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Partial differential equations. — *Intrinsic Harnack inequalities for quasi-linear singular parabolic partial differential equations*, by EMMANUELE DIBENEDETTO, UGO GIANAZZA and VINCENZO VESPRI, communicated on 6 July 2007.

ABSTRACT. — Intrinsic Harnack estimates for non-negative solutions of singular, quasi-linear, parabolic equations are established, including the prototype p-Laplacian equation (1.4) below. For p in the supercritical range $2N/(N+1) < p < 2$, the Harnack inequality is shown to hold in a parabolic form, both forward and backward in time, and in an elliptic form at fixed time. These estimates fail for the heat equation ($p \to 2$). It is shown by counterexamples that they fail for p in the subcritical range $1 < p \le 2N/(N+1)$. Thus the indicated supercritical range is optimal for a Harnack estimate to hold. The novel proofs are based on measure-theoretical arguments, as opposed to comparison principles, and are sufficiently flexible to hold for a large class of singular parabolic equations including the porous medium equation and its quasi-linear versions.

KEY WORDS: Singular parabolic equations; intrinsic Harnack estimates.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 35K65, 35B65; Secondary 35B45.

1. MAIN RESULT

Let E be an open set in $\mathbb{R}^N$ and for $T > 0$ let $E_T = E \times (0, T]$. Let u be a weak solution,

$$u \in C_{\mathrm{loc}}(0, T; L^2_{\mathrm{loc}}(E)) \cap L^p_{\mathrm{loc}}(0, T; W^{1,p}_{\mathrm{loc}}(E)), \qquad 1 < p < 2, \tag{1.1}$$

of a quasi-linear, singular parabolic equation of the type

$$u_t - \operatorname{div} \mathbf{A}(x, t, u, Du) = B(x, t, u, Du) \quad \text{weakly in } E_T \tag{1.2}$$

where the functions $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ and $B : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}$ are only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x, t, u, Du) \cdot Du \ge C_0 |Du|^p - C^p \\ |\mathbf{A}(x, t, u, Du)| \le C_1 |Du|^{p-1} + C^{p-1} \\ |B(x, t, u, Du)| \le C |Du|^{p-1} + C^p \end{cases} \quad \text{a.e. in } E_T, \tag{1.3}$$

where $p \in (1, 2)$ and C_0 and C_1 are given positive constants, and C is a given non-negative constant. If u is a weak solution of (1.1)–(1.2), the quasi-linear structure conditions (1.3) are in addition required to ensure that the truncations $\pm(u - k)_\pm$ are sub(super)solutions for all $k \in \mathbb{R}$. Namely,

$$\frac{\partial}{\partial t}(u-k)_\pm - \operatorname{div} \mathbf{A}(x, t, (u-k)_\pm, D(u-k)_\pm) \le B(x, t, (u-k)_\pm, D(u-k)_\pm) \tag{1.2$_\pm$}$$

weakly in E_T against admissible non-negative test functions. The prototype example is

$$u_t - \operatorname{div} |Du|^{p-2} Du = 0, \quad 1 < p < 2, \quad \text{weakly in } E_T. \tag{1.4}$$

Equation (1.1)–(1.2) is singular, since its modulus of ellipticity goes to ∞ as $|Du| \to 0$. We show that its non-negative weak solutions satisfy an intrinsic form of the Harnack inequality provided p is in the so called *supercritical* range

$$p_* = \frac{2N}{N+1} < p < 2. \tag{1.5}$$

The parameters $\{N, p, C_0, C_1, C\}$ are the data and we say that a generic constant $\gamma = \gamma(N, p, C_0, C_1, C)$ *depends upon the data* if it can be quantitatively determined a priori only in terms of the indicated parameters. For $\rho > 0$ let K_ρ be the cube of center the origin of $\mathbb{R}^N$ and edge 2ρ, and for $y \in \mathbb{R}^N$ let $K_\rho(y)$ denote the homothetic cube centered at y. Fix $P_0 = (x_0, t_0) \in E_T$ such that $u(x_0, t_0) > 0$, and consider cylinders of the type

$$Q_\rho(P_0) = K_\rho(x_0) \times \left\{ t_0 - \left(\frac{u(P_0)}{c^4}\right)^{2-p} \rho^p < t \le t_0 + \left(\frac{u(P_0)}{c^4}\right)^{2-p} \rho^p \right\}, \tag{1.6}$$

where c is the constant of Theorem 1.1 below. These cylinders are "intrinsic" to the solution since their time length is determined by the value of u at (x_0, t_0), and the Harnack inequality holds in such an intrinsic geometry.

THEOREM 1.1. *Let u be a non-negative, weak solution to* (1.1)–(1.3) *for p in the supercritical range* (1.5). *There exist positive constants δ_* and c, depending only upon the data, such that for all $P_0 \in E_T$ and all cylinders of the type $Q_{8\rho}(P_0) \subset E_T$, either $u(P_0) \le C\rho$, or*

$$c u(x_0, t_0) \le \inf_{K_\rho(x_0)} u(\cdot, t) \tag{1.7}$$

for all times t satisfying

$$t_0 - \delta_* [u(P_0)]^{2-p} \rho^p \le t \le t_0 + \delta_* [u(P_0)]^{2-p} \rho^p. \tag{1.8}$$

The constants c and δ_* tend to zero as either $p \to 2$ or $p \to p_*$.

This inequality is simultaneously a "forward and backward in time" Harnack estimate as well as a Harnack estimate of elliptic type. Any of these three types of inequalities would be false for non-negative solutions of the heat equation. This is reflected in (1.7)–(1.8), as the constants c and δ_* tend to zero as $p \to 2$. It turns out that these inequalities lose meaning also as p tends to the critical value p_* in (1.5). We comment on each of these aspects separately.

2. THE FORWARD IN TIME HARNACK INEQUALITY

A forward Harnack estimate can be established independently of Theorem 1.1 and it takes the following form.

THEOREM 2.1. *Let u be a non-negative weak solution to* (1.1)–(1.3) *for p in the supercritical range* (1.5)*. There exist positive constants c_+, δ_+ such that for all cylinders*

$$K_{8\rho}(x_0) \times \left\{ t_0 - \left(\frac{u(P_0)}{c_+^4} \right)^{2-p} (8\rho)^p < t \le t_0 + \left(\frac{u(P_0)}{c_+^4} \right)^{2-p} (8\rho)^p \right\}$$

contained in E_T, either

$$u(P_0) < C\rho,$$

or

$$c_+ u(x_0, t_0) \le \inf_{K_\rho(x_0)} u(x, t_0 + \delta_+ [u(P_0)]^{2-p} \rho^p). \tag{2.1}$$

The constants c_+ and δ_+ tend to zero as $p \to p_*$ but they are "stable" as $p \to 2$, in the sense that there exist positive constants $c_+(2)$ and $\delta_+(2)$, which can be determined a priori only in terms of the data, such that $c_+(p), \delta_+(p) \to c_+(2), \delta_+(2)$ as $p \to 2$. Thus by formally letting $p \to 2$ in (2.1) one recovers Moser's classical Harnack inequality of [11].

A positive waiting time is needed for a Harnack estimate to hold even for non-negative solutions of the heat equation, as pointed out by a counterexample of Moser ([11]). The novelty of (2.1) is in that such a waiting time is intrinsic to the solution itself. No forward in time Harnack estimate would be possible for non-negative solutions of (1.1)–(1.3) if the waiting time were not driven by the solution itself. Indeed, weak non-negative solutions of (1.4) in bounded domains, with homogeneous Dirichlet data on ∂E and non-negative initial data u_0, become extinct, abruptly, in finite time. That is, there exists a time T, which can be determined a priori in terms of the data and u_0, such that for all $x \in E$ ([3, Chap. VII, § 2])

$$u(x, t) > 0 \quad \text{for } t < T \quad \text{and} \quad u(x, t) = 0 \quad \text{for } t > T. \tag{2.2}$$

For such a solution, a Harnack estimate with waiting time independent of u would not hold.

3. THE ELLIPTIC HARNACK INEQUALITY

A consequence of (1.7)–(1.8) is the following elliptic form of the Harnack inequality.

COROLLARY 3.1. *Let u be a non-negative weak solution to* (1.1)–(1.3) *for p in the supercritical range* (1.5)*. There exists a positive constant c, depending only upon the data, such that for all $P_0 \in E_T$ and all cylinders of the type $Q_{8\rho}(P_0) \subset E_T$, either*

$$u(P_0) \le C\rho,$$

or

$$c u(x_0, t_0) \le \inf_{K_\rho(x_0)} u(\cdot, t_0) \tag{3.1}$$

The constant c tends to zero as either $p \to 2$ or $p \to p_*$.

While unusual, such an inequality can be understood by examining the nature of (1.4). As $|Du| \approx 0$, the modulus of ellipticity becomes large and the p.d.e. tends to favour its elliptic component. The inequality (3.1) makes this heuristic argument quantitatively precise. The parabolic component enters in that u is required to exist for a sufficiently large time interval about t_0.

4. THE BACKWARD IN TIME HARNACK INEQUALITY

Another consequence of (1.7)–(1.8) is a backward Harnack estimate in the following form.

COROLLARY 4.1. *Let u be a non-negative weak solution to* (1.1)–(1.3) *for p in the supercritical range* (1.5). *There exist positive constants δ_* and c, depending only upon the data, such that for all $P_0 \in E_T$ and all cylinders of the type $Q_{8\rho}(P_0) \subset E_T$, either*

$$u(P_0) \leq C\rho,$$

or

$$cu(x_0, t_0) \leq \inf_{K_\rho(x_0)} u(\cdot, t_0 - \delta_*[u(P_0)]^{2-p}\rho^p). \tag{4.1}$$

The constants c and δ_* tend to zero as either $p \to 2$ or $p \to p_*$.

While unexpected, this occurrence reflects the tendency of the solution to become extinct in finite time, as indicated in (2.2). Notice that we have a backward inequality, but the time is not reversed. Indeed, for (4.1) to hold, the solution u is required to exist in a large time interval about t_0. Nevertheless this remains the most intriguing aspect of these inequalities.

5. NOVELTY AND SIGNIFICANCE

In [6] a detailed discussion will be given to show that the range of p in (1.5) is optimal for the Harnack estimate (1.7)–(1.8) to hold. Indeed, for p in the subcritical range $1 < p \leq p_*$, explicit counterexamples are provided which fail to satisfy the Harnack inequality in any one of the forward, backward, or elliptic form. This raises the question of what form, if any, the Harnack estimate might take for p in such a range.

For non-negative solutions of the prototype homogeneous equation (1.4), intrinsic Harnack inequalities in the forward form (2.1) and the elliptic form (3.1) were established in a series of contributions ([7, 8]), collected and re-organized in [3]. These proofs, one way or another, had at their root the application of the maximum principle by comparing, locally, the solutions of (1.4) with either the explicit Barenblatt solutions ([3]), or some suitably constructed subsolution ([7]).

The original proofs of the parabolic Harnack inequality for non-negative solutions of the heat equation, due independently to Hadamard [9] and Pini [13], were based on local comparisons with caloric potentials. The leap forward achieved by Moser ([10, 11, 12]) consists in replacing comparison methods by measure-theoretical arguments. This is precisely one of the key novel points of this contribution, that is, the Harnack inequalities

(1.7)–(2.1) are established by entirely measure-theoretical arguments, thereby bypassing any form of comparison principle. These methods are rather different than the classical techniques of De Giorgi [2] and Moser [11], and are based on two technical tools, namely

- L^1_{loc}-L^∞_{loc} Harnack-type estimates for p in the supercritical range;
- a proper expansion of positivity based on an iteration argument originally introduced in [1].

For degenerate equations (1.1)–(1.3) with $p \geq 2$ a reasonably complete theory of the intrinsic forward Harnack inequality has recently been established in [4, 5], to which we refer for further comments.

A second key novel point is the backward inequality in the form (4.1). The latter has never been observed before, not even for the prototype equation (1.4), and it opens an intriguing issue on the local behaviour of solutions of such singular equations.

The approach is sufficiently general to apply, upon minor modifications, to non-negative weak solutions of a class of singular parabolic equations, including quasi-linear versions of the singular porous medium equations. We refer to [6] for full details and complete proofs.

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