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On the Summation of Series in Terms of Bessel Functions

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Abstract. In this article we deal with summation formulas for the series $\sum_{n=1}^{\infty} \frac{J_\mu(nx)}{n^\nu}$, referring partly to some results from our paper in J. Math. Anal. Appl. 247 (2000) 15 – 26. We show how these formulas arise from different representations of Bessel functions. In other words, we first apply Poisson's or Bessel's integral, then in the sequel we define a function by means of the power series representation of Bessel functions and make use of Poisson's formula. Also, closed form cases as well as those when it is necessary to take the limit have been thoroughly analyzed.

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1. Introduction

At first we start with integral representations of Bessel functions $J_\mu(z)$ i.e. Poisson's and Bessel's integral, and show how they give rise to different summation formulas for the series

$$\sum_{n=1}^{\infty} \frac{J_\mu(nx)}{n^\nu}. \tag{1}$$

All the aspects of relations between real parameters μ and ν arising from these representations are given in detail.

In the second part we consider another representation of Bessel functions and define a suitable function to apply Poisson's formula and find one more summation formula for (1).

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2. Summation based on Poisson's and Bessel's integral

Poisson's integral

$$J_\mu(z) = \frac{z^\mu}{2^{\mu-1}\Gamma(\frac{1}{2})\Gamma(\mu+\frac{1}{2})}\int_0^{\frac{\pi}{2}} \sin^{2\mu}\theta\cos(z\cos\theta)\,d\theta, \quad \mu > -\frac{1}{2}, \tag{2}$$

is a representation of the Bessel function $J_\mu(z)$ of the first kind and order μ. We just recall that Poisson [11] and Lommel [7] proved that for $2\mu \in \mathbb{N}_0$ Poisson's integral is a solution of Bessel's differential equation

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + (z^2-\mu^2)u = 0\,.$$

If we set $\mu = m \in \mathbb{N}_0$ in Poisson's integral, after a rearrangement, we obtain Bessel's integral

$$J_m(z) = \frac{1}{\pi}\int_0^\pi \cos(m\theta - z\sin\theta)\,d\theta = \frac{1}{\pi}\int_0^\pi g(m\theta)g(z\sin\theta)\,d\theta, \tag{3}$$

where $m = 2k+\delta$, $k \in \mathbb{N}_0$ and $g = \left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}$ $\delta = \left\{\begin{smallmatrix}1\\ 0\end{smallmatrix}\right\}$. The preceding formula is in fact Bessel's definition of the function $J_m(z)$ (see [2], p. 34).

2.1. Summation based on Poisson's integral. After placing Poisson's integral (2) in (1), we have to interchange summation and integration, i.e.,

$$\frac{2(\frac{x}{2})^\mu}{\sqrt{\pi}\Gamma(\mu+\frac{1}{2})}\sum_{n=1}^\infty \int_0^{\frac{\pi}{2}} \frac{\sin^{2\mu}\theta\cos(nx\cos\theta)}{n^{\nu-\mu}}\,d\theta$$
$$= \frac{2(\frac{x}{2})^\mu}{\sqrt{\pi}\Gamma(\mu+\frac{1}{2})}\int_0^{\frac{\pi}{2}} \sin^{2\mu}\theta\sum_{n=1}^\infty \frac{\cos(nx\cos\theta)}{n^{\nu-\mu}}\,d\theta.$$

The latter we are allowed to do because of uniform convergence of the series on the right-hand side, which we prove in

Lemma 1. *The series*

$$\sum_{n=1}^\infty \frac{\cos(nx\cos\theta)}{n^{\nu-\mu}} \qquad (\nu > \mu), \tag{4}$$

is uniformly convergent with respect to θ *on each segment* $[\varepsilon, \frac{\pi}{2}-\varepsilon] \subset (0, \frac{\pi}{2})$, $\varepsilon > 0$.

Proof. By virtue of Dirichlet's test (see [5]) the series $\sum_{n=0}^\infty a_n(x)b_n(x)$ is uniformly convergent in D, if the partial sums of $\sum_{n=0}^\infty a_n(x)$ are uniformly bounded in D, and the sequence $b_n(x)$, being monotonic for every fixed x, uniformly converges to 0.

We first note that $0 < \cos\theta < 1$, and if $0 < x < 2\pi$, we have $0 < \frac{x\cos\theta}{2} < \pi$. For each $\varepsilon > 0$ satisfying $\varepsilon \le \frac{x\cos\theta}{2} \le \pi - \varepsilon$ we know that $\sin\frac{x\cos\theta}{2} \ge \sin\varepsilon > 0$, so there follows

$$\left|1 + \sum_{k=1}^{n} \cos(kx\cos\theta)\right| = \left|\frac{\sin\frac{(n+1)x\cos\theta}{2}\cos\frac{nx\cos\theta}{2}}{\sin\frac{x\cos\theta}{2}}\right| \le \frac{1}{\sin\frac{x\cos\theta}{2}} \le \frac{1}{\sin\varepsilon},$$

meaning that partial sums $\sum_{k=1}^{n}\cos(kx\cos\theta)$ are uniformly bounded with respect to θ on each segment $[\varepsilon, \frac{\pi}{2} - \varepsilon] \subset (0, \frac{\pi}{2})$, $0 < \varepsilon < \frac{\pi}{2}$, and for $\nu > \mu$ the monotonically decreasing sequence $\frac{1}{n^{\nu-\mu}}$ tends to 0. □

Now we have to find the sum of the trigonometric series (4) in terms of Riemann's ζ-function. First of all, we start with the following

Lemma 2. *For $m \in \mathbb{N}$ there holds*

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^{2m}} = \frac{(-1)^m \pi x^{2m-1}}{2(2m-1)!} + \sum_{i=0}^{m} \frac{(-1)^i \zeta(2m-2i)}{(2i)!} x^{2i}. \tag{5}$$

Proof. Because of uniform convergence of the series

$$S_\alpha(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^\alpha}, \quad C_\alpha(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^\alpha}, \qquad \alpha > 0,\ 0 < x < 2\pi,$$

we may integrate them to get $\int_0^x S_\alpha(x)\,dx = \zeta(\alpha+1) - C_{\alpha+1}(x)$ and $\int_0^x C_\alpha(x)\,dx = S_{\alpha+1}(x)$. Since $S_1(x) = \frac{\pi - x}{2}$, we get

$$\int_0^x S_1(x)\,dx = \zeta(2) - C_2(x) \ \Rightarrow\ C_2(x) = -\frac{\pi}{2}x + \zeta(2) - \frac{\zeta(0)}{2}x^2,$$

and $S_3(x) = \int_0^x C_2(x)\,dx = -\frac{\pi}{4}x^2 + \zeta(2)x - \frac{\zeta(0)}{6}x^3$, as well as

$$\int_0^x S_3(x)\,dx = \zeta(4) - C_4(x) \ \Rightarrow\ C_4(x) = \frac{\pi}{12}x^3 + \zeta(4) - \frac{\zeta(2)}{2}x^2 + \frac{\zeta(0)}{24}x^4.$$

Continuing in the same manner, and using mathematical induction we obtain the above closed form formula (5). □

It is known that $\zeta(-2k) = 0$, $k \in \mathbb{N}$. Thus, we rewrite the formula (5) as

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^{2m}} = \frac{\pi x^{2m-1}}{2\Gamma(2m)\cos(\frac{\pi}{2}2m)} + \sum_{i=0}^{\infty} \frac{(-1)^i \zeta(2m-2i)}{(2i)!} x^{2i}, \quad m \in \mathbb{N}.$$

Now there arises a question whether the left-hand side is expressed as a non-trigonometric series in terms of ζ function, if we replace $2m$ with an arbitrary real number $s > 0$. We shall show that the same form is retained, except for s odd, when the first term changes.

Theorem 1. *For $s \in \mathbb{R}^+ \setminus 2\mathbb{N}+1$, there holds*

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^s} = \frac{\pi x^{s-1}}{2\Gamma(s)\cos\frac{\pi s}{2}} + \sum_{i=0}^{\infty} \frac{(-1)^i \zeta(s-2i)}{(2i)!}\, x^{2i}. \tag{6}$$

However, for $s \in 2\mathbb{N}+1$,

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^{2m+1}} = \Phi_{2m+1}(x) + \sum_{i=m+1}^{\infty} \frac{(-1)^i \zeta(2m+1-2i)}{(2i)!}\, x^{2i}, \quad m \in \mathbb{N}\cup\{0\},$$

where

$$\Phi_{2m+1}(x) = \frac{(-1)^m}{(2m)!}\big(\psi(2m+1)+\gamma-\ln x\big)\, x^{2m} + \sum_{i=1}^{m} \frac{(-1)^{m-i}\zeta(2i+1)}{(2m-2i)!}\, x^{2m-2i}. \tag{7}$$

Proof. We make use of the following representation

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^s} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{e^{-inx}+e^{inx}}{n^s} = \frac{1}{2}\big(\mathrm{Li}_s(e^{-ix}) + \mathrm{Li}_s(e^{ix})\big), \quad s>0, \tag{8}$$

where $\mathrm{Li}_s(z)$ is called *polylogarithm* defined by (see [6])

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{\frac{e^t}{z}-1}\, dt,$$

where the right-hand side integral converges for $z \in \mathbb{C} \setminus \{z \mid z \in \mathbb{R},\ z \geq 1\}$, and it is referred to as *Bose's integral.* We shall consider the Mellin transform of the polylogarithm in the form of Bose's integral. The Mellin transform of a function f and the inverse transform of a function φ are (see [10])

$$M\big(f(x)\big) = \int_0^{\infty} x^{u-1} f(x)\, dx, \qquad M^{-1}(\varphi(u)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-u}\varphi(u)\, du.$$

This integral transform is closely connected to the theory of Dirichlet series, and is often used in number theory and the theory of asymptotic expansions. Also, it is closely related to the Laplace and Fourier transform as well as to the theory of the gamma function and allied special functions. So we find

$$M\big(\mathrm{Li}_s(p\, e^{-x})\big) = \int_0^{\infty} x^{u-1}\mathrm{Li}_s(p\, e^{-x})\, dx = \frac{1}{\Gamma(s)} \int_0^{\infty}\int_0^{\infty} \frac{t^{s-1}x^{u-1}}{\frac{e^{t+x}}{p}-1}\, dt\, dx$$

The change of variables $x = ab$, $t = a(1-b)$ allows the integrals to be separated

$$M\big(\mathrm{Li}_s(p\, e^{-x})\big) = \frac{1}{\Gamma(s)}\int_0^1 b^{u-1}(1-b)^{s-1}\, db \int_0^{\infty} \frac{a^{s+u-1}}{\frac{e^a}{p}-1}\, da = \Gamma(u)\mathrm{Li}_{s+u}(p).$$

For $p = 1$, because $\mathrm{Li}_{s+u}(1) = \zeta(s+u)$, through the inverse Mellin transform, we have

$$\mathrm{Li}_s(e^{-x}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(u)\zeta(s+u)x^{-u}\,du,$$

where c is a constant to the right of the poles of the integrand. The path of integration may be converted into a closed contour, and the poles of the integrand are those of $\Gamma(u)$ at $u = 0, -1, -2, \ldots$, and of $\zeta(s+u)$ at $u = 1-s$. Summing the residues yields a representation of the polylogarithm as a power series

$$\mathrm{Li}_s(e^{\mu}) = (-\mu)^{s-1}\Gamma(1-s) + \sum_{k=0}^{\infty} \frac{\zeta(s-k)}{k!}\,\mu^k, \quad |\mu| < 2\pi,\ s \neq 1, 2, 3, \ldots$$

about $\mu = 0$. Further, following (8), we have

$$\frac{1}{2}\big(\mathrm{Li}_s(e^{\mu}) + \mathrm{Li}_s(e^{-\mu})\big) = \frac{1}{2}\big((-\mu)^{s-1} + \mu^{s-1}\big)\Gamma(1-s) + \sum_{k=0}^{\infty} \frac{\zeta(s-2k)}{(2k)!}\,\mu^{2k}. \quad (9)$$

If the parameter s is a positive integer, the gamma function becomes infinite, and we can not place s immediately in (9). In order to get a finite value for a positive integer n, we must take, on the right-hand side sum, the first $[\frac{n-1}{2}]$ terms if n is odd, and $[\frac{n+1}{2}]$ terms if n is even. Then we take a limit

$$\lim_{s\to n}\left(\frac{1}{2}\,((-\mu)^{s-1} + \mu^{s-1})\Gamma(1-s) + \sum_{k=0}^{[\frac{n-1}{2}]} \frac{\zeta(s-2k)}{(2k)!}\,\mu^{2k}\right).$$

We distinguish two cases: $n = 2m+1,\ m \in \mathbb{N}\cup\{0\}$, and $n = 2m,\ m \in \mathbb{N}$. So, we find

$$\lim_{s\to 2m+1}\left(\frac{1}{2}\,\big((-\mu)^{s-1} + \mu^{s-1}\big)\Gamma(1-s) + \sum_{k=0}^{m} \frac{\zeta(s-2k)}{(2k)!}\,\mu^{2k}\right)$$
$$= \frac{1}{(2m)!}\left(\psi(2m+1) + \gamma - \frac{\ln(-\mu)+\ln\mu}{2}\right)\mu^{2m} + \sum_{i=1}^{m} \frac{\zeta(2i+1)}{(2m-2i)!}\,\mu^{2m-2i}.$$

Setting $\mu = ix,\ 0 < x < 2\pi$, we obtain (7). Now we consider

$$\lim_{s\to 2m}\left(\frac{1}{2}\,\big((-\mu)^{s-1} + \mu^{s-1}\big)\Gamma(1-s) + \sum_{k=0}^{m} \frac{\zeta(s-2k)}{(2k)!}\,\mu^{2k}\right)$$
$$= -\frac{\ln(-\mu)-\ln\mu}{2(2m-1)!}\mu^{2m-1} + \sum_{k=0}^{m} \frac{\zeta(2m-2k)}{(2k)!}\,\mu^{2k}.$$

Setting $\mu = ix,\ 0 < x < 2\pi$, we obtain (5).

If the parameter s is a positive non-integer, the gamma function is finite, so replacing μ with ix, $0 < x < 2\pi$, and in view of $(\pm i)^{s-1} = e^{\pm i(s-1)\frac{\pi}{2}} = \cos\frac{\pi}{2}(s-1) \pm i\sin\frac{\pi}{2}(s-1)$, we calculate the first term in (9)

$$\frac{1}{2}\big((-\mu)^{s-1}+\mu^{s-1}\big)\Gamma(1-s) = \frac{\pi x^{s-1}}{2\Gamma(s)\sin\pi s}\big((-i)^{s-1}+i^{s-1}\big) = \frac{\pi x^{s-1}}{2\Gamma(s)\cos\frac{\pi}{2}s}.$$

Gathering all these results, taking into account (8), we complete the proof. □

Remark. We note that formula for s odd can be in fact obtained, if we let s tend to $2m+1$ ($m \in \mathbb{N}\cup\{0\}$) in (6), whose left-hand side series are related to *Clausen functions* defined by (see [6])

$$\mathrm{Cl}_{2\nu} = \sum_{n=1}^{\infty}\frac{\sin nx}{n^{2\nu}}, \qquad \mathrm{Cl}_{2\nu-1} = \sum_{n=1}^{\infty}\frac{\cos nx}{n^{2\nu-1}}, \qquad \nu\in\mathbb{N},$$

as well as to Bernoulli polynomials (see [1]). This leads to a new class of integrals

$$\sum_{n=1}^{\infty}\frac{J_\mu(nx)}{n^\nu} = \frac{z^\mu}{2^{\mu-1}\sqrt{\pi}\Gamma(\mu+\frac{1}{2})}\int_0^1\frac{\mathrm{Cl}_{\nu-\mu}(tx)}{(\sqrt{1-t^2})^{\frac{1}{2}-\mu}}\,dt. \qquad \square$$

Now, we resume deriving the summation formula for the series (1), relying on the results of Lemma 2.1 and Theorem 2.3. If we replace x with $x\cos\theta$ and s with $\nu-\mu$ in (6), we come to a non-trigonometric expression for the series (4). Then we continue

$$\begin{aligned}
&\frac{2(\frac{x}{2})^\mu}{\sqrt{\pi}\Gamma(\mu+\frac{1}{2})}\int_0^{\frac{\pi}{2}}\sin^{2\mu}\theta\sum_{n=1}^{\infty}\frac{\cos(nx\cos\theta)}{n^{\nu-\mu}}\,d\theta\\
&\quad=\frac{2(\frac{x}{2})^\mu}{\sqrt{\pi}\Gamma(\mu+\frac{1}{2})}\int_0^{\frac{\pi}{2}}\sin^{2\mu}\theta\Bigg(\frac{\pi(x\cos\theta)^{\nu-\mu-1}}{2\Gamma(\nu-\mu)\cos\frac{\pi(\nu-\mu)}{2}}\\
&\qquad+\sum_{i=0}^{\infty}\frac{(-1)^i\zeta(\nu-\mu-2i)(x\cos\theta)^{2i}}{(2i)!}\Bigg)d\theta\\
&\quad=\frac{\sqrt{\pi}x^{\nu-1}}{2^\mu\Gamma(\mu+\frac{1}{2})\Gamma(\nu-\mu)\cos\frac{\pi(\nu-\mu)}{2}}\int_0^{\frac{\pi}{2}}\sin^{2\mu}\theta\cos^{\nu-\mu-1}\theta\,d\theta\\
&\qquad+\frac{x^\mu}{2^{\mu-1}\sqrt{\pi}\Gamma(\mu+\frac{1}{2})}\sum_{i=0}^{\infty}\frac{(-1)^i\zeta(\nu-\mu-2i)x^{2i}}{\Gamma\big(2(i+\frac{1}{2})\big)}\int_0^{\frac{\pi}{2}}\sin^{2\mu}\theta\cos^{2i}\theta\,d\theta.
\end{aligned}$$

By virtue of *Legendre's duplication formula* $\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$ (see [9]) and an integral of the type (see [13])

$$2\int_0^{\frac{\pi}{2}}\sin^{2a-1}\theta\cos^{2b-1}\theta\,d\theta = \mathrm{B}(a,b),$$

we finally obtain the following summation formula (see [15])

$$\sum_{n=1}^{\infty}\frac{J_\mu(nx)}{n^\nu}=\frac{\pi(\frac{x}{2})^{\nu-1}}{2\Gamma(\frac{\nu-\mu+1}{2})\Gamma(\frac{\nu+\mu+1}{2})\cos\frac{\pi(\nu-\mu)}{2}}+\sum_{i=0}^{\infty}\frac{(-1)^i\zeta(\nu-\mu-2i)(\frac{x}{2})^{\mu+2i}}{i!\,\Gamma(\mu+i+1)}, \tag{10}$$

where the convergence region is $0<x<2\pi$ and $\nu>\mu>-\frac{1}{2}$.

2.1.1. Closed form cases. On condition that $\nu-\mu=2k$, $k\in\mathbb{N}_0$, the right-hand side series in (10) truncates because $\zeta(2k-2i)=0$ for $i>k$ (ζ equals zero if its argument is a negative even number). Thus we obtain closed form cases of the summation formula (10). That means infinite series is reduced to finite number of terms, and (10) becomes

$$\sum_{n=1}^{\infty}\frac{J_\mu(nx)}{n^{\mu+2k}}=\frac{k!\;(-1)^k\,x^{\mu+2k-1}\sqrt{\pi}}{2^{\mu+k}\;(2k)!\;\Gamma(\mu+k+\frac{1}{2})}+\sum_{i=0}^{k}\frac{(-1)^i\,\zeta(2k-2i)\;(\frac{x}{2})^{\mu+2i}}{i!\;\Gamma(\mu+i+1)}. \tag{11}$$

2.1.2. Limiting value cases. But, when $\nu-\mu=2k+1$, $k\in\mathbb{N}_0$, one should take limits. Actually, we first denote $\tau=\nu-\mu$ and replace ν with $\tau+\mu$ in (10). Afterwards we consider the limiting value

$$\lim_{\tau\to 2k+1}\left[\frac{2^{-\tau-\mu}\,\pi x^{\tau+\mu-1}}{\Gamma(\frac{\tau+1}{2})\Gamma(\mu+\frac{\tau+1}{2})\,\cos(\frac{\pi\tau}{2})}+\sum_{i=0}^{k}\frac{(-1)^i\zeta(\tau-2i)(\frac{x}{2})^{\mu+2i}}{i!\,\Gamma(\mu+i+1)}\right]=\Phi_{\mu,2k+1}(x),$$

so that we have

$$\sum_{n=1}^{\infty}\frac{J_\mu(nx)}{n^{\mu+2k+1}}=\Phi_{\mu,2k+1}(x)+\sum_{i=k+1}^{\infty}\frac{(-1)^i\zeta(2k+1-2i)(\frac{x}{2})^{\mu+2i}}{i!\,\Gamma(\mu+i+1)}, \tag{12}$$

where

$$\Phi_{\mu,2k+1}(x)=\frac{(-1)^k(\frac{x}{2})^{\mu+2k}}{2k!\Gamma(\mu+k+1)}\Big(\psi(k+1)+\psi(\mu+k+1)+2\gamma-2\ln\frac{x}{2}\Big)+\sum_{i=0}^{k-1}\frac{(-1)^i\zeta(2k+1-2i)(\frac{x}{2})^{\mu+2i}}{i!\Gamma(\mu+i+1)}.$$

Here γ is Euler's constant and ψ is the digamma function. For example, if $\nu-\mu=3$ we find

$$\sum_{n=1}^{\infty}\frac{J_\mu(nx)}{n^{\mu+3}}=\Phi_{\mu,3}(x)+\sum_{i=2}^{\infty}\frac{(-1)^i\zeta(3-2i)(\frac{x}{2})^{\mu+2i}}{i!\,\Gamma(\mu+i+1)},$$

where

$$\Phi_{\mu,3}(x)=-\frac{x^2\Gamma(\mu+1)(1+\gamma+2\ln 2-2\ln x+\psi(\mu+2))-8\Gamma(\mu+2)\zeta(3)}{2^{\mu+3}\,\Gamma(\mu+1)\,\Gamma(\mu+2)}\,x^\mu.$$

2.2. Summation based on Bessel's integral. There is, however, another summation formula of the series (1) holding when both $\nu \le \mu$ and $\nu > \mu$ providing that $\mu = 2k + \delta$, where $k \in \mathbb{N}_0$, $\delta = \left\{\begin{smallmatrix}1\\0\end{smallmatrix}\right\}$ $g = \left\{\begin{smallmatrix}\sin\\\cos\end{smallmatrix}\right\}$, and $\nu > 0$. Now we have to deal with Bessel's integral (3) and place it in (1). Then referring again to Dirichlet's test as in the case of the series (4), we similarly ascertain that $\sum_{n=1}^{\infty} \frac{g(nx\sin\theta)}{n^\nu}$ converges uniformly with respect to θ on each segment $[\varepsilon, \pi-\varepsilon] \subset (0,\pi)$, $0 < \varepsilon < \pi$, for $\nu > 0$, where the convergence region is again $0 < x < 2\pi$. Consequently, summation and integration are interchangeable

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{2}} \frac{g\big((2k+\delta)\theta\big)g(nx\cos\theta)}{n^\nu}\,d\theta = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} g\big((2k+\delta)\theta\big)\sum_{n=1}^{\infty}\frac{g(nx\cos\theta)}{n^\nu}\,d\theta.$$

Afterwards, we apply the same procedure as when deriving (10), and obtain (see [15])

$$\sum_{n=1}^{\infty}\frac{J_{2k+\delta}(nx)}{n^\nu} = (-1)^k\left[\frac{\pi x^{\nu-1}}{2^\nu g(\frac{\pi\nu}{2})\,G_k} + \sum_{i=k}^{\infty}\frac{(-1)^i\zeta(\nu-2i-\delta)(\frac{x}{2})^{2i+\delta}}{\Gamma(i-k+1)\,\Gamma(\delta+i+k+1)}\right], \tag{13}$$

where $g = \left\{\begin{smallmatrix}\sin\\\cos\end{smallmatrix}\right\}$ $\delta = \left\{\begin{smallmatrix}1\\0\end{smallmatrix}\right\}$, and $G_k = \Gamma(\frac{\nu+2k+\delta+1}{2})\Gamma(\frac{\nu-2k-\delta+1}{2})$ has been introduced for the sake of brevity. The summation index i starts from k, because for $0 \le i \le k-1$, we have $1/\Gamma(i-k+1) = 0$, implying that the first k terms vanish.

2.2.1. Closed form cases. We see that (13) takes closed form if $g = \left\{\begin{smallmatrix}\sin\\\cos\end{smallmatrix}\right\}$ $\nu = \left\{\begin{smallmatrix}2l-1\\2l\end{smallmatrix}\right\}$, $l \in \mathbb{N}$. So, if we take $g = \sin$, then $\nu = 2l-1$ and $\delta = 1$, and for $l \ge k+1$ we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{J_{2k+1}(nx)}{n^{2l-1}} &= \frac{(-1)^{k-l+1}\pi x^{2l-2}}{2^{2l-1}\Gamma(l+k+\frac{1}{2})\Gamma(l-k-\frac{1}{2})}\\ &\quad+\sum_{i=k}^{l-1}\frac{(-1)^{k+i}\zeta(2l-2i-2)(\frac{x}{2})^{2i+1}}{(i-k)!\,(i+k+1)!}.\end{aligned} \tag{14}$$

But $l < k+1$ implies $\zeta(2l-2i-2) = 0$, the whole infinite sum in (13) vanish, and we have

$$\sum_{n=1}^{\infty}\frac{J_{2k+1}(nx)}{n^{2l-1}} = \frac{(-1)^{k-l+1}\pi x^{2l-2}}{2^{2l-1}\Gamma(l+k+\frac{1}{2})\Gamma(l-k-\frac{1}{2})}. \tag{15}$$

If $g = \cos$, then $\nu = 2l$ and $\delta = 0$, then for $l \ge k$ we obtain a formula similar to (14), and for $l < k$ another one similar to (15).

2.2.2. Limiting value cases. Otherwise, if $g = \left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}$ $\nu = \left\{\begin{smallmatrix}2l\\ 2l-1\end{smallmatrix}\right\}$, we have to take limits. For instance, when $g = \cos$ and $\nu = 2l-1 > 2k = \mu$ (remember that if $g = \cos$ then $\delta = 0$) we deal with

$$\lim_{\nu\to 2l-1}\left[\frac{2^{-\nu}\pi x^{\nu-1}}{\cos(\frac{\pi\nu}{2})\Gamma(\frac{\nu+2k+1}{2})\Gamma(\frac{\nu-2k+1}{2})}+\sum_{i=k}^{l-1}\frac{(-1)^i\zeta(\nu-2i)(\frac{x}{2})^{2i}}{\Gamma(i+k+1)\Gamma(i-k+1)}\right]=\Phi_{2k,2l-1}(x),$$

where

$$\Phi_{2k,2l-1}(x) = \frac{(-1)^{l-1}(\frac{x}{2})^{2(l-1)}}{2\Gamma(l+k)\Gamma(l-k)}\left(\psi(l+k)+\psi(l-k)+2\gamma-2\ln\frac{x}{2}\right) + \sum_{i=k}^{l-2}\frac{(-1)^i\zeta(2l-1-2i)(\frac{x}{2})^{2i}}{(i+k)!(i-k)!},$$

and (13) becomes

$$\sum_{n=1}^{\infty}\frac{J_{2k}(nx)}{n^{2l-1}} = (-1)^k\left[\Phi_{2k,2l-1}(x)+\sum_{i=l}^{\infty}\frac{(-1)^i\zeta(2l-1-2i)(\frac{x}{2})^{2i}}{(i+k)!(i-k)!}\right]. \tag{16}$$

For example:

$$\sum_{n=1}^{\infty}\frac{J_4(nx)}{n^5} = \frac{x^4}{9216}(25+24\ln 2-24\ln x)+\sum_{i=3}^{\infty}\frac{(-1)^i\zeta(5-2i)(\frac{x}{2})^{2i}}{(i+2)!(i-2)!}.$$

Alternatively, if $\nu = 2l-1 < 2k = \mu$ we have $l-k < \frac{1}{2}$, implying that an integer $l-k$ is non-positive, and at these points $\Gamma(z)$ has poles, so $1/\Gamma(l-k) = 0$, i.e., $1/\Gamma(\frac{\nu-2k+1}{2}) = 0$. In this case (13) is

$$\sum_{n=1}^{\infty}\frac{J_{2k}(nx)}{n^{2l-1}} = (-1)^k\sum_{i=k}^{\infty}\frac{(-1)^i\zeta(2l-1-2i)(\frac{x}{2})^{2i}}{(i+k)!(i-k)!}. \tag{17}$$

We give an example:

$$\sum_{n=1}^{\infty}\frac{J_4(nx)}{n^3} = \sum_{i=2}^{\infty}\frac{(-1)^i\zeta(3-2i)(\frac{x}{2})^{2i}}{(i+k)!(i-k)!}.$$

3. Summation based on Poisson's formula

We shall now find sums of the series (1) by using quite a different procedure. This method is based on Poisson's formula including the Fourier transform of a suitably chosen function.

There exist several formulations of Poisson's formula in the literature (see [3], [8], [14]). We shall choose the following setting. Let $f(x)$ be a continuous function, suppose that it is smooth and absolutely integrable on $(0, +\infty)$. Then, for $\alpha\beta = 2\pi$, $\alpha > 0$,

$$\sqrt{\alpha}\left(\frac{1}{2}f(0) + \sum_{n=1}^{\infty} f(n\alpha)\right) = \sqrt{\beta}\left(\frac{1}{2}F_c(0) + \sum_{n=1}^{\infty} F_c(n\beta)\right), \tag{18}$$

holds, assuming that the left and right side of (18) converge absolutely. $F_c(\omega)$ is the Fourier transform of a function $f(x)$, i.e.,

$$\mathcal{F}_c(f(x), \omega) = F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x dx. \tag{19}$$

We shall make use of a representation of the Bessel function (see [4, p. 900])

$$J_\mu(x) = \frac{x^\mu}{2^\mu} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}\, k!\, \Gamma(\mu + k + 1)}, \qquad |\arg x| < \pi.$$

Further, on this basis we can consider a function

$$g(x) = \frac{J_\mu(x)}{x^\nu} = \frac{x^{\mu-\nu}}{2^\mu} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}\, k!\, \Gamma(\mu + k + 1)}, \quad \mu > \nu,\ x \neq 0. \tag{20}$$

If we take limit in (20), we find $\lim_{x\to 0} g(x) = 0$ for $\mu > \nu$, and $\lim_{x\to 0} g(x) = \frac{1}{2^\mu \Gamma(\mu+1)}$ for $\mu = \nu$, which we write in the form of

$$\lim_{x\to 0} g(x) = \frac{\delta_{\mu,\nu}}{2^\mu \Gamma(\mu + 1)}, \qquad \delta_{\mu,\nu} = \begin{cases} 1, & \mu = \nu \\ 0, & \mu > \nu. \end{cases}$$

Now we define a function

$$f(x) = \begin{cases} g(x), & x \neq 0 \\ \dfrac{\delta_{\mu,\nu}}{2^\mu \Gamma(\mu + 1)}, & x = 0 \end{cases} \tag{21}$$

that is continuous, smooth and absolutely integrable on $(0, +\infty)$. So we have

$$\mathcal{F}_c(f(x), \omega) = F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{J_\mu(x)}{x^\nu} \cos \omega x\, dx.$$

After substituting the right-hand side for $F_c(\omega)$ in (18) and taking $\omega = n\beta$, we get

$$\begin{aligned}\sqrt{\alpha}\left(\frac{1}{2}f(0) + \sum_{n=1}^{\infty}\frac{J_\mu(n\alpha)}{(n\alpha)^\nu}\right)\\ = \sqrt{\frac{2\beta}{\pi}}\left(\frac{1}{2}\int_0^\infty \frac{J_\mu(x)}{x^\nu}\,dx + \sum_{n=1}^{\infty}\int_0^\infty \frac{J_\mu(x)}{x^\nu}\cos(n\beta x)\,dx\right).\end{aligned} \tag{22}$$

Since $\alpha\beta = 2\pi$ in (18), there follows $\beta = \frac{2\pi}{\alpha}$, so (22) becomes

$$\sum_{n=1}^{\infty}\frac{J_\mu(n\alpha)}{(n\alpha)^\nu} = \frac{2}{\alpha}\left(\frac{1}{2}\int_0^\infty \frac{J_\mu(x)}{x^\nu}\,dx + \sum_{n=1}^{\infty}\int_0^\infty \frac{J_\mu(x)}{x^\nu}\cos\frac{2n\pi x}{\alpha}\,dx\right) - \frac{1}{2}f(0). \tag{23}$$

In [4, p. 669] we find the integral

$$\int_0^\infty \frac{J_\mu(x)}{x^\nu}\,dx = 2^{-\nu}\frac{\Gamma\left(\frac{1+\mu-\nu}{2}\right)}{\Gamma\left(\frac{1+\mu+\nu}{2}\right)}, \quad \mu+1>\nu>-\frac{1}{2}. \tag{24}$$

We also use (see [12, p. 192])

$$\begin{aligned}\int_0^\infty \frac{J_\mu(x)}{x^\nu}\cos\frac{2n\pi x}{\alpha}dx = \frac{1}{2^{2\mu-\nu+1}}\left(\frac{\alpha}{n\pi}\right)^{\mu-\nu+1}\cos\frac{(\mu-\nu+1)\pi}{2}\cdot\frac{\Gamma(\mu-\nu+1)}{\Gamma(\mu+1)}\\ \times\ {}_2F_1\left(\frac{\mu-\nu+1}{2}, \frac{\mu-\nu+2}{2}, \mu+1, \left(\frac{\alpha}{2n\pi}\right)^2\right),\end{aligned} \tag{25}$$

with $\alpha < 2n\pi,\ \mu+1>\nu>-\frac{1}{2}$.

Now we put the right-hand sides of (24), (25) and (21) instead of corresponding expressions in (23), and further use a power series representation of the Gauss hypergeometric function ${}_2F_1$:

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{z^n}{n!}, \quad |z|<1.$$

Thus we come to a double sum. Because $\alpha < 2n\pi$ is one of conditions for (25), there follows $\left(\frac{\alpha}{2n\pi}\right)^2 < 1$. Interchanging the order of these two summations, yields

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{J_\mu(n\alpha)}{n^\nu} = \frac{\alpha^{\nu-1}\Gamma(\frac{1+\mu-\nu}{2})}{2^\nu\,\Gamma(\frac{1+\mu+\nu}{2})} + \frac{\alpha^\mu\cos\left(\frac{\mu-\nu+1}{2}\pi\right)\Gamma(\mu-\nu+1)}{2^{2\mu-\nu}\pi^{\mu-\nu+1}\Gamma(\mu+1)}\\ \times\sum_{k=0}^{\infty}\frac{(\frac{\mu-\nu+1}{2})_k(\frac{\mu-\nu+2}{2})_k(\frac{\alpha}{2\pi})^{2k}}{k!\,(\mu+1)_k}\sum_{n=1}^{\infty}\frac{1}{n^{2k+\mu-\nu+1}} - \frac{\alpha^\nu\delta_{\mu,\nu}}{2^{\mu+1}\Gamma(\mu+1)}.\end{aligned} \tag{26}$$

Obviously,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k+\mu-\nu+1}} = \zeta(2k+\mu-\nu+1),$$

and if $\mu > \nu$, then $2k+\mu-\nu+1 > 1$ is valid for each $k \in \mathbb{N}_0$. We also express the Pochhammer symbol $(a)_k = a(a+1)\cdots(a+k-1)$ by means of Gamma functions: $(a)_k = \Gamma(a+k)/\Gamma(a)$, and make use of the relation $\frac{\pi}{\cos \pi z} = \Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z)$. After cancellation, (26) can be written in the form of

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{J_\mu(n\alpha)}{n^\nu} = \frac{\alpha^{\nu-1}\Gamma(\frac{1+\mu-\nu}{2})}{2^\nu\,\Gamma(\frac{1+\mu+\nu}{2})} &+ \frac{(\frac{\alpha}{2})^\mu (2\pi)^{\nu-\mu}\Gamma(\mu-\nu+1)}{\Gamma(\frac{\nu-\mu}{2})\Gamma(\frac{\mu-\nu+1}{2})\Gamma^2(\frac{\mu-\nu}{2}+1)} \\ &\times \sum_{k=0}^{\infty} \frac{\Gamma(\frac{\mu-\nu+1}{2}+k)\Gamma(\frac{\mu-\nu}{2}+1+k)(\frac{\alpha}{2\pi})^{2k}\zeta(2k+\mu-\nu+1)}{k!\,\Gamma(\mu+1+k)} \\ &- \frac{\alpha^\nu \delta_{\mu,\nu}}{2^{\mu+1}\Gamma(\mu+1)},\end{aligned} \tag{27}$$

with $0 < \alpha < 2\pi,\ \mu > \nu > -\frac{1}{2}$. By using a property of Gamma $\Gamma(z+1) = z\Gamma(z)$ and Legendre's duplication formula (see page 398), the formula (27) is reduced to

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{J_\mu(n\alpha)}{n^\nu} = \frac{\alpha^{\nu-1}\Gamma(\frac{1+\mu-\nu}{2})}{2^\nu\,\Gamma(\frac{1+\mu+\nu}{2})} &- \frac{\alpha^\nu \delta_{\mu,\nu}}{2^{\mu+1}\Gamma(\mu+1)} + \frac{\pi^{\nu-\frac{1}{2}}\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\frac{\nu-\mu}{2})} \\ \times \sum_{k=0}^{\infty} &\frac{(\frac{\alpha}{2\pi})^{2k+\mu}\zeta(2k+\mu-\nu+1)\prod_{i=1}^{k}(\frac{\mu-\nu-1}{2}+i)(\frac{\mu-\nu}{2}+i)}{k!\,\Gamma(\mu+1+k)}.\end{aligned} \tag{28}$$

3.1. Closed form cases. Unlike (10), closed form cases of (28) do not ensue because ζ-function vanishes. Its argument in this case does not take negative even integers since $\mu > \nu$. But, for $\nu - \mu = -2p$, $p \in \mathbb{N}_0$, the expression $1/\Gamma(\frac{\nu-\mu}{2})$ becomes zero since Γ has poles at non-positive integers. Because of $\delta_{\mu,\nu} = 0$ for $\mu > \nu$, (28) takes closed form

$$\sum_{n=1}^{\infty} \frac{J_\mu(n\alpha)}{n^{\mu-2p}} = \frac{\alpha^{\mu-2p-1}\Gamma(p+\frac{1}{2})}{2^{\mu-2p}\,\Gamma(\mu-p+\frac{1}{2})}. \tag{29}$$

But, for $\mu = \nu$, we have seen that $\delta_{\mu,\nu} = 1$, so in that case (28) takes closed form

$$\sum_{n=1}^{\infty} \frac{J_\mu(n\alpha)}{n^\mu} = \frac{\alpha^{\mu-1}\sqrt{\pi}}{2^\mu\,\Gamma(\mu+\frac{1}{2})} - \frac{\alpha^\mu}{2^{\mu+1}\Gamma(\mu+1)}. \tag{30}$$

For a particular choice of parameters, the closed form formula (29) can be reduced to (15) or a similar closed form case coming out of (13) (see the remark after (15)). If we take $\nu = 2l - 1$, $\mu = 2k + 1$, and require $l < k + 1$ (these are conditions of holding (15)), we further have $\mu - \nu = 2p$, where $p = k - l + 1 \in \mathbb{N}$. Replacing these values in (29), setting afterwards $z = k - l + \frac{3}{2}$ and applying the property $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, we easily come to (15). So, by using different methods we obtain the same formula. Yet, note that (29) holds whenever real numbers μ and ν ($\mu > \nu > -\frac{1}{2}$) satisfy $\mu - \nu = 2p$ ($p \in \mathbb{N}$), whereas (15) holds only for positive integers μ and ν.

Let us have a look at the formula 9 in [12, p. 678]:

$$\sum_{k=1}^{\infty} \frac{J_\nu(kx)}{k^{\nu-2n}} = \frac{\Gamma(n+\frac{3}{2})}{(2n+1)\Gamma(\nu - n + \frac{1}{2})} \left(\frac{x}{2}\right)^{\nu-2n-1},$$

where $n \in \mathbb{N}$, $\nu > 2n - \frac{1}{2}$, $0 \le x < 2\pi$. After $\Gamma(n+\frac{3}{2}) = (n+\frac{1}{2})\Gamma(n+\frac{1}{2})$ and cancellation, this formula becomes (29), which in turn holds on condition that $\mu - 2p = \nu > -\frac{1}{2}$, whence we have $\mu > 2p - \frac{1}{2}$, and we conclude that these formulas are identical.

The formula 8 in [12, p. 678]

$$\sum_{k=1}^{\infty} \frac{J_{2n+m}(kx)}{k^m} = \frac{x^{m-1}}{(2n+1)(2n+3)\cdots(2n+2m-1)}, \quad n \in \mathbb{N},\ 0 < x < 2\pi,$$

is another example of a particular case obtainable from our formula (28). If we set $\nu = m \in \mathbb{N}$ in (28) with $\mu - \nu = 2p \in \mathbb{N}$, after a rearrangement we come to the last formula, which is in fact a special case of our formula (29).

4. Concluding remarks

Now we review our results and show how each concrete case of the series (1) for given μ and ν can be handled. Actually, all possible cases of relations between parameters μ and ν are comprised.

So, if μ and ν are real numbers, $\nu > \mu > -\frac{1}{2}$, we apply the summation formula (10) obtained by using Poisson's integral. If, for these numbers, there holds $\nu - \mu = 2k$, $k \in \mathbb{N}_0$, we apply (11). If $\nu - \mu = 2k+1$, $k \in \mathbb{N}_0$, we use (12).

If μ is a non-negative integer and ν positive real number, then, irrespective of being $\nu > \mu$ or $\nu \le \mu$, we apply the summation formula (13) obtained by way of Bessel's integral. In addition, if $\nu \in \mathbb{N}$, then closed form or limit cases ensue. Some examples are given by (14) and (15) or (16) and (17) respectively.

If μ and ν are real numbers, $\mu > \nu > -\frac{1}{2}$, we use (28) obtained by means of a power series representation of Bessel functions and Poisson's formula. In particular, if there holds $\mu - \nu = 2p$, $p \in \mathbb{N}_0$, we apply (29) or (30). In special cases, when μ and ν are odd positive integers, we may use (15) or a similar formula obtainable from (13), when they are even positive integers.

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