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Existence of weak solutions for the incompressible Euler equations

Emil Wiedemann

Institut für Angewandte Mathematik, Universität Bonn, Bonn, Germany

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Abstract

Using a recent result of C. De Lellis and L. Székelyhidi Jr. (2010) [2] we show that, in the case of periodic boundary conditions and for arbitrary space dimension $d \geqslant 2$, there exist infinitely many global weak solutions to the incompressible Euler equations with initial data v_0, where v_0 may be any solenoidal L^2-vectorfield. In addition, the energy of these solutions is bounded in time.

1. Introduction

Let $Q=[0,2\pi]^d$, $d\geqslant 2$, and $L^2_{\mathrm{per}}(Q)$ be the space of Q-periodic functions in $L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, i.e. $u(x+2\pi l)=u(x)$ for a.e. $x\in\mathbb{R}^d$ and every $l\in\mathbb{Z}^d$. Then, as usual when dealing with periodic boundary conditions for fluid equations (cf. for instance [1]), we define the space

$$H^m_{\mathrm{per}}(Q)=\left\{v\in L^2_{\mathrm{per}}(Q)\colon \sum_{k\in\mathbb{Z}^d}|k|^{2m}\left|\hat{v}(k)\right|^2<\infty,\ \hat{v}(k)\cdot k=0 \text{ for every } k\in\mathbb{Z}^d,\ \text{and } \hat{v}(0)=0\right\},$$

where $\hat{v}:\mathbb{Z}^d\to\mathbb{C}^d$ denotes the Fourier transform of v. We shall write $H(Q)$ instead of $H^0_{\mathrm{per}}(Q)$ and $H_w(Q)$ for the space $H(Q)$ equipped with the weak L^2 topology.

Recall the incompressible Euler equations

$$\begin{aligned}&\partial_t v+\operatorname{div}(v\otimes v)+\nabla p=0,\\&\operatorname{div} v=0,\end{aligned}$$

where $v\otimes v$ is the matrix with entries $v_i v_j$ and the divergence is taken row-wise. A vectorfield $v\in L^\infty((0,\infty);H(Q))$ is called a *weak solution* of these equations with Q-periodic boundary conditions and initial data $v_0\in H(Q)$ if

$$\int_0^\infty\int_Q (v\cdot\partial_t\phi+v\otimes v:\nabla\phi)\,dx\,dt+\int_Q v_0(x)\phi(x,0)\,dx=0$$

for every Q-periodic divergence-free $\phi\in C_c^\infty(\mathbb{R}^d\times[0,\infty);\mathbb{R}^d)$.

E-mail address: emil.wiedemann@hcm.uni-bonn.de.

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Unlike in the case of Navier–Stokes equations, for which the existence of global weak solutions has been known since the work [3] of J. Leray, the existence problem for weak solutions of Euler has remained open so far. In this paper we show that the existence of weak solutions is a consequence of C. De Lellis' and L. Székelyhidi's work [2]. More precisely, we have

Theorem 1. *Let $v_0 \in H(Q)$. Then there exists a weak solution $v \in C([0,\infty); H_w(Q))$ (in fact, infinitely many) of the Euler equations with $v(0) = v_0$. Moreover, the kinetic energy*

$$E(t) := \frac{1}{2}\int_Q \left|v(x,t)\right|^2 dx$$

is bounded and satisfies $E(t) \to 0$ as $t \to \infty$.

Note that the condition $\hat{v}(0) = 0$ in the definition of $H(Q)$, i.e. $\int_Q v\,dx = 0$, is no actual constraint due to Galilean invariance of the Euler equations.

Our proof of this theorem is very simple: Owing to [2], it suffices to construct a suitable so-called *subsolution* with the desired initial data; we obtain such a subsolution by solving the Cauchy problem for the fractional heat equation

$$\partial_t v + (-\Delta)^{1/2} v = 0,$$
$$v(\cdot,0) = v_0,$$

which is not difficult since, owing to periodicity, we can work in Fourier space.

Although our solutions have bounded energy, they do not satisfy any form of the energy inequality. Indeed, they exhibit an increase in energy at least at time $t = 0$, and this increase will be discontinuous (this follows from $e(v_0, u_0) > \frac{1}{2}|v_0|^2$ in the proof below). If one requires, in contrast, that the energy be bounded at all times by the initial energy, then existence of such weak solutions is not known for arbitrary initial data (but only for an L^2-dense subset of initial data, see [4]). In fact, it is impossible to deduce from Theorem 2 below such an existence theorem, since for smooth initial data the existence of infinitely many weak solutions would contradict well-known local existence results and weak–strong uniqueness, see [2, Section 2.3].

2. Preliminaries

Before we prove the result of this paper, we recall some notions from [2]. Let $\mathcal{S}_0^d$ denote the space of symmetric trace-free $d \times d$-matrices. Then the *generalised energy* $e : \mathbb{R}^d \times \mathcal{S}_0^d \to \mathbb{R}$ is defined by

$$e(v,u) = \frac{d}{2}\lambda_{\max}(v \otimes v - u),$$

where $\lambda_{\max}$ denotes the largest eigenvalue. e is known to be non-negative and convex, and $\frac{1}{2}|v|^2 \leqslant e(v,u)$ for all v and u with equality if and only if $u = v \otimes v - \frac{|v|^2}{d} I_d$ (I_d being the $d \times d$ unit matrix). The following is shown in [2]:

Theorem 2. *Let $\bar{e} \in C(\mathbb{R}^d \times (0,\infty)) \cap C([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R}^d))$ be Q-periodic in the space variable and such that $\sup_{0 \leqslant t < \infty} \int_Q \bar{e}(x,t)\,dx < \infty$, and let $(\bar{v}, \bar{u}, \bar{q})$ be a smooth, Q-periodic (in space) solution of*

$$\partial_t \bar{v} + \operatorname{div} \bar{u} + \nabla \bar{q} = 0,$$
$$\operatorname{div} \bar{v} = 0 \tag{1}$$

in $\mathbb{R}^d \times (0,\infty)$ such that

$$\bar{v} \in C\big([0,\infty); H_w(Q)\big),$$
$$\bar{u}(x,t) \in \mathcal{S}_0^d$$

for every $(x,t) \in Q \times (0,\infty)$, and

$$e\big(\bar{v}(x,t), \bar{u}(x,t)\big) < \bar{e}(x,t)$$

for every $(x,t) \in Q \times (0,\infty)$.

Then there exist infinitely many weak solutions $v \in C([0,\infty); H_w(Q))$ of the Euler equations with $v(x,0) = \bar{v}(x,0)$ for a.e. $x \in Q$ and

$$\frac{1}{2}\left|v(x,t)\right|^2 = \bar{e}(x,t)$$

for every $t \in (0,\infty)$ and a.e. $x \in Q$.

Remark 3. In fact the way we stated Theorem 2 is slightly different from the original formulation in [2] (e.g. we use periodic boundary conditions). However it is easy to convince oneself that the proof in [2] applies also to the present situation with only minor modifications.

3. Proof of Theorem 1

By Theorem 2, it suffices to find suitable $(\bar{v}, \bar{u}, \bar{q})$ and $\bar{e}$.

Let us define $\bar{v}$ and $\bar{u}$ by their Fourier transforms as follows:

$$\hat{\bar{v}}(k,t) = e^{-|k|t}\hat{v}_0(k), \tag{2}$$

$$\hat{\bar{u}}_{ij}(k,t) = -i\left(\frac{k_j}{|k|}\hat{\bar{v}}_i(k,t) + \frac{k_i}{|k|}\hat{\bar{v}}_j(k,t)\right) \tag{3}$$

for every $k \neq 0$, and $\hat{\bar{u}}(0,t) = 0$. Note that $\bar{u}_{ij}$ thus defined equals $-\mathcal{R}_j\bar{v}_i - \mathcal{R}_i\bar{v}_j$, where $\mathcal{R}$ denotes the Riesz transform. Clearly, for $t > 0$, $\bar{v}$ and $\bar{u}$ are smooth. Moreover, $\bar{u}$ is symmetric and trace-free. Indeed, the latter can be seen by observing

$$\sum_{i=1}^{d}\left(\frac{k_i}{|k|}\hat{\bar{v}}_i(k,t) + \frac{k_i}{|k|}\hat{\bar{v}}_i(k,t)\right) = \frac{2}{|k|}e^{-|k|t}k\cdot\hat{v}_0(k) = 0$$

for all $k \neq 0$ (for $k = 0$ this is obvious).

Next, we can write Eqs. (1) in Fourier space as

$$\partial_t\hat{\bar{v}}_i + i\sum_{j=1}^{d}k_j\hat{\bar{u}}_{ij} + ik_i\hat{\bar{q}} = 0,$$

$$k\cdot\hat{\bar{v}} = 0 \tag{4}$$

for $k \in \mathbb{Z}^d$, $i = 1,\dots,d$. It is easy to check that $(\hat{\bar{v}}, \hat{\bar{u}}, 0)$ as defined by (2) and (3) solves (4) and hence $(\bar{v}, \bar{u}, 0)$ satisfies (1).

Concerning the energy, we have the pointwise estimate $e(\bar{v},\bar{u}) \leqslant C(|\bar{v}|^2 + |\bar{u}|)$, and because of

$$\int_Q \left|\bar{v}\right|^2 dx = \sum_{k\in\mathbb{Z}^d}|\hat{\bar{v}}|^2 = \sum_{k\in\mathbb{Z}^d}e^{-2|k|t}|\hat{v}_0|^2 \leqslant \|v_0\|^2_{L^2(Q)}$$

and, similarly,

$$\int_Q |u|\,dx \leqslant C\int_Q |u|^2\,dx \leqslant C\|v_0\|^2_{L^2(Q)},$$

we conclude that $\sup_{t>0}\|e(\bar{v}(x,t),\bar{u}(x,t))\|_{L^1(Q)} < \infty$. Moreover, from the same calculation and the dominated convergence theorem we deduce

$$\left\|e\big(\bar{v}(x,t),\bar{u}(x,t)\big)\right\|_{L^1(Q)} \to 0$$

as $t \to \infty$ as well as

$$\bar{v}(t) \to v_0$$

strongly in $L^2(Q)$ and

$$\bar{u}(t) \to u_0 := -\big(\mathcal{R}_j(v_0)_i + \mathcal{R}_i(v_0)_j\big)_{ij}$$

strongly in $L^1(Q)$. We claim that then

$$e(\bar{v}, \bar{u}) \in C\big([0, \infty); L^1(Q)\big).$$

The only issue is continuity at $t = 0$. First, one can easily check that the map

$$(v, u) \mapsto e\left(\frac{v}{\sqrt{|v|}}, u\right)$$

is Lipschitz continuous with Lipschitz constant, say, L; thus, using the inequality $||a|a - |b|b| \leqslant (|a| + |b|)|a - b|$, we have

$$\int_Q \big|e(\bar{v}, \bar{u}) - e(v_0, u_0)\big| \leqslant L \int_Q \big(\big||\bar{v}|\bar{v} - |v_0|v_0\big| + |\bar{u} - u_0|\big)\, dx$$
$$\leqslant 2L \sup_{t \geqslant 0} \big\|\bar{v}(t)\big\|_{L^2} \big\|\bar{v}(t) - v_0\big\|_{L^2} + L\|\bar{u} - u_0\|_{L^1} \to 0$$

as $t \to 0$. This proves the claim.

Therefore, $\bar{e}$ defined by

$$\bar{e}(x, t) := e\big(\bar{v}(x, t), \bar{u}(x, t)\big) + \min\left\{t, \frac{1}{t}\right\}$$

satisfies the requirements of Theorem 2 and, in addition, $\int_Q \bar{e}\, dx \to 0$ as $t \to \infty$. Theorem 2 then yields the desired weak solutions of Euler. □

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