

The isomorphism problem for residually torsion-free nilpotent groups

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Abstract. Both the conjugacy and isomorphism problems for finitely generated nilpotent groups are recursively solvable. In some recent work, the first author, with a tiny modification of work in the second author's thesis, proved that the conjugacy problem for finitely presented, residually torsion-free nilpotent groups is recursively unsolvable. Here we complete the algorithmic picture by proving that the isomorphism problem for such groups is also recursively unsolvable.

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1. Introduction

1.1. The main theorem. The objective of this paper is to prove the following theorem.

Theorem A. *There exists a recursive class of presentations of finitely presented residually torsion-free nilpotent groups with unsolvable isomorphism problem.*

More precisely we will prove the

Theorem A'. *Let Π be the collection of all presentations of extensions of the free group $F = \langle s_1, \dots, s_n \rangle$ by the free group $T = \langle t_1, \dots, t_m \rangle$ with $n \cdot m$ defining relations of the form*

$$t_i^{-1} s_j t_i = s_j c_{ij} \quad \text{where } c_{ij} \in [F, F]$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$. Then the isomorphism problem for the groups presented in Π is recursively unsolvable. Each group presented in Π has solvable word problem and is residually torsion-free nilpotent.

It is worth noting that the class Π is indeed recursive. One has only to single out, as J. Nielsen did (see, e.g., [8]), those presentations in which the images $s_j c_{ij}$ of the given generators generate the free group F . Moreover, the residual torsion-free nilpotence of the groups defined by Π follows from the fact that free groups are residually torsion-free nilpotent [9] once one observes that T acts like the identity on the abelianization of F (for details, see Lemma 2, p. 302 of [1]).

One of the by-products of the approach that we have taken here is the following theorem, which does not seem to immediately follow from the usual Adian–Rabin construction.

Theorem B. *If n is chosen sufficiently large, then there is no algorithm to determine of a finite presentation with n generating symbols whether or not the group presented is free abelian of rank n .*

So it follows that if we consider all finite presentations then there is no algorithm to decide which of the groups so presented are free abelian on the given generators.

1.2. One of the consequences of Theorem A. The isomorphism problem for finitely generated nilpotent groups is solvable [5], [6]. It follows then from Theorem A that there exist infinitely many, finitely presented, residually torsion-free nilpotent groups, no two of which are isomorphic, with the same nilpotent factor groups. The special case of the existence of finitely presented, residually nilpotent groups with the same nilpotent images as a free group, the so-called parafree groups, exist in profusion. They have been the subject of various papers (see, e.g., [2] and the references there cited).

1.3. Some general comments. Finitely generated residually nilpotent groups can be viewed as generalizations of finitely generated nilpotent groups and as such it makes sense to compare them with these finitely generated nilpotent groups. Finitely generated nilpotent groups are finitely presented and have solvable word, conjugacy and isomorphism problem. However there are continuously many finitely generated residually torsion-free nilpotent groups and so most of them are not finitely presented [1]. It follows from the fact that finitely generated nilpotent groups have solvable word problem that finitely presented residually nilpotent groups also do. However, as noted in the abstract, there exist finitely presented residually nilpotent groups with unsolvable conjugacy problem. The remaining open algorithmic problem is now settled by Theorem A.

1.4. A sketch of the proof. The basic idea involved in both the construction of a finitely presented residually nilpotent group with unsolvable conjugacy problem and a recursive family of such finitely presented residually nilpotent groups with

unsolvable isomorphism problem is contained in the thesis of the second author [12]. Miller invented a method whereby a given finite presentation

$$P = \langle s_1, \dots, s_n \mid r_1, \dots, r_m \rangle$$

of a group H is transformed, by what we term here a *word-conjugacy transformation* ω , into a presentation $\omega(P)$ of a group $\omega(H)$. The complexity of the word problem, as well as the conjugacy problem, in the group H defined by P is reflected in the conjugacy problem of the group $G = \omega(H)$. So, in particular, if H has an unsolvable word problem, G has an unsolvable conjugacy problem. This observation plays a key part in the proof of Theorem A.

In fact the proof of Theorem A is carried out in several stages. We start out with a finitely presented group with an unsolvable word problem. We then modify it in order to obtain another finitely presented group H with an unsolvable word problem all of whose defining relators are commutators. Using this group H we adapt the Adian–Rabin construction, as in [13], to produce a family H_w of finitely presented groups parametrized by the words w in the given generators of H . Each of the H_w is generated by the same number of generators and the given generators of H_w freely generate H_w modulo its derived group. The key fact here is that H_w turns out to be free abelian if and only if $w =_H 1$ and hence has a solvable word problem. In the event that $w \neq_H 1$, then H_w has an unsolvable word problem. We now transform this family of groups H_w by the word-conjugacy transformation ω . Then by Corollary 5 on p. 28 of [12], the conjugacy problem for the group $G_w = \omega(H_w)$ is unsolvable whenever $w \neq_H 1$. The proof of Theorem A is completed by proving that if $w =_H 1$, then the groups G_w are all isomorphic and have a solvable conjugacy problem; consequently the class of groups G_w has an unsolvable isomorphism problem. The proof that the G_w have a solvable conjugacy problem when $w =_H 1$ is the most difficult technical part of the proof and uses the same technology as that employed in [12].

1.5. The word-conjugacy transformation ω . The word-conjugacy transformation ω was introduced in the second author’s thesis [12]. The very nature of ω suggests that it might well have some other uses not only in combinatorial group theory but also in the study of associative and lie algebras. As already indicated it plays a key role in the proof of our main theorem.

We recall the details from [12], not only for later use but also to emphasise its importance. To this end, let

$$P = \langle s_1, \dots, s_n \mid r_1(s_1, \dots, s_n), \dots, r_m(s_1, \dots, s_n) \rangle \quad (1)$$

be a given finite presentation. Then we term the presentation $\omega(P)$ with generators

$$s_1, \dots, s_n, d_1, \dots, d_n, t_1, \dots, t_m, q$$

and defining relations

$$\begin{aligned} d_1^{-1}qd_1 &= s_1qs_1^{-1}, \dots, d_n^{-1}qd_n = s_nqs_n^{-1}, \\ d_i^{-1}s_jd_i &= s_j \quad (1 \leq i, j \leq n), \\ t_1^{-1}qt_1 &= qr_1(s_1, \dots, s_n), \dots, t_m^{-1}qt_m = qr_m(s_1, \dots, s_n), \\ t_i^{-1}s_jt_i &= s_j \quad (1 \leq i \leq m, 1 \leq j \leq n) \end{aligned}$$

the word-conjugacy transform of (1). If J is the group presented by P and if K is the group presented by $\omega(P)$, then we set $K = \omega(J)$ and we term it the word-conjugacy transform of J . It is these word-conjugacy transforms that are used in the proof of Theorem A.

2. A modification of the Adian–Rabin construction

2.1. Adjusting the presentation of a group with unsolvable word problem. We start with a finitely presented group

$$A = \langle a_1, \dots, a_n \mid r_1 = 1, \dots, r_m = 1 \rangle$$

having unsolvable word problem. Of course each r_i is a word on the a_j and we sometimes write $r_i(\vec{a})$ or $r_i(a_1, \dots, a_n)$ to make this explicit. Let F_a be the free group on a_1, \dots, a_n and similarly let F_b be the free group on b_1, \dots, b_n . Now the obvious homomorphisms map F_a and F_b onto A . As observed by Mihařlova [11] (see also [13] or [8]), the fibre product of these two homomorphisms is the subgroup S of $F_a \times F_b$ generated by the diagonal elements $a_1b_1, a_2b_2, \dots, a_nb_n$ and the elements $r_a(\vec{a}), \dots, r_m(\vec{a})$. (It follows that the $r_i(\vec{b}) \in S$ as well.) Thus $w(\vec{a}) \in S$ if and only if $w(\vec{a}) =_A 1$. Since the word problem for A is unsolvable, there is no algorithm to determine whether or not $w(\vec{a}) \in S$.

Next form the HNN extension H of $F_a \times F_b$ by adding a stable letter p which commutes with S . Then $p^{-1}w(\vec{a})p = w(\vec{a})$ if and only if $w(\vec{a}) \in S$, so the word problem for H is unsolvable. But we observe that H can be presented as

$$\begin{aligned} H = \langle a_1, \dots, a_n, b_1, \dots, b_n, p \mid [a_i, b_j] &= 1 \quad (1 \leq i, j \leq n), \\ [p, a_ib_i] &= 1 \quad (1 \leq i \leq n), [p, r_j(\vec{a})] = 1 \quad (1 \leq j \leq m) \rangle \end{aligned}$$

where we use the commutator notation $[x, y] = x^{-1}y^{-1}xy$. Thus we have established the following.

Lemma 1. *There exists a finitely presented, torsion-free group H with unsolvable word problem having a presentation in which all the defining relators are commutators.* □

2.2. Modifying the Adian–Rabin construction. In what follows we need only the weaker result that the relators all lie in the derived group of the free group on the given generators. This can be established by other methods. For instance, if we embed the group A with unsolvable word problem in a 2-generator group B using the method of Higman, Neumann and Neumann [7], then the relators of B all lie in the derived group of the free group on the generators given there.

The Adian–Rabin construction [13] yields a recursive family of finite presentations J_w parametrized by words w from a group J with unsolvable word problem. The construction is arranged so that J_w is the trivial group if and only if $w =_J 1$. Forming the free products $K_w * \mathbb{Z}^n$ with a free abelian group, we obtain a family of presentations which present a free abelian group of rank n if and only if $w =_J 1$. But we need a stronger property, namely that when $w =_J 1$ the group should be free abelian with basis all of the generating symbols in the presentation. Here is a version of the construction which yields such a family.

Lemma 2 (Modified Adian–Rabin construction). *Let H be a group given by a finite presentation*

$$H = \langle x_1, \dots, x_n \mid r_1 = 1, \dots, r_m = 1 \rangle$$

where the relators r_i all lie in the derived group of the free group on the x_j . For any word w in the given generators of H , let H_w be the group with presentation obtained from the given one for H by adding three new generators a, b, c together with defining relations

- (1) $a^{-1} [b, c] a = [w, b],$
- (2) $a^{-1} c^{-2} b c^2 a = c^{-2} b c^2,$
- (3) $a^{-1} c^{-3} b^{-1} c b c^3 a = c^{-3} b^{-1} c b c^3,$
- (4) $a^{-1} c^{-2 \cdot 5^i} [b, c] c^{2 \cdot 5^i} a = c^{-2 \cdot 5^i} [x_i, b] c^{2 \cdot 5^i}, \quad i = 1, \dots, n,$
- (5) $a^{-1} c^{-3 \cdot 5^i} [b, c] c^{3 \cdot 5^i} a = c^{-3 \cdot 5^i} b^{-1} [x_i, c] b c^{3 \cdot 5^i}, \quad i = 1, \dots, n,$
- (6) $a^{-1} c^{-5^i} x_i c^{5^i} a = c^{-5^i} x_i c^{5^i}, \quad i = 1, \dots, n$
- (7) $a^{-1} c^{-2^i \cdot 3^j} b c^{2^i \cdot 3^j} a = c^{-2^i \cdot 3^j} [x_i, x_j] b c^{2^i \cdot 3^j}, \quad 1 \leq i < j \leq n.$

Then

- (i) if $w \neq_H 1$, H is embedded in H_w by the inclusion map on generators;
- (ii) the abelianization $H_w / [H_w, H_w]$ is the free abelian group with basis the given generators x_1, \dots, x_n, a, b, c ;
- (iii) the normal closure of w in H_w is the derived group $[H_w, H_w]$; in particular, if $w =_H 1$ then H_w is free abelian on the given generators;

(iv) if H has an unsolvable word problem, then the isomorphism problem for the family of groups H_w is unsolvable.

Proof. In view of the hypothesis on the defining relators of H , its abelianization $H/[H, H]$ is free abelian with basis the images of the x_i ($i = 1, \dots, n$). In particular the x_i have infinite order. Assertion (ii) is routine to check from the given relations since both sides of each equation have the same exponent sum on each generator.

Suppose now that $w \neq_H 1$. Then in the ordinary free product $L = H * \langle c, b \mid \rangle$ of H with the free group on generators b and c , the element $[w, b]$ has infinite order. It is then easy to check that the right hand sides of the equations in (1) through (7) freely generate a free subgroup D of L . Similarly the central parts of the left hand sides of (1) through (7) being conjugated by a (that is, $[b, c]$, $c^{-2}bc^2$, ...) freely generate a free subgroup C of L . Thus if $w \neq_H 1$ we see that H_w has the structure of an HNN extension of L with stable letter a which conjugates C to D as indicated (see Subsection 3.2). So H is embedded in H_w . This proves (i).

To verify assertion (iii) we suppose $w =_H 1$. Then $[w, b] = 1$ in L and by relation (1) it follows that $[b, c] = 1$, so b and c commute. Consequently, relation (2) implies $a^{-1}ba = b$ and relation (3) implies $a^{-1}ca = c$ so that a, b, c all commute with each other. The relations of type (4) and (5) then imply that $[x_i, b] = 1$ and $[x_i, c] = 1$ for $i = 1, \dots, n$. Hence by (6) it follows that $a^{-1}x_i a = x_i$ for $i = 1, \dots, n$. Finally the relations (7) imply that $[x_i, x_j] = 1$ for $1 \leq i, j \leq n$. So all the generators of H_w commute with each other and H_w is free abelian on the given generators.

Finally to prove (iv), we observe that H_w is free abelian of rank $n + 3$ if and only if $w =_H 1$. Since H has an unsolvable word problem, we have proved (iv).

This completes the proof. \square

It follows immediately from Lemma 2, once we choose H to be a finitely presented group with unsolvable word problem in which all the relators lie in the derived group of the ambient free group, that we have proved the following theorem.

Theorem B. *If n is chosen sufficiently large, then there is no algorithm to determine of a finite presentation with n generating symbols whether or not the group presented is free abelian of rank n .*

2.3. A second look at Lemma 2. Lemma 2 will play a crucial role in the sequel. In order to formulate what we will require of Lemma 2 we adopt here the hypothesis and notation employed in its formulation. In addition we will assume that the word problem for H is recursively unsolvable. As already noted, if $w =_H 1$, then H_w is free abelian on the given generators and so has a solvable word problem. Hence there is no algorithm to determine for an arbitrary word w whether or not $H_w \cong H_1$. Finally we note that $w = 1$ (in the free group freely generated by given generators of H) if and only if $[w, b] = 1$ in H_w . Moreover, if $w = 1$, then $[w, b] = 1$ is a

consequence of the given relations $r_1 = 1, \dots, r_m = 1$ without using relations (1) through (7) in Lemma 2. Now observe that the parameter w appears in exactly one relation of H_w which can be rewritten as $[w, b]a^{-1}[c, b]a = 1$ or as $ua^{-1}[c, b]a = 1$ where $u = [w, b]$. Completely revising our notation we can summarize our results as follows:

Lemma 3. *There is a recursive collection of finite presentations of groups H_1, H_2, \dots , parametrised by the words $u_1, u_2, \dots, u_i, \dots$, of the form*

$$H_i = \langle s_1, \dots, s_n \mid R_1(u_i) = 1, R_2 = 1, \dots, R_m = 1 \rangle$$

with the following properties.

- (1) Each R_i lies in the derived group $[F_S, F_S]$ of the free group F_S on the generators s_1, \dots, s_n . Hence the groups $H_i/[H_i, H_i]$ are all free abelian on the listed generators.
- (2) The first relation $R_1(u_i) = 1$ depends on the parameter u_i and has the form $u_i u_0 = 1$ where u_i is a word depending on i and u_0 is a fixed word.
- (3) The relators R_2, \dots, R_m are fixed words, that is, they are the same for each H_i .
- (4) u_1 is the empty word and H_1 is the free abelian group on the given generators.
- (5) H_i is isomorphic to H_1 if and only if $u_i = 1$ in H_i .
- (6) If $u_i = 1$ in H_i , then this is a consequence of the relations $R_2 = 1, \dots, R_m = 1$.
- (7) If H_i is not isomorphic to H_1 , then $u_i \neq 1$ in H_i and H_i has an unsolvable word problem.
- (8) There is no algorithm to determine for an arbitrary $i \geq 1$ whether or not H_i and H_1 are isomorphic. \square

We remark that the properties of the H_i listed in the above statement are analogous to those of a corresponding class of groups for which triviality is undecidable that was used in [12] to prove that the isomorphism problem for finitely presented, residually finite groups is unsolvable.

Although not strictly necessary for what follows, it is useful to re-arrange the above presentations. We begin by transforming each of the relations (1) through (7) of Lemma 2. First we invert both sides of relations (2), (3) and (6). Then we insert

inverse pairs and conjugate to bring them into the following forms.

- (1') $a[b, w]a^{-1} [b, c] = 1,$
- (2') $[b, c^2]a^{-1}[c^2, b]a [a, b] = 1,$
- (3') $[c, bc^3]a^{-1}[bc^3, c]a [a, c] = 1,$
- (4') $c^{2 \cdot 5^i} a^{-1} c^{-2 \cdot 5^i} [c, b] c^{2 \cdot 5^i} a c^{-2 \cdot 5^i} [x_i, b] = 1, \quad i = 1, \dots, n,$
- (5') $bc^{3 \cdot 5^i} a^{-1} c^{-3 \cdot 5^i} [c, b] c^{3 \cdot 5^i} a c^{-3 \cdot 5^i} b^{-1} [x_i, c] = 1, \quad i = 1, \dots, n,$
- (6') $[x_i, c^{5^i}]a^{-1}[c^{5^i}, x_i]a [a, x_i] = 1, \quad i = 1, \dots, n,$
- (7') $bc^{2^i \cdot 3^j} a^{-1} c^{-2^i \cdot 3^j} b^{-1} c^{2^i \cdot 3^j} a c^{-2^i \cdot 3^j} [x_i, x_j] = 1, \quad 1 \leq i < j \leq n.$

Each of these relations (1') through (7') is equivalent to the corresponding relation in Lemma 2 in the sense that each has the same normal closure.

Observe that as the proof of Lemma 2, if $w =_H 1$ then relation (1') implies $[b, c] = 1$. Relations (2') through (5') have the forms $v_2[a, b] = 1$, $v_3[a, c] = 1$, $v_4[x_i, b] = 1$ and $v_5[x_i, c] = 1$ where each $v_j = 1$ is a consequence of $[b, c] = 1$. Thus $[a, b] = [a, c] = 1$ and $[x_i b,] = [x_i, c] = 1$. Then relations (6') and (7') have the form $v_6[a, x_i] = 1$ and $v_7[x_i, x_j] = 1$ where $v_6 = 1$ and $v_7 = 1$ are consequences of relations $[b, c] = 1$ and (2') through (5'). Finally the relations $r_1 = 1, \dots, r_m = 1$ are consequences of $[b, c] = 1$ and (2') through (7').

Lemma 4. *The relations R_1, \dots, R_m of the H_i in Lemma 3 can be taken to have the following additional properties where $p = \frac{n^2-n}{2}$:*

- (1) u_0 is $[s_1, s_2]$ and the relator R_1 of H_i is $u_i[s_1, s_2]$;
- (2) for each $j = 2, \dots, p$, the relation $R_j = 1$ has the form $v_j[s_k, s_\ell] = 1$ where the word v_j is a consequence of $u_0 = 1$ and the earlier relations $R_2 = 1, \dots, R_{j-1} = 1$;
- (3) for each $j = p + 1, \dots, m$, the relator R_j is a consequence of $u_0 = 1$ and the earlier relations $R_2 = 1, \dots, R_p = 1$.

In addition the first relation of the group parametrized by the identity word u_1 is $R_1 = 1$, that is, $u_0 u_1 = u_0 = [s_1, s_2] = 1$. \square

3. Construction of the residually nilpotent groups

3.1. The groups $\omega(H_i)$. We adopt now the notation and conclusions reached in the formulation of Lemma 4. Thus we have a family of groups

$$H_i = \langle s_1, \dots, s_n \mid R_1(u_i) = 1, R_2 = 1, \dots, R_m = 1 \rangle \quad (i = 1, 2, \dots)$$

satisfying the conditions described in Lemma 4. We now apply the word-conjugacy transformation ω introduced in Section 1.5, to each of these groups. We need to describe the resultant groups in a little more detail. To this end, we observe that each $\omega(H_i)$ is a semi-direct product of the free group F on the generators

$$q, s_1, \dots, s_n$$

by the free group T on the generators

$$d_1, \dots, d_n, t_1, \dots, t_m.$$

The action of T on F is described as follows:

$$\begin{aligned} t_i^{-1} q t_i &= q R_i, & 1 \leq i \leq m, \\ t_i^{-1} s_j t_i &= s_j, & 1 \leq j \leq n, \\ d_k^{-1} q d_k &= s_k q s_k^{-1}, & 1 \leq k \leq n, \\ d_k^{-1} s_j d_k &= s_j, & 1 \leq j \leq n. \end{aligned}$$

Notice that only the relation $t_1^{-1} q t_1 = q R_1(u_i)$ depends on the parameter i , or equivalently u_i , and it has the form $t_1^{-1} q t_1 = q u_i u_0$ as noted in Lemma 4.

We emphasize, as we observed earlier, the

Lemma 5. *Each $G_i = \omega(H_i)$ is the split extension of the finitely generated free group F by the finitely generated free group T . Hence each G_i has solvable word problem and is residually finite.* \square

Since each R_i belongs to $[F_S, F_S]$, it follows that the action of each of the generators of T on F induces the trivial map on $F/[F, F]$. So, as noted already, on appealing to Lemma 2, p. 302 of [1] we find that

Lemma 6. *G_i is residually torsion-free nilpotent for $i = 1, 2, \dots$* \square

3.2. HNN extensions and the lemmas of Britton and Collins. We will need to look at the G_i as HNN extensions (see [8] or [12] for discussions of HNN extensions and some relevant terminology). To this end observe that, because of the relations not involving q , the group $F_S \times T$ is a subgroup of G_i and the presentation of G_i can be obtained from one for $F_S \times T$ by adding the new generator q and the defining relations

$$\begin{aligned} q^{-1} t_i q &= t_i R_i^{-1}, & 1 \leq i \leq m, \\ q^{-1} d_k s_k q &= d_k s_k, & 1 \leq k \leq n. \end{aligned}$$

Since the subgroups generated by the left and right sides of these equations are free on the listed generators, it follows that G_i is also an HNN extension of $F_S \times T$ with stable letter q .

We can also view, somewhat trivially, the G_i as HNN extensions with base group F , stable letters $t_1, \dots, t_m, d_1, \dots, d_n$ and associated subgroups all coinciding with the base group F itself.

We remind the reader of some word and conjugacy terminology concerning HNN extensions (again, see [8] or [12]). Let $J = \langle S \mid D \rangle$ be a group and suppose that $\varphi: A \rightarrow B$ is an isomorphism between the subgroups A and B of J . Then the corresponding HNN extension E with stable letter p is presented as $E = \langle S, p \mid D, p^{-1}ap = \varphi(a), a \in A \rangle$. If w is a word of E of the form $w = w_1 p^{-1} a p w_2$ with $a \in A$ then, using the defining relations, the subword $p^{-1} a p$ can be replaced by $b = \varphi(a)$ to obtain $w_1 b w_2 =_G w$. Similarly a subword of the form $p b p^{-1}$ with $b \in B$ can be replaced by $a = \varphi^{-1}(b)$. In both cases the resulting word has 2 fewer occurrences of p or p^{-1} . A subword of either of these forms is called a p -pinch and the process of replacing them using the defining relations as indicated is called *pinching* or p -reduction. A word is p -reduced if it does not contain a p -pinch. Britton's Lemma [3] says that if w is a word of E which contains p or p^{-1} and if $w =_E 1$ then w contains a p -pinch. Recognizing p -pinches is equivalent to determining membership in the subgroups A and B . So Britton's Lemma reduces the word problem for E to the join of the word problem for J and the membership problems for A and B .

A word w is *cyclically p -reduced* if every cyclic permutation of w is p -reduced. Collins' Lemma [4] asserts that if $u, v \in E$ are cyclically p -reduced which are conjugate in E and if u begins with p , then there is a cyclic permutation \bar{v} of v which also begins with p and an element $a \in A$ such that $a^{-1} u a =_E \bar{v}$. Moreover u and \bar{v} are p -parallel meaning they have the same sequence of $p^{\pm 1}$'s appearing in them. Under similar hypotheses, in case u begins with p^{-1} then for some $b \in B$ and cyclic permutation \bar{v} of v which also begins with p^{-1} we have $b^{-1} u b =_E \bar{v}$. Similar versions of Britton's and Collins' lemmas apply when there are several stable letters.

We are now in a position to discuss the groups G_i in more detail.

3.3. Relating conjugacy in $\omega(H_i)$ to equality in H_i . The following result relates conjugacy in G_i to equality in H_i and partly explains the terminology that we have introduced here. Again the arguments given here are modeled on those in [12].

Lemma 7. *If w is any word on $\{s_1, \dots, s_n\}$, then qw is conjugate in G_i to q if and only if $w =_{H_i} 1$. Thus if H_i has unsolvable word problem, then G_i has unsolvable conjugacy problem. Moreover, $w =_{H_i} 1$ if and only if $Y^{-1} q Y =_{G_i} qw$ for some word $Y \in T$.*

Proof. In one direction this claim is easy. For observe that the mapping which sends

each s_i in G_i to the corresponding s_i in H_i and all other generators to the identity, defines a homomorphism ϕ from G_i to H_i . Then if $Y^{-1}qY =_{G_i} qw$ it follows that $\phi(Y)^{-1}\phi(Y) =_{H_i} w$ and hence $w =_{H_i} 1$.

For the converse, suppose $w =_{H_i} 1$. First observe that $t_j^{-1}qt_j = qR_j$ and hence $t_jqt_j^{-1} = qR_j^{-1}$. Also observe that $d_k^{-1}qs_kd_k = s_kq$. So if $x = x(s_i)$ is any word in the s_i and $X(d_i)$ is the corresponding word in the d_i then $X^{-1}qX = xq$. Similarly, $XqX^{-1} = x^{-1}qx$ and $X^{-1}qX = xqx^{-1}$. Combining these observations we have

$$\begin{aligned} (Xt_j^\epsilon X^{-1})^{-1}q(Xt_j^\epsilon X^{-1}) &= Xt_j^{-\epsilon}X^{-1}qXt_j^\epsilon X^{-1} = Xt_j^{-\epsilon}xqx^{-1}t_j^\epsilon X^{-1} \\ &= XxqR_j^\epsilon x^{-1}X^{-1} = qxR_j^\epsilon x^{-1}. \end{aligned}$$

Since $w =_{H_i} 1$,

$$w =_{F_S} x_1(s_i)R_{j_1}^{\epsilon_k}x_1(s_i)^{-1} \dots x_k(s_i)R_{j_k}^{\epsilon_k}x_k(s_i)^{-1}$$

for suitable words x_k in the given generators. So if we set

$$Y = X_k(d_i)t_{j_k}^{\epsilon_k}X_k(s_i)^{-1} \dots X_1(d_i)t_{j_1}^{\epsilon_k}X_1(s_i)^{-1}$$

(note that the order has been reversed) it follows that $Y^{-1}qY =_{G_i} qw$ and of course $Y \in T$ as desired. \square

Corollary 8. *If u_i is the parameter in the first relation of H_i and if $u_i \neq 1$ in H_i , then G_i has an unsolvable conjugacy problem.*

Actually the conjugacy problem for G_i is at least as difficult as the conjugacy problem for H_i as we see from the following lemma.

Lemma 9. *If u, v, x are any words on $\{s_1, \dots, s_n\}$, then qu is conjugate in G_i to qv if and only if u and v are conjugate in H_i . Moreover, $x^{-1}ux =_{H_i} v$ if and only if $Y^{-1}x^{-1}quxY =_{G_i} qv$ for some word $Y \in T$.*

Proof. Since H_i is a quotient of G_i via ϕ , qu conjugate in G_i to qv clearly implies that u and v are conjugate in H_i . For the converse suppose that $x^{-1}ux =_{H_i} v$. In G_i qu is conjugate to $x^{-1}qux$. Since x is a word in the $\{s_1, \dots, s_n\}$ we can conjugate $x^{-1}qux$ by a word in the d_j 's to $qx^{-1}ux$. Now set $w = v(x^{-1}ux)^{-1}$. Then $w =_{H_i} 1$. It follows from Lemma 7 that there is a word $Y \in T$ with $Y^{-1}qY = qw$. So we have $Y^{-1}qx^{-1}uxY = qwx^{-1}ux = qv$. This proves the result. \square

A useful variant of Lemma 7 is the following:

Lemma 10. *If x, y are any words on $\{s_1, \dots, s_n\}$, then $xy =_{G_i} 1$ if and only if there is a word $Y \in T$ with $Y^{-1}qY = xqy$. In this case $Y(t_j, d_i)x = Y(t_j, d_i s_i)$ and $Y(t_j, d_i s_i)^{-1}qY(t_j, d_i s_i) = qyx$.*

Proof. If $Y^{-1}qY = xqy$ with $Y \in T$, then applying the quotient map from G_i to H_i shows $xy =_{H_i} 1$. Conversely, if $xy =_{H_i} 1$ then by Lemma 7 for some $Y_0 \in T$ we have $Y_0^{-1}qY_0 = qxy$. Now $d_i^{-1}qs_i d_i = s_i q$ so if $x = x(s_i)$ and $X = x(d_i)$ is the corresponding word in d_i then $X^{-1}qX = xq$. So letting $Y = Y_0 X$ we have $Y^{-1}qY = X^{-1}Y_0^{-1}qY_0 X = X^{-1}qxyX = xqy$. This equation can be rewritten as $Y^{-1}q^{-1}Yxqy = 1$ and then by Britton's Lemma there is a q -pinch and so $Yx = Y(t_j, d_i s_i)$. This completes the proof. \square

We now proceed as in the proof of Theorem 26, pp. 74–77 of [12].

Lemma 11. *If u_i is the parameter in the first relation of H_i and if $u_i = 1$ in H_i , then the corresponding group G_i is isomorphic to G_1 .*

Proof. Fix $i > 1$ and suppose that $u_i =_{H_i} 1$. Then $u_i = 1$ is a consequence of the relations R_2, \dots, R_m by the properties of H_i . It follows from the proof of Lemma 7 that there is a word Y on the generators $d_1, \dots, d_n, t_2, \dots, t_m$ such that $Y^{-1}qY = qu_i$. Notice that t_1 is not included in the generators needed to express Y . Now $Y^{-1}qY = qu_i$ holds in every one of the groups G_j with $j > 1$ since only the first relation $t_1^{-1}qt_1 = qu_j u_0$ is different. Note that since Y commutes with F_S , we have $YqY^{-1} = qu_i^{-1}$.

We now define a map $\varphi: G_i \rightarrow G_1$ which sends t_1 to $t_1 Y$ and which fixes all of the other generating symbols. Note that φ induces an automorphism of T since Y does not contain t_1 . In G_1 we calculate

$$\begin{aligned} \varphi(t_1^{-1}qt_1) &= (t_1 Y)^{-1}q(t_1 Y) = Y^{-1}t_1^{-1}qt_1 Y = Y^{-1}qu_0 Y \\ &= Y^{-1}qY u_0 = qu_i u_0 = \varphi(qu_i u_0). \end{aligned}$$

Since Y commutes with F_S , all the other relations are preserved and so φ is a homomorphism.

Next we define a map $\psi: G_1 \rightarrow G_i$ by $\psi(t_1) = t_1 Y^{-1}$ and ψ fixes all the other generating symbols. This time we compute in G_i that

$$\begin{aligned} \psi(t_1^{-1}qt_1) &= (t_1 Y^{-1})^{-1}q(t_1 Y^{-1}) = Y t_1^{-1}qt_1 Y^{-1} = Yqu_i u_0 Y^{-1} \\ &= YqY^{-1}u_i u_0 = qu_i^{-1}u_i u_0 = qu_0 = \psi(qu_0). \end{aligned}$$

Again all the other relations are preserved and so ψ defines a homomorphism. Clearly φ and ψ are mutually inverse and so G_i is isomorphic to G_1 . \square

It would seem that we have nearly finished the proof of Theorem A. But it is still conceivable that G_1 is by ‘‘accident’’ isomorphic to one of the G_j with $u_j \neq 1$ in H_j . If that were the case, then G_1 would also have unsolvable conjugacy problem by Corollary 8. So to complete the proof of Theorem A it suffices to show that G_1 has a solvable conjugacy problem.

Of course conjugacy in G_1 is modelled on H_1 which is the free abelian group on the given generators. In fact, by using the technique of the previous lemma and some additional properties of the presentation for H_1 it is possible to prove the following

Lemma 12. G_1 is isomorphic to an analogous group G_0 in which all the $t_i^{-1}qt_i = qR_i$ relations have the very simple form $t_i^{-1}qt_i = q[s_j, s_k]$ or $t_i^{-1}qt_i = q$.

Since we will not use this presentation here, we omit the proof. However we note that the presentation of G_0 has only relations which say either that two symbols commute or, for the relations

$$q^{-1}t_i^{-1}qt_i = [s_\alpha, s_\beta] \quad \text{and} \quad q^{-1}d_k^{-1}qd_k = q^{-1}s_k^{-1}qs_k,$$

that the commutator of two symbols is equal to the commutator of two other symbols. It would be astounding if such a group had an unsolvable conjugacy problem. One could hope to show there are no “accidental” isomorphisms with G_1 by some more direct technique. For instance, in each G_i adding the relation $q = 1$ maps G_i onto $H_i \times T$. Now $H_i \times T$ has solvable word problem if and only if $u_i = 1$ in H_i . So if the normal closure of q were characteristic, one could rule out an “accidental” isomorphism, but it is not clear whether this is true.

4. Some elementary equations in free groups

4.1. The equation $x^kuy^k = v$. In order to solve the conjugacy problem for the group G_1 we need a number of preliminary results. We recall first the theorem of Nielsen and Schreier [8], [10] that subgroups of free groups are free. Moreover Nielsen gave an algorithm for solving the generalized word problem in a free group. That is, if $F_S = \langle s_1, \dots, s_n \mid \rangle$ is a free group, there is an algorithm to decide for words w, h_1, \dots, h_k on the s_i whether or not w belongs to the subgroup generated by h_1, \dots, h_m . An easy graphical algorithm suitable for computer use is described in [15].

If $1 \neq u \in F_S$ and we write $u = z^m$ for $m \in \mathbb{Z}$ maximal, then it follows that z generates the centralizer of u in F_S . We call z a *primitive root* of u and note that z^{-1} is the only other generator of the centralizer of u . Suppose that $1 \neq u, v \in F_S$ are two reduced words. If $uv = vu$, then v belongs to the centralizer of u and so $v = z^k$ for some k . In particular, u and v are powers of a common element and they have the same primitive root (up to inverses).

On the other hand if $uv \neq vu$ then u and v generate a non-abelian free subgroup E and, since it has rank 2, $\{u, v\}$ is a free basis for E . Thus if $w \in F_S$ and $w \in E$ then w is uniquely expressible as a word in u and v .

Lemma 13. *There is an algorithm to determine of any four given words $x, y, u, v \in F_S$ whether or not there exists an integer $k \in \mathbb{Z}$ such that*

$$x^k u y^k = v$$

in F_S , and if so, to find k . If such a k exists, it is unique except when $u = v$ and $y = u^{-1} x^{-1} u$ (and then the equation holds for all k).

Proof. Note that $x^k u y^k = v$ if and only if $x^k u y^k u^{-1} = v u^{-1}$, so if the equation has a solution for some k , then $v u^{-1}$ belongs to the subgroup E_0 generated by x and $u y u^{-1}$. We now apply the algorithm to decide membership in E_0 . If $v u^{-1}$ does not lie in E_0 , k does not exist. So we may assume that $v u^{-1}$ does lie in E_0 and that we have expressed $v u^{-1}$ as a word in the generators x and $u y u^{-1}$.

In case x and $u y u^{-1}$ do not commute, this representation of $v u^{-1}$ as a word in the generators is unique. By inspection we can see whether it has the correct form and if so determine k . In case $x = 1$ or $y = 1$ (but not both) then $v u^{-1}$ is a power of the non-trivial generator of E_0 and again k is unique and can be determined by inspection.

Next suppose x and $u y u^{-1}$ commute and neither is trivial. Let z be a primitive root of x . Then for some i, j we have $x = z^i$ and $u y u^{-1} = z^j$. To solve the desired equation we must have $z^{ik} z^{jk} = z^{k(i+j)} = v u^{-1}$. First suppose $v u^{-1} \neq 1$. Then once again, if such a k exists, it is unique and has absolute value at most the length of $v u^{-1}$. This can easily be checked. Finally suppose $v u^{-1} = 1$, that is, $u = v$. Then $j = -i$, and the original equation is $x^k u (u^{-1} x^{-1} u)^k = u$ which holds for every k . This completes the proof. \square

4.2. Solving a “simultaneous” conjugacy problem. Recall that an element $u \in F_S$ is said to be cyclically reduced if every cyclic permutation of u is freely reduced. Two cyclically reduced words are conjugate if and only if one is a cyclic permutation of the other. Of course this solves the conjugacy problem for F_S . In more detail, if $u, v \in F_S$ are conjugate, then we can write $u = u_0 u_1 u_0^{-1}$ and $v = v_0 v_1 v_0^{-1}$ where the products are freely reduced as written and u_1 and v_1 are cyclically reduced. Also $u_1 = u_2 u_3$ and $v_1 = u_3 u_2$ where the products are freely reduced as written and we choose u_2 to be the shortest initial segment of u_1 with the indicated properties. We define the *initial conjugator* of u to v to be the word $x_0 = u_0 u_2 v_0^{-1}$ and note the calculation

$$x_0^{-1} u x_0 = (v_0 u_2^{-1} u_0^{-1}) (u_0 u_2 u_3 u_0^{-1}) (u_0 u_2 v_0^{-1}) = v_0 u_3 u_2 v_0^{-1} = v.$$

Observe that the length of x_0 is at most the maximum of the lengths of u and v . Now if x is any solution of the equation $x^{-1} u x = v$ and we write $x = x_0 (x_0^{-1} x) = x_0 y$, then $x^{-1} u x = y^{-1} v y = v$. Thus y commutes with v and hence $y = z^k$ for some $k \in \mathbb{Z}$ where z is a primitive root of v . We record these observations as follows:

Lemma 14. *If the two non-trivial elements $u, v \in F_S$ are conjugate, then every solution x of the equation $x^{-1}ux = v$ has the form $x = x_0z^k$ where x_0 is the initial conjugator and z is a primitive root of v* \square

Using these observations we can now solve the “simultaneous” conjugacy problem.

Lemma 15. *There is an algorithm to determine of any two sequences u_1, \dots, u_m and v_1, \dots, v_m of m elements of F_S whether or not there exists an element $x \in F_S$ such that the equations*

$$x^{-1}u_1x = v_1, x^{-1}u_2x = v_2, \dots, x^{-1}u_mx = v_m$$

hold simultaneously. If such an x exists and $v_1 \neq 1$, then either (1) x is unique or (2) the initial conjugator x_0 for the equation $x^{-1}u_1x = v_1$ is a solution for the simultaneous system and the v_i 's are all powers of the primitive root z of v_1 .

Proof. First check to see that each pair u_i, v_i is conjugate. We can assume they are conjugate and are non-trivial. By Lemma 14, every solution of the first equation $x^{-1}u_1x = v_1$ has the form $x = x_0z^k$ where x_0 is the initial conjugator of u_1 to v_1 and z is a primitive root of v_1 . So if we put $\bar{u}_i = x_0^{-1}u_ix_0$ for $i = 2, \dots, m$, then solving the original system is equivalent to finding a $k \in \mathbb{Z}$ such that

$$z^{-k}\bar{u}_2z^k = v_2, \dots, z^{-k}\bar{u}_mz^k = v_m.$$

The elements z, \bar{u}_i and v_i are all fixed and so we can apply Lemma 13 to determine whether there is a k such that the equation $z^{-k}\bar{u}_2z^k = v_2$ has a solution. We can assume that such a k exists. Also by Lemma 13 this k is unique except when $\bar{u}_2 = v_2$ and $z = \bar{u}_2^{-1}z\bar{u}_2$, that is, except when \bar{u}_2 and v_2 are both powers of z (and then the equation holds for all k). In the first instance we just check whether z^k satisfies the remaining equations. Otherwise, we apply Lemma 13 successively to the remaining equations. Then at some stage we will either find a unique k which does or does not yield a solution to the remaining equations or else we find that $\bar{u}_i = v_i$ and that the v_i are all powers of z . In the event that the remaining v_i are all powers of z , then x conjugates the u_i to the v_i for all i . This completes the proof. \square

5. Solution of the conjugacy problem for G_1

5.1. Setting the stage for the proof. Our objective now is to find a solution of the conjugacy problem for the group G_1 . This will complete the proof of our main result, Theorem A. For simplicity of notation we put $G = G_1$ and $H = H_1$. (We could use

the even simpler presentation G_0 for G_1 , but curiously that does not seem to be of much advantage and so we have not utilized it here.)

We recall that the group H is modeled on the free abelian group on the s_i . The arguments given below work under the somewhat more general circumstances that H is abelian. So the proof that G has a solvable conjugacy problem will apply both in the case where H is free abelian and in the case where it is trivial, which was the situation envisaged in the remark on p. 77 of [12]. So for the remainder of this section we will assume without further mention that H is abelian. Consequently if $w \in [F_S, F_S]$ then $w =_H 1$; also u and v are conjugate in H if and only if $u =_H v$. Now we can restate part of Lemma 9 as follows.

Lemma 16. *Let u, v be two words of F_S . Then qu is conjugate in G to qv if and only if $u =_H v$. Moreover, $u =_H v$ if and only if $Y^{-1}quY = qv$ for some word $Y \in T$.*

5.2. Solving the conjugacy problem for G_1 . We now apply these considerations to our group $G = G_1$ modeled on $H = H_1$ which we are viewing as an HNN extension in two different ways as discussed already in Subsection 3.2. We note that the relations involving the t_i and d_j acting on F define T -pinches which preserve the number of q symbols. On the other hand the relations for the stable letter q acting on subgroups of $F_S \times T$ preserve the number of t_i and d_j symbols. We also note that there is an algorithm to recognize q -pinches since the associated subgroup is completely determined by the T portion of the word. Hence for any word w of G we can effectively find a (T, q) -reduced word \bar{w} with $w =_G \bar{w}$, that is, \bar{w} is both q -reduced and reduced with respect to the stable letters that generate T . Similarly we can find a *cyclically* (T, q) -reduced word conjugate to w in G .

We are now in a position to begin the proof of the last part of the proof of Theorem A.

Theorem 17. *If the finitely presented group H is abelian, then G has solvable conjugacy problem.*

Proof. Suppose we are given two words U, V of G and we want to determine whether or not they are conjugate. By the above discussion we can assume they are both cyclically (T, q) -reduced. There are several cases to consider, which we will deal with in turn.

5.3. Case (1): U is (T, q) -free. So here we are assuming that U does not contain any q or T symbols, so it is an element of F_S . Then V is also (T, q) -free by Collins' Lemma. Now any element W of G can be written as AY where $A \in F$ and $Y \in T$. But $WUW^{-1} = AYUY^{-1}A^{-1} = AUA^{-1}$ since the T symbols commute with the s_i 's. So if U and V are conjugate in G , they are already conjugate in F and hence in F_S . Hence we can decide whether U and V are conjugate in this case.

5.4. Case (2): U is q -free and involves at least one T symbol. So here U does not involve q but does involve some t_i or d_j . Then V is also q -free and involves a T symbol. After cyclically permuting V and taking inverses of both if necessary, we can assume U is YA and V is YB where $Y = Y(t_i, d_k) \in T$ and $A, B \in F_S$. It suffices to test for this arrangement whether there is an element $W \in F$ with $W^{-1}UW = V$. We can easily check whether U and V are conjugate in $F_S \times T$, and so can assume they are not. Hence W must involve a q symbol. We can suppose W is freely reduced and of the form $Cq^\epsilon W_1$ where $C \in F_S$ and $\epsilon = \pm 1$. Now if

$$W^{-1}UW = W_1^{-1}q^{-\epsilon}C^{-1}YACq^\epsilon W_1 = V$$

then $q^{-\epsilon}C^{-1}YACq^\epsilon$ is a q -pinch by Britton's Lemma. Hence, if $\epsilon = +1$, then we must have $C^{-1}YAC = Y(t_i, d_k s_k)$ and $q^{-1}C^{-1}YACq = Y(t_i R_i^{-1}, d_k s_k)$ because of the form of the relations for q as a stable letter. Similarly if $\epsilon = -1$, then we must have $C^{-1}YAC = Y(t_i R_i^{-1}, d_k s_k)$ and $qC^{-1}YACq^{-1} = Y(t_i, d_k s_k)$. But similar considerations apply to $V = YB$. So in order for $U = YA$ and $V = YB$ to be conjugate in G , one of them must be conjugate in $F_S \times T$ to $Y(t_i, d_k s_k)$ and the other to $Y(t_i R_i^{-1}, d_k s_k)$, and then the conjugation can be done by q or q^{-1} . Hence we can decide whether U and V are conjugate.

5.5. Case (3): U is T -free but involves q . Then the same is true of V , and by taking inverses and cyclic permutations as necessary we may suppose they have the forms

$$\begin{aligned} U &= qu_1q^{\epsilon_2}u_2 \dots q^{\epsilon_j}u_j, \\ V &= qv_1q^{\epsilon_2}v_2 \dots q^{\epsilon_j}v_j \end{aligned}$$

where $u_i, v_i \in F_S$. Moreover we can arrange that, if circularly U contains two consecutive q symbols with the same sign, then $\epsilon_2 = +1$. Otherwise either there is only one q or the signs alternate.

By Collins' Lemma it suffices to determine whether these words are conjugate by a word of the form $W(t_i, d_k s_k)$. Now for any such word we can write $W(t_i, d_k s_k) = W(t_i, d_k)x$ where $x = W(1, s_k) \in F_S$ and $W(t_i, d_k)^{-1}qW(t_i, d_k) = xqy$. Note that $xy =_H 1$.

Subcase (3.1): $j = 1$ and so $U = qu_1$. Then V is just qv_1 and, by Lemma 16, U and V are conjugate if and only if $u_1 =_H v_1$ which is decidable since H is abelian.

Subcase (3.2): There is more than one q symbol and $\epsilon_2 = +1$. If hypothetically $W(t_i, d_k s_k)$ conjugates U to V then

$$V = W(t_i, d_k s_k)^{-1}UW(t_i, d_k s_k) = qyu_1xqyu_2 \dots (xqy)^{\epsilon_j}u_jx$$

and hence $v_1 =_{F_S} yu_1x$. Then $u_1 =_H v_1$ since $xy =_H 1$. Hence we can assume that $u_1 =_H v_1$ and hence by Lemma 16 find a word $Y \in T$ such that $Y^{-1}qu_1Y = qv_1$. Now we conjugate U by Y to obtain

$$\bar{U} = Y^{-1}UY = qv_1q\bar{u}_2q^{\epsilon_3} \dots q^{\epsilon_j}\bar{u}_j$$

where $\bar{u}_i \in F_S$ are determined by the u_i and the action of Y on q . So we are reduced to determining whether \bar{U} is conjugate to V by a W of the desired form.

If now hypothetically $W(t_i, d_k s_k)$ conjugates \bar{U} to V then

$$V = W(t_i, d_k s_k)^{-1} \bar{U} W(t_i, d_k s_k) = qy v_1 x q y \bar{u}_2 (x q y)^{\epsilon_3} \dots (x q y)^{\epsilon_j} \bar{u}_j x$$

and hence $v_1 =_{F_S} y v_1 x$ or equivalently $y = v_1 x^{-1} v_1^{-1}$. So this equation becomes

$$V = q v_1 q v_1 x^{-1} v_1^{-1} \bar{u}_2 (x q v_1 x^{-1} v_1^{-1})^{\epsilon_3} \dots (x q v_1 x^{-1} v_1^{-1})^{\epsilon_j} \bar{u}_j x.$$

Comparing the right hand side with V gives a series of equations depending on the signs ϵ_i . For instance if $\epsilon_3 = -1$ we would have $v_2 = v_1 x^{-1} v_1^{-1} \bar{u}_2 v_1 x v_1^{-1}$ while if $\epsilon_3 = +1$ we would have $v_2 = v_1 x^{-1} v_1^{-1} \bar{u}_2 x$. But for any sequence of signs ϵ_i these observations give a sequence of equations over F_S of the form $\hat{v}_i = x^{-1} \hat{u}_i x$ for $i = 2, \dots, j$. By Lemma 15 we can determine whether a solution $x \in F_S$ to this system exists and if so find one. If no such solution exists, then U and V are not conjugate in G . On the other hand if x is a solution, we also know $x v_1 x^{-1} v_1^{-1} =_H 1$ since H is abelian and so by Lemma 10 we can find a $W \in T$ such that $W^{-1} q W = x q v_1 x^{-1} v_1^{-1}$ and hence $W(t_i, d_k s_k)$ conjugates U to V . Thus we can decide conjugacy in this subcase.

Subcase (3.3): There are an even number of q 's with alternating signs. If now hypothetically $W(t_i, d_k s_k)$ conjugates \bar{U} to V then

$$V = W(t_i, d_k s_k)^{-1} U W(t_i, d_k s_k) = q y u_1 y^{-1} q^{-1} x^{-1} u_2 x q y \dots q^{-1} x^{-1} u_{2j} x.$$

This gives two systems of conjugacy equations: for odd indices $y u_i y^{-1} = v_i$ and for even indices $x^{-1} u_i x = v_i$. Notice that none of the u_i or v_i can be equal to 1 since U and V are q -reduced.

Applying Lemma 15 we can decide whether solutions $x, y \in F_S$ to these systems exist. Hence we may assume we have found such solutions. If they are both unique, then U and V are conjugate if and only if $xy =_H 1$ which is decidable.

If one, say y , is unique but x is not then the initial conjugator x_0 for the equation $x^{-1} u_2 x = v_2$ solves the x -system and all the v_{2i} are powers of the primitive root z_2 of v_2 . So U and V are conjugate if and only if $x_0 z_2^k y =_H 1$ hold for some k , that is, if and only if $x_0 y$ belongs to the subgroup of H generated by z_2 . Since H is a finitely generated abelian group, there is an algorithm to determine whether such a k exists. Similar considerations apply if x is unique but y is not unique.

We are left with the possibility that neither solution x or y is unique. In this case, the initial conjugators x_0, y_0 solve the respective systems and the v_{2i} are powers of the primitive root z_2 of v_2 and the v_{2i+1} are powers of the primitive root z_1 of v_1 . Then U and V are conjugate if and only if $x_0 z_2^k z_1^\ell y_0 =_H 1$, that is, if and only if $x_0 y_0$ belongs to the subgroup of H generated by z_1 and z_2 . Again this is decidable since H is finitely generated and abelian. This completes this subcase.

5.6. Case (4): U involves both q and T symbols. Then the same is true of V and by taking inverses and cyclic permutations as necessary we may suppose they have the forms

$$\begin{aligned} U &= qu_1q^{\epsilon_2}u_2 \dots q^{\epsilon_j}u_jY, \\ V &= qv_1q^{\epsilon_2}v_2 \dots q^{\epsilon_j}v_jY \end{aligned}$$

where $u_i, v_i \in F_S$ and $Y \in T$ is cyclically reduced. By Collins' Lemma it suffices to determine whether these words are conjugate by a word of the form $W(t_i, d_k s_k)$.

Now if we hypothetically suppose that $W(t_i, d_k s_k)^{-1}UW(t_i, d_k s_k) = V$ then we have $W(t_i, d_k)^{-1}YW(t_i, d_k) = Y$ and so W is a power of the primitive root $Z \in T$ of Y . Thus U and V are conjugate if and only

$$Z(t_i, d_k s_k)^{-k}U(t_i, d_k s_k)^k = V$$

for some $k \in \mathbb{Z}$. Here $Z \in T$ is fixed and $Z^{-1}qZ = xqy$ for fixed words $x, y \in F_S$. Also $Z^{-k}qZ^k = x^kqy^k$. So U and V are conjugate if and only if the equation

$$qv_1q^{\epsilon_2}v_2 \dots q^{\epsilon_j}v_j =_{F_S} qy^k u_1(x^k qy^k)^{\epsilon_2}u_2 \dots (x^k qy^k)^{\epsilon_j}u_j x^k$$

has a solution $k \in \mathbb{Z}$ (x and y are known). Now this gives a system of equations of the form $y^k u_i x^k = v_i$ or $y^k u_i y^{-k} = v_i$ or $x^{-k} u_i y^{-k} = v_i$ or $x^{-k} u_i x^k = v_i$ depending on the signs of the e_i . By Lemma 13, there is an algorithm to determine for each of these whether such a k exists. For each equation the k is unique or it holds for all k . Hence there is an algorithm to determine whether there is a k for which all hold simultaneously, and then U and V will be conjugate. Either it is unique or the equations holds for all k , and then $U = V$. This completes the proof in this case and hence the proof of the theorem. \square

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