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Completely Positive Invariant Conjugate-Bilinear Maps in Partial *-Algebras

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Abstract. The notion of completely positive invariant conjugate-bilinear map in a partial *-algebra is introduced and a generalized Stinespring theorem is proven. Applications to the existence of integrable extensions of *-representations of commutative, locally convex quasi*-algebras are also discussed.

Keywords. Completely positive maps, partial *-algebras, partial O*-algebras

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1. Introduction

Completely positive linear maps on *-algebras play a relevant role in many applications such as quantum theory, quantum information, quantum probability theory (see [5, 9], for overviews). In quantum physics, for instance, these maps describe the passage from the dynamics of a system to that of its subsystems and they act on the observable algebra of the system itself which is usually taken to be a C*-algebra and then represented by bounded operators on some Hilbert space.

It is now a long time that the C*-algebraic approach to quantum theory has been shown to be a too rigid scheme to include in its framework all objects of physical interest and several possible generalizations have been proposed: quasi *-algebras, partial *-algebras and so on. It is then natural to try and extend the notion of complete positivity to these different situations that become relevant when unbounded operators occur.

From a mathematical point of view the most classical result on this topic is the Stinespring dilation theorem, that essentially says that a linear map

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$T : \mathfrak{A} \mapsto \mathfrak{B}$ where $\mathfrak{A}$ is a C*-algebra with unit and $\mathfrak{B}$ is a C*-algebra of bounded operators in Hilbert space $\mathcal{H}$, is completely positive if and only if it has the form

$$T(a) = V^*\pi(a)V, \quad a \in \mathfrak{A}$$

where π is a bounded representation of $\mathfrak{A}$ in the Hilbert space $\mathcal{K}$ and V is a bounded linear map of $\mathcal{H}$ into $\mathcal{K}$.

A more general set-up was considered by Schmüdgen in [8, Ch.11] where he considered completely positive maps from an arbitrary *-algebra $\mathfrak{A}$ into a vector space $\mathfrak{X}$ and showed that a Stinespring-like representation holds for all completely positive mappings of $\mathfrak{A}$ into a vector space $\mathfrak{X}$. This result found applications in the study of integrable extensions of *-representations of both commutative *-algebras and enveloping algebras.

This paper is devoted to the possibility of extending Schmüdgen's results to the case where $\mathfrak{A}$ is a partial *-algebra [2]. The lack of an everywhere defined multiplication makes impossible to adapt the usual notion of complete positivity for a linear map T, since in this case products of the form a^*b, $a, b \in \mathfrak{A}$ need not be defined. For this reason, we consider instead of linear maps, *conjugate-bilinear* maps defined on a subspace of $\mathfrak{A} \times \mathfrak{A}$. But, in the same fashion as Antoine and two of us did in [1, 2] for generalizing the GNS costruction to partial *-algebras, also in this case, in order to obtain what will be called a *Stinespring dilation* of the given completely positive conjugate-bilinear map, we need to suppose the existence of a subspace (the *core*) of the space of universal right multipliers $R\mathfrak{A}$ of $\mathfrak{A}$ enjoying certain conditions of *quasi-invariance.*

The paper is organized as follows. After giving some preliminaries (Section 2), we prove, in Section 3, a generalized Stinespring theorem for completely positive, conjugate bilinear, quasi-invariant maps on a partial *-algebra $\mathfrak{A}$, with values in a vector space $\mathfrak{X}$ and we examine the relationships of the related representations when different cores are considered. In Section 4 we consider completely positive invariant linear maps on partial O*-algebras that are the natural framework were *-representations of abstract partial *-algebras are defined. In Section 5, we discuss applications to the existence of integrable extensions of *-representations of commutative, locally convex quasi*-algebras.

2. Preliminaries

In this Section we will collect some basic definitions needed in what follows.

A *partial *-algebra* is a complex vector space $\mathfrak{A}$, endowed with an involution $x \mapsto x^*$ (that is, a bijection such that $x^{**} = x$) and a partial multiplication defined by a set $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$ (a binary relation) such that:

(i) $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$;

(ii) $(x, y_1), (x, y_2) \in \Gamma$ implies $(x, \lambda y_1 + \mu y_2) \in \Gamma$, for all $\lambda, \mu \in \mathbb{C}$;

(iii) for any $(x, y) \in \Gamma$, there is defined a product $x \cdot y \in \mathfrak{A}$, which is distributive w.r.t. the addition and satisfies the relation $(x \cdot y)^* = y^* \cdot x^*$.

We shall assume the partial *-algebra $\mathfrak{A}$ contains a unit 1, i.e., $1^* = 1, (1, x) \in \Gamma$, for all $x \in \mathfrak{A}$, and $1 \cdot x = x \cdot 1 = x$, for all $x \in \mathfrak{A}$. (If $\mathfrak{A}$ has no unit, it may always be embedded into a larger partial *-algebra with unit, in the standard fashion.) Given the defining set Γ, spaces of multipliers are defined in the obvious way:

$$\begin{aligned}(x, y) \in \Gamma &\iff x \in L(y) \text{ or } x \text{ is a left multiplier of } y \\ &\iff y \in R(x) \text{ or } y \text{ is a right multiplier of } x.\end{aligned}$$

A partial *-algebra $\mathfrak{A}$ is said to be *semi-associative* if $y \in R(x)$ implies $y \cdot z \in R(x)$ for every $z \in R\mathfrak{A}$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Let $\mathfrak{A}[\tau]$ be a partial *-algebra, which is a locally convex space for the locally convex topology τ. Then $\mathfrak{A}[\tau]$ is called a *locally convex partial *-algebra* if the following two conditions are satisfied:

(i) the involution $x \mapsto x^*$ is τ-continuous;

(ii) the maps $x \mapsto ax$ and $x \mapsto xb$ are τ-continuous for all $a \in L\mathfrak{A}$ and $b \in R\mathfrak{A}$.

A *quasi *-algebra* is a couple $(\mathfrak{A}, \mathfrak{A}_0)$, where $\mathfrak{A}$ is a vector space with involution *, $\mathfrak{A}_0$ is a *-algebra and a vector subspace of $\mathfrak{A}$ and $\mathfrak{A}$ is an $\mathfrak{A}_0$-bimodule whose module operations and involution extend those of $\mathfrak{A}_0$ [8]. Of course, any quasi *-algebra is a partial *-algebra.

A quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is said to be a *locally convex quasi *-algebra* if $\mathfrak{A}$ is endowed with a locally convex topology τ such that

(i) the involution $x \mapsto x^*$ is τ-continuous;

(ii) the maps $x \mapsto ax$ and $x \mapsto xb$ are τ-continuous, for all $a, b \in \mathfrak{A}_0$.

(iii) $\mathfrak{A}_0$ is τ-dense in $\mathfrak{A}$.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all (closable) linear operators X such that $\mathcal{D}(X) = \mathcal{D}$, $\mathcal{D}(X^*) \supseteq \mathcal{D}$. The set $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a *partial *-algebra* [2] with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger = X^* \lceil \mathcal{D}$ and the *(weak)* partial multiplication $X_1 \,\Box\, X_2 = {X_1}^{\dagger *} X_2$, defined by

$$\begin{aligned}&(X_1, X_2) \in \Gamma \Leftrightarrow X_2\mathcal{D} \subset D(X_1^{\dagger *}) \text{ and } X_1^\dagger \mathcal{D} \subset D(X_2^*) \\ &(X_1 \,\Box\, X_2)\xi := X_1^{\dagger *} X_2 \xi, \quad \forall \xi \in \mathcal{D}.\end{aligned}$$

If $(X_1, X_2) \in \Gamma$, we say that X_2 is a weak right multiplier of X_1 or, equivalently, that X_1 is a weak left multiplier of X_2 (we write $X_2 \in R^{\mathrm{w}}(X_1)$ or $X_1 \in L^{\mathrm{w}}(X_2)$).

A †-invariant subset (resp. subspace) of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is said to be an *O^*-family* (resp. *O^*-vector space) on $\mathcal{D}$.*

A *partial O^*-algebra* on $\mathcal{D}$ is a *-subalgebra $\mathfrak{M}$ of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, that is, $\mathfrak{M}$ is a subspace of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, containing the identity and such that $X^\dagger \in \mathfrak{M}$ whenever $X \in \mathfrak{M}$ and $X_1 \square X_2 \in \mathfrak{M}$ for any $X_1, X_2 \in \mathfrak{M}$ such that $X_2 \in R^{\mathrm{w}}(X_1)$.

Let

$$\mathcal{L}^\dagger(\mathcal{D}) = \{X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) : X\mathcal{D} \subseteq D,\ X^\dagger\mathcal{D} \subseteq D\}.$$

Then $\mathcal{L}^\dagger(\mathcal{D})$ is a *-algebra w.r.to $\square$ and $X_1 \square X_2\xi = X_1(X_2\xi)$ for each $\xi \in \mathcal{D}$. A *-subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$ is called an O*-algebra [8].

The following topologies on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ will be used in this paper:

- the *weak topology* $\tau_{\mathrm{w}}^{\mathcal{D}}$: defined by the seminorms $p_{\xi,\eta}$, $\xi, \eta \in \mathcal{D}$ where $p_{\xi,\eta}(X) = |\langle X\xi | \eta\rangle|$, $X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$;
- the *strong topology* $\tau_s^{\mathcal{D}}$: defined by the seminorms p_ξ, $\xi \in \mathcal{D}$ where $p_\xi(X) = \|X\xi\|$, $X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$;
- the *strong* topology* $\tau_{s^*}^{\mathcal{D}}$: defined by the seminorms p_ξ^*, $\xi \in \mathcal{D}$ where $p_\xi^*(X) = \max\{\|X\xi\|, \|X^\dagger\xi\|\}$, $X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$.

A **-representation* of a partial *-algebra $\mathfrak{A}$ is a *-homomorphism of $\mathfrak{A}$ into $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, for some pair $(\mathcal{D},\mathcal{H})$, $\mathcal{D}$ a dense subspace of $\mathcal{H}$, that is, a linear map $\pi : \mathfrak{A} \mapsto \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ such that:

(i) $\pi(a^*) = \pi(a)^\dagger$ for every $a \in \mathfrak{A}$;

(ii) If $a, b \in \mathfrak{A}$ and $a \in L(b)$ then $\pi(a) \in L^{\mathrm{w}}(\pi(b))$ and $\pi(a) \square \pi(b) = \pi(ab)$.

If (ii) holds only when $a \in \mathfrak{A}$ and $b \in R\mathfrak{A}$, we say that π is a *quasi* *-representation.

If π is a *-representation of the partial *-algebra $\mathfrak{A}$, then $\pi(\mathfrak{A})$ need not be a partial O*-algebra, but, in general, it is only an O*-vector space.

If $\mathfrak{M}$ is an O*-family on $\mathcal{D}$, the *graph topology* on $\mathcal{D}$ is the locally convex topology defined by the family $\{\|\cdot\|_X; X \in \mathfrak{M}\}$ of seminorms: $\|\xi\|_X \equiv \|X\xi\|, \xi \in \mathcal{D}$ and it is denoted by $t_{\mathfrak{M}}$. We denote by $\widetilde{\mathcal{D}}(\mathfrak{M})$ the completion of the locally convex space $\mathcal{D}[t_{\mathfrak{M}}]$ and put

$$\widehat{\mathcal{D}}(\mathfrak{M}) = \bigcap_{X \in \mathfrak{M}} \mathcal{D}(\overline{X}).$$

An O*-family $\mathfrak{M}$ on $\mathcal{D}$ is said to be *closed* if $\mathcal{D} = \widetilde{\mathcal{D}}(\mathfrak{M})$; and it is said to be *fully closed* if $\mathcal{D} = \widehat{\mathcal{D}}(\mathfrak{M})$. Now, put

$$\mathcal{D}^*(\mathfrak{M}) = \bigcap_{X \in \mathfrak{M}} \mathcal{D}(X^*).$$

Then $\mathfrak{M}$ is said to be *selfadjoint* if $\mathcal{D} = \mathcal{D}^*(\mathfrak{M})$. Finally, $\mathfrak{M}$ is said to be *integrable* if $\mathfrak{M}$ is fully closed and each $X \in \mathfrak{M}$ such that $X = X^\dagger$ is essentially selfadjoint. The set

$$\mathfrak{M}'_\sigma = \{C \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) : \langle X\xi | C^*\eta\rangle = \langle C\xi | X^\dagger\eta\rangle,\ \forall X \in \mathfrak{M},\ \forall \xi, \eta \in \mathcal{D}\},$$

is called the *weak unbounded commutant* of $\mathfrak{M}$. Its bounded part $\mathfrak{M}'_{\mathrm{w}}$ is the *weak bounded commutant* of $\mathfrak{M}$.

A fully closed partial O*-algebra $\mathfrak{M}$ on $\mathcal{D}$ is called a *partial GW*-algebra* if $\mathfrak{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$ and $\mathfrak{M} = \mathfrak{M}''_{\mathrm{w}\sigma}$.

A *-representation π of a partial *-algebra $\mathfrak{A}$ is called closed (respectively, fully closed, self-adjoint, integrable) if $\pi(\mathfrak{A})$ is closed (respectively, fully closed, self-adjoint, integrable).

3. Generalized Stinespring theorem

Let $\mathfrak{A}$ be a partial *-algebra with identity *1* and $\mathfrak{X}$ a vector space. We denote by $\mathbb{S}(\mathfrak{X})$ the involutive vector space of all sesquilinear forms on $\mathfrak{X} \times \mathfrak{X}$ with involution $\varphi \to \varphi^+$ where $\varphi^+(\xi, \eta) = \overline{\varphi(\eta, \xi)}$, $\xi, \eta \in \mathfrak{X}$.

A map $\Phi : \mathcal{D}(\Phi) \times \mathcal{D}(\Phi) \mapsto \mathbb{S}(\mathfrak{X})$ is said to be *conjugate-bilinear* if

- $\mathcal{D}(\Phi)$ is a subspace of $\mathfrak{A}$;
- $\Phi(x, y)^+ = \Phi(y, x), \quad \forall x, y \in \mathcal{D}(\Phi)$;
- $\Phi(\alpha x + \beta y, z) = \alpha\Phi(x, z) + \beta\Phi(y, z), \quad \forall x, y, z \in \mathcal{D}(\Phi), \forall \alpha, \beta \in \mathbb{C}$.

In particular, if $\mathcal{D}(\Phi) = \mathfrak{A}$, then Φ is said to be *conjugate-bilinear map on* $\mathfrak{A}\times\mathfrak{A}$. It is clear that Φ is a sesquilinear map, i.e.,

- $\Phi(x, \alpha y + \beta z) = \overline{\alpha}\Phi(x, y) + \overline{\beta}\Phi(x, z), \quad \forall x, y, z \in \mathcal{D}(\Phi), \forall \alpha, \beta \in \mathbb{C}$.

Definition 3.1. A conjugate-bilinear map $\Phi : \mathcal{D}(\Phi) \times \mathcal{D}(\Phi) \mapsto \mathbb{S}(\mathfrak{X})$ is said to be *quasi-invariant* if there exists a subspace B_Φ of $\mathcal{D}(\Phi)$ such that

$(\mathrm{I})_1$: $B_\Phi \subset R\mathfrak{A}$;

$(\mathrm{I})_2$: $\mathfrak{A}B_\Phi \subset \mathcal{D}(\Phi)$;

$(\mathrm{I})_3$: $\Phi(ax, y) = \Phi(x, a^*y), \quad \forall a \in \mathfrak{A}, \forall x, y \in B_\Phi$;

$(\mathrm{I})_4$: B_Φ satisfies the following density condition: for all $x \in D(\Phi)$, for all $\xi \in \mathfrak{X}$, there exists a sequence $\{x_n\} \subset B_\Phi$ such that

$$\lim_{n\to\infty} \Phi(x_n - x, x_n - x)(\xi, \xi) = 0.$$

Furthermore, if

$(\mathrm{I})'_3$: $\Phi(a^*x, by) = \Phi(x, (ab)y), \quad \forall a, b \in \mathfrak{A} : a \in L(b), \forall x, y \in B_\Phi$,

then Φ is said to be *invariant.*

A subspace B_Φ satisfying the above requirements is called a *core* for Φ. If $R\mathfrak{A}$ is a core for Φ, then Φ is said to be *totally invariant.*

In analogy with [1, 3, 8], we give the following

Definition 3.2. A conjugate-bilinear map $\Phi : \mathcal{D}(\Phi) \times \mathcal{D}(\Phi) \mapsto \mathbb{S}(\mathfrak{X})$ is said to be *positive* if $\Phi(x, x) \geq 0$ (i.e., $\Phi(x, x)(\xi, \xi) \geq 0$ for every $\xi \in \mathfrak{X}$) for each $x \in \mathcal{D}(\Phi)$; the map Φ is said to be *completely positive* if, for each $n \in \mathbb{N}$,

$$\sum_{k,l=1}^{n} \Phi(x_k, x_l)(\xi_k, \xi_l) \geq 0, \quad \forall\{x_1, \ldots, x_n\} \subset \mathcal{D}(\Phi),\ \{\xi_1, \ldots, \xi_n\} \subset \mathfrak{X}.$$

We now give some examples of completely positive, invariant conjugate-bilinear maps.

Example 3.3. Let $\mathfrak{A}$ be a partial *-algebra and $\mathfrak{X}$ a vector space. Let π be a (quasi) *-representation of $\mathfrak{A}$ on the domain $\mathcal{D}(\pi)$. Let $V : \mathfrak{X} \mapsto \mathcal{D}(\pi)$ be a linear map. We define a map $\Phi_{\{\pi,V\}}$ of $\mathfrak{A} \times \mathfrak{A}$ into $\mathbb{S}(\mathfrak{X})$ by

$$\Phi_{\{\pi,V\}}(a,b)(\xi,\eta) = \langle \pi(a)V\xi \,|\pi(b)V\eta \rangle \,, \quad a,b \in \mathfrak{A},\ \xi,\eta \in \mathfrak{X}.$$

Then $\Phi_{\{\pi,V\}}$ is a completely positive conjugate-bilinear map on $\mathfrak{A} \times \mathfrak{A}$. We put

$$B_{\{\pi,V\}} = \{x \in R\mathfrak{A}; \pi(x)V\mathfrak{X} \subset \mathcal{D}(\pi)\}.$$

If $\pi(B_{\{\pi,V\}})$ is $\tau_s^{\mathcal{D}}$-dense in $\pi(\mathfrak{A})$, then $\Phi_{\{\pi,V\}}$ is (quasi-)invariant with core $B_{\{\pi,V\}}$.

Example 3.4. Let $\mathfrak{A}$ be a partial *-algebra and π a *-representation of $\mathfrak{A}$. We define a map Φ_π of $\mathfrak{A} \times \mathfrak{A}$ into $\mathbb{S}(\mathcal{D}(\pi))$ by

$$\Phi_\pi(a,b)(\xi,\eta) = \langle \pi(a)\xi \,|\pi(b)\eta \rangle \quad a,b \in \mathfrak{A},\ \xi,\eta \in \mathcal{D}(\pi).$$

Then Φ_π is a completely positive conjugate-bilinear map on $\mathfrak{A} \times \mathfrak{A}$. We put

$$B_\pi = \{x \in R\mathfrak{A}; \pi(x)\mathcal{D}(\pi) \subset \mathcal{D}(\pi)\}.$$

If $\pi(B_\pi)$ is $\tau_s^{\mathcal{D}}$-dense in $\pi(\mathfrak{A})$, then Φ_π is invariant with core B_π. Furthermore, if π is selfadjoint, then $B_\pi = R\mathfrak{A}$ and Φ_π is totally invariant.

Example 3.5. Let $\mathfrak{A}$ be a partial *-algebra and π a (quasi) *-representation of $\mathfrak{A}$. Let $\mathfrak{X}$ be a vector space and $\mathfrak{A} \otimes \mathfrak{X}$ the algebraic tensor product of $\mathfrak{A}$ and $\mathfrak{X}$. A linear map λ defined on a subspace $\mathcal{D}(\lambda)$ of $\mathfrak{A} \otimes \mathfrak{X}$ into $\mathcal{H}_\pi$ is said to be a *strongly cyclic vector representation* of $\mathfrak{A} \otimes \mathfrak{X}$ for π if there exists a subspace B_λ of $\mathcal{D}_\lambda := \{x \in \mathfrak{A}; x \otimes \xi \in \mathcal{D}(\lambda), \forall \xi \in \mathfrak{X}\}$ such that $\mathfrak{A}B_\lambda \subset \mathcal{D}_\lambda, \pi(a)\lambda(x \otimes \xi) = \lambda(ax \otimes \xi)$ for each $a \in \mathfrak{A}$, $x \in B_\lambda$ and $\xi \in \mathfrak{X}$, and $\lambda(B_\lambda \otimes \mathfrak{X})$ is dense in $\mathcal{D}(\pi)[t_\pi]$. We define a map $\Phi_{\{\pi,\lambda\}} : \mathcal{D}_\lambda \times \mathcal{D}_\lambda \mapsto \mathbb{S}(\mathfrak{X})$ by

$$\Phi_{\{\pi,\lambda\}}(x,y)(\xi,\eta) = \langle \lambda(x \otimes \xi) \,|\lambda(y \otimes \eta) \rangle \,, \quad x,y \in \mathcal{D}_\lambda,\ \xi,\eta \in \mathfrak{X}.$$

Then $\Phi_{\{\pi,\lambda\}}$ is a completely positive conjugate-bilinear map on $\mathfrak{A} \times \mathfrak{A}$ such that

$$\Phi_{\{\pi,\lambda\}}(ax,by)(\xi,\eta) = \langle \pi(a)\lambda(x \otimes \xi) \,|\pi(b)\lambda(y \otimes \eta) \rangle$$

for each $a,b \in \mathfrak{A}$, $x,y \in B_\lambda$, $\xi,\eta \in \mathfrak{X}$. Furthermore, if $\lambda(B_\lambda \otimes \xi)$ is dense in $\lambda(\mathcal{D}_\lambda \otimes \xi)$, for each $\xi \in \mathfrak{X}$, then $\Phi_{\{\pi,\lambda\}}$ is (quasi-)invariant with core B_λ.

Example 3.6. Let $\mathfrak{A}[\tau]$ be a locally convex semi-associative partial *-algebra. Then $M\mathfrak{A} = L\mathfrak{A} \cap R\mathfrak{A}$ is a *-algebra. Let $\Phi_0 : M\mathfrak{A} \mapsto \mathbb{S}(\mathfrak{X})$ be a completely positive *linear* map on $M\mathfrak{A}$. We assume that $\mathbb{S}(\mathfrak{X})$ is endowed with the topology t_{S} of simple convergence, defined by the seminorms $p_{\xi,\eta}(\varphi) = |\Phi_0(\xi,\eta)|$. We assume that

- $M\mathfrak{A}$ is dense in $\mathfrak{A}[\tau]$;
- the map $(x, y) \in M\mathfrak{A} \times M\mathfrak{A} \mapsto \Phi_0(y^*x) \in \mathbb{S}(\mathfrak{X})$ is continuous with respect to the product topology defined by τ on $M\mathfrak{A}$ and the topology t_{S} of $\mathbb{S}(\mathfrak{X})$.

For $a, b \in \mathfrak{A}$ we define a map Φ of $\mathfrak{A} \times \mathfrak{A}$ into $\mathbb{S}(\mathfrak{X})$ by

$$\Phi(a,b)(\xi,\eta) = \lim_{\alpha,\beta} \Phi_0(y_\beta^* x_\alpha)(\xi,\eta), \quad \xi,\eta \in \mathfrak{X},$$

where $\{x_\alpha\}$ and $\{y_\beta\}$ are nets in $M\mathfrak{A}$ that converge to a and b, respectively. Then Φ is a completely positive quasi-invariant conjugate bilinear map on $\mathfrak{A} \times \mathfrak{A}$ with core $M\mathfrak{A}$. In particular, if $\mathfrak{A}$ is a locally convex quasi*-algebra over $\mathfrak{A}_0$ (in this case $M\mathfrak{A} = \mathfrak{A}_0$), then Φ is a completely positive totally invariant conjugate bilinear map on $\mathfrak{A} \times \mathfrak{A}$ with core $\mathfrak{A}_0$.

Example 3.7. Let $\mathfrak{A}_0[\|\cdot\|]$ be a unital C*-algebra with C*-norm $\|\cdot\|$ and τ a locally convex topology on $\mathfrak{A}_0$ which is finer than the C*-norm $\|\cdot\|$-topology such that $\mathfrak{A}_0[\tau]$ is a locally convex $*$-algebra. Let F_0 be a completely positive linear map of $\mathfrak{A}_0$ into the $*$-algebra $\mathfrak{B}(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$.

(1) Suppose that the map : $(x, y) \in \mathfrak{A}_0[\tau] \times \mathfrak{A}_0[\tau] \mapsto F_0(y^*x) \in \mathfrak{B}(\mathcal{H})[\tau_{\mathrm{w}}^{\mathcal{D}}]$ is continuous for some dense subspace $\mathcal{D}$ in $\mathcal{H}$. Then we put

$$F(a,b)(\xi,\eta) = \lim_{\alpha,\beta} \left\langle F_0(y_\beta^* x_\alpha)\xi \,|\eta \right\rangle, \quad \xi,\eta \in \mathcal{D},$$

where $\{x_\alpha\}$ and $\{y_\beta\}$ are nets in $\mathfrak{A}_0$ which converge to a and b w.r.t. the topology τ, respectively. Then F is a completely positive totally invariant conjugate-bilinear map of the locally convex quasi $*$-algebra $\widetilde{\mathfrak{A}_0}[\tau]$ over $\mathfrak{A}_0$ constructed from the completion of $\mathfrak{A}_0[\tau]$ with core $\mathfrak{A}_0$.

(2) Suppose that the map : $x \in \mathfrak{A}_0[\tau] \to F_0(x) \in \mathfrak{B}(\mathcal{H})[\tau_{s^*}^{\mathcal{D}}]$ is continuous. Then we put

$$F(a)\xi = \lim_{\alpha} F_0(x_\alpha)\xi, \quad \xi \in \mathcal{D},$$

where $\{x_\alpha\}$ is a net in $\mathfrak{A}_0$ which converges to a w.r.t. τ.

(i) If the multiplication of $\mathfrak{A}_0[\tau]$ is jointly continuous, then F is a completely positive linear map of the locally convex $*$-algebra $\widetilde{\mathfrak{A}_0}[\tau]$ into $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$.

(ii) If the multiplication of $\mathfrak{A}_0[\tau]$ is not jointly continuous, then we can't even define the notion of complete positivity of F. In this case, the results of Section 4 can be used.

Example 3.8. The previous example suggests a possible physical application concerning the time evolution of a quantum system. Let $\mathfrak{A}_0$ be the C*-algebra of local observables of some physical system, in the sense of [9]. Let α^t be the automorphisms group that describes the time evolution of the elements of $\mathfrak{A}_0$. Then the completion $\mathfrak{A}$ of $\mathfrak{A}_0$ w.r.t. the *physical* topology [6] is a locally convex quasi *-algebra over $\mathfrak{A}_0$, which needs to be introduced because it contains physically relevant observables as well as their time evolutions. Then, if we define

$$F_0(x, y) = \alpha^t(y^* x), \quad x, y \in \mathfrak{A}_0,$$

F_0 enjoys all conditions required in the previous example, so that the corresponding F is a completely positive totally invariant conjugate-bilinear map.

We now show that Example 3.5 completely covers the general situation; that is, for any completely positive (quasi) invariant conjugate bilinear map $\Phi : \mathcal{D}(\Phi) \times \mathcal{D}(\Phi) \mapsto \mathbb{S}(\mathfrak{X})$ there exists a couple $\{\pi, \lambda\}$ consisting of a *-representation π of $\mathfrak{A}$ and of a strongly cyclic vector representation λ of $\mathfrak{A} \otimes \mathfrak{X}$ for π such that $\Phi = \Phi_{\{\pi,\lambda\}}$. This is a generalization of Stinespring's theorem for completely positive linear maps on von Neumann algebras [10]. Generalizations of Stinespring's theorem have been studied by Powers [7] and Schmüdgen [8] for O*-algebras and by Ekhaguere and Odiobala [3] and Ekhaguere [4] for partial *-algebras. This paper is aimed to generalize Schmüdgen's results to partial *-algebras. The outcome is also a generalization of the studies of Ekhaguere and Odiobala.

Let $\mathfrak{A}$ be a partial *-algebra with identity *1*, $\mathfrak{X}$ a vector space and Φ a completely positive invariant conjugate bilinear map of $\mathcal{D}(\Phi) \times \mathcal{D}(\Phi)$ into $\mathbb{S}(\mathfrak{X})$. By the complete positivity of Φ, a semidefinite inner product $\langle\,|\,\rangle$ on the algebraic tensor product $\mathcal{D}(\Phi) \otimes \mathfrak{X}$ of $\mathcal{D}(\Phi)$ and $\mathfrak{X}$ can be defined by

$$\left\langle \sum_{k=1}^{n} x_k \otimes \xi_k \,\middle|\, \sum_{l=1}^{m} y_l \otimes \eta_l \right\rangle = \sum_{k=1}^{n} \sum_{l=1}^{m} \Phi(x_k, y_l)(\xi_k, \eta_l),$$

for $\{x_k\}, \{y_l\} \subset \mathcal{D}(\Phi)$ and $\{\xi_k\}, \{\eta_l\} \subset \mathfrak{X}$. We define a subpace $\mathcal{N}$ of $\mathcal{D}(\Phi) \otimes \mathfrak{X}$ by

$$\mathcal{N} = \left\{ \sum_{k=1}^{n} x_k \otimes \xi_k \in \mathcal{D}(\Phi) \otimes \mathfrak{X};\ \left\langle \sum_{k=1}^{n} x_k \otimes \xi_k \,\middle|\, \sum_{k=1}^{n} x_k \otimes \xi_k \right\rangle = 0 \right\}$$

and the coset

$$\lambda_\Phi \left(\sum_{k=1}^{n} x_k \otimes \xi_k \right) = \sum_{k=1}^{n} x_k \otimes \xi_k + \mathcal{N}$$

of $\sum_{k=1}^{n} x_k \otimes \xi_k$. Then the quotient space $\lambda_\Phi(\mathcal{D}(\Phi) \otimes \mathfrak{X}) \equiv \mathcal{D}(\Phi) \otimes \mathcal{N}$ is a pre-Hilbert space and its completion is denoted by $\mathcal{H}_\Phi$. By condition (I_4) of

Definition 3.1 it is easily seen that $\lambda_\Phi(B_\Phi \otimes \xi)$ is dense in $\lambda_\Phi(\mathcal{D}(\Phi) \otimes \xi)$, for each $\xi \in \mathfrak{X}$ and $\lambda_\Phi(B_\Phi \otimes \mathfrak{X})$ is dense in $\mathcal{H}_\Phi$. We put

$$\pi_0(a)\lambda_\Phi\left(\sum_{k=1}^n x_k \otimes \xi_k\right) = \lambda_\Phi\left(\sum_{k=1}^n ax_k \otimes \xi_k\right)$$

for $a \in \mathfrak{A}$ and $\sum_{k=1}^n x_k \otimes \xi_k \in B_\Phi \otimes \mathfrak{X}$. Then π_0 is a *-representation of $\mathfrak{A}$ in $\mathcal{H}_\Phi$ with $\mathcal{D}(\pi_0) = \lambda_\Phi(B_\Phi \otimes \mathfrak{X})$. Indeed, take arbitrary $a, b \in \mathfrak{A}$ with $a \in \mathrm{Ł}(b)$. We have

$$\begin{aligned}
\left\langle \pi_0(a^*)\lambda_\Phi\left(\sum_{k=1}^n x_k \otimes \xi_k\right) \middle| \pi_0(b)\lambda_\Phi\left(\sum_{l=1}^m y_l \otimes \eta_l\right) \right\rangle \\
&= \left\langle \lambda_\Phi\left(\sum_{k=1}^n a^*x_k \otimes \xi_k\right) \middle| \lambda_\Phi\left(\sum_{l=1}^m by_l \otimes \eta_l\right) \right\rangle \\
&= \sum_{k=1}^n \sum_{l=1}^m \Phi(a^*x_k, by_l)(\xi_k, \eta_l) \\
&= \sum_{k=1}^n \sum_{l=1}^m \Phi(x_k, (ab)y_l)(\xi_k, \eta_l) \\
&= \left\langle \left(\sum_{k=1}^n x_k \otimes \xi_k\right) \middle| \pi_0(ab)\lambda_\Phi\left(\sum_{l=1}^m by_l \otimes \eta_l\right) \right\rangle
\end{aligned}$$

for each $\sum_{k=1}^n x_k \otimes \xi_k$, $\sum_{l=1}^m y_l \otimes \eta_l \in B_\Phi \otimes \mathfrak{X}$, which implies that π_0 is well-defined and that it is a *-representation of $\mathfrak{A}$. We denote by π its closure. Then it is clear that λ_Φ is a strongly cyclic vector representation of $\mathfrak{A} \otimes \mathfrak{X}$ for π with core B_Φ and that $\Phi = \Phi_{\{\pi,\lambda_\Phi\}}$. In particular, suppose that $B_\Phi \ni \mathit{1}$. We put

$$V : \xi \in \mathfrak{X} \mapsto \mathit{1} \otimes \xi \in B_\Phi \otimes \mathfrak{X}.$$

Then V is a linear map of $\mathfrak{X}$ into $\mathcal{D}(\pi)$ such that $\lambda_\Phi(B_\Phi \otimes \mathfrak{X}) = \pi(B_\Phi)V\mathfrak{X}$ and Φ equals the completely positive invariant conjugate bilinear map $\Phi_{\{\pi,V\}}$ of Example 3.3. The maps π and V above are denoted with π_{B_ϕ} and V_Φ, respectively, since they are determined, respectively, by the core B_Φ and by Φ only.

In the case that Φ is quasi-invariant, π_{B_ϕ} is a quasi *-representation of $\mathfrak{A}$ and λ_Φ and V_Φ are defined in similar way as above.

Thus we have proved the following

Theorem 3.9. *Let $\mathfrak{A}$ be a partial *-algebra with identity $\mathit{1}$, $\mathfrak{X}$ a vector space and Φ a completely positive (quasi-) invariant conjugate bilinear map of $\mathcal{D}(\Phi) \otimes \mathcal{D}(\Phi)$ into $\mathbb{S}(\mathfrak{X})$. Then there exists a couple $(\pi_{B_\Phi}, \lambda_\Phi)$ consisting of a closed (quasi-)*

**-representation π_{B_Φ} of $\mathfrak{A}$ and a strongly cyclic vector representation λ_Φ of $\mathfrak{A}\otimes\mathfrak{X}$ for π_{B_Φ} with core B_Φ such that*

$$\Phi(ax, by)(\xi, \eta) = \langle \pi_{B_\Phi}(a)\lambda_\Phi(x \otimes \xi) \,|\pi_{B_\Phi}(b)\lambda_\Phi(y \otimes \eta)\rangle$$

for every $a, b \in \mathfrak{A}$, $x, y \in B_\Phi$ and $\xi, \eta \in \mathfrak{X}$. In particular, if $B_\Phi \ni 1$, then there exist a linear map V_Φ of $\mathfrak{X}$ into $\mathcal{D}(\pi_{B_\Phi})$ such that $\pi_{B_\Phi}(B_\Phi)V\mathfrak{X} = \lambda_\Phi(B_\phi \otimes \mathfrak{X})$.

Corollary 3.10. *Let Φ be a completely positive totally (quasi-) invariant conjugate-bilinear map of $\mathfrak{A} \times \mathfrak{A}$ into $\mathbb{S}(\mathfrak{X})$. Then the couple (π, V) of Theorem* 3.9 *is uniquely determined up to unitary equivalence.*

Proof. Let (ρ, W) be another couple consisting of a *-representation ρ of $\mathfrak{A}$ and a linear map W of $\mathfrak{X}$ into $\mathcal{D}(\rho)$ such that

(i) $\Phi(a, b)(\xi, \eta) = \langle \rho(a)W\xi \,|\rho(b)W\eta\rangle$ for every $a, b \in \mathfrak{A}$ and $\xi, \eta \in \mathfrak{X}$;
(ii) $\rho(R\mathfrak{A})W\mathfrak{X}$ is dense in $\mathcal{D}(\rho)[t_\rho]$.

We put

$$U\pi(a)V\xi = \rho(a)W\xi, \quad a \in \mathfrak{A}, \xi \in \mathfrak{X}.$$

Then U can be extended to a unitary operator of $\mathcal{H}_\pi$ onto $\mathcal{H}_\rho$. We denote this extension with the same symbol U. Since $\pi(R\mathfrak{A}))V\mathfrak{X}$ and $\rho(R\mathfrak{A}))W\mathfrak{X}$ are dense in $\mathcal{D}(\pi)[t_\pi]$ and $\mathcal{D}(\rho)[t_\rho]$, respectively, it is easily shown that $UV = W$, $U\mathcal{D}(\pi) = \mathcal{D}(\rho)$ and $\pi(a) = U^{-1}\rho(a)U$, for each $a \in \mathfrak{A}$. This completes the proof. □

The couples $(\pi_{B_\Phi}, \lambda_\Phi)$ and (π_{B_Φ}, V_Φ) for a completely positive (quasi-) invariant conjugate-bilinear map Φ with core B_Φ are called the *Stinespring dilations* of Φ determined by the core B_Φ.

In the case of a completely positive totally invariant conjugate-bilinear map Φ, $\pi_{R\mathfrak{A}}$ is determined by Φ only and so we denote it by π_Φ and (π_Φ, V_Φ) is called the *Stinespring dilation* of Φ.

Let Φ be a a completely positive (quasi-) invariant conjugate-bilinear map of $\mathcal{D}(\Phi) \times \mathcal{D}(\Phi)$ into $\mathbb{S}(\mathfrak{X})$ and denote by $\mathfrak{B}_\Phi$ the set of all cores for Φ. It may happen that $\pi_{B_\Phi} = \pi_{B'_\Phi}$ for $B_\Phi \neq B'_\Phi$, $B_\Phi, B'_\Phi \in \mathfrak{B}_\Phi$. However the set of all cores that yield the same representation has a maximal element. Indeed, we have:

Proposition 3.11. *Let Φ be a a completely positive (quasi-) invariant conjugate-bilinear map of $\mathcal{D}(\Phi) \times \mathcal{D}(\Phi)$ into $\mathbb{S}(\mathfrak{X})$ with core B_Φ. We put*

$$B^L_\Phi = \big\{x \in \mathcal{D}(\Phi) \cap R\mathfrak{A}; \lambda_\Phi(x \otimes \xi) \in \mathcal{D}(\pi_{B_\Phi}), \forall \xi \in \mathfrak{X};\, ax \in \mathcal{D}(\Phi),\\ \lambda_\Phi(ax \otimes \xi) = \pi_{B_\Phi}(a)\lambda_\Phi(x \otimes \xi), \forall a \in \mathfrak{A}, \xi \in \mathfrak{X}\big\}.$$

Then B^L_Φ is the largest among all cores B'_Φ for which $\pi_{B'_\Phi} = \pi_{B_\Phi}$.

Proof. It is easily shown that B^L_Φ is a core for Φ such that $\lambda_\Phi(B_\Phi \otimes \mathfrak{X}) \subset \lambda_\Phi(B^L_\Phi \otimes \mathfrak{X}) \subset \mathcal{D}(\pi_{B_\Phi})$ and $\pi_{B^L_\Phi} \lceil_{\lambda_\Phi(B^L_\Phi \otimes \mathfrak{X})} = \pi_{B_\Phi} \lceil_{\lambda_\Phi(B^L_\Phi \otimes \mathfrak{X})}$, which implies $\pi_{B^L_\Phi} = \pi_{B_\Phi}$. Take an arbitrary core B'_Φ for Φ such that $\pi_{B'_\Phi} = \pi_{B_\Phi}$. By the definition of B^L_Φ we have $B^L_\Phi \supset B'_\Phi$. Thus, B^L_Φ is the largest among the cores for Φ having the mentioned properties. This completes the proof. □

We put

$$\mathfrak{B}^L_\Phi = \{B_\Phi \in \mathfrak{B}_\Phi; B_\Phi = B^L_\Phi\}.$$

We obtain a unique characterization of a *-representation π_{B_Φ} in terms of a core B_Φ.

Proposition 3.12. *Let Φ be a a completely positive (quasi-) invariant conjugate-bilinear map of $\mathcal{D}(\Phi) \times \mathcal{D}(\Phi)$ into $\mathbb{S}(\mathfrak{X})$ and $B_\Phi, B'_\Phi \in \mathfrak{B}_\Phi$. Then the following statements hold:*

(1) $\pi_{B_\Phi} \subset \pi_{B'_\Phi}$ *if and only if* $B_\Phi \subset B'_\Phi$.

(2) $\pi_{B_\Phi} = \pi_{B'_\Phi}$ *if and only if* $B_\Phi = B'_\Phi$.

We now specialize the generalized Stinespring theorem that we have obtained to some particular cases. The first one is the case where $\mathfrak{A}$ is a locally convex quasi *-algebra. The second is the case of completely positive totally invariant conjugate-bilinear maps into partial O*-algebras.

Corollary 3.13. *Let $\mathfrak{A}$ be locally convex quasi*-algebra over $\mathfrak{A}_0$. Let Φ be the completely positive totally invariant conjugate-bilinear map of $\mathfrak{A} \times \mathfrak{A}$ into $\mathbb{S}(\mathfrak{X})$ defined in Example 3.6. Then the following statements hold:*

(1) $\lambda_\Phi(\mathfrak{A} \otimes \mathfrak{X}) = \pi_\Phi(\mathfrak{A}_0)V_\Phi\mathfrak{X}$ *is dense in* $\mathcal{D}(\pi_\Phi)[t_{\pi_\Phi}]$.

(2) $\pi_\Phi(\mathfrak{A}_0)$ *is an O*-algebra on* $\mathcal{D}(\pi_\Phi)$ *and* $\pi_\Phi \lceil_{\mathfrak{A}_0}$ *is a *-representation of the *-algebra* $\mathfrak{A}_0$ *with* $\mathcal{D}(\pi_\Phi \lceil_{\mathfrak{A}_0}) \subset \mathcal{D}(\pi_\Phi) = \mathcal{D}(\widetilde{\pi_\Phi \lceil_{\mathfrak{A}_0}})$.

(3) $\pi_\Phi(\mathfrak{A})'_{\mathrm{w}} = \pi_\Phi(\mathfrak{A}_0)'_{\mathrm{w}}$.

Let T be a conjugate-bilinear map of $\mathfrak{A} \times \mathfrak{A}$ into $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. If $a, b \in \mathfrak{A}$, we define a sesquilinear form on $\mathcal{D} \times \mathcal{D}$ by $\Phi_T(a,b)(\xi, \eta) = \langle T(a,b)\xi \,|\eta\rangle$, $\xi, \eta \in \mathcal{D}$. Then T is said to completely positive if Φ_T is completely positive. The notion of (quasi-) invariance for T is defined in similar way.

If T is completely positive and totally invariant, then it determines a couple $(\pi_{\Phi_T}, V_{\Phi_T})$ as described in Theorem 3.9. For shortness, we put $\pi_{\Phi_T} \equiv \pi_T$ and $V_{\Phi_T} = V_T$.

Corollary 3.14. *Let $\mathfrak{A}$ be a partial *-algebra with identity 1. Let $\mathcal{D}$ be a dense subspace of Hilbert space $\mathcal{H}$ and T a completely positive totally invariant conjugate-bilinear map of $\mathfrak{A} \times \mathfrak{A}$ into $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Then:*

(i) *$T(1,1)$ is a bounded operator if, and only if $\overline{V}_T$ is a bounded linear operator of $\mathcal{H}$ into $\mathcal{H}_{\pi_T}$.*

(ii) *$T(1,1)=I$ if, and only if $\overline{V}_T$ is an isometry of $\mathcal{H}$ into $\mathcal{H}_{\pi_T}$.*
Moreover, $T(a,1)=V_T^*\pi_T(a)V_T$, for all $a\in\mathfrak{A}$.

Proof. By Theorem 3.9, we have

$$\langle T(a,b)\xi\,|\eta\rangle = \langle \pi_T(a)V_T\xi\,|\pi_T(b)V_T\eta\rangle\,,\quad \forall a,b\in\mathfrak{A},\ \forall\xi,\eta\in\mathcal{D}.$$

Hence $\|V_T\xi\|^2=\langle T(1,1)\xi\,|\xi\rangle\,,\quad \forall\xi\in\mathcal{D}$. It is then easily shown that (i) and (ii) hold. Moreover

$$\langle T(a,1)\xi\,|\eta\rangle = \langle \pi_T(a)V_T\xi\,|V_T\eta\rangle = \langle V^*\pi_T(a)V_T\xi\,|\eta\rangle\,,\quad \forall a\in\mathfrak{A},\ \forall\xi,\eta\in\mathcal{D}.$$

Hence $T(a,1)=V_T^*\pi_T(a)V_T$, for all $a\in\mathfrak{A}$. □

4. Completely positive linear maps on partial O*-algebras

In this section we define and investigate completely positive invariant linear maps on partial O*-algebras. Let $\mathfrak{M}$ be a partial O*-algebra on $\mathcal{D}$ in $\mathcal{H}$ with identity operator I.

Definition 4.1. Let F be a linear map of $\mathfrak{M}$ into $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$. If there exists a completely positive conjugate-bilinear map $\mathring{F}$ of $\mathfrak{M}\times\mathfrak{M}$ into $\mathbb{S}(\mathcal{D})$ such that $\mathring{F}(A,I)=F(A)$ for all $A\in\mathfrak{M}$, then F is said to be *completely positive.* If $\mathring{F}$ is (totally) invariant, then F is said to be *(totally) invariant.*

By Theorem 3.9 and Corollary 3.14 we have the generalized Stinespring theorem for completely positive invariant linear maps on partial O*-algebras.

Theorem 4.2. *Suppose that F is a completely positive totally invariant linear map of $\mathfrak{M}$ into $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ such that $F(I)\in\mathfrak{B}(\mathcal{H})$ (resp. $F(I)=I$). Then there exists a couple (π_F,V_F) consisting of a closed $*$-representation π_F of $\mathfrak{M}$ and a bounded linear map (resp. an isometry) V_F of $\mathcal{D}$ into $\mathcal{D}(\pi_F)$ such that $F(A)=V_F^*\pi_F(A)V_F$ for all $A\in\mathfrak{M}$.*

We construct completely positive invariant linear maps on partial O*-algebras.

Proposition 4.3. *Let $\mathfrak{M}$ be a self-adjoint partial O*-algebra on $\mathcal{D}$ in $\mathcal{H}$ with identity operator I and F a linear map of $\mathfrak{M}$ into $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$. Suppose that*

(i) *$M(\mathfrak{M})\equiv R^{\mathrm{w}}(\mathfrak{M})^\dagger\cap R^{\mathrm{w}}(\mathfrak{M})$ is $\tau_{s^*}^{\mathcal{D}}$-dense in $\mathfrak{M}$;*

(ii) *F is $\tau_{\mathrm{w}}^{\mathcal{D}}$-continuous;*

(iii) *the restriction $F\lceil_{M(\mathfrak{M})}$ of F to the O*-algebra $M(\mathfrak{M})$ is completely positive.*

Then F is a completely positive invariant linear map on $\mathfrak{M}$ with core $M(\mathfrak{M})$.

Proof. For any $A, B \in \mathfrak{M}$ we put

$$\overset{\circ}{F}(A,B)(\xi,\eta) = \lim_{\alpha,\beta} \langle F(Y_\beta^\dagger X_\alpha)\xi \mid \eta \rangle, \quad \forall \xi, \eta \in \mathcal{D},$$

where $\{X_\alpha\}$ and $\{Y_\beta\}$ are nets in $M(\mathfrak{M})$ which converge to A and B with respect to the topology $\tau_{s^*}^{\mathcal{D}}$, respectively. Then it is shown that $\overset{\circ}{F}$ is a completely positive invariant conjugate-bilinear map on $\mathfrak{M} \times \mathfrak{M}$ with core $M(\mathfrak{M})$ such that $\overset{\circ}{F}(A, I) = F(A)$ for all $A \in \mathfrak{M}$. Hence F is a completely positive invariant linear map on $\mathfrak{M}$ with core $M(\mathfrak{M})$.

Corollary 4.4. *Let $\mathcal{L}^\dagger(\mathcal{D})_b$ be the *-algebra of all bounded operators in $\mathcal{L}^\dagger(\mathcal{D})$, and $\mathfrak{M}_0$ a *-subalgebra of $\mathcal{L}^\dagger(\mathcal{D})_b$ with identity operator I. Suppose that $\mathfrak{M}_0'\mathcal{D} \subset \mathcal{D}$ and $\widetilde{\mathfrak{M}_0[\tau_{s^*}^{\mathcal{D}}]}$ is fully closed. Then every $\tau_{\mathrm{w}}^{\mathcal{D}}$-continuous completely positive linear map F_0 of $\mathfrak{M}_0$ into $\mathfrak{B}(\mathcal{H})$ extends to a completely positive invariant linear map F on the partial GW*-algebra $\widetilde{\mathfrak{M}_0[\tau_{s^*}^{\mathcal{D}}]}$ with core $\mathfrak{M}_0$.*

Proof. By [1, Corollary 2.5.13] $\widetilde{\mathfrak{M}_0[\tau_{s^*}^{\mathcal{D}}]}$ is a partial GW*-algebra over $\mathfrak{M}_0''$ and $\mathfrak{M}_0 \subset R^{\mathrm{w}}(\widetilde{\mathfrak{M}_0[\tau_{s^*}^{\mathcal{D}}]})^\dagger \cap R^{\mathrm{w}}(\widetilde{\mathfrak{M}_0[\tau_{s^*}^{\mathcal{D}}]})$. Since F_0 is $\tau_{\mathrm{w}}^{\mathcal{D}}$-continuous, it extends to a $\tau_{\mathrm{w}}^{\mathcal{D}}$-continuous linear map F on $\widetilde{\mathfrak{M}_0[\tau_{s^*}^{\mathcal{D}}]}$. Thus $\widetilde{\mathfrak{M}_0[\tau_{s^*}^{\mathcal{D}}]}$ and F satisfy conditions (i)–(iii) in Proposition 4.3. Hence F is a completely positive invariant linear map on $\widetilde{\mathfrak{M}_0[\tau_{s^*}^{\mathcal{D}}]}$ with core $\mathfrak{M}_0$.

Example 4.5. Let $\mathfrak{M}_0$ be a von Neumann algebra on $\mathcal{H}$. Let T be a positive self-adjoint operator in $\mathcal{H}$ affiliated with $\mathfrak{M}_0$ and $\mathcal{D}^\infty(T) \equiv \bigcap_{n=1}^\infty \mathcal{D}(T^n)$. Every $\tau_{\mathrm{w}}^{\mathcal{D}^\infty(T)}$-continuous completely positive linear map F_0 of $\mathfrak{M}_0$ into $\mathfrak{B}(\mathcal{H})$ extends to a completely positive invariant linear map on $\widetilde{\mathfrak{M}_0\big[\tau_{s^*}^{\mathcal{D}^\infty(T)}\big]}$. Indeed, let $T = \int_0^\infty \lambda dE_T(\lambda)$ be a spectral resolution of T and $\mathfrak{N}_0$ a *-subalgebra generated by I and $\{E_T(m)XE_T(n); m,n \in \mathbb{N}, X \in \mathfrak{M}_0\}$. Then $\mathfrak{N}_0$ is a *-subalgebra of $\mathcal{L}^\dagger(\mathcal{D}^\infty(T))_b$ such that $\mathfrak{N}_0' = \mathfrak{M}_0'$, $\mathfrak{N}_0'\mathcal{D}^\infty(T) \subset \mathcal{D}^\infty(T)$ and $\widetilde{\mathfrak{N}_0\big[\tau_{s^*}^{\mathcal{D}^\infty(T)}\big]} = \widetilde{\mathfrak{M}_0\big[\tau_{s^*}^{\mathcal{D}^\infty(T)}\big]}$ is fully closed. Hence it follows from Corollary 4.4 that F_0 extends to a completely positive invariant linear map on $\widetilde{\mathfrak{M}_0\big[\tau_{s^*}^{\mathcal{D}^\infty(T)}\big]}$

In particular, every $\tau_{\mathrm{w}}^{\mathcal{D}^\infty(T)}$-continuous completely positive linear map of $\mathfrak{B}(\mathcal{H})$ into $\mathfrak{B}(\mathcal{H})$ extends to a completely positive invariant linear map on $\mathcal{L}^\dagger(\mathcal{D}^\infty(T), \mathcal{H})$.

5. Integrable extensions of *-representations of commutative locally convex quasi *-algebras

Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a locally convex quasi *-algebras with unit *1*. Let τ be the topology of $\mathfrak{A}$. Let also π be a closed *-representation of $\mathfrak{A}_0$ which is continuous from $\mathfrak{A}_0[\tau]$ to $\pi(\mathfrak{A}_0)[\tau_{s^*}^{\mathcal{D}(\pi)}]$. Then, for any $a \in \mathfrak{A}$ we put

$$\overline{\pi}(a)\xi = \lim_\alpha \pi(x_\alpha)\xi, \quad \xi \in \mathcal{D}(\pi),$$

where $\{x_\alpha\} \subset \mathfrak{A}_0$ is a net τ-converging to a. Then we have the following

Lemma 5.1. *$\overline{\pi}$ is a closed *-representation of $\mathfrak{A}$ with $\mathcal{D}(\overline{\pi}) = \mathcal{D}(\pi)$ such that:*

(i) $\overline{\pi}(x) = \pi(x), \quad \forall x \in \mathfrak{A}_0$;

(ii) $\overline{\pi}(\mathfrak{A})'_{\mathrm{w}} = \overline{\pi}(\mathfrak{A}_0)'_{\mathrm{w}}$.

Proof. First of all we observe that $\overline{\pi}$ is a *-representation of $\mathfrak{A}$ and the closedness of π implies the closedness of $\overline{\pi}$.

(ii) In general we have $\overline{\pi}(\mathfrak{A})'_{\mathrm{w}} \subset \overline{\pi}(\mathfrak{A}_0)'_{\mathrm{w}} = \pi(\mathfrak{A}_0)'_{\mathrm{w}}$. Viceversa, for all $C \in \pi(\mathfrak{A}_0)'_{\mathrm{w}}$ we have

$$\langle C\overline{\pi}(a)\xi \,|\eta\rangle = \lim_\alpha \langle C\pi(x_\alpha)\xi \,|\eta\rangle = \lim_\alpha \langle C\xi \,|\pi(x^*_\alpha)\eta\rangle = \langle C\xi \,|\overline{\pi}(a^*)\eta\rangle\,,$$

for all $a \in \mathfrak{A}$ and $\xi, \eta \in \mathcal{D}(\overline{\pi})$. Therefore $C \in \overline{\pi}(\mathfrak{A})'_{\mathrm{w}}$. □

In this section we investigate under which conditions $\overline{\pi}$ has an integrable extension, as an application of the results of the previous section. In other words, we generalize Schmüdgen's result ([8, Theorem 11.3.4]), originally given for *-algebras, to the case of partial *-algebras.

We denote by $M_n(\mathbb{C}[x_1, \dots, x_m])$ the set of all $n \times n$-matrices $(P_{kl}(x_1, \dots, x_m))$ of polynomials in the m variables $x_1, \dots, x_m$. An element (P_{kl}) of $M_n(\mathbb{C}[x_1, \dots, x_m])$ is said to be *positive definite* if, for any $(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}^m$, the matrix $(P_{kl}(\lambda_1, \lambda_2, \dots, \lambda_m))$ is positive semi-definite, that is $\sum_{k,l=1}^n P_{kl}(\lambda_1, \lambda_2, \dots, \lambda_m)\alpha_l\,\overline{\alpha}_k \geq 0$, for every $(\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{C}^m$. We now put $M(\mathbb{C}[x_1, \dots, x_m]) = \bigcup_{n\in\mathbb{N}} M_n(\mathbb{C}[x_1, \dots, x_m])$.

Definition 5.2. Let $\mathfrak{B}_0 = \{b_j;\, j \in J\}$ be a subset of $(\mathfrak{A}_0)_h = \{x \in \mathfrak{A}_0 : x^* = x\}$ such that $\mathfrak{B}_0 \cup \{1\}$ generates $\mathfrak{A}_0$. Let $M(\mathfrak{A}_0, \mathrm{int})_+$ be the set of all matrices in $M(\mathfrak{A}_0)_h$ of the form $(P_{kl}(b_{j1}, \dots, b_{jm}))$, where $m \in \mathbb{N}$, (P_{kl}) is a positive definite matrix of $M(\mathbb{C}[x_1, \dots, x_m])$ and $j_1, \dots, j_m \in J$.

By [8, Lemma 11.3.2], $M(\mathfrak{A}_0, \mathrm{int})_+$ is independent of $\mathfrak{B}_0$ and it is an m-admissible cone in $\mathfrak{A}_0$, that is:

- $M(\mathfrak{A}_0, \mathrm{int})_+ + M(\mathfrak{A}_0, \mathrm{int})_+ \subset M(\mathfrak{A}_0, \mathrm{int})_+$;
- $\lambda M(\mathfrak{A}_0, \mathrm{int})_+ \subset M(\mathfrak{A}_0, \mathrm{int})_+$ for all $\lambda \geq 0$;
- $M(\mathfrak{A}_0, \mathrm{int})_+ \cap (-M(\mathfrak{A}_0, \mathrm{int})_+) = \{0\}$;
- $\mathfrak{P}(\mathfrak{A}_0) \equiv \{\sum_{k=1}^n {}^*{}_k x^* x_k;\, x_k \in \mathfrak{A}_0\, (k = 1, \dots, n),\, n \in \mathbb{N}\} \subset M(\mathfrak{A}_0, \mathrm{int})_+$ and $x^* M(\mathfrak{A}_0, \mathrm{int})_+ x \subset M(\mathfrak{A}_0, \mathrm{int})_+$, for all $x \in \mathfrak{A}_0$.

Definition 5.3. A *-representation π of $\mathfrak{A}_0$ is said to be completely positive w.r.t. $M(\mathfrak{A}_0, \mathrm{int})_+$ if the sesquilinear form $\langle \pi(x)\cdot\,|\cdot\rangle$ on $\mathcal{D}(\pi) \times \mathcal{D}(\pi)$ for $x \in \mathfrak{A}_0$ is completely positive w.r.t. $M(\mathfrak{A}_0, \mathrm{int})_+$, that is if

$$\left\langle \sum_{k,l=1}^n (\pi(P_{kl}(b_{j1}, \dots, b_{jm}))\xi_k \,|\xi_l \right\rangle \geq 0$$

for each positive definite $(P_{kl}(b_{j1},\dots,b_{jm}))\in M_n(\mathbb{C}[x_1,\dots,x_m])$ and $\{\xi_1,\dots,\xi_n\} \subset \mathcal{D}(\pi)$, for each $n,m\in\mathbb{N}$.

Theorem 5.4. *Let $\mathfrak{A}=\widetilde{\mathfrak{A}_0}[\tau]$ be a commutative locally convex quasi *-algebra with identity 1 and π a closed *-representation of the *-algebra $\mathfrak{A}_0$ which is continuous from $\mathfrak{A}_0[\tau]$ to $\pi(\mathfrak{A}_0)\big[\tau_{s^*}^{\mathcal{D}(\pi)}\big]$. Then the following statements are equivalent:*

(i) *π is completely positive with respect to the cone $M(\mathfrak{A}_0, int)_+$.*

(ii) *There exists an integrable *-representation of $\mathfrak{A}$ in a larger Hilbert space which is an extention of $\overline{\pi}$.*

Proof. Theorem 11.3.4 of [8] ensures us of the existence of an integrable *-representation π_1 in a larger Hilbert space $\mathcal{H}_1$ such that:

(5.1) $\pi\subset\pi_1$;

(5.2) $(\pi_1(\mathfrak{A}_0)'_{\mathrm{w}})'$ is a commutative von Neumann algebra, see [7];

(5.3) $\pi_1(\mathfrak{A}_0)'_{\mathrm{w}}\mathcal{D}(\pi)$ is dense in $\mathcal{D}(\pi_1)[t_{\pi_1}]$.

We put

$$\rho(a)C\xi = C\overline{\pi}(a)\xi,$$

for $a\in\mathfrak{A}$, $C\in\pi_1(\mathfrak{A}_0)'_{\mathrm{w}}$ and $\xi\in\mathcal{D}(\pi)$. By (5.2) and (5.3), $\mathcal{H}_\rho$, the norm closure of $\pi_1(\mathfrak{A}_0)'_{\mathrm{w}}\mathcal{D}(\pi)$, equals $\mathcal{H}_{\pi_1}$.

First, we show that ρ is a *-representation of $\mathfrak{A}$ in $\mathcal{H}_\rho=\mathcal{H}_{\pi_1}$. Indeed, we have:

$$\langle\rho(a)C\xi\,|C'\eta\rangle = \langle C\xi\,|\rho(a^*)C'\eta\rangle, \quad \forall\, a\in\mathfrak{A},\ \forall\, C,\ C'\in\pi_1(\mathfrak{A}_0)'_{\mathrm{w}},\ \forall\xi,\ \eta\in\mathcal{D}(\pi).$$

This follows from the equalities

$$\begin{aligned}\langle\rho(a)C\xi\,|C'\eta\rangle &= \langle C'^*C\overline{\pi}(a)\xi\,|\eta\rangle\\ &= \lim_\alpha\langle C'^*C\pi_1(x_\alpha)\xi\,|\eta\rangle\\ &= \lim_\alpha\langle C'^*C\xi\,|\pi_1(x_\alpha^*)\eta\rangle\\ &= \langle C\xi\,|C'\overline{\pi}(a^*)\eta\rangle\\ &= \langle C\xi\,|\rho(a^*)C'\eta\rangle\end{aligned}$$

Moreover $\rho(a)\in\mathcal{L}^\dagger(\mathcal{D}(\rho),\mathcal{H}_\rho)$ is well-defined, where $\mathcal{D}(\rho)=\pi_1(\mathfrak{A}_0)'_{\mathrm{w}}\mathcal{D}(\pi)$.

If $a\in L(b)$, then $\rho(a)\,\Box\,\rho(b)=\rho(ab)$. Indeed, let $C,C'\in\pi_1(\mathfrak{A}_0)'_{\mathrm{w}}$ and $\xi,\eta\in\mathcal{D}(\pi)$ and assume, for the moment, that $a\in\mathfrak{A}_0$. Then, since π_1 is integrable, we have

$$\begin{aligned}
\langle \rho(ab)C\xi \,|C'\eta\rangle &= \langle C'^*C\overline{\pi}(ab)\xi \,|\eta\rangle \\
&= \langle C'^*C\overline{\pi}(a^*)^*\overline{\pi}(b)\xi \,|\eta\rangle \\
&= \langle C'^*C\pi(a^*)^*\overline{\pi}(b)\xi \,|\eta\rangle \\
&= \left\langle C'^*C\overline{\pi_1(a)}\,\overline{\pi}(b)\xi \,|\,\eta\right\rangle \\
&= \langle C'^*C\overline{\pi}(b)\xi \,|\pi_1(a^*)\eta\rangle \\
&= \langle C\overline{\pi}(b)\xi \,|C'\pi_1(a^*)\eta\rangle \\
&= \langle \rho(b)C\xi \,|\rho(a^*)C'\eta\rangle \,.
\end{aligned}$$

In the case where $b \in \mathfrak{A}_0$ the proof is slightly different. In this case, since $\pi(b)\xi$ belongs to $\mathcal{D}(\pi)$ we have

$$\begin{aligned}
\langle \rho(ab)C\xi \,|C'\eta\rangle &= \langle C'^*C\overline{\pi}(a^*)^*\overline{\pi}(b)\xi \,|\eta\rangle \\
&= \langle C'^*C\overline{\pi}(a)\pi(b)\xi \,|\eta\rangle \\
&= \lim_\alpha \langle C'^*C\overline{\pi}(x_\alpha)\pi(b)\xi \,|\eta\rangle \\
&= \lim_\alpha \langle C'^*C\overline{\pi}(b)\xi \,|\pi(x_\alpha^*)\eta\rangle \\
&= \langle C\overline{\pi}(b)\xi \,|C'\overline{\pi}(a^*)\eta\rangle \\
&= \langle \rho(b)C\xi \,|\rho(a^*)C'\eta\rangle \,.
\end{aligned}$$

Let us now prove that ρ is integrable. Indeed, we can first prove that $\pi_1(\mathfrak{A}_0)'_{\mathrm{w}} = \rho(\mathfrak{A})'_{\mathrm{w}}$. Let, in fact, $C \in \pi_1(\mathfrak{A}_0)'_{\mathrm{w}}$. Then, for all $a \in \mathfrak{A}$, $C_1, C_2 \in \pi_1(\mathfrak{A}_0)'_{\mathrm{w}}$ and for all $\xi, \eta \in \mathcal{D}(\pi)$, we have

$$\begin{aligned}
\langle C\rho(a)C_1\xi \,|C_2\eta\rangle &= \langle C\,C_1\overline{\pi}(a)\xi \,|C_2\eta\rangle \\
&= \lim_\alpha \langle C\,C_1\pi(x_\alpha)\xi \,|C_2\eta\rangle \\
&= \lim_\alpha \langle C\,C_1\xi \,|C_2\pi(x_\alpha^*)\eta\rangle \\
&= \langle C\,C_1\,\xi \,|C_2\overline{\pi}(a^*)\eta\rangle \\
&= \langle C\,C_1\xi \,|\rho(a^*)C_2\eta\rangle \,.
\end{aligned}$$

Therefore $\pi_1(\mathfrak{A}_0)'_{\mathrm{w}} \subset \rho(\mathfrak{A})'_{\mathrm{w}}$. Conversely, take an arbitrary $K \in \rho(\mathfrak{A})'_{\mathrm{w}}$, $C_1, C_2 \in \pi_1(\mathfrak{A}_0)'_{\mathrm{w}}$, $\xi_1, \xi_2 \in \mathcal{D}(\pi)$ and a generic element $x \in \mathfrak{A}_0$ we have:

$$\begin{aligned}
\langle K\pi_1(x)C_1\xi_1 \,|C_2\xi_2\rangle &= \langle K\,C_1\pi_1(x)\xi_1 \,|C_2\xi_2\rangle \\
&= \langle K\,C_1\overline{\pi}(x)\xi_1 \,|C_2\xi_2\rangle \\
&= \langle K\,\rho(x)C_1\xi_1 \,|C_2\xi_2\rangle \\
&= \langle K\,C_1\xi_1 \,|\rho(x^*)C_2\xi_2\rangle \\
&= \langle K\,C_1\xi_1 \,|\pi_1(x^*)C_2\xi_2\rangle \,.
\end{aligned}$$

Since $(\pi_1(\mathfrak{A}_0)'_{\mathrm{w}})'\mathcal{D}(\pi) \subset \pi_1(\mathfrak{A}_0)'_{\mathrm{w}}\mathcal{D}(\pi)$ is dense in $\mathcal{D}(\pi_1)[t_{\pi_1}]$, it follows that $K \in \pi_1(\mathfrak{A}_0)'_{\mathrm{w}}$. We finally show that the closure $\tilde{\rho}$ of ρ is integrable. Indeed, the equality $(\rho(\mathfrak{A})'_{\mathrm{w}})' = (\pi_1(\mathfrak{A}_0)'_{\mathrm{w}})'$ implies that $(\rho(\mathfrak{A})'_{\mathrm{w}})'$ is commutative and since $\rho(\mathfrak{A})'_{\mathrm{w}}\mathcal{D}(\rho) \subset \mathcal{D}(\rho)$, by [2, Theorem 3.1.3] it follows that $\tilde{\rho}$ is integrable.

Let us now prove the converse implication: (ii) $\Rightarrow$ (i). For this we consider an integrable *-representation ρ of $\mathfrak{A}$ in a larger Hilbert space which is an extension of $\overline{\pi}$. Since $\pi \subset \overline{\pi}$, $\widetilde{\rho \upharpoonright \mathfrak{A}_0}$ is an integrable *-representation of $\mathfrak{A}_0$ which is an extension of π, so that, by [8, Theorem 11.3.4], π is completely positive w.r.t. $M(\mathfrak{A}_0, \mathrm{int})_+$. This completes the proof. $\square$

Let f_0 be a positive linear functional on $\mathfrak{A}_0$ such that the sesquilinear form

$$(x, y) \in (\mathfrak{A}_0 \times \mathfrak{A}_0) \longrightarrow f(y^* x) \in \mathbb{C}$$

is continuous. We put

$$f(a, b) = \lim_{\lambda,\mu} f_0(y_\mu^* x_\lambda), \quad a, b \in \mathfrak{A}.$$

Then f is a positive sesquilinear form on $\mathfrak{A} \times \mathfrak{A}$, which is *a completely positive totally invariant conjugate-bilinear map on* $\mathfrak{A} \times \mathfrak{A}$ *into* $\mathbb{C}$. Then let (π_f, λ_f) be the GNS-construction relative to f, that is, the Stinespring dilation. By Theorem 5.4, we get the following

Corollary 5.5. *The following statements are equivalent:*

(i) $\pi_f \upharpoonright \mathfrak{A}_0$ *is completely positive w.r.t. the cone* $M(\mathfrak{A}_0, \mathrm{int})_+$.

(ii) *There exists an integrable *-representation of* $\mathfrak{A}$ *in a large Hilbert space* $\mathcal{H}$ *which is an extension of* π_f.

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