign patterns of length k, then

Corollary 11. *For any $A \geq 1$ there exists a constant $\delta = \delta(A)$ such that $s(k) \geq \delta k^A$ for every $k \in \mathbb{N}$.*

The previous record [8] had $A = 2$.

4 Refinements and further applications

In a recent pre-print [3], Radziwiłł and the author revisited the problem of multiplicative functions in short intervals. As explained above already, the work in [2] led to further progress and many applications. However, there are certain drawbacks in it as well. In [3], we extended the results to sparsely supported functions, improved the quantitative bounds and extended to the complex case with the correctly twisted main term.

A key application of these new developments concerns the distribution of norm forms in short intervals. Let us discuss the simplest possible case, the characteristic function $\mathbf{1}_{n\in\mathcal{N}}$ of the set $\mathcal{N}$ of numbers that can be represented as a sum of two squares. Then it is well known that $\mathbf{1}_{n\in\mathcal{N}}$ is multiplicative and furthermore

$$\mathbf{1}_{p^k\in\mathcal{N}} = \begin{cases} 0 & \text{if } p \equiv 3 \pmod 4 \text{ and } k \text{ is odd;} \\ 1 & \text{otherwise.} \end{cases}$$

Hence $\mathbf{1}_{p\in\mathcal{N}} = 0$ for essentially half of the primes, which implies that the density of $\mathcal{N}$ is asymptotically $C/(\log X)^{1/2}$, i.e.

$$\sum_{\substack{n\leq X \\ n\in\mathcal{N}}} 1 = C\frac{X}{\sqrt{\log X}} + o\left(\frac{X}{\sqrt{\log X}}\right)$$

for certain constant $C > 0$. In other words, the average gap of two elements of $\mathcal{N}$ is of size $\sqrt{\log X}/C$. Consequently, one cannot expect $\mathcal{N}$ to be regularly distributed in intervals shorter than this.

In [2] we obtained a quantitative version of Theorem 2 but even it is completely trivial for sparsely supported functions such as $f(n) = \mathbf{1}_{n\in\mathcal{N}}$ and so does not tell us anything about the behaviour of $\mathbf{1}_{n\in\mathcal{N}}$ in short intervals. But fortunately, the method can be adapted to this situation and we proved

Theorem 12. *As soon as $h \to \infty$ with $X \to \infty$, one has*

$$\Big| \sum_{x<n\leq x+h(\log X)^{1/2}} \mathbf{1}_{n\in\mathcal{N}} - Ch \Big| = o(h) \qquad (13)$$

for almost all $x \leq X$.

Previously Hooley [1] and Plaksin [10, 11] had shown that there exist constants c_1 and C_1 such that, as soon as $h \to \infty$ with $X \to \infty$, one has, for almost all $x \leq X$,

$$c_1 h \leq \sum_{x<n\leq x+h(\log X)^{1/2}} \mathbf{1}_{n\in\mathcal{N}} \leq C_1 h.$$

Their methods were based on an asymptotic formula for

$$\sum_{n\leq X} r_K(n) r_K(n+h),$$

where $r_K(n)$ are the coefficients of the Dedekind ζ-function for $K = \mathbb{Q}(i)$ ($r_{\mathbb{Q}(i)}(n)$ counts the number of representations of n as a sum of two squares).

Such an asymptotic formula is known for $K = \mathbb{Q}(i)$, but is completely open for non-quadratic number fields. Hence Hooley and Plaksin's methods have no chance of generalising to higher degree number fields.

In [3] we only use multiplicativity and get much more general results: we say that an integer n is a norm-form of a number field K over $\mathbb{Q}$ if n is equal to the norm of an algebraic integer in K. In the case $K = \mathbb{Q}(i)$, the norm forms are simply the sums of two squares. In [3] we have a much more general version of Theorem 12 for norm forms of number fields of any degree.

5 How do we attack short intervals?

Let us next discuss a common strategy for attacking arithmetic questions in short intervals. We would like to show that

$$\sum_{x<n\leq x+H} \lambda(n) = o(H)$$

for almost all $x \leq X$.

A typical way in analytic number theory to pick up the condition $x < n \leq x + H$ is to use the contour integration formula

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{y^s}{s} ds = \begin{cases} 0 & \text{if } y < 1; \\ 1 & \text{if } y > 1. \end{cases} \qquad (14)$$

This formula follows by moving the integration far to the right in case $y < 1$ and far to the left in case $y > 1$; in the second case we obtain the main term 1 from the residue of the pole at $s = 0$.

Applying (14) twice (when $x, x + H \notin \mathbb{N}$, but this small technicality is easy to deal with), we have, for any $x \leq X$ and $H \leq X$,

$$\sum_{x<n\leq x+H} \lambda(n) = \sum_{n\leq 2X} \lambda(n)\left(\mathbf{1}_{\frac{x+H}{n}\geq 1} - \mathbf{1}_{\frac{x}{n}\geq 1}\right) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \sum_{n\leq 2X} \frac{\lambda(n)}{n^s}\cdot\frac{(x+H)^s - x^s}{s} ds + O(1). \tag{15}$$

Objects of the form

$$F(s) = \sum_{n\leq N} \frac{a_n}{n^s}$$

are called Dirichlet polynomials and they are very important tools in analytic number theory.

Recall that we aim to prove (10) only for almost all $x \leq X$. Thus it suffices to show that

$$\int_1^X \left|\sum_{x<n\leq x+H} \lambda(n)\right|^2 dx = o(H^2 X). \tag{16}$$

Using (15) one can show that in the language of Dirichlet polynomials this essentially reduces to the claim

$$\int_{-X/H}^{X/H} \left|N(1+it)\right|^2 dt = o(1), \tag{17}$$

where

$$N(s) := \sum_{X/2<n\leq X} \frac{\lambda(n)}{n^s}.$$

A fundamental tool for studying mean squares of Dirichlet polynomials is the mean value theorem for Dirichlet polynomials which gives that, for any complex coefficients a_n and any $T, N \geq 2$, one has

$$\int_{-T}^{T} \left|\sum_{N/2<n\leq N} \frac{a_n}{n^{1+it}}\right|^2 dt = O\left(\left(\frac{T}{N}+1\right)\frac{1}{N}\sum_{N/2<n\leq N} |a_n|^2\right). \tag{18}$$

Let us motivate this in the case of coefficients with $|a_n| = 1$ for all n in which case one simply gets the bound $O(T/N + 1)$.

The term $O(T/N)$ reflects the expected average behaviour – from a random model one expects that for a typical t one has square-root cancellation, i.e. something like

$$\sum_{N/2<n\leq N} \frac{a_n}{n^{1+it}} \asymp N^{-1/2}$$

leading in the mean square to $O(T(N^{-1/2})^2) = O(T/N)$.

On the other hand, the term $O(1)$ reflects possible peaks of the polynomial – for some values of t one might have

$$\sum_{N/2<n\leq N} \frac{a_n}{n^{1+it}} \asymp 1; \tag{19}$$

surely this holds, e.g., if, for some t_0, one has $a_n = n^{it_0}$ for every n. At any rate, (18) is in general best possible.

If we now apply the mean value theorem (18) to the left hand side of (17), we obtain the bound

$$O\left(\frac{X/H}{X} + 1\right) = O(1) \tag{20}$$

which barely fails to produce the desired $o(1)$; this bound $O(1)$ for the left hand side of (17) gives the trivial bound $O(H^2 X)$ for the left hand side of (16). Note that since this mean value theorem argument did not utilise any properties of $\lambda(n)$ except that $|\lambda(n)| \leq 1$, it had no chance of leading to $o(H^2 X)$ for (16).

Now it is the second term on the left hand side of (20) that is not $o(1)$. In the case of $a_n = \lambda(n)$, it is known that (19) cannot happen; for any $|t| \leq N$ we have

$$\sum_{n\leq N} \frac{\lambda(n)}{n^{1+it}} = O\left(\frac{1}{(\log N)^{1000}}\right)$$

by a known zero-free region for the Riemann zeta-function. Hence there seems to be some hope.

Often when one deals with Dirichlet polynomials, it is helpful if there is some bilinear structure, i.e. one can write the relevant Dirichlet polynomial (in our case $N(s)$) as a product of two or more Dirichlet polynomials. There are several classical ways to obtain such a decomposition, such as the identities of Vaughan and Heath-Brown.

These techniques work equally well for the primes and the Liouville function, and this is one of the reasons why many results are of similar quality in these two cases.

In our case, when we study very short intervals (such as $H = X^\varepsilon$ and smaller), it is of benefit to have a decomposition where one of the factors is very short.

Indeed, a crucial step in the proof is to use the multiplicativity of $\lambda(n)$ and take out a small prime factor utilising the fact that almost all integers $n \leq X$ have a prime factor from the interval $(P, Q]$ as soon as $\log Q/\log P \to \infty$ with $X \to \infty$,

This can be done rigorously though, using either the Turán-Kubilius inequality or a Ramáre type identity. The latter gives better quantitative results and we use it in our research papers, but let us use the former here, which yields the following.

Lemma 13. *Let $X \geq H \geq 2$ and $3P \leq Q \leq H^{1/2}$. Then*

$$\sum_{X<n\leq X+H} \lambda(n) = \frac{1}{\sum_{P<p\leq Q} 1/p} \sum_{\substack{m,p \\ X<mp\leq X+H \\ P<p\leq Q}} \lambda(mp) + O\left(\frac{H}{\left(\log\frac{\log Q}{\log P}\right)^{1/2}}\right).$$

Here

$$\sum_{P<p\leq Q} \frac{1}{p} = \log\frac{\log Q}{\log P} + O(1)$$

is a normalising factor corresponding to the average number of representations that n has as mp with $p \in (P, Q]$.

Note that utilising this idea that almost all integers $n \leq X$ have a prime factor from $(P, Q]$ as soon as $\log Q/\log P \to \infty$ with $X \to \infty$ fundamentally fails in the case of primes, as primes $p > Q$ never have such a prime factor.

6 Intervals of length $H \geq X^\varepsilon$

In this section we sketch the proof of Theorem 1 in case $H \geq X^\varepsilon$. We start by applying Lemma 13 with

$$P = \exp((\log X)^{3/4}) \quad \text{and} \quad Q = \exp((\log X)^{7/8})$$

so that

$$\log\frac{\log Q}{\log P} = \frac{1}{8}\log\log X.$$

Hence by Lemma 13 it suffices to show that, for almost all $x \le X$, one has

$$\sum_{P<p\le Q}\ \sum_{x<mp\le x+H} \lambda(mp) = o(H \log\log X). \tag{21}$$

Note that here by multiplicativity $\lambda(mp) = -\lambda(m)$. We split the summation over p into dyadic ranges $p \in (P_1, 2P_1]$, so that we wish to show, for any $P_1 \in (P, Q]$,

$$\sum_{P_1<p\le 2P_1}\ \sum_{x<mp\le x+H} \lambda(m) = o\left(\frac{H}{\log P_1}\right);$$

summing this over $P_1 = 2^j$ with $P < 2^j \le Q$ gives essentially (21).

We can run a similar argument as in the previous section but with $N(s)$ replaced by

$$P_1(s)M(s) := \sum_{P_1<p\le 2P_1} \frac{1}{p^s} \sum_{X/(4P_1)<m\le 4X/P_1} \frac{\lambda(m)}{m^s}, \tag{22}$$

so that we need to show that

$$I := \int_{-X/H}^{X/H} \left|P_1(1+it)M(1+it)\right|^2 dt = o\left(\frac{1}{(\log P_1)^2}\right). \tag{23}$$

Now (18) still fails to do this, but we have the additional advantage of having a bilinear structure. The known zero-free region for the Riemann zeta-function yields

$$|P_1(1+it)| = O\left((\log X)^{-1000}\right) \tag{24}$$

for every $|t| \le X$. Using this we get that

$$I = O\left((\log X)^{-2000} \int_{-X/H}^{X/H} |M(1+it)|^2 dt\right).$$

Now we are in the position to apply (18) to the polynomial $M(s)$, giving the bound

$$I = O\left((\log X)^{-2000}\left(\frac{X/H}{X/P_1} + 1\right)\right) = O((\log X)^{-2000}) \tag{25}$$

since $P_1 \le H$. Hence (23) follows.

7 Shorter intervals

When $H \le \exp((\log X)^{2/3})$, a new issue arises: to make the last step in (25) work, we need to have $P_1 \le H$. However, for such short $P_1(s)$, we do not know (24) for all $|t| \le X$ any more. Fortunately, in [2] we were able to develop an iterative argument to rescue us.

Let us explain the rough idea. For simplicity, we pretend that Lemma 13 implies that, for $j = 1, 2, \ldots, J$, we have

$$N(s) = P_j(s)M_j(s) := \sum_{P_j<p\le 2P_j} \frac{1}{p^s} \sum_{X/(4P_j)<m\le 4X/P_j} \frac{\lambda(m)}{m^s},$$

with $P_1 = H, P_{j+1} = P_j^{\log P_j}$ for $1 \le j \le J-1$ and $P_J = \exp((\log X)^{3/4})$. For those t for which $|P_1(1+it)| \le P_1^{-1/10}$ the earlier argument works.

For those t for which $|P_1(1+it)| > P_1^{-1/10}$ and $|P_2(1+it)| \le P_2^{-11/100}$ we note that $1 \le (|P_1(1+it)|P_1^{1/10})^{2k}$ with $k = \lfloor \log P_1 \rfloor$ and in this case it suffices to show that

$$P_1^{k/5} P_2^{-11/50} \int_{-X/H}^{X/H} |P_1(1+it)^k M_2(1+it)|^2 dt = o\left(\frac{1}{(\log P_2)^2}\right)$$

which follows from the mean value theorem which is efficient as $M_2(s)P_1(s)^k$ has length about X.

Now we are left with t, for which $|P_2(1+it)| > P_2^{-11/100}$. Continuing the recursion, we are eventually left with t for which $|P_{J-1}(1+it)| \ge P_{J-1}^{-1/8}$, say. But now P_{J-1} is so large that this can only happen rarely, and we can use (24) for $P_J(s)$ together with a large value theorem for Dirichlet polynomials.

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