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Partial differential equations. — *Nonradial symmetric bound states for a system of coupled Schrödinger equations*, by JUNCHENG WEI and TOBIAS WETH, communicated on 9 March 2007.

ABSTRACT. — We consider bound state solutions of the coupled elliptic system

$$\begin{aligned} &\Delta u - u + u^3 + \beta v^2 u = 0 \quad \text{in } \mathbb{R}^N,\\ &\Delta v - v + v^3 + \beta u^2 v = 0 \quad \text{in } \mathbb{R}^N,\\ &u > 0, \quad v > 0, \quad u, v \in \mathbb{H}^1(\mathbb{R}^N), \end{aligned}$$

where $N = 2, 3$. It is known ([13]) that when $\beta < 0$, there are no ground states, i.e., no least energy solutions. We show that, for certain finite subgroups of $O(N)$ acting on $\mathbb{H}^1(\mathbb{R}^N)$, least energy solutions can be found within the associated subspaces of symmetric functions. For $\beta \le -1$ these solutions are nonradial. From this we deduce, for every $\beta \le -1$, the existence of infinitely many nonradial bound states of the system.

KEY WORDS: Bound states; coupled Schrödinger equations; least energy solutions.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 35B40, 35B45; Secondary 35J40, 92C40.

1. INTRODUCTION

In this paper, we study solitary wave solutions of time-dependent coupled nonlinear Schrödinger equations given by

$$\text{(1.1)} \qquad \begin{cases} -i\dfrac{\partial}{\partial t}\Phi_1 = \Delta\Phi_1 + \mu_1|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1 & \text{for } y \in \mathbb{R}^N,\ t > 0,\\ -i\dfrac{\partial}{\partial t}\Phi_2 = \Delta\Phi_2 + \mu_2|\Phi_2|^2\Phi_2 + \beta|\Phi_1|^2\Phi_2 & \text{for } y \in \mathbb{R}^N,\ t > 0,\\ \Phi_j = \Phi_j(y,t) \in \mathbb{C}, \quad j = 1, 2, \\ \Phi_j(y,t) \to 0 \quad \text{as } |y| \to +\infty,\ t > 0,\ j = 1, 2, \end{cases}$$

where μ_1, μ_2 are positive constants, $n \le 3$, and β is a coupling constant.

The system (1.1) arises in many physical problems, especially in the study of incoherent solitons in nonlinear optics. We refer to [19, 20] for experimental results, and [1, 6, 10–12] for a comprehensive list of references. Physically, the solution Φ_j denotes the j-th component of the beam in Kerr-like photorefractive media. The positive constant μ_j is for self-focusing in the j-th component of the beam. The coupling constant β is the *interaction* between the first and the second component of the beam. The interaction is attractive if $\beta > 0$, and repulsive if $\beta < 0$.

Problem (1.1) also arises in the Hartree–Fock theory for a double condensate, i.e. a binary mixture of Bose–Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$

([8]). Physically, Φ_1 and Φ_2 are the corresponding condensate amplitudes, μ_j and β are the intraspecies and interspecies scattering lengths. The sign of the scattering length β determines whether the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive (when $\beta < 0$, see [24]) or attractive (when $\beta > 0$). The interactions of atoms of the single state $|j\rangle$ are attractive when $\mu_j > 0$.

To obtain solitary wave solutions of system (1.1), we set $\Phi_1(x,t) = e^{i\lambda_1 t}u(x)$, $\Phi_2(x,t) = e^{i\lambda_2 t}v(x)$ and transform (1.1) into a coupled elliptic system given by

$$\begin{cases} \Delta u - \lambda_1 u + \mu_1 u^3 + \beta v^2 u = 0 & \text{in } \mathbb{R}^N, \\ \Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{1.2}$$

An important class of solutions are *bound states*, that is, solutions (u,v) satisfying (1.2) and the following conditions:

$$u, v > 0 \quad \text{in } \mathbb{R}^N, \qquad u(y), v(y) \to 0 \quad \text{as } |y| \to +\infty. \tag{1.3}$$

In [2–4, 18, 22], the existence of bound states is proved when $\beta > 0$ under various additional assumptions. Notice that in this case all solutions of (1.2), (1.3) are radially symmetric up to translation (see [25]). When $\beta < 0$, this is no longer true: a result in [16] says that if

$$N = 2, \qquad \min\left(\sqrt{\frac{\lambda_1}{\lambda_2}}, \sqrt{\frac{\lambda_2}{\lambda_1}}\right) < \sin\frac{\pi}{k} \qquad \text{for some } k \geq 2, \tag{1.4}$$

then, for $\beta < 0$ with $|\beta|$ sufficiently small, there are positive solutions to (1.2) with one component concentrating at the center, and the other component concentrating around a regular k-polygon.

The main purpose of the present paper is to study the existence of *nonradial* solutions in the case where $\beta < 0$ and $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$. Note that in this case (1.4) fails, so that the result of [16] does not apply. Without loss of generality, we may assume that $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$. That is, we consider the following system of elliptic equations:

$$\begin{cases} \Delta u - u + u^3 + \beta v^2 u = 0 & \text{in } \mathbb{R}^N, \\ \Delta v - v + v^3 + \beta u^2 v = 0 & \text{in } \mathbb{R}^N, \\ u, v > 0 \quad \text{in } \mathbb{R}^N, & u(y), v(y) \to 0 \quad \text{as } |y| \to +\infty. \end{cases} \tag{1.5}$$

Solutions of (1.5) are critical points of the energy functional $E : (\mathbb{H}^1(\mathbb{R}^N))^2 \to \mathbb{R}$ defined by

$$E[u,v] = \frac{1}{2}(\|u\|^2 + \|v\|^2) - \frac{1}{4}\int_{\mathbb{R}^N}(u^4 + v^4) - \frac{\beta}{2}\int_{\mathbb{R}^N} u^2 v^2,$$

where $\|u\|^2 := \int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)\,dx$ for $u \in \mathbb{H}^1(\mathbb{R}^N)$. All nontrivial solutions of (1.5) belong to the *Nehari set*

$$\mathbf{N} = \Big\{(u,v) \in (\mathbb{H}^1(\mathbb{R}^N))^2 : u, v \geq 0,\ u, v \not\equiv 0,$$
$$\|u\|^2 = \int_{\mathbb{R}^N}(u^4 + \beta u^2 v^2),\ \|v\|^2 = \int_{\mathbb{R}^N}(v^4 + \beta u^2 v^2)\Big\}.$$

A solution $(\bar{u}, \bar{v})$ of (1.5) is called a *ground state* if $E(\bar{u}, \bar{v}) = c_0$, where

$$c_0 = \inf_{(u,v)\in \mathbf{N}} E[u, v]. \tag{1.6}$$

In particular, $E(\bar{u}, \bar{v}) \le E(u, v)$ for any nontrivial solution (u, v) of (1.5). Concerning the existence of ground states, it was proved in [14] that c_0 is attained for $\beta > 0$ small, whereas c_0 is not attained for any $\beta < 0$. To explain this phenomenon, it is worth pointing out that E fails to satisfy the Palais–Smale condition since the embedding $\mathbb{H}^1(\mathbb{R}^N) \hookrightarrow L^4(\mathbb{R}^N)$ is not compact. Moreover, for $\beta < 0$ the interaction of the two species is repulsive. Therefore a spatial separation of u and v in $\mathbb{R}^N$ is observed for $(u, v) \in \mathbf{N}$ with energy close to c_0. In fact, the repulsion of u and v seems to be closely related to the repulsion of positive and negative bumps in the study of sign changing solutions of the single equation $-\Delta u + u = u^3$ in $\mathbb{R}^N$ (see e.g. [26]).

For $\beta > -1$, (1.5) admits the scalar solutions

$$(u, v) = \frac{1}{\sqrt{1+\beta}}(w_0, w_0) \tag{1.7}$$

(and their translations), where $w_0 \in \mathbb{H}^1(\mathbb{R}^N)$ is the unique solution of the scalar elliptic problem

$$\begin{cases} -\Delta w + w = w^3, \quad w > 0 \quad \text{in } \mathbb{R}^N, \\ w(0) = \max\limits_{y\in\mathbb{R}^N} w(y), \quad w \in \mathbb{H}^1(\mathbb{R}^N), \end{cases} \tag{1.8}$$

(cf. [7, 9]). As remarked above, these solutions are not ground states for $-1 < \beta < 0$. For $\beta \le -1$, (1.5) does not admit any solutions with $u = v$. Indeed, for $\beta \le -1$, it is evident that

$$u \ne v \quad \text{for every } (u, v) \in \mathbf{N}. \tag{1.9}$$

In the present paper we prove, for any $\beta < 0$, the existence of ground states within spaces of functions invariant under the action of a finite subgroup $\mathcal{G}$ of $O(N)$. In these spaces, E still fails to satisfy the Palais–Smale condition, but we recover compactness of energy minimizing sequences by balancing the self-attraction of the single species with the repulsion of different species and by applying concentration-compactness arguments.

To state our main results, we recall some notation for a (nontrivial) finite subgroup $\mathcal{G} \le O(N)$. We set $\mathcal{G}x = \{Ax : A \in \mathcal{G}\} \subset \mathbb{R}^N$ and $\mathcal{G}^x := \{A \in \mathcal{G} : Ax = x\} \subset \mathcal{G}$ for $x \in \mathbb{R}^N$, and we denote by $|\mathcal{G}x|$ resp. $|\mathcal{G}^x|$ the number of elements in $\mathcal{G}x$, $\mathcal{G}^x$, respectively. Moreover, we set $\mathrm{Fix}(\mathcal{G}) = \{x \in \mathbb{R}^N : \mathcal{G}x = \{x\}\}$, which is a subspace of $\mathbb{R}^N$, and we write $V_{\mathcal{G}} = \mathrm{Fix}(\mathcal{G})^\perp$ for the orthogonal complement of $\mathrm{Fix}(\mathcal{G})$ in $\mathbb{R}^N$. Finally, we set $l(\mathcal{G}) = \min\{|\mathcal{G}y| : y \in V_{\mathcal{G}} \setminus \{0\}\}$. If $u \in \mathbb{H}^1(\mathbb{R}^N)$, we say that u is *$\mathcal{G}$-symmetric* if $u(Ax) = u(x)$ for every $A \in \mathcal{G}$, and we put

$$H_{\mathcal{G}} = \{u \in \mathbb{H}^1(\mathbb{R}^N) : u \text{ is } \mathcal{G}\text{-symmetric}\}.$$

We introduce the following definition.

DEFINITION 1.1. *Let $B \in O(N)$, and let $\mathcal{G} \le O(N)$ be a finite subgroup of $O(N)$. We call the pair $(B, \mathcal{G})$* admissible *if*

(a) *B is contained in the normalizer of $\mathcal{G}$, and $B^2 \in \mathcal{G}$.*
(b) *$Bx = x$ for every $x \in$* Fix$(\mathcal{G})$.
(c) *There exists $x_0 \in V_{\mathcal{G}} \setminus \{0\}$ with*

 (c1) $|\mathcal{G}x_0| = l(\mathcal{G})$,
 (c2) $\min_{A\in\mathcal{G}\setminus\mathcal{G}^{x_0}} |x_0 - Ax_0| < 2\min_{A\in\mathcal{G}} |x_0 - BAx_0|$.

Condition (c2) in particular implies that $B \notin \mathcal{G}$. Condition (a) ensures that $\mathcal{G}_B = \mathcal{G} \cup B\mathcal{G}$ is a subgroup of $O(N)$, and that an action $*$ of $\mathcal{G}_B$ on $(\mathbb{H}^1(\mathbb{R}^N))^2$ is well defined by $A * (u, v) := (u \circ A^{-1}, v \circ A^{-1})$ for $A \in \mathcal{G}$ and $B * (u, v) := (v \circ B^{-1}, u \circ B^{-1})$. The $*$-invariant elements of $(\mathbb{H}^1(\mathbb{R}^N))^2$ are precisely of the form $(u, u \circ B)$ with $u \in H_{\mathcal{G}}$. We define

$$\mathbf{N}(B, \mathcal{G}) := \{u \in H_{\mathcal{G}} : (u, u \circ B) \in \mathbf{N}\}$$

and

$$c(B, \mathcal{G}) = \inf_{u\in\mathbf{N}(B,\mathcal{G})} E(u, u \circ B). \tag{1.10}$$

Now we state our main result.

THEOREM 1.2. *Let $N = 2$ or $N = 3$, let $(B, \mathcal{G})$ be an admissible pair, and let $\beta < 0$. Then:*

(a) *$\mathbf{N}(B, \mathcal{G})$ is nonempty and $c(B, \mathcal{G})$ is attained. Moreover, every minimizer $u \in \mathbf{N}(B, \mathcal{G})$ for* (1.10) *gives rise to a $\mathcal{G}$-symmetric solution $(u, u\circ B)$ of* (1.5) *with $u > 0$ everywhere on $\mathbb{R}^N$.*
(b) *If $|\beta| < 1$ is small, then $c(B, \mathcal{G}) = \frac{1}{2(1+\beta)}\|w_0\|^2$, and this value is attained only at the solutions* (1.7) *and their translations.*

From part (a) and (1.9), we directly deduce the following.

COROLLARY 1.3. *Under the assumptions of Theorem* 1.2, *for $\beta \le -1$, there exists a $\mathcal{G}$-symmetric solution $(u, u \circ B)$ of* (1.5) *with $u \ne u \circ B$. Hence u is not $\mathcal{G}_B$-symmetric and therefore nonradial.*

We briefly comment on Definition 1.1. Part (a) of this definition is clearly related to the action $*$ defined above. Part (b) ensures that $\mathbf{N}(B, \mathcal{G})$ and the reduced energy $u \mapsto E(u, u\circ B)$ are invariant under translations of the form $u \mapsto u(\cdot - y)$ for $y \in$ Fix(G). Part (c) will be crucial for estimating the value of $c(B, \mathcal{G})$ and thus for finding a relatively compact energy minimizing sequence in $\mathbf{N}(B, \mathcal{G})$. A classification of admissible pairs $(B, \mathcal{G})$ in arbitrary dimension seems out of reach. In dimensions $N \le 3$ the finite subgroups of $O(N)$ and their properties are well known (see e.g. [5]), and therefore we can determine all admissible pairs. In combination with Corollary 1.3, the following list highlights the rich structure of the solution set of (1.5) for $\beta \le -1$.

EXAMPLE 1.4. (i) *Polygonal symmetry in* $\mathbb{R}^2$: Let $N = 2$, fix $k \in \mathbb{N}$, $k \geq 2$ and let $B_k \in O(2)$ denote the (counter-clockwise) rotation by $\theta_k = \pi/k$, i.e.,

$$B_k(x) = (x_1 \cos\theta_k - x_2 \sin\theta_k, x_1 \sin\theta_k + x_2 \cos\theta_k) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2.$$

We set $\mathcal{G}_k = \{\mathrm{Id}, B_k^2, B_k^4, \dots, B_k^{2k-2}\}$. Then the admissibility condition (a) is clearly satisfied for the pair $(B_k, \mathcal{G}_k)$. Note also that $\mathrm{Fix}(\mathcal{G}_k) = \{0\}$. Moreover, for every $x \in \mathbb{R}^2 \setminus \{0\}$ we have $|\mathcal{G}_k x| = l(\mathcal{G}_k) = k$ and

$$\min_{\substack{A \in \mathcal{G}_k \\ A \neq \mathrm{Id}}} |x - Ax| = |x - B_k^2 x| = 2 \sin\theta_k < 4 \sin\left(\frac{\theta_k}{2}\right) = 2|x - B_k x| = 2 \min_{A \in \mathcal{G}_k} |x - B_k A x|.$$

Hence the pair $(B_k, \mathcal{G}_k)$ is admissible.

(ii) *Polygonal symmetry in* $\mathbb{R}^3$: Let $N = 3$, fix $k \in \mathbb{N}$, $k \geq 2$ and let $B_k \in O(3)$ denote the rotation of (x_1, x_2) by $\theta_k = \pi/k$, i.e.,

$$B_k(x) = (x_1 \cos\theta_k - x_2 \sin\theta_k, x_1 \sin\theta_k + x_2 \cos\theta_k, 0) \quad \text{for } x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

With this choice of B_k we may define $\mathcal{G}_k$ as in (i), and again the admissibility condition (a) is satisfied for the pair $(B_k, \mathcal{G}_k)$. In contrast to (i) we now have a nontrivial space of fixed points $\mathrm{Fix}(\mathcal{G}_k) = \{(0, 0, \xi) : \xi \in \mathbb{R}\}$. Nevertheless, for every $x \in V_{\mathcal{G}_k} \setminus \{0\}$ we still have $|\mathcal{G}_k x| = l(\mathcal{G}_k) = k$ and

$$\min_{\substack{A \in \mathcal{G}_k \\ A \neq \mathrm{Id}}} |x - Ax| = |x - B_k^2 x| = 2 \sin\theta_k < 4 \sin\left(\frac{\theta_k}{2}\right) = 2|x - B_k x| = 2 \min_{A \in \mathcal{G}_k} |x - B_k A x|.$$

Hence the pair $(B_k, \mathcal{G}_k)$ is admissible.

(iii) *Tetrahedral symmetry in* $\mathbb{R}^3$: Let $N = 3$, and consider the group $\mathcal{G} \leq O(3)$ generated by the coordinate permutations $(x_1, x_2, x_3) \mapsto (x_{\pi_1}, x_{\pi_2}, x_{\pi_3})$ and $F \in O(3)$ defined by $F(x) = (x_1, -x_2, -x_3)$. Then $|\mathcal{G}| = 24$. Let $B \in O(3)$ be defined by $B(x) = -x$. Then $B^2 = \mathrm{Id} \in \mathcal{G}$, and since B commutes with permutations and with F, the admissibility condition (a) is satisfied for the pair $(B, \mathcal{G})$. We also note that $\mathrm{Fix}(\mathcal{G}) = \{0\}$. For $x_0 = (1, 1, 1)$ we have

$$\mathcal{G}x_0 = \{(1, 1, 1),\ (-1, -1, 1),\ (1, -1, -1),\ (-1, 1, -1)\},$$

so that $|\mathcal{G}x_0| = 4 = l(\mathcal{G})$. Moreover, since

$$B\mathcal{G}x_0 = \{(-1, -1, -1),\ (1, 1, -1),\ (-1, 1, 1),\ (1, -1, 1)\},$$

we have

$$\min_{A \in \mathcal{G} \setminus \mathcal{G}^{x_0}} |x_0 - Ax_0| = 2 < 2\sqrt{2} = 2 \min_{A \in \mathcal{G}} |x_0 - BAx_0|.$$

Hence the pair $(B, \mathcal{G})$ is admissible. Note that the group $\mathcal{G}$ leaves the tetrahedron with vertices $(1, 1, 1)$, $(-1, -1, 1)$, $(1, -1, -1)$ and $(-1, 1, -1)$ fixed.

By choosing $k_j = 2^j$, $j \in \mathbb{N}$ in Examples 1.4 (i) and (ii) above, Corollary 1.3 implies the existence of bound states (u_j, v_j) of (1.5) which are G_{k_j}-symmetric but not $G_{k_{j+1}}$-symmetric. In particular, (u_j, v_j), $j \in \mathbb{N}$, are pairwise different nonradial solutions. Thus we deduce our last main result.

COROLLARY 1.5. *For $N = 2, 3$ and $\beta \le -1$, the system* (1.5) *admits infinitely many nonradial bound states.*

The paper is organized as follows. In Section 2 we recall known facts and collect preliminary results. In Section 3 we prove Theorem 1.2(a), and Section 4 contains the proof of Theorem 1.2(b).

2. PRELIMINARIES

Throughout the remainder of this paper, we assume that $\beta \le 0$. We fix some notation. As usual, we endow the Hilbert space $\mathbb{H}^1(\mathbb{R}^N)$ with the scalar product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + uv)\, dx, \qquad u, v \in \mathbb{H}^1(\mathbb{R}^N),$$

and we set $\|u\|^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\, dx$ as before. Moreover, for $1 \le p \le \infty$ and $u \in L^p(\mathbb{R}^N)$ we denote by $|u|_p$ the usual L^p-norm of u. It is well known (see [7]) that the (unique) solution w_0 of the scalar problem (1.8) is a radial and radially decreasing function which minimizes the Sobolev quotient of the embedding $\mathbb{H}^1(\mathbb{R}^N) \hookrightarrow L^4(\mathbb{R}^N)$, i.e.,

$$\|w_0\| = \frac{\|w_0\|^2}{|w_0|_4^2} = \min_{u \in \mathbb{H}^1(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2}{|u|_4^2}. \tag{2.11}$$

We recall the following asymptotic estimates for w_0 (see e.g. [9, 17]):

$$\left\{ \begin{aligned} w_0(y) &= a_N |y|^{-(N-1)/2} e^{-|y|}(1 + o(1)) \\ \frac{\partial w_0}{\partial r}(y) &= -a_N |y|^{-(N-1)/2} e^{-|y|}(1 + o(1)) \end{aligned} \right\} \quad \text{as } |y| \to \infty. \tag{2.12}$$

Here $a_N > 0$ is a constant depending only on the dimension N. Similarly to [14, Lemma 2.6] we deduce some integral estimates.

LEMMA 2.1. *As $y \to \infty$,*

$$\frac{1}{w_0(y)} \int_{\mathbb{R}^N} w_0^3(x) w_0(x - y)\, dx \to b_N > 0, \tag{2.13}$$

where $b_N = a_N \int_{\mathbb{R}^N} w_0^3\, dx$. Moreover, for $0 < \delta < 2$,

$$\frac{1}{w_0(\delta y)} \int_{\mathbb{R}^N} w_0^2(x) w_0^2(x - y)\, dx \to 0 \quad \textit{as } y \to \infty. \tag{2.14}$$

PROOF. By (2.12),

$$\frac{w_0(x - y)}{w_0(y)} \to a_N \quad \text{as } |y| \to \infty \qquad \text{for every } x \in \mathbb{R}^N. \tag{2.15}$$

Moreover, there is $c > 1$ such that

$$c^{-1} \min\{1, |y|^{-(N-1)/2}\} e^{-|y|} \le w_0(y) \le c \min\{1, |y|^{-(N-1)/2}\} e^{-|y|} \tag{2.16}$$

for every $y \in \mathbb{R}^N$. Let $|y| \geq 1$, and put $\bar{c} = c^5 2^{(N-1)/2}$. If $|x| \geq |y|/2$, then

$$w_0^3(x)\frac{w_0(x-y)}{w_0(y)} \leq c^5\left(\frac{|y|}{|x|}\right)^{(N-1)/2} e^{-3|x|-|x-y|+|y|} \leq \bar{c}e^{-3|x|-|x-y|+|y|} \leq \bar{c}e^{-2|x|},$$

and for $|x| \leq |y|/2$ we also have

$$w_0^3(x)\frac{w_0(x-y)}{w_0(y)} \leq c^5\left(\frac{|y|}{|x-y|}\right)^{(N-1)/2} e^{-3|x|-|x-y|+|y|} \leq \bar{c}e^{-2|x|}.$$

Consequently,

$$w_0^3(x)\frac{w_0(x-y)}{w_0(y)} \leq \bar{c}e^{-2|x|} \qquad \text{for } |y| \geq 1 \text{ and every } x.$$

Hence, by (2.15) and Lebesgue's theorem,

$$\lim_{|y|\to\infty} \frac{1}{w_0(y)} \int_{\mathbb{R}^N} w_0^3(x) w_0(x-y)\, dx = a_N \int_{\mathbb{R}^N} w_0^3(x)\, dx = b_N.$$

Next we consider (2.14), and we may assume that $\delta \geq 1$. Using (2.16) we estimate, for $|y| \geq 1$,

$$\begin{aligned}\frac{w_0^2(x)w_0^2(x-y)}{w_0(\delta y)} &\leq c^5(\delta|y|)^{(N-1)/2} e^{-2|x|-2|x-y|+\delta|y|}\\ &\leq c^5(\delta|y|)^{(N-1)/2} e^{-2|x|-(2+\delta)|x-y|/2+\delta|y|}\\ &\leq c^5(\delta|y|)^{(N-1)/2} e^{-(2-\delta)(|x|+|y|)/2} = f_\delta(y)e^{-(2-\delta)|x|/2}.\end{aligned}$$

where $f_\delta(y) := c^5(\delta|y|)^{(N-1)/2}e^{-(2-\delta)|y|/2} \to 0$ as $|y| \to \infty$. Hence

$$\frac{1}{w_0(\delta y)} \int_{\mathbb{R}^N} w_0^2(x) w_0^2(x-y)\, dx \leq f_\delta(y) \int_{\mathbb{R}^N} e^{-(2-\delta)|x|/2}\, dx \to 0$$

as $|y| \to \infty$, as claimed. □

Next, we fix an admissible pair $(B, \mathcal{G})$ in the sense of Definition 1.1. We consider the reduced energy functional

$$E_{\mathcal{G}} \in C^2(H_{\mathcal{G}}, \mathbb{R}), \qquad E_{\mathcal{G}}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}|u|_4^4 - \beta Q(u),$$

where the C^2-functional $Q : \mathbb{H}^1(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$Q(u) = \frac{1}{4}\int_{\mathbb{R}^N} u^2(x)u^2(Bx)\, dx = \frac{1}{4}|u \cdot (u \circ B)|_2^2.$$

LEMMA 2.2. *For $u \in H_{\mathcal{G}},\ v \in H^1(\mathbb{R}^N)$ we have*

$$\langle \nabla Q(u), v\rangle = \int_{\mathbb{R}^N} u^2(Bx)u(x)v(x)\, dx.$$

PROOF. For $u, v \in H^1(\mathbb{R}^N)$ we find

$$\begin{aligned}\langle \nabla Q(u), v\rangle &= \frac{1}{2}\int_{\mathbb{R}^N} (u^2(x)u(Bx)v(Bx) + u^2(Bx)u(x)v(x))\,dx\\ &= \frac{1}{2}\int_{\mathbb{R}^N} (u^2(B^{-1}x) + u^2(Bx))u(x)v(x)\,dx.\end{aligned}$$

For $u \in H_{\mathcal{G}}$ we have $u \circ B = u \circ B^{-1}$, since $B^2 \in \mathcal{G}$ and $u \circ A = u$ for every $A \in \mathcal{G}$. We thus conclude that

$$\langle \nabla Q(u), v\rangle = \int_{\mathbb{R}^N} u^2(Bx)u(x)v(x)\,dx \quad \text{for } u \in H_{\mathcal{G}},\ v \in \mathbb{H}^1(\mathbb{R}^N). \tag{2.17}$$ □

COROLLARY 2.3. *If $u \in H_{\mathcal{G}}$ is a nontrivial and nonnegative critical point of $E_{\mathcal{G}}$, then $(u, u \circ B)$ is a solution of* (1.5).

PROOF. For $v \in \mathbb{H}^1(\mathbb{R}^N)$ we have, by Lemma 2.2,

$$\begin{aligned}0 = \langle \nabla E_{\mathcal{G}}(u), v\rangle &= \langle u, v\rangle - \int_{\mathbb{R}^N} u^3 v\,dx - \beta\langle \nabla Q(u), v\rangle\\ &= \langle u, v\rangle - \int_{\mathbb{R}^N} u^3 v\,dx - \beta\int_{\mathbb{R}^N} (u\circ B)^2 uv\,dx.\end{aligned}$$

Thus u is a weak solution of the equation $-\Delta u + u - u^3 = \beta(u\circ B)^2 u$. By standard elliptic regularity, u is in fact a classical solution. Moreover, since $u \geq 0$ and $u \not\equiv 0$, it follows from the strong maximum principle that $u > 0$ in $\mathbb{R}^N$. Now $u \circ B$ solves

$$-\Delta(u\circ B) + (u\circ B) - (u\circ B)^3 = \beta(u\circ B^2)^2(u\circ B) = \beta u^2(u\circ B),$$

since $B^2 \in \mathcal{G}$. Hence $(u, u\circ B)$ is a classical solution of (1.5). □

Next we put

$$\mathbf{N}_{\mathcal{G}} = \{u \in H_{\mathcal{G}} : u \neq 0,\ E'_{\mathcal{G}}(u)u = 0\} = \{u \in H_{\mathcal{G}} : u \neq 0,\ \|u\|^2 = |u|_4^4 + \beta|u\cdot(u\circ B)|_2^2\},$$

where the second equality follows from Lemma 2.2. We note that $\mathbf{N}(B, \mathcal{G}) = \{u \in \mathbf{N}_{\mathcal{G}} : u \geq 0\}$. We need the following lemma.

LEMMA 2.4. (i) *$|u|_4^2 \geq \|u\| \geq \kappa$ for some constant $\kappa > 0$ (independent of $\beta \leq 0$) and every $u \in \mathbf{N}_{\mathcal{G}}$.*

(ii) *$\mathbf{N}_{\mathcal{G}} \subset H_{\mathcal{G}}$ is a closed C^1-manifold.*

(iii) *$E_{\mathcal{G}}(u) = \|u\|^2/4$ for $u \in \mathbf{N}_{\mathcal{G}}$.*

(iv) *If $u \in H_{\mathcal{G}} \setminus \{0\}$ satisfies $|u|_4^4 > |\beta|\,|u\cdot(u\circ B)|_2^2$, then $\sqrt{t(u)}\,u \in \mathbf{N}_{\mathcal{G}}$ for*

$$t(u) = \frac{\|u\|^2}{|u|_4^4 + \beta|u\cdot(u\circ B)|_2^2} > 0.$$

PROOF. (i) By definition and Sobolev embeddings, we have $\|u\|^2 \le |u|_4^4 \le \kappa_0\|u\|^4$ for some $\kappa_0 > 0$ and $u \in \mathbf{N}_{\mathcal{G}}$, so that $|u|_4^2 \ge \|u\| \ge \kappa$ for $\kappa = \sqrt{\kappa_0^{-1}}$.

(ii) By (i), $\mathbf{N}_{\mathcal{G}}$ is closed in $H_{\mathcal{G}}$. Moreover, $\mathbf{N}_{\mathcal{G}}$ is the zero set of the functional

$$F \in C^1(H_{\mathcal{G}}, \mathbb{R}), \qquad F(u) = \|u\|^2 - |u|_4^4 - 4\beta Q(u). \tag{2.18}$$

Since for $u \in \mathbf{N}_{\mathcal{G}}$ we have

$$F'(u)u = 2\|u\|^2 - 4(|u|_4^4 + \beta|u \cdot (u \circ B)|_2^2) = -2\|u\|^2 \neq 0, \tag{2.19}$$

$\mathbf{N}_{\mathcal{G}}$ is a C^1-submanifold of $H_{\mathcal{G}}$.

(iii) For $u \in \mathbf{N}_{\mathcal{G}}$ we have

$$E_{\mathcal{G}}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}(|u|_4^4 + \beta|u \cdot (u \circ B)|_2^2) = \frac{1}{4}\|u\|^2.$$

(iv) This also follows by direct computation. □

3. EXISTENCE OF MINIMIZERS

In this section we prove part (a) of Theorem 1.2, which is an immediate consequence of the following proposition. Here we set

$$\tilde{c} := \inf_{u \in \mathbf{N}_{\mathcal{G}}} E_{\mathcal{G}}(u). \tag{3.20}$$

PROPOSITION 3.1. (i) *The value* $\tilde{c}$ *is attained.*

(ii) $\tilde{c} = c(B, \mathcal{G})$, *and if* $u \in \mathbf{N}_{\mathcal{G}}$ *is a minimizer for* (3.20), *then either* $(u, u \circ B)$ *or* $(-u, -u \circ B)$ *is a solution of* (1.5). *In particular, either* $u \in \mathbf{N}(B, \mathcal{G})$ *or* $-u \in \mathbf{N}(B, \mathcal{G})$.

The remainder of this section is devoted to the proof of Proposition 3.1. The proof consists of two steps; first we obtain an estimate for the value of $\tilde{c}$ in terms of $\|w_0\|$, and then we analyze minimizing sequences for (3.20) via concentration-compactness arguments. The strict inequality in the following estimate is crucial.

PROPOSITION 3.2. *We have* $\tilde{c} < \frac{k}{4}\|w_0\|^2$, *where* $k = l(\mathcal{G}) = |\mathcal{G}x_0|$ *and* x_0 *is given by Definition* 1.1.

PROOF. Let $A_1 = \mathrm{Id} \in O(N)$, and let $A_2, \dots, A_k \subset \mathcal{G} \setminus \mathcal{G}^{x_0}$ be such that $\mathcal{G}x_0 = \{A_1x_0, \dots, A_kx_0\}$. We put

$$\mu = \min_{j \neq 1} |x_0 - A_jx_0| = \min_{i \neq j} |A_ix_0 - A_jx_0| > 0$$

and

$$\nu = \min_{j} |x_0 - BA_jx_0| = \min_{i,j} |A_ix_0 - A_jBx_0|,$$

so that $\mu < 2\nu$ by Definition 1.1(c2). For $r > 0$ and $j = 1, \dots, k$ we set $w_r^j = w_0(\cdot - rA_jx_0)$, and we consider $U_r = \sum_{j=1}^k w_r^j \in H_{\mathcal{G}}$. As $r \to \infty$, (2.13) implies that

$$d_r := \sum_{i\neq j} \int_{\mathbb{R}^N} (w_r^i)^3 w_r^j \, dx = (b_N + o(1)) \sum_{i\neq j} w_0(r[A_i x_0 - A_j x_0]),$$

hence

$$(b_N + o(1))(\mu r)^{-(N-1)/2} e^{-\mu r} \le d_r \le \frac{k(k-1)}{2}(b_N + o(1))(\mu r)^{-(N-1)/2} e^{-\mu r}.$$

Moreover, (2.14) yields for $1 \le i, j \le k$ and $\delta = \mu/\nu < 2$ the estimate

$$\begin{aligned}\int_{\mathbb{R}^N} (w_r^i(x))^2 (w_r^j(Bx))^2 \, dx &= o(w_0(\delta r[A_i x_0 - A_j B x_0])) \\ &= o((\delta \nu r)^{-(N-1)/2} e^{-\delta \nu r}) = o(d_r)\end{aligned} \tag{3.21}$$

as $r \to \infty$. We also have

$$\begin{aligned}\|U_r\|^2 &= k\|w_0\|^2 + \sum_{i\neq j} \int_{\mathbb{R}^N} (\nabla w_r^i \nabla w_r^j + w_r^i w_r^j) \, dx \\ &= k\|w_0\|^2 + \sum_{i\neq j} \int_{\mathbb{R}^N} (w_r^i)^3 w_r^j \, dx = k\|w_0\|^2 + d_r,\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}|U_r|_4^4 = \int_{\mathbb{R}^N} \Big(\sum_{j=1}^k w_r^j\Big)^4 dx &\ge \sum_{j=1}^k \int_{\mathbb{R}^N} (w_r^j)^4 \, dx + 4 \sum_{i\neq j} \int_{\mathbb{R}^N} (w_r^i)^3 w_r^j \, dx \\ &= k|w_0|_4^4 + 4d_r = k\|w_0\|^2 + 4d_r.\end{aligned} \tag{3.23}$$

Furthermore we estimate

$$\begin{aligned}\int_{\mathbb{R}^N} U_r^2(x) U_r^2(Bx) \, dx &= \int_{\mathbb{R}^N} \Big(\sum_{i,j} w_r^i(x) w_r^j(x)\Big)\Big(\sum_{i,j} w_r^i(Bx) w_r^j(Bx)\Big) dx \\ &\le \frac{1}{4} \int_{\mathbb{R}^N} \Big(\sum_{i,j} [(w_r^i)^2(x) + (w_r^j)^2(x)]\Big)\Big(\sum_{i,j} [(w_r^i)^2(Bx) + (w_r^j)^2(Bx)]\Big) dx \\ &\le k^2 \sum_{i,j} \int_{\mathbb{R}^N} (w_r^i)^2(x) (w_r^j)^2(Bx) \, dx \; = \; o(d_r)\end{aligned} \tag{3.24}$$

by (3.21). Let

$$t_r := t(U_r) = \frac{\|U_r\|^2}{|U_r|_4^4 + \beta |U_r(U_r \circ B)|_2^2},$$

so that $\sqrt{t_r} U_r \in \mathbf{N}_{\mathcal{G}}$ by Lemma 2.4(iv). Combining (3.22), (3.23) and (3.24), we obtain

$$\begin{aligned}E_{\mathcal{G}}(\sqrt{t_r} U_r) &= \frac{1}{4} \|\sqrt{t_r} U_r\|^2 = \frac{1}{4} \cdot \frac{\|U_r\|^4}{|U_r|_4^4 + \beta |U_r(U_r \circ B)|_2^2} \\ &\le \frac{1}{4} \cdot \frac{(k\|w_0\|^2 + d_r)^2}{k\|w_0\|^2 + 4d_r + o(d_r)} = \frac{k}{4}\|w_0\|^2 \cdot \frac{k\|w_0\|^2 + 2d_r + o(d_r)}{k\|w_0\|^2 + 4d_r + o(d_r)},\end{aligned}$$

so that $\tilde{c} \le E_{\mathcal{G}}(\sqrt{t_r} U_r) < \frac{k}{4}\|w_0\|^2$ for r large. □

LEMMA 3.3. *There exists a sequence $(u_n)_n \subset \mathbf{N}_{\mathcal{G}}$ with $E_{\mathcal{G}}(u_n) \to \tilde{c}$ and $E'_{\mathcal{G}}(u_n) \to 0$ in $H^*_{\mathcal{G}}$.*

PROOF. Since $\mathbf{N}_{\mathcal{G}}$ is a C^1-manifold, we may invoke Ekeland's variational principle (see e.g. [23]) to deduce the existence of a sequence $(u_n)_n \subset \mathbf{N}_{\mathcal{G}}$ such that $E_{\mathcal{G}}(u_n) \to \tilde{c}$ and

$$o(1) = (E_{\mathcal{G}}|_{\mathbf{N}_{\mathcal{G}}})'(u_n) = E'_{\mathcal{G}}(u_n) - \lambda_n F'(u_n) \quad \text{in } H^*_{\mathcal{G}} \tag{3.25}$$

for a sequence $(\lambda_n)_n \subset \mathbb{R}$, where F is defined in (2.18). Since $u_n \in \mathbf{N}_{\mathcal{G}}$, (2.19) and (3.25) imply that

$$o(1)\|u_n\| = \lambda_n F'(u_n)u_n = -2\lambda_n \|u_n\|^2, \tag{3.26}$$

and therefore $\lambda_n \to 0$ as $n \to \infty$ by Lemma 2.4(i). Thus (3.25) yields $E'_{\mathcal{G}}(u_n) \to 0$ as $n \to \infty$, as claimed. □

PROOF OF PROPOSITION 3.1 (COMPLETED). (i) Let $(u_n)_n \subset \mathbf{N}_{\mathcal{G}}$ be a sequence as provided by Lemma 3.3, and let $y_n \in \mathbb{R}^N,\ n \in \mathbb{N}$, satisfy

$$\int_{B_1(y_n)} u_n^4\, dx = \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} u_n^4\, dx.$$

Since $\mathcal{N}_{\mathcal{G}}$ and $E_{\mathcal{G}}$ are invariant and $\nabla E_{\mathcal{G}}$ is equivariant under translations $u \mapsto u(\cdot + y)$ with $y \in \mathrm{Fix}(\mathcal{G})$ (cf. Definition 1.1(b)), we may assume that $y_n \in V_{\mathcal{G}} = \mathrm{Fix}(\mathcal{G})^\perp$ for every n. We recall that u_n is bounded in $\mathbb{H}^1(\mathbb{R}^N)$ and $|u_n|_4^2 \geq \kappa > 0$ for every n by Lemma 2.4, so a result of Lions [17, Lemma I.1] implies that

$$\liminf_{n\to\infty} \int_{B_1(y_n)} u_n^4\, dx > 0. \tag{3.27}$$

We claim that

$$(y_n)_n \text{ is bounded.} \tag{3.28}$$

Suppose this is false. Then we may pass to a subsequence with $|y_n| \to \infty$ and $y_n/|y_n| \to y \in V_{\mathcal{G}} \setminus \{0\}$. Since $k := l(\mathcal{G}) \leq |\mathcal{G}y|$, there are $A_1, \ldots, A_k \in \mathcal{G}$ such that

$$\text{the points } A_j y,\ j = 1, \ldots, k, \text{ are pairwise different.} \tag{3.29}$$

Let $\hat{u}_n = u_n(\cdot + y_n)$. Up to a subsequence, $\hat{u}_n \rightharpoonup \hat{u} \in \mathbb{H}^1(\mathbb{R}^N)$ weakly, where $\hat{u} \neq 0$ by (3.27). Since $\nabla E_{\mathcal{G}}(u_n) \to 0$ in $\mathbb{H}^1(\mathbb{R}^N)$,

$$\begin{aligned}
o(1) &= \langle \nabla E_{\mathcal{G}}(u_n), \hat{u}(\cdot - y_n)\rangle \\
&= \langle u_n, \hat{u}(\cdot - y_n)\rangle - \int_{\mathbb{R}^N} u_n^3 \hat{u}(\cdot - y_n)\, dx - \beta \langle Q(u_n), \hat{u}(\cdot - y_n)\rangle \\
&= \langle \hat{u}_n, \hat{u}\rangle - \int_{\mathbb{R}^N} \hat{u}_n^3 \hat{u}\, dx - \beta \int_{\mathbb{R}^N} u_n^2(Bx) u_n(x) \hat{u}(x - y_n)\, dx \\
&= \langle \hat{u}_n, \hat{u}\rangle - \int_{\mathbb{R}^N} \hat{u}_n^3 \hat{u}\, dx + |\beta| \int_{\mathbb{R}^N} u_n^2(Bx + y_n) \hat{u}_n(x) \hat{u}(x)\, dx \\
&\geq \|\hat{u}\|^2 - |\hat{u}|_4^4 + |\beta| \int_{\mathbb{R}^N} u_n^2(Bx + y_n)(\hat{u}_n(x) - \hat{u}(x))\hat{u}(x)\, dx + o(1),
\end{aligned}$$

while

$$\left|\int_{\mathbb{R}^N} u_n^2(Bx+y_n)(\hat{u}_n(x)-\hat{u}(x))\hat{u}(x)\,dx\right|$$
$$\leq \left(\int_{\mathbb{R}^N} u_n^4(Bx+y_n)\,dx\right)^{1/2}\left(\int_{\mathbb{R}^N}(\hat{u}_n(x)-\hat{u}(x))^2\hat{u}^2(x)\,dx\right)^{1/2} \to 0 \quad \text{as } n\to\infty.$$

We therefore conclude that $0 < \|\hat{u}\|^2 \leq |\hat{u}|_4^4$, and thus

$$\|\hat{u}\|^2 \geq \frac{\|\hat{u}\|^4}{|\hat{u}|_4^4} \geq \frac{\|w_0\|^4}{|w_0|_4^4} = \|w_0\|^2$$

by (2.11). Using Proposition 3.2, we may choose $R = R(\varepsilon) > 0$ such that

$$\int_{B_R(0)} (|\nabla \hat{u}|^2 + \hat{u}^2)\,dx > \frac{4\tilde{c}}{k}. \tag{3.30}$$

By (3.29), the balls $B_R(A_j y_n)$, $j = 1,\dots,k$, are disjoint for n large. Therefore

$$E_{\mathcal{G}}(u_n) = \frac{1}{4}\|u_n\|^2 \geq \frac{1}{4}\sum_{j=1}^{k}\int_{B_R(A^j y_n)} (|\nabla u_n|^2 + u_n^2)\,dx$$
$$\geq \frac{k}{4}\int_{B_R(y_n)} (|\nabla u_n|^2 + u_n^2)\,dx = \frac{k}{4}\int_{B_R(0)} (|\nabla \hat{u}_n|^2 + \hat{u}_n^2)\,dx.$$

Since $\hat{u}_n \rightharpoonup \hat{u}$ weakly, (3.30) yields

$$\liminf_{n\to\infty} E_{\mathcal{G}}(u_n) \geq \frac{k}{4}\int_{B_R(0)} (|\nabla \hat{u}|^2 + \hat{u}^2)\,dx > \tilde{c},$$

which contradicts the fact that $(u_n)_n$ is a minimizing sequence for (3.20).

Thus (3.28) holds. Consequently, we may pass to a subsequence such that $u_n \rightharpoonup u$ weakly in $H_{\mathcal{G}}$, where $u \in H_{\mathcal{G}} \setminus \{0\}$. Since $E'_{\mathcal{G}} : H_{\mathcal{G}} \to H^*_{\mathcal{G}}$ is weak-to-weak continuous, we conclude that u is a critical point of $E_{\mathcal{G}}$, so that $u \in \mathbf{N}_{\mathcal{G}}$. Moreover,

$$\tilde{c} = \lim_{n\to\infty} E_{\mathcal{G}}(u_n) = \frac{1}{4}\lim_{n\to\infty}\|u_n\|^2 \geq \frac{1}{4}\|u\|^2 = E_{\mathcal{G}}(u),$$

so that u is a minimizer of (3.30). Hence $\tilde{c}$ is attained, and the proof of (a) is finished.

(ii) If $u \in \mathbf{N}_{\mathcal{G}}$ is a minimizer for (3.20), then

$$0 = (E_{\mathcal{G}}|_{\mathbf{N}_{\mathcal{G}}})'(u) = E'_{\mathcal{G}}(u) - \lambda F'(u) \tag{3.31}$$

for some $\lambda \subset \mathbb{R}$, since $\mathbf{N}_{\mathcal{G}} = F^{-1}(0)$ is a C^1-manifold. Hence

$$0 = E'_{\mathcal{G}}(u)u - \lambda F'(u)u = -\lambda F'(u)u = -2\lambda\|u\|^2$$

by (2.19), which yields $\lambda = 0$ and therefore $E'_{\mathcal{G}}(u) = 0$. We consider $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\} \in H_{\mathcal{G}}$. Then

$$\begin{aligned}0 = E'_{\mathcal{G}}(u)u^{\pm} &= \|u^{\pm}\|^2 - |u^{\pm}|_4^4 - \beta Q'(u)u^{\pm}\\ &= \|u^{\pm}\|^2 - |u^{\pm}|_4^4 + |\beta| \int_{\mathbb{R}^N} u^2(Bx)u(x)u^{\pm}(x)\,dx\\ &\geq \|u^{\pm}\|^2 - |u^{\pm}|_4^4 + |\beta| \int_{\mathbb{R}^N} [u^{\pm}(Bx)]^2[u^{\pm}(x)]^2\,dx.\end{aligned}$$

Hence, if $0 < \|u^+\| < \|u\|$, then $t(u^+) \leq 1$ (cf. Lemma 2.4(iv)) and

$$E_{\mathcal{G}}(\sqrt{t(u^+)}u^+) \leq \frac{1}{4}\|u^+\|^2 < \frac{1}{4}\|u\|^2 = E_{\mathcal{G}}(u),$$

contradicting the assumption that u is a minimizer for (3.20). Similarly, $0 < \|u^-\| < \|u\|$ leads to a contradiction. We therefore conclude that u does not change sign. By Lemma 2.3, either $(u, u \circ B)$ or $(-u, -u \circ B)$ is a solution of (1.5). The proof is finished. □

4. Proof of (b) of Theorem 1.2

Here we prove part (b) of Theorem 1.2. For this we consider $\beta_n < 0$, $n \in \mathbb{N}$, with $\beta_n \to 0$ and a sequence of corresponding minimizers $(u_n, u_n \circ B)_n$ of (1.10). We only need to show that $u_n = u_n \circ B$ for large n, because then the uniqueness result in [7] for solutions of $-\Delta u + u = (1 + \beta_n)u^3$ implies that $u_n = u_n \circ B = \frac{1}{\sqrt{1+\beta_n}} w_0$ up to translation in $\mathrm{Fix}(\mathcal{G})$. So we assume by contradiction that, for a subsequence,

$$u_n \neq u_n \circ B \quad \text{for every } n. \tag{4.1}$$

The minimization property and (2.11) imply that

$$\begin{aligned}\frac{1}{4}\|u_n\|^2 &= \inf\left\{\frac{\|u\|^2}{4} : u \in \mathbb{H}^1(\mathbb{R}^N) \setminus \{0\} : \|u\|^2 = |u|_4^4 + \beta_n |u \cdot (u \circ B)|_2^2\right\}\\ &= \inf\left\{\frac{\|u\|^2}{4} : u \in \mathbb{H}^1(\mathbb{R}^N) \setminus \{0\} : \|u\|^2 = |u|_4^4\right\} + o(1)\\ &= \frac{1}{4}\|w_0\|^2 + o(1).\end{aligned} \tag{4.2}$$

Hence $(u_n)_n$ is bounded in $\mathbb{H}^1(\mathbb{R}^N)$, and $|u_n|_4^2 \geq \kappa > 0$ by Lemma 2.4(i). Similarly to the proof of Proposition 3.1, we may assume that

$$\int_{B_1(y_n)} u_n^4\,dx = \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} u_n^4\,dx \geq c > 0 \tag{4.3}$$

for points $y_n \in V_{\mathcal{G}}$, $n \in \mathbb{N}$, and a constant $c > 0$. Setting $\hat{u}_n = u(\cdot + y_n)$, we have $\hat{u}_n \rightharpoonup \hat{u} \neq 0$ (after passing to a subsequence), where $\hat{u}$ is a solution of the scalar problem

$$-\Delta\hat{u} + \hat{u} = \hat{u}^3, \quad u \in \mathbb{H}^1(\mathbb{R}^N),\ u > 0, \tag{4.4}$$

so that $\hat{u}$ equals w_0 up to translation. By (4.2) we thus have $\|\hat{u}\| = \|w_0\| = \lim_{n\to\infty}\|u_n\| = \lim_{n\to\infty}\|\hat{u}_n\|$, hence $\hat{u}_n \to \hat{u}$ strongly in $H^1(\mathbb{R}^N)$. Since $u_n \in H_{\mathcal{G}}$,

$$\hat{u}_n(x) = u_n(x+y_n) = u_n(Ax+Ay_n) = \hat{u}_n(Ax+(Ay_n-y_n)) \quad \text{for } A \in \mathcal{G}, x \in \mathbb{R}^N,$$

so that the relative compactness of $(\hat{u}_n)_n$ in $\mathbb{H}^1(\mathbb{R}^N)$ implies the boundedness of the sequence $(Ay_n - y_n)_n \subset \mathbb{R}^N$ for every $A \in \mathcal{G}$. Recalling that $(y_n)_n \subset V_{\mathcal{G}} = \mathrm{Fix}(\mathcal{G})^{\perp}$, we conclude that $(y_n)_n$ is bounded. Since y_n is bounded, we infer that $u_n \to u$ in $H_{\mathcal{G}}$, where $u \in H_{\mathcal{G}}$ is a nontrivial solution of (4.4). This then implies that $u = w_0(\cdot - z_0)$ for some $z_0 \in \mathrm{Fix}(\mathcal{G})$. Since $\sup_{y\in\mathbb{R}^N}\int_{B_1(y)} w_0^4\,dx$ is attained precisely at $y = 0$, we deduce from (4.3) that $z_0 = 0$, so that $u_n \to w_0$ in $H_{\mathcal{G}}$. Combining this information with elliptic estimates as in [26, Sec. 2], we find that

$$u_n \to w_0 \quad \text{uniformly on } \mathbb{R}^N. \tag{4.5}$$

We set $\varphi_n = u_n - u_n \circ B \in \mathcal{H}_{\mathcal{G}}$ and note that φ_n satisfies

$$\Delta\varphi_n - \varphi_n + 3w_0^2\varphi_n + c_n(x)\varphi_n = 0, \tag{4.6}$$

where

$$c_n = u_n^2 + (u_n\circ B)^2 + u_n(u_n\circ B) - 3w_0^2 - \beta_n u_n(u_n\circ B) \to 0 \quad \text{as } n \to \infty \tag{4.7}$$

uniformly in $\mathbb{R}^N$. By (4.1), we may choose $x_n \in \mathbb{R}^N$ with $\varphi_n(x_n) = \max_{x\in\mathbb{R}^3}|\varphi_n(x)| > 0$. Using (4.6) and (4.7), we deduce that $(x_n)_n \subset \mathbb{R}^N$ is a bounded sequence. We consider $\hat{\varphi}_n = \varphi_n/|\varphi_n(x_n)|$ which satisfies

$$\Delta\hat{\varphi}_n - \hat{\varphi}_n + 3w_0^2\hat{\varphi}_n + c_n(x)\hat{\varphi}_n = 0, \qquad \hat{\varphi}_n(x_n) = 1. \tag{4.8}$$

Using elliptic estimates we derive that, for a subsequence, $x_n \to x_0 \in \mathbb{R}^N$ and $\hat{\varphi}_n \to \hat{\varphi}_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $n \to \infty$, where $\hat{\varphi}_0 \in H_{\mathcal{G}} \cap C^2(\mathbb{R}^N)$ is a solution of $\Delta\hat{\varphi}_0 - \hat{\varphi}_0 + 3w_0^2\hat{\varphi}_0 = 0$ with $\hat{\varphi}_0(x_0) = 1$. It follows from Appendix C of [21] that

$$\hat{\varphi}_0 = \frac{\partial w_0}{\partial \tau} \quad \text{for some vector } \tau \in \mathbb{R}^N. \tag{4.9}$$

Since w_0 is radial, the $\mathcal{G}$-symmetry of $\partial w_0/\partial\tau = \hat{\varphi}_0 \in H_{\mathcal{G}}$ implies that $\tau \in \mathrm{Fix}(\mathcal{G})$. But then $B\tau = \tau = B^{-1}\tau$ by Definition 1.1(b), and therefore

$$\hat{\varphi}_0(Bx) = \frac{\partial w_0}{\partial\tau}(Bx) = \frac{\partial w_0}{\partial\tau}(x) = \hat{\varphi}_0(x) \quad \text{for all } x \in \mathbb{R}^N.$$

On the other hand, by definition we have $\varphi_n \circ B = -\varphi_n$ and therefore $\hat{\varphi}_0 \circ B = -\hat{\varphi}_0$. Hence $\hat{\varphi}_0 \equiv 0$, contradicting $\hat{\varphi}_0(x_0) = 1$. We conclude that $u_n = u_n \circ B$ for n large, as required. The proof of Theorem 1.2(b) is finished.

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