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Maximizers for the Strichartz inequality

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Abstract. We compute explicitly the best constants and, by solving some functional equations, we find all maximizers for homogeneous Strichartz estimates for the Schrödinger equation and for the wave equation in the cases when the Lebesgue exponent is an even integer.

Keywords. Strichartz estimates, Schrödinger equation, wave equation, functional equations

1. Introduction

Let n be a positive integer and let $p = p(n) = 2 + 4/n$. The Strichartz inequality for the homogeneous Schrödinger equation in n spatial dimensions states that there exists a constant $S > 0$ such that

$$\|u\|_{L^{p(n)}(\mathbb{R}^{1+n})} \leq S \|f\|_{L^2(\mathbb{R}^n)}, \quad (1)$$

whenever $u(t, x)$ is the solution of the equation

$$i \partial_t u = \Delta u \quad (2)$$

with initial data $u(0, x) = f(x)$; see [8] for the original proof by Strichartz. We denote by $S(n)$ the best constant for the estimate (1),

$$S(n) = \sup_{f \in L^2(\mathbb{R}^n)} \frac{\|e^{-it\Delta} f\|_{L^{p(n)}(\mathbb{R}^{1+n})}}{\|f\|_{L^2(\mathbb{R}^n)}}.$$

If $n \geq 2$, we can also consider the Strichartz inequality for the homogeneous wave equation in n spatial dimensions which states that there exists a constant $W > 0$ such that

$$\|u\|_{L^{p(n-1)}(\mathbb{R}^{1+n})} \leq W \|(f, g)\|_{\dot{H}^{1/2}(\mathbb{R}^n) \times \dot{H}^{-1/2}(\mathbb{R}^n)} \quad (3)$$

whenever $u(t, x)$ is the solution of the equation

$$\partial_t^2 u = \Delta u \quad (4)$$

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with initial data

$$u(0, x) = f(x), \quad \partial_t u(0, x) = g(x). \tag{5}$$

This was also proved in [8]. We denote by $W(n)$ the best constant for the estimate (3),

$$W(n) = \sup_{\substack{f \in \dot{H}^{1/2}(\mathbb{R}^n) \\ g \in \dot{H}^{-1/2}(\mathbb{R}^n)}} \frac{\|\cos(t\sqrt{-\Delta})f + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g\|_{L^{p(n-1)}(\mathbb{R}^{1+n})}}{\|(f, g)\|_{\dot{H}^{1/2}(\mathbb{R}^n) \times \dot{H}^{-1/2}(\mathbb{R}^n)}}.$$

Kunze [6] has recently proved the existence of a maximizing function $f_* \in L^2(\mathbb{R})$ for the estimate (1) in the special case $n = 1$ and $p = 6$, which means that for the corresponding solution u_* ,

$$i \partial_t u_* = \partial_x^2 u_*, \quad u_*(0, x) = f_*(x),$$

we have the equality $\|u_*\|_{L^6(\mathbb{R} \times \mathbb{R})} = S(1)\|f_*\|_{L^2(\mathbb{R})}$. The proof in [6] is based on an elaborate application of the concentration compactness principle and does not provide an explicit expression for a maximizer.

Here, we present a more direct and elementary approach which allows us to explicitly determine the families of maximizers and compute the best constants for the estimates (1) and (3) when the exponent $p = p(n)$ is an even integer. We show that the classes of maximizers are unique modulo the natural geometric invariance properties of the equations. Moreover, maximizers turn out to be smooth solutions to some functional equations which can be solved explicitly.

For the Schrödinger equation we have:

Theorem 1.1. *In the case $n = 1$ and $p = 6$, we have $S(1) = 12^{-1/12}$; in the case $n = 2$ and $p = 4$, we have $S(2) = 2^{-1/2}$. In both cases an example of a maximizer $f_* \in L^2(\mathbb{R}^n)$ for which*

$$\|e^{-it\Delta} f_*\|_{L^p(\mathbb{R} \times \mathbb{R}^n)} = S(n)\|f_*\|_{L^2(\mathbb{R}^n)} \tag{6}$$

is provided by the Gaussian function $f_(x) = \exp(-|x|^2)$.*

The geometric invariance properties of the equation (2) suggest a way to completely characterize the class of all maximizers.

Definition 1.2. *Let \mathcal{G} be the Lie group of transformations generated by:*

- *space-time translations: $u(t, x) \rightsquigarrow u(t + t_0, x + x_0)$ with $t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n$;*
- *parabolic dilations: $u(t, x) \rightsquigarrow u(\lambda^2 t, \lambda x)$ with $\lambda > 0$;*
- *changes of scale: $u(t, x) \rightsquigarrow \mu u(t, x)$ with $\mu > 0$;*
- *space rotations: $u(t, x) \rightsquigarrow u(t, Rx)$ with $R \in SO(n)$;*
- *phase shifts: $u(t, x) \rightsquigarrow e^{i\theta} u(t, x)$ with $\theta \in \mathbb{R}$;*
- *Galilean transformations:*

$$u(t, x) \rightsquigarrow \exp\left(\frac{i}{4}(|v|^2 t + 2v \cdot x)\right) u(t, x + tv)$$

with $v \in \mathbb{R}^n$.

If u solves (2) and $g \in \mathcal{G}$ then $v = g \cdot u$ is still a solution to (2). Moreover, the ratio $\|u\|_{L^p(n)} / \|u(0)\|_{L^2}$ is left unchanged by the action of \mathcal{G} .

Remark 1.3. We should mention that there exists another important (discrete) symmetry for the Schrödinger equation given by the *pseudo-conformal inversion*:

$$u(t, x) \rightsquigarrow t^{-n/2} e^{i|x|^2/(4t)} u\left(-\frac{1}{t}, \frac{x}{t}\right).$$

Combining the inversion with translations and dilations, we find that the Schrödinger equation is invariant under the representation of $SL(2, \mathbb{R})$ given by

$$u(t, x) \rightsquigarrow (a + bt)^{-n/2} e^{ib|x|^2/(4(a+bt))} u\left(\frac{c + dt}{a + bt}, \frac{x}{a + bt}\right), \quad ad - bc = 1.$$

These transformations have many important applications. However, we do not really need them in the context of our analysis and for simplicity we are not including them in the list of generators of the group \mathcal{G} .

Theorem 1.4. Let $(n, p) = (1, 6)$ or $(n, p) = (2, 4)$. Let $f_*(x) = \exp(-|x|^2)$ and $u_*(t, x) = e^{-it\Delta} f_*(x)$ be the corresponding solution to the Schrödinger equation (2). Then the set of maximizers for which the equality (6) holds coincides with the set of initial data of solutions to (2) in the orbit of u_* under the action of the group \mathcal{G} . In particular, all maximizers are given by L^2 functions of the form

$$f_*(x) = \exp(A|x|^2 + b \cdot x + C)$$

with $A, C \in \mathbb{C}$, $b \in \mathbb{C}^n$ and $\Re(A) < 0$.

For the wave equation we have:

Theorem 1.5. In the case $n = 2$ and $p = 6$, we have $W(2) = (25/(64\pi))^{1/6}$; in the case $n = 3$ and $p = 4$, we have $W(3) = (3/(16\pi))^{1/4}$. In both cases an example of a maximizer pair $(f_*, g_*) \in \dot{H}^{1/2}(\mathbb{R}^n) \times \dot{H}^{-1/2}(\mathbb{R}^n)$ for which we have

$$\left\| \cos(t\sqrt{-\Delta})f_* + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g_* \right\|_{L^p(\mathbb{R}^{1+n})} = W(n)\|(f_*, g_*)\|_{\dot{H}^{1/2}(\mathbb{R}^n) \times \dot{H}^{-1/2}(\mathbb{R}^n)} \quad (7)$$

is provided by the functions $f_*(x) = (1 + |x|^2)^{-(n-1)/2}$, $g_*(x) = 0$.

The geometric invariance properties of the equation (4) suggest a way to completely characterize the class of all maximizers.

Definition 1.6. Let \mathcal{L} be the Lie group of transformations acting on solutions of the wave equation and generated by:

- *space-time translations:* $u(t, x) \rightsquigarrow u(t + t_0, x + x_0)$ with $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$;
- *isotropic dilations:* $u(t, x) \rightsquigarrow u(\lambda t, \lambda x)$ with $\lambda > 0$;
- *changes of scale:* $u(t, x) \rightsquigarrow \mu u(t, x)$ with $\mu > 0$;

- *space rotations*: $u(t, x) \rightsquigarrow u(t, Rx)$ with $R \in SO(n)$;
- *phase shifts*: $u(t, x) \rightsquigarrow e^{i\theta_+}u_+(t, x) + e^{i\theta_-}u_-(t, x)$ with $\theta_+, \theta_- \in \mathbb{R}$ (for the meaning of u_+ and u_- see the next section);
- *Lorentzian boosts*:

$$u(t, x_1, x') \rightsquigarrow u(\cosh(a)t + \sinh(a)x_1, \sinh(a)t + \cosh(a)x_1, x')$$

with $a \in \mathbb{R}$.

If u solves (4) and $g \in \mathcal{L}$ then $v = g \cdot u$ is still a solution to (4). Moreover, the ratio $\|u\|_{L^p(n-1)} / \|(u(0), \partial_t u(0))\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}}$ is left unchanged by the action of \mathcal{L} .

Theorem 1.7. *Let $(n, p) = (2, 6)$ or $(n, p) = (3, 4)$. We consider the initial data $f_*(x) = (1 + |x|^2)^{-(n-1)/2}$, $g_*(x) = 0$, and let u_* be the corresponding solution to the wave equation (4). Then the set of maximizers for which the equality (7) holds coincides with the set of initial data of solutions to (4) in the orbit of u_* under the action of the group \mathcal{L} .*

In order to understand how to construct maximizers, we first present sharp proofs of the Strichartz estimates, based on the space-time Fourier transform in the spirit of Klainerman and Machedon’s work on bilinear estimates [5], [2]. We then optimize each step of the proof by imposing conditions under which all inequalities become equalities. What we find are functional equations for the Fourier transform of maximizers; their solutions are given by particular exponential functions with linear or quadratic exponents.

The key tool is the following well-known simple fact about Cauchy–Schwarz’s inequality for inner products.

Lemma 1.8. *Let $\langle \cdot, \cdot \rangle$ be a (complex) inner product on a vector space V and let $u, v \in V$ be two non-zero vectors. Cauchy–Schwarz’s inequality says that*

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle;$$

moreover, equality holds if and only if $u = \alpha v$ for some scalar $\alpha \in \mathbb{C}$.

Remark 1.9. The uniqueness of maximizers modulo the transformation groups described in Definitions 1.2 and 1.6 will be checked *a posteriori*, after we obtain explicit formulae for maximizers, and it is not used in the proof. While our proof relies heavily on the fact that p is an even integer, the geometric characterization can be stated also in higher dimensions when p is not an even integer. It would be interesting to prove our results without making use of the Fourier transform. For the moment, we formulate the following natural conjectures.

Conjecture 1.10. *For any integer $n \geq 1$, let $p = 2 + 4/n$, let $f_*(x) = \exp(-|x|^2)$ and $u_*(t, x) = e^{-it\Delta} f_*(x)$ be the corresponding solution to the Schrödinger equation (2). Then the set of maximizers for which the equality (6) holds coincides with the set of initial data of solutions to (2) in the orbit of u_* under the action of the group \mathcal{G} .*

Conjecture 1.11. For any integer $n \geq 2$, let $p = 2 + 4/(n - 1)$ and let f_* be the function on \mathbb{R}^n whose Fourier transform is $\widehat{f}_*(\xi) = |\xi|^{-1} \exp(-|\xi|)$. Let u_* be the solution to the wave equation (4) corresponding to the initial data $u_*(0) = f_*$, $\partial_t u_*(0) = 0$. Then the set of maximizers for which the equality (7) holds coincides with the set of initial data of solutions to (4) in the orbit of u_* under the action of the group \mathcal{L} .

2. Notation and preliminaries

For $1 \leq p < \infty$, $L^p(\mathbb{R}^n)$ is the usual Lebesgue space with norm

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}.$$

The homogeneous Sobolev spaces $\dot{H}^{1/2}(\mathbb{R}^n)$ and $\dot{H}^{-1/2}(\mathbb{R}^n)$ are defined by the norms

$$\|f\|_{\dot{H}^{1/2}(\mathbb{R}^n)} = \|D^{1/2}f\|_{L^2(\mathbb{R}^n)}, \quad \|f\|_{\dot{H}^{-1/2}(\mathbb{R}^n)} = \|D^{-1/2}f\|_{L^2(\mathbb{R}^n)},$$

where $D = \sqrt{-\Delta}$. In the context of the wave equation we set

$$\|(f, g)\|_{\dot{H}^{1/2}(\mathbb{R}^n) \times \dot{H}^{-1/2}(\mathbb{R}^n)} = (\|f\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 + \|g\|_{\dot{H}^{-1/2}(\mathbb{R}^n)}^2)^{1/2}.$$

If $f(x)$ is an integrable function defined on \mathbb{R}^n , we define its (spatial) Fourier transform by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix \cdot \xi} dx.$$

If $F(t, x)$ is an integrable function defined on $\mathbb{R} \times \mathbb{R}^n$, we define its space-time Fourier transform by

$$\widetilde{F}(\tau, \xi) = \int_{\mathbb{R} \times \mathbb{R}^n} F(t, x)e^{-i(t\tau + x \cdot \xi)} dt dx.$$

These definitions extend in the usual way to tempered distributions. The Fourier transform acts like an isometry on L^2 and, with our definition for the Fourier transform, Plancherel's theorem states that

$$\|\widehat{f}\|_{L^2(\mathbb{R}^n)} = (2\pi)^{n/2} \|f\|_{L^2(\mathbb{R}^n)}, \quad \|\widetilde{F}\|_{L^2(\mathbb{R}^{1+n})} = (2\pi)^{(1+n)/2} \|F\|_{L^2(\mathbb{R}^{1+n})}.$$

We recall also that the Fourier transform of a pointwise product (when it is defined) is given by the convolution product of the Fourier transform of each factor,

$$\begin{aligned} \widehat{fg}(\xi) &= \frac{1}{(2\pi)^n} \widehat{f} * \widehat{g}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} \widehat{f}(\eta) \widehat{g}(\zeta) \delta(\xi - \eta - \zeta) d\eta d\zeta, \\ \widetilde{FG}(\tau, \xi) &= \frac{1}{(2\pi)^{n+1}} \widetilde{F} * \widetilde{G}(\tau, \xi) \\ &= \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n} \widetilde{F}(\lambda, \eta) \widetilde{G}(\mu, \zeta) \delta \left(\begin{matrix} \tau - \lambda - \mu \\ \xi - \eta - \zeta \end{matrix} \right) d\lambda d\eta d\mu d\zeta. \end{aligned}$$

Here $\delta(\cdot)$ denotes Dirac’s delta measure concentrated at 0, $\int \delta(x)f(x) dx = f(0)$. We also denote the tensor product of two delta functions by

$$\delta \begin{pmatrix} a \\ b \end{pmatrix} = \delta(a)\delta(b).$$

If $u(t, x)$ is the solution of the Schrödinger equation (2) then its space-time Fourier transform is

$$\tilde{u}(\tau, \xi) = 2\pi \delta(\tau - |\xi|^2) \widehat{f}(\xi),$$

where f is the initial data at time $t = 0$. This shows that \tilde{u} is a measure supported on the paraboloid $\tau = |\xi|^2$. We notice, in connection with the invariance of equation (2) under Galilean transformations, that the measure $\delta(\tau - |\xi|^2)$ is invariant under the volume preserving affine change of variables

$$(\tau, \xi) \rightsquigarrow (\tau + 2v \cdot \xi + |v|^2, \xi + v), \tag{8}$$

for any $v \in \mathbb{R}^n$.

If $u(t, x)$ is the solution of the wave equation (4) with initial data (5), then we can split it as $u = u_+ + u_-$, where

$$\begin{aligned} u_+(t) &= e^{itD} D^{-1/2} f_+, & f_+ &= \frac{1}{2}(D^{1/2} f - i D^{-1/2} g), \\ u_-(t) &= e^{-itD} D^{-1/2} f_-, & f_- &= \frac{1}{2}(D^{1/2} f + i D^{-1/2} g). \end{aligned}$$

We call u_+ a (+)-wave with data f_+ and u_- a (-)-wave with data f_- . Observe that, by the parallelogram law,

$$\|(f, g)\|_{\dot{H}^{1/2}(\mathbb{R}^n) \times \dot{H}^{-1/2}(\mathbb{R}^n)}^2 = 2(\|f_+\|_{L^2(\mathbb{R}^n)}^2 + \|f_-\|_{L^2(\mathbb{R}^n)}^2).$$

The space-time Fourier transforms of u_+ and u_- are

$$\tilde{u}_+(\tau, \xi) = 2\pi |\xi|^{-1/2} \delta(\tau - |\xi|) \widehat{f}_+(\xi), \quad \tilde{u}_-(\tau, \xi) = 2\pi |\xi|^{-1/2} \delta(\tau + |\xi|) \widehat{f}_-(\xi).$$

Hence, \tilde{u}_+ and \tilde{u}_- are measures supported on the null cones $\tau = |\xi|$ and $\tau = -|\xi|$, respectively. We also notice that the measures $|\xi|^{-1} \delta(\tau \mp |\xi|)$ are invariant under proper Lorentz transformations. Indeed, we can write

$$\frac{\delta(\tau \mp |\xi|)}{|\xi|} = 2\delta(\tau^2 - |\xi|^2) \chi(\pm\tau > 0).$$

The invariance properties of these delta measures on paraboloids and on null cones later will help us in the computation of some convolution integrals. Eventually we will need the following simple property of convolutions.

Lemma 2.1. *Let A be an $n \times n$ invertible matrix and b a vector in \mathbb{R}^n . Suppose f is a function (or a distribution) on \mathbb{R}^n which is invariant under the linear affine change of variable $x \rightsquigarrow Ax + b$, in the sense that $f(x) = f(Ax + b)$ for all $x \in \mathbb{R}^n$. Then, if the convolution $f * f$ is well defined, we have*

$$f * f(x) = \frac{1}{\det A} f * f(Ax + 2b),$$

and, if the convolution $f * f * f$ is well defined, we have

$$f * f * f(x) = \frac{1}{(\det A)^2} f * f * f(Ax + 3b).$$

For a complex number $z \in \mathbb{C}$, we denote its real and imaginary parts by $\Re(z)$ and $\Im(z)$ and its complex conjugate by \bar{z} . Whenever they are mentioned, $\log(z)$ and \sqrt{z} are the branches of the complex logarithm and of the complex square root defined on $\mathbb{C} \setminus \mathbb{R}_-$ which extend analytically the standard real logarithm and the standard square root of positive real numbers.

For a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we write $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$, so that $x = (x_1, x')$.

If E is a subset of \mathbb{R}^n we denote its closure with respect to the usual topology by \bar{E} .

3. Schrödinger equation in dimension $n = 2$

Consider the case $n = 2, p = 4$ for estimate (1). By Plancherel's theorem, $u \in L^4$ if and only if $\tilde{u}^2 \in L^2$ and

$$\|u\|_{L^4(\mathbb{R}^3)}^2 = \|u^2\|_{L^2(\mathbb{R}^3)} = (2\pi)^{-3/2} \|\tilde{u}^2\|_{L^2(\mathbb{R}^3)}. \tag{9}$$

The Fourier transform of u^2 reduces to

$$\tilde{u}^2(\tau, \xi) = \frac{1}{(2\pi)^3} \tilde{u} * \tilde{u}(\tau, \xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \hat{f}(\eta) \hat{f}(\zeta) \delta\left(\begin{matrix} \tau - |\eta|^2 - |\zeta|^2 \\ \xi - \eta - \zeta \end{matrix}\right) d\eta d\zeta. \tag{10}$$

When $\xi = \eta + \zeta$ and $\tau = |\eta|^2 + |\zeta|^2$, by the parallelogram law we have

$$2\tau = |\eta + \zeta|^2 + |\eta - \zeta|^2 \geq |\xi|^2.$$

It follows that \tilde{u}^2 is supported in the closure of the region

$$\mathcal{P}_2 = \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^2 : 2\tau > |\xi|^2\}.$$

For each choice of $(\tau, \xi) \in \mathcal{P}_2$, we denote by $\langle \cdot, \cdot \rangle_{(\tau, \xi)}$ the L^2 inner product associated with the measure

$$\mu_{(\tau, \xi)} = \delta\left(\begin{matrix} \tau - |\eta|^2 - |\zeta|^2 \\ \xi - \eta - \zeta \end{matrix}\right) d\eta d\zeta, \tag{11}$$

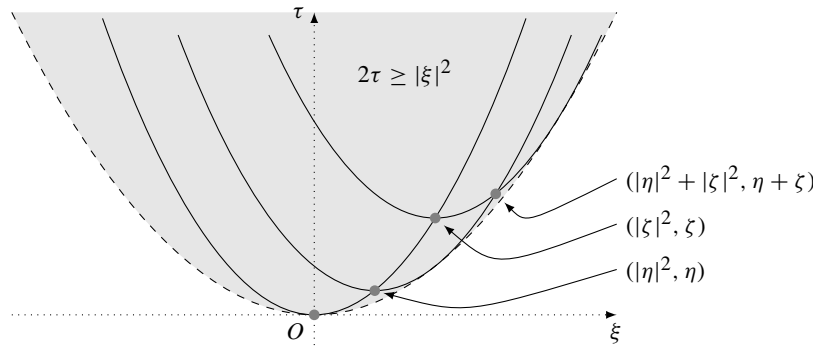


Fig. 1. The region \mathcal{P}_2 .

and by $\|\cdot\|_{(\tau,\xi)}$ the corresponding norm; more precisely, we set

$$\langle F, G \rangle_{(\tau,\xi)} = \int_{\mathbb{R}^2 \times \mathbb{R}^2} F(\eta, \zeta) \overline{G(\eta, \zeta)} \delta \left(\begin{matrix} \tau - |\eta|^2 - |\zeta|^2 \\ \xi - \eta - \zeta \end{matrix} \right) d\eta d\zeta,$$

$$\|F\|_{(\tau,\xi)} = \left(\int_{\mathbb{R}^2 \times \mathbb{R}^2} |F(\eta, \zeta)|^2 \delta \left(\begin{matrix} \tau - |\eta|^2 - |\zeta|^2 \\ \xi - \eta - \zeta \end{matrix} \right) d\eta d\zeta \right)^{1/2}.$$

Remark 3.1. The measure $\mu_{(\tau,\xi)}$ defined in (11) is the pull-back of the Dirac delta on $\mathbb{R} \times \mathbb{R}^2$ by the function $\Phi_{(\tau,\xi)} : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \times \mathbb{R}^2$ given by

$$\Phi_{(\tau,\xi)}(\eta, \zeta) = (\tau - |\eta|^2 - |\zeta|^2, \xi - \eta - \zeta).$$

This pull-back is well defined as long as the differential of $\Phi_{(\tau,\xi)}$ is surjective at the points where $\Phi_{(\tau,\xi)}$ vanishes (we refer to [3, Theorem 6.1.2 and Example 6.1.3] for more details about pull-backs of distributions). A quick computation shows that the differential of $\Phi_{(\tau,\xi)}$ is surjective at a point (η, ζ) if and only if $\eta \neq \zeta$. On the other hand, if $\Phi_{(\tau,\xi)}(\eta, \eta) = 0$ we must have

$$2\tau = 2(|\eta|^2 + |\eta|^2) = |\eta + \eta|^2 = |\xi|^2.$$

This tells us that $\mu_{(\tau,\xi)}$ is not well defined on the boundary of \mathcal{P}_2 , when $2\tau = |\xi|^2$. However, we can safely ignore the problems at this boundary and observe instead that for any locally integrable function $F(\eta, \zeta)$ defined on $\mathbb{R}^2 \times \mathbb{R}^2$ the integral $G(\tau, \xi) = \int F d\mu_{(\tau,\xi)}$ defines a locally integrable function on $\mathbb{R} \times \mathbb{R}^2$. Indeed, if K is a compact set in $\mathbb{R} \times \mathbb{R}^2$, we have

$$\begin{aligned} \iint_K |G(\tau, \xi)| d\tau d\xi &\leq \iint_{(\tau,\xi) \in K} \iint |F(\eta, \zeta)| \delta \left(\begin{matrix} \tau - |\eta|^2 - |\zeta|^2 \\ \xi - \eta - \zeta \end{matrix} \right) d\eta d\zeta d\tau d\xi \\ &= \iint_{(|\eta|^2 + |\zeta|^2, \eta + \zeta) \in K} |F(\eta, \zeta)| d\eta d\zeta, \end{aligned}$$

and $\{(\eta, \zeta) : (|\eta|^2 + |\zeta|^2, \eta + \zeta) \in K\}$ is a compact set in $\mathbb{R}^2 \times \mathbb{R}^2$.

We can now write (10) as

$$\tilde{u}^2(\tau, \xi) = \frac{1}{2\pi} \langle \widehat{f} \otimes \widehat{f}, 1 \otimes 1 \rangle_{(\tau, \xi)},$$

where the tensor product is defined by $(f \otimes g)(\eta, \zeta) = f(\eta)g(\zeta)$. By Cauchy–Schwarz’s inequality we obtain

$$|\tilde{u}^2(\tau, \xi)| \leq \frac{1}{2\pi} \|\widehat{f} \otimes \widehat{f}\|_{(\tau, \xi)} \|1 \otimes 1\|_{(\tau, \xi)}. \tag{12}$$

Hence,

$$\|\tilde{u}^2\|_{L^2(\mathbb{R}^3)} \leq \frac{1}{2\pi} \left(\sup_{(\tau, \xi) \in \mathcal{P}_2} \|1 \otimes 1\|_{(\tau, \xi)} \right) \left(\int_{\mathcal{P}_2} \|\widehat{f} \otimes \widehat{f}\|_{(\tau, \xi)}^2 d\tau d\xi \right)^{1/2}. \tag{13}$$

The next lemma shows that $\|1 \otimes 1\|_{(\tau, \xi)}$ is not only uniformly bounded with respect to (τ, ξ) , but actually constant on the support of \tilde{u}^2 .

Lemma 3.2. *For each $(\tau, \xi) \in \mathcal{P}_2$ we have $\|1 \otimes 1\|_{(\tau, \xi)} = \sqrt{\pi/2}$.*

Proof. The quantity

$$I(\tau, \xi) = \|1 \otimes 1\|_{(\tau, \xi)}^2 = \int_{\mathbb{R}^2} \delta(\tau - |\xi - \eta|^2 - |\eta|^2) d\eta$$

is just the convolution of the measure $\delta(\tau - |\xi|^2)$ with itself. The invariance of this measure with respect to the transformation (8) together with Lemma 2.1 implies that

$$I(\tau, \xi) = I(\tau + 2v \cdot \xi + |v|^2, \xi + v)$$

for any $v \in \mathbb{R}^2$. If we take $v = -\xi/2$ we obtain $I(\tau, \xi) = I(\tau^*, 0)$, where $\tau^* = \tau - |\xi|^2/2$. Moreover, it is evident from the definition that, by homogeneity, I is invariant under parabolic dilations, $I(\tau, \xi) = I(\lambda^2\tau, \lambda\xi)$. Hence, when $\tau^* > 0$ we have

$$\begin{aligned} I(\tau, \xi) &= I(\tau^*, 0) = I(1, 0) = \int_{\mathbb{R}^2} \delta(1 - 2|\eta|^2) d\eta \\ &= 2\pi \int_0^\infty \delta(1 - 2r^2)r dr = \pi \int_0^\infty \delta(1 - 2s) ds = \frac{\pi}{2}. \quad \square \end{aligned}$$

We also have

$$\begin{aligned} &\int_{\mathcal{P}_2} \|\widehat{f} \otimes \widehat{f}\|_{(\tau, \xi)}^2 d\tau d\xi \\ &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} |\widehat{f}(\eta)\widehat{f}(\zeta)|^2 \int_{\mathcal{P}_2} \delta\left(\begin{matrix} \tau - |\eta|^2 - |\zeta|^2 \\ \xi - \eta - \zeta \end{matrix}\right) d\tau d\xi d\eta d\zeta \\ &= \|\widehat{f} \otimes \widehat{f}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 = \|\widehat{f}\|_{L^2(\mathbb{R}^2)}^4 = (2\pi)^4 \|f\|_{L^2(\mathbb{R}^2)}^4. \end{aligned} \tag{14}$$

It follows from (9), (13), (14) and Lemma 3.2 that

$$\|u\|_{L^4(\mathbb{R}^3)} \leq \frac{1}{\sqrt{2}} \|f\|_{L^2(\mathbb{R}^2)}. \tag{15}$$

This proves that for $n = 2$ the best constant $S(2)$ in (1) is no larger than $1/\sqrt{2}$.

Remark 3.3. We observe that in the above computations the only place where we have used an inequality instead of an equality is in (13) as a consequence of the Cauchy–Schwarz inequality (12). If we can find a function f for which we have equality in (12) for all $(\tau, \xi) \in \mathcal{P}_2$ then there will be equality also in (15). This will show that f is a maximizer for the estimate and that $S(2) = 1/\sqrt{2}$.

We have equality in (13) if there is equality in (12) for almost all $(\tau, \xi) \in \mathcal{P}_2$. By Lemma 1.8, this happens if there exists a scalar function $F : \mathcal{P}_2 \rightarrow \mathbb{C}$ such that

$$(\widehat{f} \otimes \widehat{f})(\eta, \zeta) = F(\tau, \xi)(1 \otimes 1)(\eta, \zeta)$$

for almost all (η, ζ) (with respect to the measure (11)) in the support of the measure (11) and for almost all (τ, ξ) in \mathcal{P}_2 (with respect to the Lebesgue measure on $\mathbb{R} \times \mathbb{R}^2$). This means that we are looking for functions f and F such that

$$\widehat{f}(\eta)\widehat{f}(\zeta) = F(|\eta|^2 + |\zeta|^2, \eta + \zeta) \tag{16}$$

for almost all $(\eta, \zeta) \in \mathbb{R}^2 \times \mathbb{R}^2$. An example of such functions is given by the pair $\widehat{f}(\xi) = e^{-|\xi|^2}$, $F(\tau, \xi) = e^{-\tau}$.

If f is a maximizer, \widehat{f} must solve the equation (16) and it follows from Proposition 7.15 that

$$\widehat{f}(\xi) = \exp(\widehat{A}|\xi|^2 + \widehat{b} \cdot \xi + \widehat{C}), \quad \xi \in \mathbb{R}^2, \tag{17}$$

for some constants $\widehat{A} \in \mathbb{C}$, $\widehat{b} = (\widehat{b}_1, \widehat{b}_2) \in \mathbb{C}^2$, $\widehat{C} \in \mathbb{C}$, with $\Re(\widehat{A}) < 0$ in order to have $f \in L^2(\mathbb{R}^2)$. The inverse Fourier transform of (17) is again a function of the same class

$$f(x) = \exp(A|x|^2 + b \cdot x + C), \quad x \in \mathbb{R}^2, \tag{18}$$

where the relations between the parameters $A \in \mathbb{C}$, $b \in \mathbb{C}^2$, $C \in \mathbb{C}$ and the parameters \widehat{A} , \widehat{b} , \widehat{C} are given by

$$A = \frac{1}{4\widehat{A}}, \quad b = -\frac{i\widehat{b}}{4\widehat{A}}, \quad C = \widehat{C} - \frac{\widehat{b}_1^2 + \widehat{b}_2^2}{4\widehat{A}} - \log(-4\pi\widehat{A}).$$

The class of initial data of the form (18) is invariant under the action of the group \mathcal{G} described in Definition 1.2. The coefficients change according to the following rules:

- space-time translations: $(\widehat{A}, \widehat{b}, \widehat{C}) \rightsquigarrow (\widehat{A} + it_0, \widehat{b} + ix_0, \widehat{C})$;
- parabolic dilations: $(\widehat{A}, \widehat{b}, \widehat{C}) \rightsquigarrow (\widehat{A}/\lambda^2, \widehat{b}/\lambda, \widehat{C} - n \log \lambda)$;
- changes of scale: $(\widehat{A}, \widehat{b}, \widehat{C}) \rightsquigarrow (\widehat{A}, \widehat{b}, \widehat{C} + \log \mu)$;
- space rotations: $(\widehat{A}, \widehat{b}, \widehat{C}) \rightsquigarrow (\widehat{A}, R\widehat{b}, \widehat{C})$;

- phase shifts: $(\widehat{A}, \widehat{b}, \widehat{C}) \rightsquigarrow (\widehat{A}, \widehat{b}, \widehat{C} + i\theta)$;
- Galilean transformations:

$$(\widehat{A}, \widehat{b}, \widehat{C}) \rightsquigarrow \left(\widehat{A}, \widehat{b} + Av, \widehat{C} - \frac{\widehat{A}}{4}|v|^2 - \frac{\widehat{b}}{2} \cdot v \right).$$

Hence, after a translation and a phase shift we can make all coefficients real; by a Galilean transformation we can make $\widehat{b} = 0$; then, by a parabolic dilation we can have $\widehat{A} = -1/4$; finally, a change of scale gives $\widehat{C} = \log(-\pi)$. This would correspond to the case $A = -1, b = 0, C = 0$, which is the function $f_*(x) = e^{-|x|^2}$. Thus, we have proved that any maximizer is connected to f_* by the action of \mathcal{G} .

4. Schrödinger equation in dimension $n = 1$

Consider the case $n = 1, p = 6$ for estimate (1). This case was considered in [6]. By Plancherel’s theorem, $u \in L^6$ if and only if $\widetilde{u}^3 \in L^2$ and

$$\|u\|_{L^6(\mathbb{R}^2)}^3 = \|u^3\|_{L^2(\mathbb{R}^2)} = (2\pi)^{-1} \|\widetilde{u}^3\|_{L^2(\mathbb{R}^2)}. \tag{19}$$

The Fourier transform of u^3 reduces to

$$\begin{aligned} \widetilde{u}^3(\tau, \xi) &= \frac{1}{(2\pi)^4} \widetilde{u} * \widetilde{u} * \widetilde{u}(\tau, \xi) \\ &= \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}} \widehat{f}(\eta_1) \widehat{f}(\eta_2) \widehat{f}(\eta_3) \delta \left(\begin{matrix} \tau - \eta_1^2 - \eta_2^2 - \eta_3^2 \\ \xi - \eta_1 - \eta_2 - \eta_3 \end{matrix} \right) d\eta_1 d\eta_2 d\eta_3 \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^3} \widehat{f}(\eta_1) \widehat{f}(\eta_2) \widehat{f}(\eta_3) \delta \left(\begin{matrix} \tau - |\eta|^2 \\ \xi - (1, 1, 1) \cdot \eta \end{matrix} \right) d\eta, \end{aligned} \tag{20}$$

where now $\eta = (\eta_1, \eta_2, \eta_3)$. When $\xi = (1, 1, 1) \cdot \eta$ and $\tau = |\eta|^2$, we have $3\tau \geq \xi^2$. It follows that \widetilde{u}^3 is supported in the closure of the region

$$\mathcal{P}_1 = \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R} : 3\tau > \xi^2\}.$$

For each choice of $(\tau, \xi) \in \mathcal{P}_1$, we denote by $\langle \cdot, \cdot \rangle_{(\tau, \xi)}$ the L^2 inner product associated with the measure

$$\delta \left(\begin{matrix} \tau - |\eta|^2 \\ \xi - (1, 1, 1) \cdot \eta \end{matrix} \right) d\eta, \tag{21}$$

and by $\| \cdot \|_{(\tau, \xi)}$ the corresponding norm. We can then write (20) as

$$\widetilde{u}^3(\tau, \xi) = \frac{1}{2\pi} \langle \widehat{f} \otimes \widehat{f} \otimes \widehat{f}, 1 \otimes 1 \otimes 1 \rangle_{(\tau, \xi)},$$

where the tensor product is defined by $(f \otimes g \otimes h)(\eta) = f(\eta_1)g(\eta_2)h(\eta_3)$. By Cauchy–Schwarz’s inequality we obtain

$$|\widetilde{u}^3(\tau, \xi)| \leq \frac{1}{2\pi} \|\widehat{f} \otimes \widehat{f} \otimes \widehat{f}\|_{(\tau, \xi)} \|1 \otimes 1 \otimes 1\|_{(\tau, \xi)}. \tag{22}$$

Hence,

$$\|\tilde{u}^3\|_{L^2(\mathbb{R}^2)} \leq \frac{1}{2\pi} \left(\sup_{(\tau, \xi) \in \mathcal{P}_1} \|1 \otimes 1 \otimes 1\|_{(\tau, \xi)} \left(\int_{\mathcal{P}_1} \|\widehat{f} \otimes \widehat{f} \otimes \widehat{f}\|_{(\tau, \xi)}^2 d\tau d\xi \right)^{1/2} \right). \quad (23)$$

The next lemma shows that $\|1 \otimes 1 \otimes 1\|_{(\tau, \xi)}$ is not only bounded, but actually constant on the support of \tilde{u}^3 .

Lemma 4.1. *For each $(\tau, \xi) \in \mathcal{P}_1$ we have $\|1 \otimes 1 \otimes 1\|_{(\tau, \xi)} = \sqrt{\pi/\sqrt{3}}$.*

Proof. The quantity

$$I(\tau, \xi) = \|1 \otimes 1 \otimes 1\|_{(\tau, \xi)}^2 = \int_{\mathbb{R}^3} \delta \left(\begin{matrix} \tau - |\eta|^2 \\ \xi - (1, 1, 1) \cdot \eta \end{matrix} \right) d\eta$$

is just a twofold convolution of the measure $\delta(\tau - \xi^2)$ with itself. From Lemma 2.1 and the invariance of this measure with respect to the transformation (8), it follows that $I(\tau, \xi) = I(\tau^*, 0)$ where $\tau^* = \tau - \xi^2/3$. Moreover, by homogeneity I is invariant under parabolic dilations, $I(\lambda^2\tau, \lambda\xi) = I(\tau, \xi)$. Hence, when $\tau^* > 0$ we have

$$\begin{aligned} I(\tau, \xi) &= I(\tau^*, 0) = I(1, 0) = \int_{\mathbb{R}^3} \delta \left(\begin{matrix} 1 - |\eta|^2 \\ -(1, 1, 1) \cdot \eta \end{matrix} \right) d\eta = \int_{\mathbb{R}^3} \delta \left(\begin{matrix} 1 - |\eta|^2 \\ |(1, 1, 1)|\eta_1 \end{matrix} \right) d\eta \\ &= \frac{1}{\sqrt{3}} \int_{\mathbb{R}^2} \delta(1 - |\zeta|^2) d\zeta = \frac{2\pi}{\sqrt{3}} \int_0^1 \delta(1 - r^2) r dr = \frac{\pi}{\sqrt{3}}. \quad \square \end{aligned}$$

We also have

$$\begin{aligned} &\int_{\mathcal{P}_1} \|\widehat{f} \otimes \widehat{f} \otimes \widehat{f}\|_{(\tau, \xi)}^2 d\tau d\xi \\ &= \int_{\mathbb{R}^3} |\widehat{f}(\eta_1)\widehat{f}(\eta_2)\widehat{f}(\eta_3)|^2 \int_{\mathcal{P}_1} \delta \left(\begin{matrix} \tau - |\eta|^2 \\ \xi - (1, 1, 1) \cdot \eta \end{matrix} \right) d\tau d\xi d\eta \\ &= \|\widehat{f} \otimes \widehat{f} \otimes \widehat{f}\|_{L^2(\mathbb{R}^3)}^2 = \|\widehat{f}\|_{L^2(\mathbb{R})}^6 = (2\pi)^3 \|f\|_{L^2(\mathbb{R})}^6. \quad (24) \end{aligned}$$

It follows from (19), (23), (24) and Lemma 4.1 that

$$\|u\|_{L^6(\mathbb{R}^2)} \leq 12^{-1/12} \|f\|_{L^2(\mathbb{R})}. \quad (25)$$

This proves that for $n = 1$ the best constant $S(1)$ in (1) is no larger than $12^{-1/12}$.

As before, we observe that if we could find a function f for which we have equality in (22) for all $(\tau, \xi) \in \mathcal{P}_1$ then we would have equality in (25) and we would have found a maximizer for the estimate. We have equality in the Cauchy–Schwarz inequality (22) for (almost) all $(\tau, \xi) \in \mathcal{P}_1$ if there exists a scalar function $F : \mathcal{P}_1 \rightarrow \mathbb{C}$ such that

$$(\widehat{f} \otimes \widehat{f} \otimes \widehat{f})(\eta) = F(\tau, \xi)(1 \otimes 1 \otimes 1)(\eta)$$

for (almost) all η in the support of the measure (21). This means that we are looking for functions f and F such that

$$\widehat{f}(\eta_1)\widehat{f}(\eta_2)\widehat{f}(\eta_3) = F(\eta_1^2 + \eta_2^2 + \eta_3^2, \eta_1 + \eta_2 + \eta_3) \tag{26}$$

for (almost) all $\eta \in \mathbb{R}^3$. Again, an example of such functions is given by the pair $\widehat{f}(\xi) = e^{-\xi^2}$, $F(\tau, \xi) = e^{-\tau}$.

If f is a maximizer, \widehat{f} must solve the equation (26) and it follows from Proposition 7.10 that

$$\widehat{f}(\xi) = \exp(\widehat{A}\xi^2 + \widehat{B}\xi + \widehat{C}), \quad \xi \in \mathbb{R}, \tag{27}$$

for some complex constants $\widehat{A}, \widehat{B}, \widehat{C}$, with $\Re(\widehat{A}) < 0$ in order to have $f \in L^2(\mathbb{R})$. The inverse Fourier transform of (27) is again a function of the same class

$$f(x) = \exp(Ax^2 + Bx + C), \quad x \in \mathbb{R}, \tag{28}$$

where the relations between the parameters A, B, C and $\widehat{A}, \widehat{B}, \widehat{C}$ are

$$A = \frac{1}{4\widehat{A}}, \quad B = -\frac{i\widehat{B}}{4\widehat{A}}, \quad C = \widehat{C} - \frac{\widehat{B}^2}{4\widehat{A}} - \frac{1}{2} \log(-4\pi\widehat{A}).$$

As we have seen at the end of Section 3, the class of initial data of the form (28) is invariant under the action of the group \mathcal{G} and any maximizer is connected to the function $f_*(x) = e^{-x^2}$ by the action of \mathcal{G} .

5. Wave equation in dimension $n = 3$

Consider the case $n = 3, p = 4$ for estimate (3). We have

$$\widetilde{u_+u_+}(\tau, \xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\widehat{f_+}(\eta)\widehat{f_+}(\zeta)}{|\eta|^{1/2}|\zeta|^{1/2}} \delta\left(\begin{matrix} \tau - |\eta| - |\zeta| \\ \xi - \eta - \zeta \end{matrix}\right) d\eta d\zeta. \tag{29}$$

In particular, $\widetilde{u_+u_+}$ is supported in the closure of the region

$$\mathcal{C}_{++} = \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^3 : \tau > |\xi|\}.$$

Similarly, $\widetilde{u_+u_-}$ is supported in the closure of

$$\mathcal{C}_{+-} = \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^3 : |\tau| < |\xi|\};$$

and $\widetilde{u_-u_-}$ is supported in the closure of

$$\mathcal{C}_{--} = \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^3 : \tau < -|\xi|\}.$$

We remark that formulae like (29) are the starting point for the bilinear estimates studied in [2].

We first prove the estimate for u_+ . By Plancherel’s theorem we have

$$\|u_+\|_{L^4(\mathbb{R}^4)}^2 = \|u_+\|_{L^2(\mathbb{R}^4)}^2 = (2\pi)^{-2} \|\widetilde{u_+^2}\|_{L^2(\mathbb{R}^4)}. \tag{30}$$

For each choice of $(\tau, \xi) \in \mathcal{C}_{++}$, we denote by $\langle \cdot, \cdot \rangle_{(\tau, \xi)}$ the L^2 inner product associated with the measure

$$\mu_{(\tau, \xi)} = \delta \left(\begin{matrix} \tau - |\eta| - |\zeta| \\ \xi - \eta - \zeta \end{matrix} \right) d\eta d\zeta \tag{31}$$

and by $\| \cdot \|_{(\tau, \xi)}$ the corresponding norm.

Remark 5.1. The measure $\mu_{(\tau, \xi)}$ defined in (31) is the *pull-back* of the Dirac delta on \mathbb{R}^{1+3} by the function $\Phi_{(\tau, \xi)} : (\mathbb{R}^3 \setminus 0) \times (\mathbb{R}^3 \setminus 0) \rightarrow \mathbb{R} \times \mathbb{R}^3$ given by

$$\Phi_{(\tau, \xi)}(\eta, \zeta) = (\tau - |\eta| - |\zeta|, \xi - \eta - \zeta).$$

A quick computation shows that the differential of $\Phi_{(\tau, \xi)}$ is surjective at a point (η, ζ) if and only if $\eta/|\eta| \neq \zeta/|\zeta|$. On the other hand, if $\eta/|\eta| = \zeta/|\zeta|$ and $\Phi_{(\tau, \xi)}(\eta, \zeta) = 0$ we must have $\tau = |\eta| + |\zeta| = |\eta + \zeta| = |\xi|$. This tells us that $\mu_{(\tau, \xi)}$ is not well defined on the boundary of \mathcal{C}_{++} , when $\tau = |\xi|$. However, we can safely ignore the problems at this boundary and observe instead that for any locally integrable function $F(\eta, \zeta)$ defined on $\mathbb{R}^3 \times \mathbb{R}^3$ the integral $G(\tau, \xi) = \int F d\mu_{(\tau, \xi)}$ defines a locally integrable function on $\mathbb{R} \times \mathbb{R}^3$. Indeed, if K is a compact set in $\mathbb{R} \times \mathbb{R}^3$, we have

$$\begin{aligned} \iint_K |G(\tau, \xi)| d\tau d\xi &\leq \iint_{(\tau, \xi) \in K} \iint |F(\eta, \zeta)| \delta \left(\begin{matrix} \tau - |\eta| - |\zeta| \\ \xi - \eta - \zeta \end{matrix} \right) d\eta d\zeta d\tau d\xi \\ &= \iint_{(|\eta| + |\zeta|, \eta + \zeta) \in K} |F(\eta, \zeta)| d\eta d\zeta, \end{aligned}$$

and $\{(\eta, \zeta) : (|\eta| + |\zeta|, \eta + \zeta) \in K\}$ is a compact set in $\mathbb{R}^3 \times \mathbb{R}^3$.

We can now write (29) as

$$\widetilde{u}_+^2(\tau, \xi) = \frac{1}{(2\pi)^2} \langle \widehat{f}_+ \otimes \widehat{f}_+, |\cdot|^{-1/2} \otimes |\cdot|^{-1/2} \rangle_{(\tau, \xi)}. \tag{32}$$

The quantity we want to compute this time is $\| |\cdot|^{-1/2} \otimes |\cdot|^{-1/2} \|_{(\tau, \xi)}$.

Lemma 5.2. For each $(\tau, \xi) \in \mathcal{C}_{++}$ we have $\| |\cdot|^{-1/2} \otimes |\cdot|^{-1/2} \|_{(\tau, \xi)} = (2\pi)^{1/2}$.

Proof. The quantity

$$I(\tau, \xi) = \| |\cdot|^{-1/2} \otimes |\cdot|^{-1/2} \|_{(\tau, \xi)}^2 = \int_{\mathbb{R}^3} \frac{\delta(\tau - |\xi - \eta| - |\eta|)}{|\xi - \eta| |\eta|} d\eta$$

is just the convolution of the measure $|\xi|^{-1} \delta(\tau - |\xi|)$ with itself. If $\tau > |\xi|$, from the invariance of this measure with respect to proper Lorentz transformations and from the fact that it is always possible to find a proper Lorentz transformation which takes (τ, ξ) to the point $(\tau^*, 0)$ where $\tau^* = (\tau^2 - |\xi|^2)^{1/2}$, it follows that $I(\tau, \xi) = I(\tau^*, 0)$. Moreover, it is evident from the definition that by homogeneity I is invariant under isotropic dilations, $I(\lambda\tau, \lambda\xi) = I(\tau, \xi)$. Hence, when $\tau > |\xi|$,

$$I(\tau, \xi) = I(1, 0) = \int_{\mathbb{R}^3} \frac{\delta(1 - 2|\xi|)}{|\xi|^2} d\xi = 4\pi \int_0^\infty \delta(1 - 2r) dr = 2\pi. \quad \square$$

Cauchy–Schwarz’s inequality applied to (32) together with Lemma 5.2 gives

$$\begin{aligned} \|\widetilde{u}_+\|_{L^2(\mathbb{R}^4)}^2 &\leq \frac{1}{(2\pi)^3} \int_{\mathcal{C}_{++}} \|\widehat{f}_+ \otimes \widehat{f}_+\|_{L^2(\tau, \xi)}^2 d\tau d\xi = \frac{1}{(2\pi)^3} \|\widehat{f}_+ \otimes \widehat{f}_+\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \\ &= \frac{1}{(2\pi)^3} \|\widehat{f}_+\|_{L^2(\mathbb{R}^3)}^4 = (2\pi)^3 \|f_+\|_{L^2(\mathbb{R}^3)}^4. \end{aligned} \tag{33}$$

Hence, combining (30) and (33) we obtain

$$\|u_+\|_{L^4(\mathbb{R}^4)} \leq (2\pi)^{-1/4} \|f_+\|_{L^2(\mathbb{R}^3)}. \tag{34}$$

This time, equality holds if there exists a function $F : \mathcal{C}_{++} \rightarrow \mathbb{C}$ such that

$$(\widehat{f}_+ \otimes \widehat{f}_+)(\eta, \zeta) = F(\tau, \xi) |\eta|^{-1/2} |\zeta|^{-1/2}$$

for all (η, ζ) in the support of the measure (31). This means that

$$|\eta|^{1/2} \widehat{f}_+(\eta) |\zeta|^{1/2} \widehat{f}_+(\zeta) = F(|\eta| + |\zeta|, \eta + \zeta)$$

for almost all $\eta, \zeta \in \mathbb{R}^3$. An example of such functions is given by the pair

$$\widehat{f}_+(\xi) = |\xi|^{-1/2} e^{-|\xi|}, \quad F(\tau, \xi) = e^{-\tau}.$$

It follows from Proposition 7.23 that any maximizer for the estimate (34) is a function whose Fourier transform has the form

$$\widehat{f}(\xi) = |\xi|^{-1/2} \exp(A|\xi| + b \cdot \xi + C), \tag{35}$$

with $A, C \in \mathbb{C}, b \in \mathbb{C}^3, \Im(C) \in [0, 2\pi[$ and $|\Re(b)| < -\Re(A)$ (in order to have $f \in L^2(\mathbb{R}^3)$). In the next lemma we compute an explicit expression for homogeneous waves with data of the form (35).

Lemma 5.3. *Let u be the (+)-wave corresponding to an L^2 data of the form (35),*

$$u(t, x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \exp((A + it)|\xi| + (b + ix) \cdot \xi + C) \frac{d\xi}{|\xi|}. \tag{36}$$

Then we have the explicit formula

$$\begin{aligned} &2\pi^2 e^{-i\Im(C)} u(t - \Im(A), x - \Im(b)) \\ &= \frac{e^{\Re(C)}}{(\Re(A))^2 - |\Re(b)|^2 + |x|^2 - t^2 + 2i(\Re(A)t - \Re(b) \cdot x)}. \end{aligned} \tag{37}$$

Proof. The integral

$$F(t, x) = \int_{\mathbb{R}^3} \exp(t|\xi| + x \cdot \xi) \frac{d\xi}{|\xi|}$$

is well defined for $t \in \mathbb{C}$ and $x \in \mathbb{C}^3$ when $\Re t < -|\Re x|$. For $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$ with $t < -|x|$, using polar coordinates, $r = |\xi|$ and $u = (x/|x|) \cdot (\xi/|\xi|)$, we find

$$\begin{aligned} F(t, x) &= 2\pi \int_{-1}^1 \int_0^\infty \exp((t + |x|u)r) r \, dr \, du = \int_{-1}^1 \frac{2\pi \, du}{(t + |x|u)^2} \\ &= \frac{4\pi}{t^2 - x_1^2 - x_2^2 - x_3^2}. \end{aligned}$$

By analytic continuation this formula remains valid for complex t and x with $\Re t < -|\Re x|$. Formula (37) follows from the identity

$$e^{-i\Im(C)} u(t - \Im(A), x - \Im(b)) = \frac{e^{\Re(C)}}{(2\pi)^3} F(\Re(A) + it, \Re(b) + ix). \quad \square$$

Remark 5.4. If u is the (+)-wave corresponding to an L^2 data of the form (35), then the knowledge of $|u(t, x)|$ uniquely determines the value of the coefficients A, b and $\Re(C)$. Indeed, by Lemma 5.3 the imaginary parts $\Im(A)$ and $\Im(b)$ are determined by the fact that $|u(t, x)|$ has a unique maximum at the point $t = -\Im(A), x = -\Im(b)$, while the real parts $\Re(A) < 0, \Re(b)$ and $\Re(C)$ are determined by the coefficients of the polynomial

$$\begin{aligned} &|u(t - \Im(A), x - \Im(b))|^{-2} \\ &= 4\pi^4 e^{-2\Re(C)} ((\Re(A))^2 - |\Re(b)|^2 + |x|^2 - t^2)^2 + 4(\Re(A)t - \Re(b) \cdot x)^2. \end{aligned}$$

We can repeat the above procedure for the term u_-^2 (the only difference is that τ must be replaced by $-\tau$):

$$\|u_-\|_{L^4(\mathbb{R}^4)} \leq (2\pi)^{-1/4} \|f_-\|_{L^2(\mathbb{R}^3)}, \tag{38}$$

with equality if and only if f_- is of the form (35).

For the term u_+u_- , we observe that by Hölder’s inequality we have

$$\|u_+u_-\|_{L^2(\mathbb{R}^4)} \leq \|u_+\|_{L^4(\mathbb{R}^4)} \|u_-\|_{L^4(\mathbb{R}^4)} \leq (2\pi)^{-1/2} \|f_+\|_{L^2(\mathbb{R}^3)} \|f_-\|_{L^2(\mathbb{R}^3)}. \tag{39}$$

The first inequality in (39) is an equality if there is a constant $\mu \in \mathbb{R}$ such that $|u_+(t, x)| = \mu|u_-(t, x)|$ for (almost all) $(t, x) \in \mathbb{R} \times \mathbb{R}^3$. The second inequality in (39) is an equality if f_+ and f_- are functions of the form (35).

Combining the L^2 orthogonality of the terms u_+^2, u_-^2 and u_+u_- (due to the disjointness of the supports of their Fourier transforms) with (34), (38) and (39), we obtain

$$\begin{aligned} \|u\|_{L^4}^4 &= \|(u_+ + u_-)^2\|_{L^2}^2 = \|u_+\|_{L^4}^4 + \|u_-\|_{L^4}^4 + 4\|u_+u_-\|_{L^2}^2 \\ &\leq \frac{1}{2\pi} (\|f_+\|_{L^2}^4 + \|f_-\|_{L^2}^4 + 4\|f_+\|_{L^2(\mathbb{R}^3)}^2 \|f_-\|_{L^2(\mathbb{R}^3)}^2) \\ &\leq \frac{3}{4\pi} (\|f_+\|_{L^2}^2 + \|f_-\|_{L^2}^2)^2 = \frac{3}{16\pi} \|(f, g)\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}}^4, \end{aligned} \tag{40}$$

where we have used the sharp inequality

$$X^2 + Y^2 + 4XY \leq \frac{3}{2}(X + Y)^2, \quad X, Y \geq 0,$$

for which equality holds if and only if $X = Y$. This proves that for $n = 3$ the best constant $W(3)$ in (3) is no larger than $(3/(16\pi))^{1/4}$. The next proposition tells us that maximizers exist and that the inequalities in (40) are sharp; hence, $W(3) = (3/(16\pi))^{1/4}$.

Proposition 5.5. *We have $\|u\|_{L^4} = (3/(16\pi))^{1/4} \|(f, g)\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}}$ if and only if*

$$\widehat{f}_+(\xi) = |\xi|^{-1/2} \exp(A|\xi| + b \cdot \xi + C), \quad \widehat{f}_-(\xi) = |\xi|^{-1/2} \exp(\bar{A}|\xi| - \bar{b} \cdot \xi + D), \quad (41)$$

where $A, C, D \in \mathbb{C}$ and $b \in \mathbb{C}^3$ with $|\Re(b)| < -\Re(A)$ and $\Re(D) = \Re(C)$.

Proof. By the above discussion, we have equalities in (40) if and only if f_+, f_- are both functions of the form (35) and $|u_+(t, x)| = |u_-(t, x)|$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^3$. Observe that if u_- is a $(-)$ -wave with data f_- , where

$$\widehat{f}_-(\xi) = |\xi|^{-1/2} \exp(A_-|\xi| + b_- \cdot \xi + C_-),$$

then its complex conjugate \bar{u}_- is a $(+)$ -wave with data

$$\overline{\widehat{f}_-(-\xi)} = |\xi|^{-1/2} \exp(\bar{A}_-|\xi| - \bar{b}_- \cdot \xi + \bar{C}_-).$$

By Remark 5.4, if two $(+)$ -waves with initial data of the form (35) have the same absolute value at every point of the space-time then they must have the same coefficients A, b and $\Re(C)$. □

A particular case of (41), corresponding to $A = -1, b = 0, C = D = \log(2\pi^2)$, is given by the initial data

$$f_*(x) = \frac{1}{1 + |x|^2}, \quad g_*(x) = 0, \quad x \in \mathbb{R}^3. \quad (42)$$

The class of initial data of the form (41) is invariant under the action of the group \mathcal{L} described in Definition 1.6. The coefficients change according to the following rules:

- space-time translations: $(A, b, C, D) \rightsquigarrow (A + it_0, b + ix_0, C, D)$;
- isotropic dilations: $(A, b, C, D) \rightsquigarrow (A/\lambda, b/\lambda, C - n \log \lambda, D - n \log \lambda)$;
- changes of scale: $(A, b, C, D) \rightsquigarrow (A, b, C + \log \mu, D + \log \mu)$;
- space rotations: $(A, b, C, D) \rightsquigarrow (A, Rb, C, D)$;
- phase shifts: $(A, b, C, D) \rightsquigarrow (A, b, C + i\theta_+, D + i\theta_-)$;
- Lorentzian boosts:

$$(A, (b_1, b'), C, D) \rightsquigarrow (A \cosh(a) - b_1 \sinh(a), (-A \sinh(a) + b_1 \cosh(a), b'), C, D).$$

Hence, after a translation and a phase shift we can make all coefficients real; by a rotation we can make $b' = 0$ and by a Lorentzian boost we can make $b_1 = 0$; then, by an isotropic dilation we can have $A = -1$; finally, a change of scale gives $C = D = \log(2\pi^2)$. This would correspond to the functions

$$\widehat{f}_+(\xi) = \widehat{f}_-(\xi) = -2\pi^2|\xi|^{-1/2} \exp(-|\xi|),$$

which are the Fourier transforms of the (+) and (-) parts of the initial data (42). Thus, we have proved that any maximizer is connected to (f_*, g_*) by the action of \mathcal{L} .

6. Wave equation in dimension $n = 2$

Consider the case $n = 2, p = 6$ for estimate (3). We decompose u into its (+) and (-) parts and treat the L^6 norm of u as an L^2 norm of u^3 . By expanding the products we find

$$\begin{aligned} \|u\|_{L^6}^6 &= \|(u_+ + u_-)^3\|_{L^2}^2 = \|u_+^3 + 3u_+^2u_- + 3u_+u_-^2 + u_-^3\|^2 \\ &= \|u_+^3\|^2 + \|u_-^3\|^2 + 9\|u_+^2u_-\|^2 + 9\|u_+u_-^2\|^2 \\ &\quad + 6\Re\langle u_+^3, u_+^2u_- \rangle + 6\Re\langle u_+u_-^2, u_-^3 \rangle + 18\Re\langle u_+^2u_-, u_+u_-^2 \rangle \\ &\quad + 6\Re\langle u_+^3, u_+u_-^2 \rangle + 2\Re\langle u_+^3, u_-^3 \rangle + 6\Re\langle u_+^2u_-, u_-^3 \rangle, \end{aligned}$$

where $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ now stand for the standard norm and inner product in $L^2(\mathbb{R} \times \mathbb{R}^2)$. We shall study one term at a time, but first we compute some integrals which will be needed later.

Lemma 6.1. For $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^2$ with $\tau > |\xi|$, we define

$$\begin{aligned} I_2(\tau, \xi) &= \int_{(\mathbb{R}^2)^2} \delta\left(\begin{matrix} \tau - |\eta_1| - |\eta_2| \\ \xi - \eta_1 - \eta_2 \end{matrix}\right) \frac{d\eta_1 d\eta_2}{|\eta_1| |\eta_2|}, \\ I_3(\tau, \xi) &= \int_{(\mathbb{R}^2)^3} \delta\left(\begin{matrix} \tau - |\eta_1| - |\eta_2| - |\eta_3| \\ \xi - \eta_1 - \eta_2 - \eta_3 \end{matrix}\right) \frac{d\eta_1 d\eta_2 d\eta_3}{|\eta_1| |\eta_2| |\eta_3|}. \end{aligned}$$

Then $I_2(\tau, \xi) = 2\pi/\sqrt{\tau^2 - |\xi|^2}$ and $I_3(\tau, \xi) = 4\pi^2$.

Proof. The fact that I_2 and I_3 are well defined locally integrable functions on $\mathbb{R} \times \mathbb{R}^2$ follows from considerations similar to the ones made at the end of Remark 5.1. Let us define μ to be the measure $\mu(\tau, \xi) = |\xi|^{-1} \delta(\tau - |\xi|)$; we have $I_2 = \mu * \mu$ and $I_3 = \mu * \mu * \mu$. The measure μ is invariant under proper Lorentz transformations, and given (τ, ξ) such that $\tau > |\xi|$, there always exists a proper Lorentz transformation which takes (τ, ξ) to the point $(\tau^*, 0)$ where $\tau^* = \sqrt{\tau^2 - |\xi|^2}$. By Lemma 2.1, it follows that $I_k(\tau, \xi) = I_k(\tau^*, 0)$ for $k = 2, 3$. The integral I_2 is homogeneous of degree -1 while I_3 is homogeneous of degree 0. Hence, for $\tau > |\xi|$ we have

$$I_2(\tau, \xi) = \frac{I_2(1, 0)}{\tau^*} = \frac{1}{\tau^*} \int_{\mathbb{R}^2} \frac{\delta(1 - 2|\eta|)}{|\eta|^2} d\eta = \frac{2\pi}{\tau^*} \int_0^\infty \frac{\delta(1 - 2r)}{r} dr = \frac{2\pi}{\sqrt{\tau^2 - |\xi|^2}}$$

and

$$\begin{aligned}
 I_3(\tau, \xi) &= I_3(1, 0) = \int_{2|\eta| \leq 1} I_2(1 - |\eta|, -\eta) \frac{d\eta}{|\eta|} = \int_{2|\eta| < 1} \frac{2\pi d\eta}{((1 - |\eta|)^2 - |\eta|^2)^{1/2} |\eta|} \\
 &= 4\pi^2 \int_0^{1/2} \frac{dr}{\sqrt{1 - 2r}} = 4\pi^2. \quad \square
 \end{aligned}$$

Let us now begin the proof of the estimate for the term $\|u_+^3\|$. The Fourier transform of u_+^3 is

$$\widetilde{u_+^3}(\tau, \xi) = \frac{1}{(2\pi)^3} \int_{(\mathbb{R}^2)^3} \frac{\widehat{f_+}(\eta_1)\widehat{f_+}(\eta_2)\widehat{f_+}(\eta_3)}{|\eta_1|^{1/2}|\eta_2|^{1/2}|\eta_3|^{1/2}} \delta\left(\begin{matrix} \tau - |\eta_1| - |\eta_2| - |\eta_3| \\ \xi - \eta_1 - \eta_2 - \eta_3 \end{matrix}\right) d\eta_1 d\eta_2 d\eta_3. \tag{43}$$

The support of $\widetilde{u_+^3}$ is contained in the closure of the region $\mathcal{C}_{+++} = \{(\tau, \xi) : \tau > |\xi|\}$. For each choice of $(\tau, \xi) \in \mathcal{C}_{+++}$, we denote by $\langle \cdot, \cdot \rangle_{(\tau, \xi)}$ the L^2 inner product associated with the measure

$$\delta\left(\begin{matrix} \tau - |\eta_1| - |\eta_2| - |\eta_3| \\ \xi - \eta_1 - \eta_2 - \eta_3 \end{matrix}\right) d\eta_1 d\eta_2 d\eta_3. \tag{44}$$

We can then write

$$\widetilde{u_+^3}(\tau, \xi) = \frac{1}{(2\pi)^3} \langle \widehat{f_+} \otimes \widehat{f_+} \otimes \widehat{f_+}, |\cdot|^{-1/2} \otimes |\cdot|^{-1/2} \otimes |\cdot|^{-1/2} \rangle_{(\tau, \xi)}, \tag{45}$$

Lemma 6.2. *For each $(\tau, \xi) \in \mathcal{C}_{+++}$ we have $\| |\cdot|^{-1/2} \otimes |\cdot|^{-1/2} \otimes |\cdot|^{-1/2} \|_{(\tau, \xi)} = 2\pi$.*

Proof. The square of the norm we want to compute is the integral I_3 of Lemma 6.1,

$$\| |\cdot|^{-1/2} \otimes |\cdot|^{-1/2} \otimes |\cdot|^{-1/2} \|_{(\tau, \xi)}^2 = I_3(\tau, \xi) = 4\pi^2. \quad \square$$

Cauchy–Schwarz’s inequality applied to (45) and Lemma 6.2 give

$$\begin{aligned}
 \|u_+^3\|^2 &= \frac{1}{(2\pi)^3} \|\widetilde{u_+^3}\|^2 \leq \frac{1}{(2\pi)^7} \int_{\mathbb{R} \times \mathbb{R}^2} \|\widehat{f_+} \otimes \widehat{f_+} \otimes \widehat{f_+}\|_{(\tau, \xi)}^2 d\tau d\xi \\
 &= \frac{1}{(2\pi)^7} \|\widehat{f_+} \otimes \widehat{f_+} \otimes \widehat{f_+}\|_{L^2((\mathbb{R}^2)^3)}^2 = \frac{1}{(2\pi)^7} \|\widehat{f_+}\|^6 = \frac{1}{2\pi} \|f_+\|^6. \tag{46}
 \end{aligned}$$

This time, equality holds if there exists a function $F : \mathcal{C}_{+++} \rightarrow \mathbb{C}$ such that

$$(\widehat{f_+} \otimes \widehat{f_+} \otimes \widehat{f_+})(\eta_1, \eta_2, \eta_3) = F(\tau, \xi) |\eta_1|^{-1/2} |\eta_2|^{-1/2} |\eta_3|^{-1/2}$$

for all (η_1, η_2, η_3) in the support of the measure (44). This means that

$$|\eta_1|^{1/2} \widehat{f_+}(\eta_1) |\eta_2|^{1/2} \widehat{f_+}(\eta_2) |\eta_3|^{1/2} \widehat{f_+}(\eta_3) = F(|\eta_1| + |\eta_2| + |\eta_3|, \eta_1 + \eta_2 + \eta_3)$$

for $\eta_1, \eta_2, \eta_3 \in \mathbb{R}^2$. Examples of such functions are again $\widehat{f_+}(\xi) = |\xi|^{-1/2} e^{-|\xi|}$, $F(\tau, \xi) = e^{-\tau}$. More generally, by Proposition 7.19 all maximizers for the estimate (46) are given by the family

$$\widehat{f}(\xi) = |\xi|^{-1/2} \exp(A|\xi| + b \cdot \xi + C) \tag{47}$$

with $A, C \in \mathbb{C}$, $b \in \mathbb{C}^2$ and $|\Re(b)| < -\Re(A)$ (in order to have an L^2 function).

Lemma 6.3. *Let u be the (+)-wave corresponding to an L^2 function of the form (47),*

$$u(t, x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp((A + it)|\xi| + (b + ix) \cdot \xi + C) \frac{d\xi}{|\xi|}.$$

Then we have the explicit formula

$$\begin{aligned} & 2\pi e^{-i\Im(C)} u(t - \Im(A), x - \Im(b)) \\ &= \frac{e^{\Re(C)}}{\sqrt{(\Re(A))^2 - |\Re(b)|^2 + |x|^2 - t^2 + 2i(\Re(A)t - \Re(b) \cdot x)}}. \end{aligned} \tag{48}$$

Proof. The integral

$$F(t, x) = \int_{\mathbb{R}^2} \exp(t|\xi| + x \cdot \xi) \frac{d\xi}{|\xi|}$$

is well defined for $t \in \mathbb{C}$ and $x \in \mathbb{C}^2$ when $\Re(t) < -|\Re(x)|$. For $t \in \mathbb{R}$ and $x \in \mathbb{R}^2$ with $t < -|x|$, using polar coordinates we find

$$F(t, x) = \int_0^{2\pi} \int_0^\infty \exp((t + |x| \cos \theta)r) dr d\theta = \int_0^{2\pi} \frac{d\theta}{-t - |x| \cos \theta} = \frac{2\pi}{\sqrt{t^2 - |x|^2}}.$$

By analytic continuation this formula remains valid for complex t and x with $\Re(t) < -|\Re(x)|$. Formula (48) follows from the identity

$$e^{-i\Im(C)} u(t - \Im(A), x - \Im(b)) = \frac{e^{\Re(C)}}{(2\pi)^2} F(\Re(A) + it, \Re(b) + ix). \quad \square$$

Remark 6.4. If u is the (+)-wave corresponding to an L^2 function of the form (47), then the knowledge of $|u(t, x)|$ uniquely determines the value of the coefficients A, b and $\Re(C)$. The proof of this fact is similar to the one outlined in Remark 5.4.

Similarly, for the term $\|u_-^3\|$ we have

$$\|u_-^3\|^2 \leq \frac{1}{2\pi} \|f_-\|^6,$$

with equality when f_- takes the form (47).

For the term $\|u_+^2 u_-\|$, we observe that by Hölder’s inequality we have

$$\|u_+^2 u_-\|^2 \leq \|u_+\|^{4/3} \|u_-\|^{2/3} \leq \frac{1}{2\pi} \|f_+\|^4 \|f_-\|^2. \tag{49}$$

The second inequality in (49) is an equality if f_+ and f_- are functions of the form (47). The first inequality in (49) is an equality if there is a constant $\mu \geq 0$ such that $|u_+(t, x)| = \mu |u_-(t, x)|$ for (almost all) $(t, x) \in \mathbb{R} \times \mathbb{R}^2$.

Lemma 6.5. *Let u_+ be a (+)-wave and u_- be a (-)-wave corresponding to initial data f_+ and f_- of the form (47). If there exists $\mu \geq 0$ such that $|u_+(t, x)| = \mu|u_-(t, x)|$ for all t and x , then*

$$\widehat{f}_+(\xi) = |\xi|^{-1/2} \exp(A|\xi| + b \cdot \xi + C), \quad \widehat{f}_-(\xi) = |\xi|^{-1/2} \exp(\bar{A}|\xi| - \bar{b} \cdot \xi + D) \quad (50)$$

for some $A, C, D \in \mathbb{C}$ and $b \in \mathbb{C}^2$.

This lemma follows by the same argument used in the proof of Proposition 5.5.

Similarly, for the term $\|u_+u_-^2\|$ we have

$$\|u_+u_-^2\|^2 \leq \frac{1}{2\pi} \|f_+\|^2 \|f_-\|^4,$$

with equality if and only if f_+ and f_- are functions of the form (50).

Let us now consider the term $\Re\langle u_+^3, u_+^2u_- \rangle$. We have

$$\Re\langle u_+^3, u_+^2u_- \rangle \leq |\langle u_+^3, u_+^2u_- \rangle| \leq \|u_+^3\| \|u_+^2u_-\| \leq \frac{1}{2\pi} \|f_+\|^5 \|f_-\|.$$

Equality in the second and third inequalities here implies that f_+ and f_- are of the form (50), while we must have $\Im(C) = \Im(D)$ to have equality in the first inequality.

Similarly for the terms $\Re\langle u_+u_-^2, u_-^3 \rangle$ and $\Re\langle u_+^2u_-, u_+u_-^2 \rangle$ we have

$$|\Re\langle u_+u_-^2, u_-^3 \rangle| \leq \frac{1}{2\pi} \|f_+\| \|f_-\|^5, \quad |\Re\langle u_+^2u_-, u_+u_-^2 \rangle| \leq \frac{1}{2\pi} \|f_+\|^3 \|f_-\|^3,$$

with equality when f_+ and f_- are of the form (50) with $\Im(C) = \Im(D)$.

The terms $\Re\langle u_+^3, u_+u_-^2 \rangle$, $\Re\langle u_+^3, u_-^3 \rangle$, and $\Re\langle u_+^2u_-, u_-^3 \rangle$ are always zero. Indeed, the Fourier transform of the cubic terms $u_+^3, u_+^2u_-, u_+u_-^2, u_-^3$ are L^2 functions supported on the closures of the regions

$$\begin{aligned} \mathcal{C}_{+++} &= \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^2 : \tau > |\xi|\}, \\ \mathcal{C}_{++-} &= \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^2 : \tau > -|\xi|\}, \\ \mathcal{C}_{+--} &= \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^2 : \tau > |\xi|\}, \\ \mathcal{C}_{---} &= \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^2 : \tau > -|\xi|\}, \end{aligned}$$

respectively, and the intersections $\bar{\mathcal{C}}_{+++} \cap \bar{\mathcal{C}}_{+--}, \bar{\mathcal{C}}_{+++} \cap \bar{\mathcal{C}}_{---}, \bar{\mathcal{C}}_{++-} \cap \bar{\mathcal{C}}_{---}$ are of measure zero.

We put together all the estimates for each single term and obtain

$$\begin{aligned} 2\pi \|u\|_{L^6}^6 &\leq \|f_+\|^6 + \|f_-\|^6 + 9\|f_+\|^4 \|f_-\|^2 + 9\|f_+\|^2 \|f_-\|^4 \\ &\quad + 6\|f_+\|^5 \|f_-\| + 6\|f_+\| \|f_-\|^5 + 18\|f_+\|^3 \|f_-\|^3. \end{aligned} \quad (51)$$

Lemma 6.6. For $X \geq 0$ and $Y \geq 0$ we have the sharp polynomial inequality

$$X^6 + Y^6 + 9X^4Y^2 + 9X^2Y^4 + 6X^5Y + 6XY^5 + 18X^3Y^3 \leq \frac{25}{4}(X^2 + Y^2)^3$$

with equality if and only if $X = Y$.

Proof. By homogeneity we can assume that $Y = 1$. Let

$$P(X) = X^6 + 1 + 9X^4 + 9X^2 + 6X^5 + 6X + 18X^3, \quad Q(X) = X^2 + 1.$$

We want to prove that $4P(X) \leq 25Q(X)^3$ for $X \geq 0$, with equality if and only if $X = 1$. Since we have the identity

$$4P(X) = 4Q(X)^3 + 24(XQ(X)^2 + X^2Q(X) + X^3),$$

our inequality is equivalent to

$$24(XQ(X)^2 + X^2Q(X) + X^3) \leq 21Q(X)^3,$$

which reduces to

$$8(Z + Z^2 + Z^3) \leq 7, \quad Z = \frac{X}{Q(X)} = \frac{X}{X^2 + 1} \in [0, 1/2].$$

On the interval $[0, 1/2]$ the polynomial $Z + Z^2 + Z^3$ is strictly increasing and takes its maximum value $7/8$ when $Z = 1/2$, which corresponds to $X = 1$. \square

We apply Lemma 6.6 to (51) and finally obtain

$$\|u\|_{L^6} \leq \left(\frac{25}{8\pi}\right)^{1/6} (\|f_+\|_{L^2}^2 + \|f_-\|^2)^{1/2} = \left(\frac{25}{64\pi}\right)^{1/6} \|(f, g)\|_{\dot{H}^{1/2}(\mathbb{R}^n) \times \dot{H}^{-1/2}(\mathbb{R}^n)}, \tag{52}$$

which proves that $W(2) \geq (25/(64\pi))^{1/6}$. The next proposition tells us that maximizers exist and that all the inequalities are sharp; hence, $W(2) = (25/(64\pi))^{1/6}$.

Proposition 6.7. We have $\|u\|_{L^6} = (25/(64\pi))^{1/6} \|(f, g)\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}}$ if and only if

$$\widehat{f}_+(\xi) = |\xi|^{-1/2} \exp(A|\xi| + b \cdot \xi + C), \quad \widehat{f}_-(\xi) = |\xi|^{-1/2} \exp(\bar{A}|\xi| - \bar{b} \cdot \xi + C), \tag{53}$$

where $A, C \in \mathbb{C}$ and $b \in \mathbb{C}^2$ with $|\Re b| < -\Re A$.

Proof. From the above discussion we have equalities in the estimates for each single cubic term if and only if f_+ and f_- are of the form (50) with $\Im(C) = \Im(D)$. To have equality in (52) we also need $\|f_+\| = \|f_-\|$, which in this case implies $\Re(C) = \Re(D)$. \square

A particular case of (53), corresponding to $A = -1, b = 0, C = \log(2\pi)$, is given by the initial data

$$f_*(x) = \frac{1}{\sqrt{1 + |x|^2}}, \quad g_*(x) = 0, \quad x \in \mathbb{R}^3. \tag{54}$$

As we have seen at the end of Section 5, the class of initial data of the form (53) is invariant under the action of the group \mathcal{L} and any maximizer is connected to the functions (f_*, g_*) by the action of \mathcal{L} .

7. Functional equations

In this section we study the functional equations which characterize the families of maximizers that we have found in the previous sections. They are:

$$f(x)f(y)f(z) = F(x^2 + y^2 + z^2, x + y + z) \quad \text{for a.e. } (x, y, z) \in \mathbb{R}^3, \tag{55}$$

$$f(x)f(y) = F(|x|^2 + |y|^2, x + y) \quad \text{for a.e. } (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2, \tag{56}$$

$$f(x)f(y)f(z) = F(|x| + |y| + |z|, x + y + z) \quad \text{for a.e. } (x, y, z) \in (\mathbb{R}^2)^3, \tag{57}$$

$$f(x)f(y) = F(|x| + |y|, x + y), \quad \text{for a.e. } (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3, \tag{58}$$

where f and F are unknown complex-valued measurable functions and the identities are supposed to hold almost everywhere with respect to the Lebesgue measure. We are going to show that locally integrable solutions to these equations are actually smooth functions; this is a general principle which holds for a large class of functional equations (actually even assuming only measurability implies continuity, see the work of A. J arai in [4]), but for the sake of completeness we include a direct proof adapted to our equations. Once the smoothness of f and F is established, it is not difficult to solve the equation using geometric or differential methods. It turns out that in all cases the function F must be an exponential function of the form

$$F(t, x) = \exp(At + b \cdot x + C).$$

A simpler model for the above functional equations is provided by the exponential law, $f(x)f(y) = f(x + y)$, which is one of the four basic Cauchy functional equations. We refer the reader to [1] for a general introduction to the subject of functional equations. Here, we only require the following result which is a simple exercise in real analysis.

Lemma 7.1. *Let Ω be an open subset of \mathbb{R}^n such that $x + y \in \Omega$ whenever $x, y \in \Omega$. Let $f : \Omega \rightarrow \mathbb{C}$ be a non-trivial locally integrable solution of the Cauchy functional equation*

$$f(x)f(y) = f(x + y) \quad \text{for a.e. } (x, y) \in \Omega \times \Omega. \tag{59}$$

Then there exists a vector $b \in \mathbb{C}^n$ such that $f(x) = \exp(b \cdot x)$ for a.e. $x \in \Omega$.

Proof. Let Q be a cube contained in Ω such that $\int_Q f(y) dy \neq 0$. If we integrate (59) with respect to $y \in Q$ we see that $f(x)$ must coincide (almost everywhere) with the continuous function

$$x \mapsto \frac{\int_Q f(x + y) dy}{\int_Q f(y) dy}.$$

If f is continuous the above function is differentiable. Hence, we may assume that f is differentiable. Fix $y_0 \in \Omega$ and let $b = (\nabla f(y_0))/f(y_0)$. If we differentiate (59) with respect to y and set $y = y_0$ we obtain the differential equation

$$\nabla f(x + y_0) = f(x)\nabla f(y_0) = f(x)f(y_0)b = f(x + y_0)b,$$

whose non-trivial solutions have the form $f(z) = \exp(b \cdot z + C)$ for some constant $C \in \mathbb{C}$. Substituting this expression for f into (59), we see that $\exp(C)$ must be 1. □

As was done in the lemma, regularity properties of solutions to functional equations can be obtained by (partial) integration of the equation. The following lemmata, although not expressed in their most general form, are what we need to deduce continuity from local integrability in our equations.

Lemma 7.2. *Let A, B, Ω be open subsets of \mathbb{R}^n . Let $f \in L^p_{\text{loc}}(\Omega)$ and let $\varphi : A \times B \rightarrow \Omega$ be a smooth map such that*

$$\det \left| \frac{\partial \varphi}{\partial y}(x, y) \right| \neq 0 \quad \text{for } (x, y) \in A \times B. \quad (60)$$

Let K be a compact subset of B . For each $x \in A$, let $g_x : B \rightarrow \Omega$ be the function $g_x(y) = f(\varphi(x, y))$. Then the map $x \mapsto g_x$ is a continuous map from A to $L^p(K)$.

Proof. The case of f continuous is immediate. The general case follows by density. The condition on the partial Jacobian of φ is enough to apply, at least locally, the change of variable $y \rightsquigarrow z = \varphi(x, y)$ in the integration over K . We leave the details to the reader. \square

Remark 7.3. Lemma 7.2 does not hold if we remove condition (60). For example, if $A = B = \Omega$ and $\varphi(x, y) = x$ then we would have $g_x(y) = f(x)$ and the map $x \mapsto \|g_x\|_{L^p(K)} = |K|^{1/p} |f(x)|$ may not be continuous.

Lemma 7.4. *Let A, B, Ω be open subsets of \mathbb{R}^n . Let $f_j \in L^{p_j}_{\text{loc}}(\Omega)$, $j = 1, \dots, N$, with $\sum_j 1/p_j = 1$. Let $\varphi_1, \dots, \varphi_N : A \times B \rightarrow \Omega$ be smooth maps such that*

$$\det \left| \frac{\partial \varphi_j}{\partial y}(x, y) \right| \neq 0, \quad (x, y) \in A \times B, \quad j = 1, \dots, N.$$

Let $\psi : A \times B \rightarrow \mathbb{C}$ be a continuous function. Let K be a compact subset of B . Then the function

$$F(x) = \int_K \prod_{j=1}^N f_j(\varphi_j(x, y)) \psi(x, y) dy, \quad x \in A,$$

is continuous.

Proof. The continuity of F follows from the previous lemma and the continuity of the functional

$$(x, g_1, \dots, g_N) \mapsto \int_K \prod_j g_j(y) \psi(x, y) dy$$

defined on $A \times \prod_j L^{p_j}_{\text{loc}}(\Omega)$. \square

Proposition 7.5. *Let Ω be an open subset of $\mathbb{R}^n \times \mathbb{R}^n$ such that the section $\Omega_x = \{y \in \mathbb{R}^n : (x, y) \in \Omega\}$ is dense in \mathbb{R}^n for each $x \in \mathbb{R}^n$. Let $P, Q : \Omega \rightarrow \mathbb{R}^n$ be smooth maps such that*

$$\det \left| \frac{\partial P}{\partial y}(x, y) \right| \neq 0, \quad \det \left| \frac{\partial Q}{\partial y}(x, y) \right| \neq 0 \quad \text{for } (x, y) \in \Omega. \quad (61)$$

If $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is a locally integrable solution of the functional equation

$$f(x)f(y) = f(P(x, y))f(Q(x, y)) \quad \text{for a.e. } (x, y) \in \Omega, \tag{62}$$

then f is continuous.

Proof. We may assume that f is non-trivial. Let $g(x) = \sqrt{|f(x)|}$. When $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ we have $g \in L^2_{\text{loc}}(\mathbb{R}^n)$. Hence, by (61), it follows that for every $x \in \mathbb{R}^n$ the function $y \mapsto g(P(x, y))g(Q(x, y))$ is locally integrable on Ω . Fix $x_0 \in \mathbb{R}^n$ and choose a compact domain D in Ω_{x_0} such that $\int_D f \neq 0$.

We integrate the square root of the absolute value of equation (62) with respect to y over the domain D and obtain

$$g(x) \int_D g(y) dy = \int_D g(P(x, y))g(Q(x, y)) dy.$$

By Lemma 7.4, the right hand side is a continuous function of x for x in a neighborhood of x_0 . Since $\int_D g \neq 0$, we deduce that g is continuous at x_0 . This proves that $|f|$ is a continuous function. In particular it follows that $f \in L^2_{\text{loc}}$. We can bootstrap the argument: if we integrate (62) with respect to y over the domain D , we obtain

$$f(x) \int_D f(y) dy = \int_D f(P(x, y))f(Q(x, y)) dy,$$

from which it follows that f is continuous. □

In the following subsections we will study in detail each of our four functional equations. In each case we will adopt the following strategy:

- (1) Local integrable solutions are continuous.
- (2) Non-trivial continuous solutions never vanish.
- (3) Continuous solutions which never vanish are of exponential form.

Remark 7.6. It is interesting to observe that there are functional equations which formally look very similar to the ones we are considering, but for which each of the above three steps fails. Take for instance the functional equation

$$f(x)f(y) = F(x^2 + y^2, x + y) \quad \text{for a.e. } (x, y) \in \mathbb{R}^2.$$

In this case, given *any* function f we can always construct a solution by setting

$$F(s, t) = f\left(\frac{t + \sqrt{2s - t^2}}{2}\right)f\left(\frac{t - \sqrt{2s - t^2}}{2}\right).$$

Even if the function f is locally integrable, it does not need to be an exponential, it can vanish on any set and does not need to be continuous.

7.1. Equation (55)

As in Section 4 we let $\mathcal{P}_1 = \{(s, t) \in \mathbb{R} \times \mathbb{R} : 3s > t^2\}$.

Lemma 7.7. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ and $F : \overline{\mathcal{P}}_1 \rightarrow \mathbb{C}$ be functions which solve equation (55). If f is locally integrable then f and F are continuous functions.*

Proof. We first prove that f locally integrable implies F locally integrable. Indeed,

$$\begin{aligned} \iiint_{|x|^2+|y|^2+|z|^2 \leq R^2} |f(x)f(y)f(z)| dx dy dz &= \int_{\substack{v \in \mathbb{R}^3 \\ |v| \leq R}} |F(|v|^2, v \cdot (1, 1, 1))| dv \\ &= 2\pi \int_0^R \int_{-1}^1 |F(r^2, \sqrt{3}ru)| r^2 du dr = \frac{\pi}{\sqrt{3}} \iint_{t^2/3 \leq s \leq R^2} |F(s, t)| ds dt \end{aligned}$$

for any $R > 0$. Moreover, using the change of variables

$$(y, z) \rightsquigarrow (s, t) = \Phi(y, z) = (y^2 + z^2, y + z)$$

from the region $\{y > z\}$ to $\{s > t^2/2\}$, with $ds dt = 2(y - z) dy dz$, we have

$$\begin{aligned} f(x) \int_{\Omega} f(y)f(z)(y - z) dy dz &= \int_{\Omega} F(x^2 + y^2 + z^2, x + y + z)(y - z) dy dz \\ &= \frac{1}{2} \int_{\Phi(\Omega)} F(x^2 + s, x + t) ds dt \end{aligned}$$

for any bounded domain $\Omega \subseteq \{y > z\}$. The local integrability of F implies that the function $x \mapsto \int_{\Phi(\Omega)} F(x^2 + s, x + t) ds dt$ is continuous. We choose Ω so that the integral $\int_{\Omega} f(y)f(z)(y - z) dy dz$ is not zero and it follows that $f(x)$ is continuous.

The continuity of F comes easily from the equation and the continuity of f , since we have

$$F(s, t) = f(0)f\left(\frac{t + \sqrt{2s - t^2}}{2}\right)f\left(\frac{t - \sqrt{2s - t^2}}{2}\right). \quad \square$$

Remark 7.8. We can write (55) as

$$(f \otimes f \otimes f)(v) = F(s, t), \quad v \in \mathbb{R}^3, \quad s = |v|^2, \quad t = v \cdot (1, 1, 1),$$

which shows that the tensor product $f \otimes f \otimes f$, as a function on \mathbb{R}^3 , is constant along any circle $\Gamma(s, t)$ obtained as the intersection of the sphere of radius \sqrt{s} centered at the origin and the plane orthogonal to the vector $(1, 1, 1)$ passing through the point $(t/3, t/3, t/3)$.

Lemma 7.9. *If f and F are continuous functions which solve equation (55) and f vanishes at one point then f and F vanish everywhere.*

Proof. By continuity it is enough to prove that the set of all points where f vanishes is open. Suppose f vanishes at a point x_* . If f is not identically zero then there exists some open set A in \mathbb{R} on which f never vanishes. Let y_* and z_* be two distinct points in A . Consider the map

$$(y, z) \mapsto (s, t) = (x_*^2 + y^2 + z^2, x_* + y + z).$$

Its Jacobian determinant at the point (y_*, z_*) is $2|y_* - z_*| \neq 0$. Hence, the map is invertible from a neighborhood $U \subset A \times A$ of (y_*, z_*) to a neighborhood V of $(s_*, t_*) = (x_*^2 + y_*^2 + z_*^2, x_* + y_* + z_*)$. It follows that, for each x sufficiently close to x_* so that $(x^2 + y_*^2 + z_*^2, x + y_* + z_*)$ lies in V , there exists a pair $(y, z) \in U$ such that

$$(x^2 + y_*^2 + z_*^2, x + y_* + z_*) = (x_*^2 + y^2 + z^2, x_* + y + z).$$

Using the functional equation (55) we obtain

$$f(x)f(y_*)f(z_*) = f(x_*)f(y)f(z) = 0,$$

and we know that $f(y_*)f(z_*) \neq 0$. Hence, $f(x)$ vanishes for x in a neighborhood of x_* . \square

Proposition 7.10. *If $f : \mathbb{R} \rightarrow \mathbb{C}$ and $F : \overline{\mathcal{P}}_1 \rightarrow \mathbb{C}$ are non-trivial locally integrable functions which satisfy the functional equation (55) for all $x, y, z \in \mathbb{R}$, then there exists complex constants A, B, C such that*

$$f(x) = \exp(Ax^2 + Bx + C), \quad F(t, x) = \exp(At + Bx + 3Cx)$$

for (almost) all $(t, x) \in \mathcal{P}_1$.

Proof. By Lemma 7.7, we may assume that f and F are continuous. By Lemma 7.9, we may assume that f and F never vanish. We define

$$\begin{aligned} g(x) &= \frac{f(x)}{f(-x)}, & G(s, t) &= \frac{F(s, t)}{F(s, -t)}, \\ h(x) &= f(x)f(-x), & H(s, t) &= F(s, t)F(s, -t). \end{aligned}$$

The function g corresponds to the *odd* component of f , $g(x)g(-x) = 1$, $g(0) = 1$, and satisfies the same equation

$$g(x)g(y)g(z) = G(x^2 + y^2 + z^2, x + y + z), \quad x, y, z \in \mathbb{R}.$$

In particular,

$$G(2s, 0) = g(\sqrt{s})g(-\sqrt{s})g(0) = 1$$

for all $s \geq 0$. It follows that

$$g(x)g(y) = g(x+y)g(-x-y)g(x)g(y) = g(x+y)G((x+y)^2 + x^2 + y^2, 0) = g(x+y).$$

By Lemma 7.1, g must be an exponential function of the form $g(x) = \exp(2Bx)$ for some complex constant B .

The function h corresponds to the *even* component of f , $h(x) = h(-x)$ and satisfies the equation

$$h(x)h(y)h(z) = H(x^2 + y^2 + z^2, x \pm y \pm z), \quad x, y, z \in \mathbb{R},$$

for any combination of signs. By the same argument used in Remark 7.8, we deduce that $h \otimes h \otimes h$ is constant along circles obtained by intersecting spheres centered at the origin with planes perpendicular to the four vectors $(1, \pm 1, \pm 1)$. It follows that $h \otimes h \otimes h$ must be constant on any sphere centered at the origin. In fact, any two points at the same distance from the origin can be connected by a finite sequence of arcs of the above circles. This means that there exists a function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{C}$ such that

$$\frac{h(x)h(y)h(z)}{h(0)^3} = \varphi(x^2 + y^2 + z^2), \quad x, y, z \in \mathbb{R}.$$

In particular, for $s \geq 0$ and $t \geq 0$, we have

$$\varphi(s)\varphi(t) = \frac{h(\sqrt{s})h(0)^2}{h(0)^3} \cdot \frac{h(\sqrt{t})h(0)^2}{h(0)^3} = \frac{h(\sqrt{s})h(\sqrt{t})h(0)}{h(0)^3} = \varphi(s+t).$$

By Lemma 7.1, φ must be an exponential function of the form $\varphi(s) = \exp(2As)$ for some complex constant A . Hence, $h(x) = h(0) \exp(2Ax^2)$.

We conclude the proof of the lemma by observing that

$$f(x)^2 = g(x)h(x) = \exp(2Bx)h(0) \exp(Ax^2) = \exp(2Ax^2 + 2Bx + 2C),$$

where C is a complex constant such that $f(0) = e^C$. □

7.2. Equation (56)

If we integrate equation (56) with respect to y on a domain of \mathbb{R}^2 , we cannot apply directly the regularity results of Lemma 7.4, because the domain of F is a region in \mathbb{R}^3 and the image of the map $y \mapsto (|x|^2 + |y|^2, x + y)$ is a set of measure zero in \mathbb{R}^3 . To overcome this difficulty we exploit the geometric invariance properties of the equation.

Remark 7.11. Let $I : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the identity on \mathbb{R}^2 and let $H : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $H(x_1, x_2) = (-x_2, x_1)$, be a counterclockwise rotation by $\pi/2$ of the plane. Given two points x and y in \mathbb{R}^2 the functions

$$P(x, y) = \frac{x + y}{2} + H\left(\frac{x - y}{2}\right) = \frac{I + H}{2}x + \frac{I - H}{2}y,$$

$$Q(x, y) = \frac{x + y}{2} - H\left(\frac{x - y}{2}\right) = \frac{I - H}{2}x + \frac{I + H}{2}y,$$

determine two other points $p = P(x, y)$ and $q = Q(x, y)$ so that x, y and p, q are the opposite vertices of a square (see Figure 2) and we have

$$p + q = x + y, \quad |p|^2 + |q|^2 = |x|^2 + |y|^2. \tag{63}$$

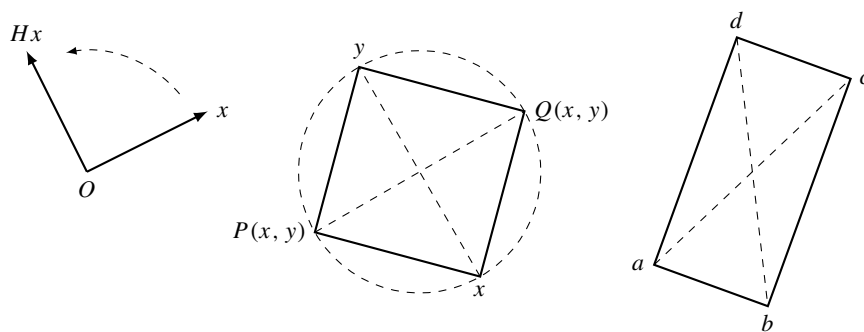


Fig. 2. The functions H (left) and P, Q (center) described in Remark 7.11 and a rectangle (right) used in Remark 7.14.

Moreover, the linear map $(x, y) \mapsto (p, q)$ is an isometry on $\mathbb{R}^2 \times \mathbb{R}^2$. By property (63), it follows that the equation (56) implies the equation

$$f(x)f(y) = f(P(x, y))f(Q(x, y)) \quad \text{for a.e. } (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2. \quad (64)$$

Observe also that $\partial P/\partial y = (I - H)/2$ and $\partial Q/\partial y = (I + H)/2$ are non-singular matrices.

As in Section 3 we let $\mathcal{P}_2 = \{(s, v) \in \mathbb{R} \times \mathbb{R}^2 : 2s > |v|^2\}$.

Lemma 7.12. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ and $F : \overline{\mathcal{P}}_2 \rightarrow \mathbb{C}$ be solutions of equation (56). If f is locally integrable then f and F are continuous functions.*

Proof. By Remark 7.11, the lemma becomes a corollary of Proposition 7.5. □

Lemma 7.13. *If f is a continuous solution of equation (64) and f vanishes at one point then f vanishes everywhere.*

Proof. Let $y \in \mathbb{R}^2$. If f vanishes at the point x_0 , equation (64) implies that f vanishes at the point x_1 , where x_1 is either $P(x_0, y)$ or $Q(x_0, y)$, and $|x_1 - y| = (1/\sqrt{2})|x_0 - y|$. By iterating this argument, we can construct a sequence of points x_n such that $f(x_n) = 0$ and $\lim_n x_n = y$. By continuity it follows that $f(y) = 0$. □

Remark 7.14. It follows from equation (56) that f solves the rectangular functional equation

$$f(a)f(c) = f(b)f(d),$$

whenever the points a, c and b, d are the opposite vertices of a rectangle (see Figure 2). Indeed, when $a - b = d - c$ and $a - b \perp c - b$, we have

$$0 = (a - b) \cdot (c - b) = a \cdot c - (a + c) \cdot b + |b|^2 = a \cdot c - (b + d) \cdot b + |b|^2 = a \cdot c - d \cdot b.$$

$$\text{Hence, } a \cdot c = b \cdot d \text{ and } |a|^2 + |c|^2 = |a + c|^2 - 2a \cdot c = |b + d|^2 - 2b \cdot d = |b|^2 + |d|^2.$$

Proposition 7.15. *If $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ and $F : \overline{\mathcal{P}}_2 \rightarrow \mathbb{C}$ are nontrivial locally integrable functions which satisfy the functional equation (56) then there exist constants $A \in \mathbb{C}$, $b \in \mathbb{C}^2$, $C \in \mathbb{C}$ such that*

$$f(x) = \exp(A|x|^2 + b \cdot x + C), \quad F(t, x) = \exp(At + b \cdot x + 2C)$$

for (almost) all $(t, x) \in \mathcal{P}_2$.

In the proof of the proposition we follow a geometric construction which is an adaptation of the one for odd orthogonally additive mappings found in [7].

Proof. By Lemma 7.12, we may assume that f and F are continuous. By Lemma 7.13, we may assume that f and F never vanish. We define

$$g(x) = \frac{f(x)}{f(-x)}, \quad G(t, z) = \frac{F(t, z)}{F(t, -z)},$$

$$h(x) = f(x)f(-x), \quad H(t, z) = F(t, z)F(t, -z).$$

The function g corresponds to the *odd* component of f , $g(x)g(-x) = 1$ and $g(0) = 1$. By Remark 7.14 we know it satisfies the rectangular equation $g(a)g(c) = g(b)g(d)$ whenever the points a, c and b, d are the opposite vertices of a rectangle. Given two vectors x and y in \mathbb{R}^2 it is always possible to find a third vector z such that $z \perp x + y$ and $x + z \perp y - z$. Let p and $-p$ be the components of x and y perpendicular to $x + y$. Consider the three rectangles formed by $(0, x + z, x + y, y - z)$, $(x, x + z, p + z, p)$ and $(y, y - z, -p - z, -p)$ (see Figure 3); using the rectangular equation we have

$$g(x + y)g(0) = g(x + z)g(y - z),$$

$$g(x)g(p + z) = g(x + z)g(p),$$

$$g(y)g(-p - z) = g(y - z)g(-p);$$

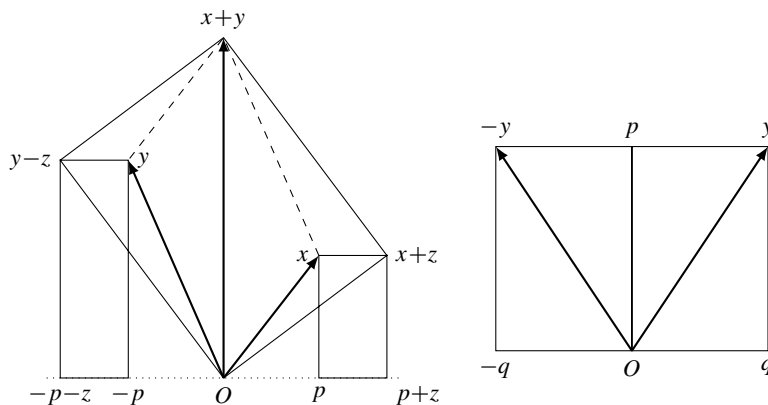


Fig. 3. Constructions for the function g (left) and the function h (right) in the proof of Proposition 7.15.

and using the parity properties of g we obtain

$$\begin{aligned} g(x + y) &= g(x + y)g(0) = g(x + z)g(y - z) = g(x + z)g(p)g(y - z)g(-p) \\ &= g(x)g(p + z)g(y)g(-p - z) = g(x)g(y). \end{aligned}$$

By Lemma 7.1, g must be an exponential function of the form $g(x) = \exp(2b \cdot x)$ for some complex vector $b \in \mathbb{C}^2$.

The function h corresponds to the *even* component of f , $h(x) = h(-x)$, and satisfies the rectangular equation $h(a)h(c) = h(b)h(d)$ whenever the points a, c and b, d are the opposite vertices of a rectangle. Given two points x and y with $|x| = |y|$, let $p = (x + y)/2$, $q = (x - y)/2$ and consider the rectangles $(0, p, x, q)$, $(0, p, y, -q)$ (see Figure 3); by the rectangular equation and the parity of h we have

$$h(x)h(0) = h(p)h(q) = h(p)h(-q) = h(y)h(0).$$

Hence, $|x| = |y|$ implies $h(x) = h(y)$. This means that h is spherically symmetric and there exists a function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{C}$ such that $h(x) = h(0)\varphi(|x|^2)$ for $x \in \mathbb{R}^2$. Given $s \geq 0$ and $t \geq 0$, let x and y be two points in \mathbb{R}^2 such that $|x|^2 = s$, $|y|^2 = t$ and $x \perp y$; by the Pythagorean theorem we have $|x + y|^2 = s + t$. It follows that

$$\varphi(s)\varphi(t) = \frac{h(x)h(y)}{h(0)^2} = \frac{h(x + y)h(0)}{h(0)^2} = \varphi(s + t).$$

By Lemma 7.1, φ must be an exponential function of the form $\varphi(s) = \exp(2As)$ for some complex constant A . Hence, $h(x) = h(0) \exp(2A|x|^2)$.

We conclude the proof of the lemma by observing that

$$f(x)^2 = g(x)h(x) = \exp(2b \cdot x)h(0) \exp(A|x|^2) = \exp(2Ax^2 + 2b \cdot x + 2C),$$

where C is a complex constant such that $f(0) = e^C$. □

7.3. Equation (57)

As in Section 6 we let $\mathcal{C}_{+++} = \{(t, v) \in \mathbb{R} \times \mathbb{R}^2 : t > |v|\}$.

Lemma 7.16. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ and $F : \bar{\mathcal{C}}_{+++} \rightarrow \mathbb{C}$ be functions which solve equation (57). If f is locally integrable then f and F are continuous functions.*

Proof. Suppose first that $f \in L^p_{loc}(\mathbb{R}^2)$ for some $p > 2$. Using the results of Lemma 6.1 we can see that $F \in L^1_{loc}(\mathcal{C}_{+++})$; indeed,

$$\begin{aligned} \int_{|v|^2 \leq t \leq R} |F(t, v)| dv dt &= \frac{1}{8\pi^2} \int_{|v|^2 \leq t \leq R} |F(t, v)| I_3(t, v) dv dt \\ &= \frac{1}{8\pi^2} \int_{t \leq R} \frac{|F(|x| + |y| + |z|, x + y + z)|}{|x| |y| |z|} \delta \left(\begin{matrix} t - |x| - |y| - |z| \\ v - x - y - z \end{matrix} \right) dx dy dz dv dt \\ &= \frac{1}{8\pi^2} \int_{|x| + |y| + |z| \leq R} \frac{|f(x)f(y)f(z)|}{|x| |y| |z|} dx dy dz \leq \frac{1}{8\pi^2} \left(\int_{|x| \leq R} \frac{|f(x)|}{|x|} dx \right)^3 \\ &\leq C(R^{1-2/p})^3 \|f\|^3_{L^p(B(0, R))}. \end{aligned}$$

We now choose a bounded domain $\Omega \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ such that the integral

$$C_\Omega = \int_\Omega \frac{f(y)f(z)}{|y||z|I_2(|y|+|z|, y+z)} dy dz$$

is finite and not zero (here I_2 is the function defined in Lemma 6.1). This is possible when f is not trivial, since $f(x)/|x|$ is locally integrable and $1/I_2$ is bounded on compact subsets of \mathcal{C}_{+++} . We divide both sides of (57) by $|y||z|I_2(|y|+|z|, y+z)$ and integrate with respect to $(y, z) \in \Omega$ to obtain

$$f(x)C_\Omega = \int_D F(|x|+t, x+v)\varphi_\Omega(t, v) dv dt \tag{65}$$

for almost every $x \in \mathbb{R}^2$, where

$$\varphi_\Omega(t, v) = \frac{1}{I_2(t, v)} \int_\Omega \delta \left(\begin{matrix} t - |y| - |z| \\ v - y - z \end{matrix} \right) \frac{dy dz}{|y||z|} \leq 1$$

is a bounded continuous function and the region D is its support. The continuity of f now follows from the continuity of the right hand side in (65) by Lemma 7.4.

Suppose now that $f \in L^1_{\text{loc}}(\mathbb{R}^2)$. Then for $p > 2$ the functions $g = |f|^{1/p} \in L^p_{\text{loc}}(\mathbb{R}^2)$, $G = |F|^{1/p}$ also solve equation (57) and it follows from the previous argument that g is continuous. Hence, $|f|$ is also continuous and so $f \in L^p_{\text{loc}}(\mathbb{R}^2)$ for any p .

The continuity of F comes easily from the equation and the continuity of f : if $t \geq r \geq 0$ and ω is a unit vector, we have

$$F(t, r\omega) = f(0)f\left(\frac{r+t}{2}\omega\right)f\left(\frac{r-t}{2}\omega\right). \quad \square$$

Lemma 7.17. *If f and F are continuous functions which solve equation (57) and f vanishes at one point then f and F vanish everywhere.*

Proof. Equation (57) implies that

$$f\left(\frac{x}{3}\right)^3 = F(|x|, x) = f(x)f(0)^2. \tag{66}$$

Suppose $f(x_0) = 0$. Then $f(x_1) = 0$ for $x_1 = x_0/3$. By iterating this argument, $x_{k+1} = x_k/3$, we can construct a sequence of points x_n such that $f(x_n) = 0$ and $\lim_n x_n = 0$. By continuity it follows that $f(0) = 0$, and by (66), f must vanish everywhere. \square

Lemma 7.18. *Let $n \geq 1$. Let $\mathcal{N} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t = |x|\}$ be the cone of future null vectors and $\mathcal{C} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t > |x|\}$ the cone of future time-like vectors. Observe that $\mathcal{N} + \mathcal{N} = \overline{\mathcal{C}} = \mathcal{N} \cup \mathcal{C}$. If $F : \mathcal{N} \cup \mathcal{C} \rightarrow \mathbb{C}$ is a continuous solution of the conditional functional equation*

$$U, V \in \mathcal{N} \Rightarrow F(U)F(V) = F(U + V) \tag{67}$$

then F is also a solution of the unconditional functional equation

$$F(X)F(Y) = F(X + Y), \quad \forall X, Y \in \mathcal{C}.$$

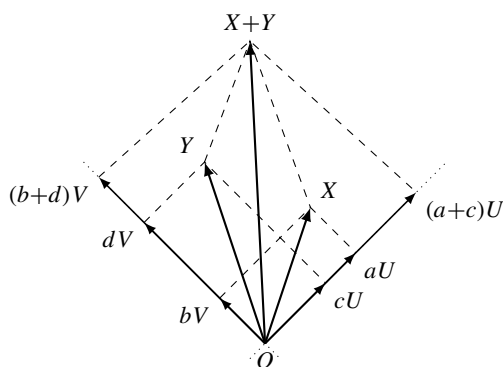


Fig. 4. Construction for the proof of Lemma 7.18.

Proof. Let X and Y be two vectors in $\mathcal{N} \cup \mathcal{C}$ which are not both in \mathcal{N} . Let Π be a two-dimensional plane through the origin which contains X and Y ; the intersection of the plane Π with the cone \mathcal{N} is the union of two null directed half lines,

$$\Pi \cap \mathcal{N} = (\mathbb{R}_+U) \cup (\mathbb{R}_+V),$$

where U and V are two linearly independent vectors in \mathcal{N} . We write X and Y as linear combinations of U and V ,

$$X = aU + bV, \quad Y = cU + dV,$$

for some non-negative coefficients a, b, c, d . Then, by (67),

$$\begin{aligned} F(X + Y) &= F((aU + bV) + (cU + dV)) = F((a + c)U + (b + d)V) \\ &= F((a + c)U)F((b + d)V) = (F(aU)F(cU))(F(bV)F(dV)) \\ &= (F(aU)F(bV))(F(cU)F(dV)) \\ &= F(aU + bV)F(cU + dV) = F(X)F(Y). \end{aligned} \quad \square$$

Proposition 7.19. *If $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ and $F : \bar{\mathcal{C}}_{++++} \rightarrow \mathbb{C}$ are non-trivial locally integrable functions which satisfy the functional equation (57) then there exist constants $A \in \mathbb{C}$, $b \in \mathbb{C}^2$, $C \in \mathbb{C}$ such that*

$$f(x) = \exp(A|x| + b \cdot x + C), \quad F(t, x) = \exp(At + b \cdot x + 3C)$$

for (almost) all $(t, x) \in \bar{\mathcal{C}}_{++++}$.

Proof. By Lemma 7.16, we may assume that f and F are continuous. By Lemma 7.17, we may assume that f and F never vanish. Setting $y = 0$ and $z = 0$ in (57) we obtain $F(|x|, x) = f(x)f(0)^2$. We define $G(t, x) = F(t, x)/F(0, 0)$. Then

$$G(|x|, x) = \frac{F(|x|, x)}{F(0, 0)} = \frac{f(x)}{f(0)}, \quad x \in \mathbb{R}^2.$$

We also have

$$G(|x|, x)G(|y|, y) = \frac{f(x)f(y)f(0)}{f(0)^3} = \frac{F(|x| + |y|, x + y)}{F(0, 0)} = G(|x| + |y|, x + y).$$

We can apply first Lemma 7.18 and then Lemma 7.1 to the function G to deduce that $G(t, x) = \exp(At + b \cdot x)$ for some constants $A \in \mathbb{C}$ and $b \in \mathbb{C}^2$. The result then follows by choosing C so that $F(0, 0) = \exp(3C)$. \square

7.4. Equation (58)

Lemma 7.20. *It is possible to construct an open set $\Omega \subset \mathbb{R}^3 \times \mathbb{R}^3$ whose sections $\Omega_x = \{y : (x, y) \in \Omega\}$ are dense in \mathbb{R}^3 for every $x \in \mathbb{R}^3$, and a pair of smooth maps $P, Q : \Omega \rightarrow \mathbb{R}^3$ such that, for every $(x, y) \in \Omega$,*

$$|P(x, y)| + |Q(x, y)| = |x| + |y|, \tag{68}$$

$$P(x, y) + Q(x, y) = x + y, \tag{69}$$

$$\det \left| \frac{\partial P}{\partial y}(x, y) \right| \neq 0, \quad \det \left| \frac{\partial Q}{\partial y}(x, y) \right| \neq 0.$$

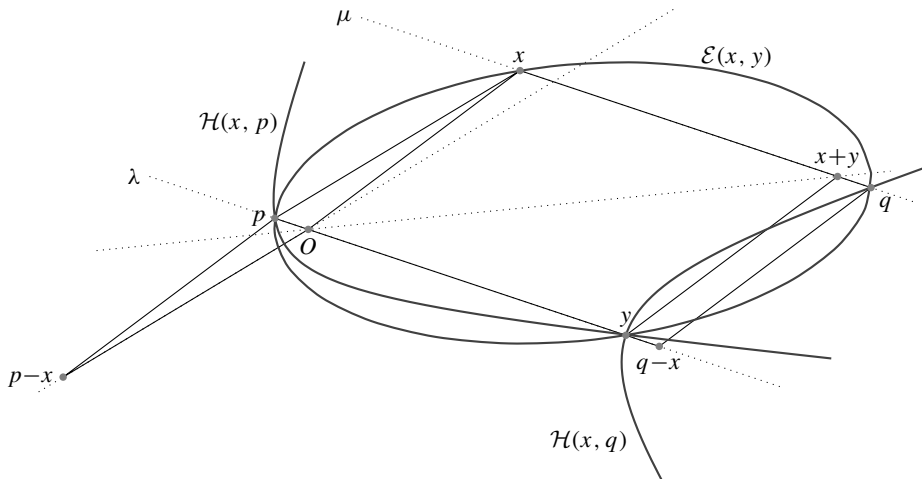


Fig. 5. Constructions of the functions P, Q and their inverses as described in Lemma 7.20.

Proof. The set $\Omega = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : x \times y \neq 0\}$ of linearly independent pairs of vectors clearly has sections Ω_x dense in \mathbb{R}^3 for every x . Given $(x, y) \in \Omega$, the ellipsoid of revolution

$$\mathcal{E}(x, y) = \{u \in \mathbb{R}^3 : |u| + |x + y - u| = |x| + |y|\},$$

with foci at 0 and $x + y$ and which contains the points x and y , is non-degenerate, and any line passing through one of the foci intersects the ellipsoid in exactly two points.

In particular, the line λ passing through y and 0 intersects $\mathcal{E}(x, y)$ in y and in another point p ; similarly, the line μ passing through x and $x + y$ intersects $\mathcal{E}(x, y)$ in x and in another point q . By symmetry we have $p + q = x + y$ and from the definition of \mathcal{E} it follows that

$$|p| + |q| = |p| + |x + y - p| = |x| + |y|.$$

It is evident from the geometric construction that the correspondence $(x, y) \mapsto (p, q)$ is a smooth map as long as the vectors x and y remain linearly independent; moreover, when x and y are linearly independent, also (x, p) and (x, q) are pairs of linearly independent vectors. Setting $P(x, y) = p$ and $Q(x, y) = q$, we obtain two smooth maps $P, Q : \Omega \rightarrow \mathbb{R}^3$ which satisfy (68) and (69).

To verify that, for fixed $x \in \mathbb{R}^3$, the maps $y \mapsto P(x, y)$ and $y \mapsto Q(x, y)$ are locally invertible we provide a smooth geometric construction of their inverses.

Given a pair of points $(x, p) \in \Omega$, we define $\mathcal{H}(x, p)$ to be the branch of the hyperboloid with foci at 0 and $p - x$ passing through the point p ,

$$\mathcal{H}(x, p) = \{u \in \mathbb{R}^3 : |u| - |p - x - u| = |p| - |x|\},$$

and we notice that it is non-degenerate since p does not belong to the line passing through 0 and $p - x$. The line passing through p and 0 intersects $\mathcal{H}(x, p)$ in p and in another point y_* . The map $(x, p) \mapsto y_*$ is smooth as long as x and p remain linearly independent. We claim that $P(x, y_*) = p$; indeed, p belongs to the line passing through y_* and 0 , and from the definition of $\mathcal{H}(x, p)$ it follows that

$$|p| + |p - x - y_*| = |x| + |y_*|,$$

which means that $p \in \mathcal{E}(x, y_*)$.

Similarly, given a pair of points $(x, q) \in \Omega$, we consider $\mathcal{H}(x, q)$, the branch of the hyperboloid with foci at 0 and $q - x$ passing through the point q . The line passing through $q - x$ and 0 intersects $\mathcal{H}(x, q)$ in one point y_{**} , the vertex of the hyperboloid. The map $(x, q) \mapsto y_{**}$ is smooth as long as x and q remain linearly independent, and it is easy to check that $Q(x, y_{**}) = q$. Indeed, since $q - x$ belongs to the line passing through y_{**} and 0 , by a translation we find that q belongs to the line passing through $x + y_{**}$ and x , moreover from the definition of $\mathcal{H}(x, q)$ it follows that

$$|q| + |q - x - y_{**}| = |x| + |y_{**}|,$$

which means that $q \in \mathcal{E}(x, y_{**})$. □

Remark 7.21. Explicit formulae for the functions P and Q constructed in the previous lemma are given by

$$P(x, y) = \left(\frac{x \cdot y - |x||y|}{x \cdot y + |x||y| + 2|y|^2} \right) y, \quad Q(x, y) = x + y - P(x, y).$$

As in Section 5 we let $\mathcal{C}_{++} = \{(t, v) \in \mathbb{R} \times \mathbb{R}^3 : t > |v|^2\}$.

Lemma 7.22. *Let $f : \mathbb{R}^3 \rightarrow \mathbb{C}$ and $F : \overline{\mathcal{C}}_{++} \rightarrow \mathbb{C}$ be functions which solve equation (58). If f is locally integrable then f and F are continuous functions.*

Proof. Let P and Q be the functions constructed in Lemma 7.20. If f and F are solutions to (58) it follows that

$$f(x)f(y) = f(P(x, y))f(Q(x, y)) \quad \text{for a.e. } (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3,$$

and the lemma then becomes a corollary of Proposition 7.5. \square

Once the continuity of locally integrable solutions to (58) is established, one proceeds in the same manner as in the previous subsection and obtains the following result.

Proposition 7.23. *If $f : \mathbb{R}^3 \rightarrow \mathbb{C}$ and $F : \overline{\mathcal{C}}_{++} \rightarrow \mathbb{C}$ are non-trivial locally integrable functions which satisfy the functional equation*

$$f(x)f(y) = F(|x| + |y|, x + y)$$

for all $x, y \in \mathbb{R}^3$, then there exist constants $A \in \mathbb{C}$, $b \in \mathbb{C}^3$, $C \in \mathbb{C}$ such that

$$f(x) = \exp(A|x| + b \cdot x + C), \quad F(t, x) = \exp(At + b \cdot x + 2C)$$

for (almost) all $(t, x) \in \mathcal{C}_{++}$.

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