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On the Limiting Regularity Result of some Nonlinear Elliptic Equations

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Abstract. We shall prove the limiting regularity $W_0^{1,\frac{N(p-1)}{N-1}}(\Omega)$ of solutions of some nonlinear elliptic problems with right hand side in $LLog^\alpha L(\Omega)$ and $\alpha \geq \frac{N-1}{N}$. Also, an improved regularity is given when $\alpha < \frac{N-1}{N}$.

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1. Introduction

We deal with boundary value problems

$$\begin{cases} \mathcal{A}(u) := -\operatorname{div}(a(\cdot, u, \nabla u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{E}$$

where Ω is a regular bounded domain of $\mathbb{R}^N, N \geq 2, a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, measurable with respect to x in Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω). We assume that there exist a real positive constant $\nu > 0$, a nonnegative function k in $L^{p'}(\Omega), p' = \frac{p}{p-1}$, where $2 - \frac{1}{N} < p \leq N$, such that for almost every x in Ω , for every s in $\mathbb{R}$, for every ξ and ξ^* in $\mathbb{R}^N$:

$$a(x, s, \xi)\xi \geq \nu|\xi|^p \tag{1.1}$$

$$[a(x, s, \xi) - a(x, s, \xi^*)][\xi - \xi^*] > 0, \quad \xi \neq \xi^* \tag{1.2}$$

$$|a(x, s, \xi)| \leq k(x) + |s|^{p-1} + |\xi|^{p-1}. \tag{1.3}$$

The use of the $LLog^\alpha L(\Omega)$ space to study the problem (E) in the linear case, is

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early introduced by G. Stampacchia in [14] (for the case $\alpha = \frac{N-1}{N}$), by A. Passareli di Napoli and C. Sbordonne in [13] (for $0 < \alpha \leq 1$) and recently by A. Fiorenza and M. Krebec in [7] for the case $\alpha \geq \frac{N-1}{N}$. In the nonlinear case, the particular situations were given in [4]. Another approach to reach the limiting regularity was given in [3].

Our main result consists in reaching the limiting regularity $W_0^{1,\bar{q}}(\Omega), \bar{q} = \frac{N(p-1)}{N-1}$ with f belonging to the space $LLog^{\alpha}L(\Omega), \alpha \geq \frac{N-1}{N}$ in the nonlinear case.

For the sake of simplicity, we restrict our studies to the p-Laplacian problem model, i.e., $a(\cdot, u, \nabla u) = |\nabla u|^{p-2}\nabla u$.

2. Preliminaries

We list some well known results about Orlicz and Orlicz–Sobolev spaces.

2.1. Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-*function*, i.e., M is continuous, convex with $M(t) > 0$ for $t > 0, \frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. Equivalently, M admits the representation $M(t) = \int_0^t a(s)\,ds$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t) \to \infty$ as $t \to \infty$. The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \overline{a}(s)\,ds$, $\overline{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\overline{a}(t) = \sup\{s : a(s) \leq t\}$ (see [1, 10]). The N-function is said to satisfy the Δ_2-*condition* if, for some $k > 0$,

$$M(2t) \leq kM(t) \quad \forall t \geq 0. \tag{2.1}$$

If (2.1) holds only for $t \geq t_0 > 0$, then M is said to satisfy the Δ_2-condition near infinity.

We will extend these N-functions into even functions on all $\mathbb{R}$.

2.2. Let Ω be an open subset of $\mathbb{R}^N$. The *Orlicz class* $K_M(\Omega)$ (resp. the *Orlicz space* $L_M(\Omega)$) is defined as the set of (equivalences classes of) real valued measurable functions u on Ω such that $\int_\Omega M(u(x))\,dx < +\infty$ (resp. $\int_\Omega M(\frac{u(x)}{\lambda})\,dx < +\infty$ for some $\lambda > 0$). $L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx \leq 1\right\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2-condition, for all t or for t large according to whether Ω has infinite measure or not. The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of pairing $\int_\Omega u(x)v(x)\,dx$ and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalently to $\|u\|_{\overline{M},\Omega}$.

The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2-condition, for all t or for t large according to whether Ω has infinite measure or not.

2.3. We now turn to the *Orlicz–Sobolev space.* $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_{M,\Omega}.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of N + 1 copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\prod L_M, \prod L_{\overline{M}})$ (see [8, 9]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined.

We denote by $LLog^\alpha L(\Omega)$ the Orlicz space $L_M(\Omega)$ where $M(t) \sim t\ln^\alpha(t)$ as $t \to \infty$.

The following abstract lemma will be applied in the following.

Lemma 2.1 ([2]). *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian with $F(0) = 0$. Let M be an N-function and let $u \in W_0^1L_M(\Omega)$ (resp. $W_0^1E_M(\Omega)$). Then $F(u) \in W_0^1L_M(\Omega)$ (resp. $W_0^1E_M(\Omega)$). Moreover, if the set of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \ in \ \{x \in \Omega : u(x) \notin D\} \\ 0 & a.e. \ in \ \{x \in \Omega : u(x) \in D\}. \end{cases}$$

3. Main result

Let M be an N-function such that

$$\text{(H)} \quad \begin{cases} K(s) = \left(M^{-1}(s)\right)^p \text{ is convex and} \\ \int_0^{\cdot} M\circ B^{-1}\left(\frac{1}{r^{1-\frac{1}{N}} Log^\alpha(\frac{1}{r})}\right) dr < +\infty, \ B(t) = t^{p-1}. \end{cases}$$

Theorem 3.1. *Under the assumptions* (1.1)–(1.3), $2-\frac{1}{N}<p\leq N$ *and* f *in* $LLog^{\alpha}L(\Omega)$ *with* $\alpha\geq\frac{N-1}{N}$, *there exists at least a weak solution* $u\in W_0^{1,\bar{q}}(\Omega)$ *of problem* (E) *where* $\bar{q}=\frac{N(p-1)}{N-1}$. *Moreover, if* $\alpha>\frac{N-1}{N}$, *then* $u\in W_0^1L_M(\Omega)$ *for every N-function* M *satisfying* (H).

Remark 3.2. The proof of the last theorem allows us to obtain an improved regularity of the solution u of (E) in the Orlicz–Sobolev spaces. For example,

$$\begin{array}{ll} u\in W_0^1L_M(\Omega), M(t)=\frac{t^{\bar{q}}}{Log^{\sigma}(e+t)}, & \text{for all } \sigma>1-\frac{\alpha N}{N-1} \text{ if } \alpha\in[0,\frac{N-1}{N}[\\ u\in W_0^1L_M(\Omega), M(t)=t^{\bar{q}}Log^{\sigma}(e+t), & \text{for all } \sigma<\frac{\alpha N}{N-1}-1 \text{ if } \alpha>\frac{N-1}{N}. \end{array}$$

For the case $\alpha=\frac{N-1}{N}, p<N$, the regularity $W_0^{1,\bar{q}}(\Omega)$ is optimal.

Hereafter, we denote by $\mathcal{X}_N$ the real number defined by $\mathcal{X}_N=NC_N^{\frac{1}{N}}$, C_N is the measure of the unit ball of $\mathbb{R}^N$, $\mu(t)=\text{meas}\,\{|u|>t\}$.

The following lemma (see [15] for the general case) plays an essential role for estimation of the approximate solutions of the problem .

Lemma 3.3. *Let* $u\in W_0^{1,p}(\Omega), 1<p<+\infty$. *Then*

$$-\mu'(t)\geq\mathcal{X}_N\mu(t)^{1-\frac{1}{N}}\left(-\frac{1}{\mathcal{X}_N\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}}|\nabla u|^p\,dx\right)^{-\frac{1}{p-1}}.$$

Proof of Theorem 3.1. If $\alpha>\frac{N-1}{N}, 2-\frac{1}{N}<p\leq N$, then we consider the approximate problem

$$\begin{cases}\mathcal{A}(u_n):=-\operatorname{div}(a(\cdot,u_n,\nabla u_n))=f_n \quad \text{in } \Omega\\ \qquad\qquad\qquad\qquad u_n\in W_0^{1,p}(\Omega),\end{cases}\tag{3.1}$$

where (f_n) is a smooth sequence of functions satisfying $f_n\to f$ in $L_H(\Omega)$ for the modular convergence, $H(t)=tLog^{\alpha}(1+t)$.

Let φ be a truncation defined by

$$\varphi(\xi)=\begin{cases}0, & 0\leq\xi\leq t\\ \frac{1}{h}(\xi-t), & t<\xi<t+h\\ 1, & \xi\geq t+h\\ -\varphi(-\xi), & \xi<0,\end{cases}\tag{3.2}$$

for all t, $h>0$. Without loss of generality, we omit the index n. Using $v=\varphi(u)$ as a test function in (3.1), we obtain

$$\int_{\Omega}a(\cdot,u,\nabla u)\nabla u\varphi'(u)\,dx=\int_{\Omega}f\varphi(u)\,dx$$

$$\frac{1}{h}\int_{\{t<|u|<t+h\}}|\nabla u|^p\,dx\leq C\int_{\{|u|\geq t+h\}}f\,dx\,.$$

And letting $h \to 0$, we have

$$-\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|^p \, dx \le C \int_{\{|u|\ge t\}} f \, dx \,. \tag{3.3}$$

By using Lemma 3.3 we obtain (supposing $-\mu'(t) > 0$ which does not affected the proof)

$$\frac{1}{\mu'(t)}\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|^p \, dx \le \Big(-\frac{1}{\mathcal{X}_N \mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|^p \, dx\Big)^{\frac{p}{p-1}}$$

or equivalently

$$\Big(\frac{1}{\mu'(t)}\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|^p \, dx\Big)^{\frac{1}{p}} \le \Big(-\frac{1}{\mathcal{X}_N \mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|^p \, dx\Big)^{\frac{1}{p-1}}$$

Let M be an N-function satisfying (H). Jensen's inequality involves

$$K\left(\frac{\int_{\{t<|u|<t+h\}} M(|\nabla u|)}{-\mu(t+h)+\mu(t)}\right) \le \frac{\int_{\{t<|u|<t+h\}} (|\nabla u|)^p}{-\mu(t+h)+\mu(t)}.$$

Then

$$\begin{aligned} M^{-1}\left(\frac{1}{\mu'(t)}\frac{d}{dt}\int_{\{|u|>t\}} M(|\nabla u|)\, dx\right) &\le \left(\frac{1}{\mu'(t)}\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|^p \, dx\right)^{\frac{1}{p}} \\ &\le \Big(-\frac{1}{\mathcal{X}_N \mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|^p \, dx\Big)^{\frac{1}{p-1}}. \end{aligned}$$

Therefore we have

$$-\frac{d}{dt}\int_{\{|u|>t\}} M(|\nabla u|)\, dx \le (-\mu'(t)) M\left(\Big(-\frac{1}{\mathcal{X}_N \mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|^p \, dx\Big)^{\frac{1}{p-1}}\right).$$

Combining with (3.3) and the fact that the function $t \to \int_{\{|u|>t\}} M(|\nabla u|)\, dx$ is absolutely continuous, we obtain

$$\begin{aligned} \int_\Omega M(|\nabla u|)\, dx &= \int_0^{+\infty} \Big(-\frac{d}{dt}\int_{\{|u|>t\}} M(|\nabla u|)\, dx\Big)\, dt \\ &\le \int_0^{+\infty} (-\mu'(t)) M\left(\left(\frac{C\int_{\{|u|\ge t\}} f\, dx}{\mathcal{X}_N \mu(t)^{1-\frac{1}{N}}}\right)^{\frac{1}{p-1}}\right) dt \\ &\le \frac{1}{C'}\int_0^{C'|\Omega|} M\left(\left(\frac{C}{r^{1-\frac{1}{N}} Log^{\alpha}(\frac{1}{r})}\right)^{\frac{1}{p-1}}\right) dr < \infty, \end{aligned}$$

where $C' = (\frac{\mathcal{X}_N}{C})^{N'}$, and the last inequality is obtained by using the Hölder inequality on $\int_{\{|u|\geq t\}} f\,dx$.

Finally, we deduce that $(\nabla u_n)_{n\geq 0}$ is bounded in $(L_M(\Omega))^N$ for every N-function satisfying (H). In particular, $(\nabla u_n)_{n\geq 0}$ is bounded in $(L^{\bar{q}}(\Omega))^N$. As in [4], the almost everywhere convergence of the gradients can be obtained and the proof of theorem follows with the same way.

We deal now with the case $\alpha = \frac{N-1}{N}$ and $2 - \frac{1}{N} < p < N$. We recall that the authors in [4] have proved some regularity result but by assuming that $\alpha = \frac{N}{N-1}$ and $p = N$. Consider now the following approximate problems:

$$\begin{cases} -\operatorname{div}\big(a(x,u_n,\nabla u_n)\big) - \dfrac{1}{n}\operatorname{div}\big(|\nabla u_n|^{N-2}\nabla u_n\big) = f_n \quad \text{in } \Omega \\ \hfill u_n \in W_0^{1,N}(\Omega)\,. \end{cases} \tag{3.4}$$

The solutions u_n exist thanks to the Leray–Lions theorem (see [11]). Taking $v = u_n$ as test function in the problem (3.4), we have

$$\int_\Omega |\nabla u_n|^p\,dx + \frac{1}{n}\int_\Omega |\nabla u_n|^N\,dx \leq 2\|f_n\|_A\|u_n\|_{\bar{A}} \leq C\|u_n\|_{W_0^{1,N}}\,,$$

where we have used the continuous and optimal injection $W_0^{1,N}(\Omega) \hookrightarrow L_{\bar{A}}(\Omega)$ with $\bar{A}(t) = e^{t^{N'}} - 1$ (see [5]). Then we deduce $\frac{1}{n}(\int_\Omega |\nabla u_n|^N\,dx)^{\frac{N-1}{N}} \leq C$. Let now $\phi \in W_0^{1,N}(\Omega)$ as test function, one has

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\phi\,dx + \frac{1}{n}\int_\Omega |\nabla u_n|^{N-2}\nabla u_n\nabla\phi\,dx = \int_\Omega f_n\phi\,dx\,,$$

so

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\phi\,dx \leq \left|\frac{1}{n}\int_\Omega |\nabla u_n|^{N-2}\nabla u_n\nabla\phi\,dx\right| + C\|\phi\|_{W_0^{1,N}} \leq C\|\phi\|_{W_0^{1,N}}$$

which implies, thanks to [6, Theorem 4.1], that $\int_\Omega |\nabla u_n|^{\bar{q}}\,dx \leq C$, where here and below C denote positive constants not depending on n. Therefore, we can see that there exist a measurable function $u \in W_0^{1,\bar{q}}(\Omega)$ and a subsequence also denoted $(u_n)_n$,

$$\begin{aligned} u_n &\to u && \text{weakly in } W_0^{1,\bar{q}}(\Omega) \\ T_k(u_n) &\to T_k(u) && \text{weakly in } W_0^{1,p}(\Omega)\,, \end{aligned}$$

where T_k is the usual truncation defined by $T_k(s) = \max(-k, \min(k,s))$, for all $s \in \mathbb{R}$, for all $k \geq 0$. Let $v \in \mathcal{D}(\Omega)$ and choose the test function $T_k(u_n - v), n > k + \|v\|_\infty$, in the approximate problem, we have

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla T_k(u_n - v)\,dx + \frac{1}{n}\int_\Omega |\nabla u_n|^{N-2}\nabla u_n\nabla T_k(u_n - v)\,dx \\ &= \int_\Omega f_n T_k(u_n - v)\,dx \end{aligned}$$

which we rewrite as follows:

$$
\begin{aligned}
&\int_\Omega \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla v|^{p-2}\nabla v\big)\nabla T_k(u_n - v)\, dx \\
&+ \int_\Omega |\nabla v|^{p-2}\nabla v \nabla T_k(u_n - v)\, dx \\
&+ \frac{1}{n}\int_\Omega \big(|\nabla u_n|^{N-2}\nabla u_n - |\nabla v|^{N-2}\nabla v\big)\nabla T_k(u_n - v)\, dx \\
&+ \frac{1}{n}\int_\Omega |\nabla v|^{N-2}\nabla v \nabla T_k(u_n - v)\, dx \\
= &\int_\Omega f_n T_k(u_n - v)\, dx\,.
\end{aligned}
$$

This obviously gives

$$\int_\Omega |\nabla v|^{p-2}\nabla v\nabla T_k(u_n - v) dx + \frac{1}{n}\int_\Omega |\nabla v|^{N-2}\nabla v \nabla T_k(u_n - v) dx \le \int_\Omega f_n T_k(u_n - v) dx.$$

By using the fact that $T_k(u_n - v) \to T_k(u - v)$ weakly in $W_0^{1,p}(\Omega)$, we obtain

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla T_k(u - v)\, dx \le \int_\Omega f T_k(u - v)\, dx, \quad \forall v \in \mathcal{D}(\Omega),$$

By the density argument the last inequality remains true for all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

To prove that u is a weak solution of the problem (E), we follow the technique used in [12]. Let h and k be positive real numbers, let t belong to $(-1, 1)$ and let ψ be a function in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Choose $\phi = T_h(u) + tT_k(u - \psi)$ in the previous inequality, we have u is a so-called entropy solution of (E) which completes the proof of the theorem. □

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